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A linear-time algorithm for finding Hamiltonian (s, t)-paths in even-sized rectangular grid graphs with a rectangular hole

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ABSTRACT

The Hamiltonian path problem for general grid graphs is NP-complete. In this paper, we give the necessary conditions for the existence of a Hamiltonian path between two given vertices in a rectangular grid graph with a rectangular hole; where the size of graph is even. In addition, we show that the Hamiltonian path in these graphs can be computed in linear-time.

1. Introduction

The Hamiltonian path problem is NP-complete for general graphs and it is also NP-complete for some special classes of graphs such as general grid graphs [15], planar graphs [7], circle graphs [3], and directed path graphs [6] (see [4,9,25] for more results on this topic). Only a few polynomial-time algorithms are known for the Hamiltonian path problem for special classes of graphs. The focus of this paper is on the results about Hamiltonian paths on grid graphs. The two-dimensional integer grid G^∞ is an infinite undirected graph in which the vertices are all points of the plane with integer coordinates and two vertices are connected by an edge if and only if the Euclidean distance between them is equal to one. A grid graph G_g is a finite vertex-induced subgraph of the two-dimensional integer grid G^∞.

The Hamiltonian path in grid graphs have many applications in various fields such as robotic motion planning [8] and pathogen biology [26]. In the offline exploration problem [13], which is a kind of motion planning, a mobile robot with limited sensor should visit every cell in a known cellular room without obstacles in order to explore it and return to start point such that the number of multiple cell visits is small. In this problem, let the vertices correspond to the center of each cell and edges connect adjacent cells, then we have a grid graph. Finding a Hamiltonian cycle in the grid graph corresponds to visiting each cell exactly once (i.e., a cycle containing all the vertices of the grid graph).

Studies dealing with the dynamics of pathogen transmission can be accelerated through the modeling of animals movements or in other words, traveling of animals in their routes. In a recent study, Srinivasa Rao et al. [26] tried to investigate

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the movement of domestic chicken in association with the transmission of its protozoan parasite; Eimeria, considering physical, rather than airborne contact. They suggested a configuration for maximum possible distance of walking in straight and non-overlapping paths on square grid graphs. Their results led to the better understanding of non-airborne pathogen transmission. The modeling of the repetitive overlapping walks of domestic chicken can be supported by combined individual non-overlapping walks within a grid delineated area. This in turn can provide a framework to model fecal deposition and the following distribution of pathogen through fecal/host contact. A maximum possible walk between two cells on square grid graphs can be considered as a Hamiltonian path between these two cells.

As other applications we can mention the following. In the problem of embedding a graph in a given grid [5], the first step is to recognize if there are enough rooms in the host grid for the guest graph. If the guest graph is a path, then the problem makes relation to the well-known longest path and Hamiltonian path problems. If we would like to see if a given solid grid graph has a Hamiltonian path we reach to the problem of finding a Hamiltonian path between two given vertices. In the picturesque maze generation problem [11], we are given a rectangular black-and-white raster image and want to randomly generate a maze in which the solution path fills up the black pixels. The solution path is a Hamiltonian path of a subgraph induced by the vertices that correspond to the black cells.

A solid grid graph is a grid graph without holes. A rectangular grid graph $R(m,n)$ is a grid graph with horizontal size m and vertical size n. A rectangular grid graph with a rectangular hole is a rectangular grid graph $R(m,n)$ such that a rectangular grid subgraph $R(k,l)$ is removed from it. Lenhart and Umans [22] have presented a polynomial-time algorithm for finding Hamiltonian cycles in quad–quad graphs, a non-trivial superclass of solid grid graphs. They are raised the question whether a polynomial-time algorithm exists for solid grid graphs with some holes of restricted form. Itai et al. [15] show that the Hamiltonian path problem for general grid graphs, with or without specified endpoints, is NP-complete. They proved that the problem can be computed in linear-time for rectangular grid graphs. Later, Chen et al. [2] improved the algorithm of [15] and presented a parallel algorithm for the problem in mesh architecture.

Recently, Islam et al. [14] showed that the Hamiltonian cycle problem in hexagonal grid graphs is NP-complete. Gordon et al. [10] proved that all connected, locally connected triangular grid graphs (with one exception) are Hamiltonian and also showed that the Hamiltonian cycle problem for triangular grid graphs is NP-complete. Salman [27] introduced a family of grid graphs, i.e. alphabet grid graphs, and determined classes of alphabet grid graphs that contain Hamiltonian cycles. Some other results about grid graphs are investigated in [1,12,16–18,23,24,28,29].

To our knowledge, the question of whether a polynomial-time algorithm exists for finding a Hamiltonian path in restricted grid graphs with some holes has remained open. In this paper, we first obtain necessary conditions for the existence of a Hamiltonian path between two given vertices in even-sized rectangular grid graphs $R(m,n)$ with a rectangular hole $R(k,l)$ while $R(m,n)$ has no border side in common with $R(k,l)$. Then we give a linear-time algorithm for finding a Hamiltonian path in these graphs, this solves a special case of the open problem posed by Lenhart and Umans [22]. In order to shorten the paper, the case of odd-sized graphs is considered in a separate paper [20].

The paper is organized as follows. In Section 2, some necessary definitions and previous results are given. Necessary conditions for the existence of a Hamiltonian path are given in Section 3. In Section 4, we present a linear-time algorithm to solve the Hamiltonian path problem. The conclusion is given in Section 5.

2. Preliminary definitions and previous results

In this section, we give a few definitions and introduce the corresponding notations. We then gather some previously established results on the Hamiltonian path problem in grid graphs which have been presented in [2,15,19,21].

The *two-dimensional integer grid* G^∞ is an undirected graph in which vertices are all points of the plane with integer coordinates and two vertices are connected by an edge if and only if the Euclidean distance between them is equal to 1. For a vertex v of this graph, let v_x and v_y denote x and y coordinates of its corresponding point, respectively (sometimes we use (v_x, v_y) instead of v). We color the vertices of the two-dimensional integer grid as black and white. A vertex v is colored *white* if $v_x + v_y$ is even, otherwise it is colored *black*.

A *grid graph* G_g is a finite vertex-induced subgraph of the two-dimensional integer grid G^∞. In a grid graph G_g, each vertex has degree at most four. It is obvious that there is no edge between any two vertices of the same color. Thus, G_g is a bipartite graph. Notice that any cycle or path in a bipartite graph alternates between black and white vertices. Suppose that $G = (V(G), E(G))$ is a graph with vertex set $V(G)$ and edge set $E(G)$. Let $v \in V(G)$, then the number of edges incident to v in G is called the degree of v in G and is denoted by $degree(v)$.

A *rectangular grid graph*, denoted by $R(m,n)$ (or R for short), is a grid graph whose vertex set is $V(R) = \{v \mid 1 \le v_x \le m,\ 1 \le v_y \le n\}$. The graph $R(5,4)$ is illustrated in Fig. 1(a). The size of $R(m,n)$ is defined to be $m \times n$. $R(m,n)$ is called *odd-sized* if $m \times n$ is odd, otherwise it is called *even-sized*. $R(m,n)$ is called a k-rectangle, where $k = m$ or n.

A *rectangular grid graph with a rectangular hole* is a rectangular grid graph $R(m,n)$ such that a rectangular grid subgraph $R(k,l)$ is removed from it, where $k,l \ge 1$ and $m,n > 1$. The size of this graph is defined by $m \times n - k \times l$, and is called *even-sized* if $m \times n - k \times l$ is even, otherwise it is called *odd-sized*. If $R(m,n)$ has no border side in common with $R(k,l)$ (see Fig. 1(b)), then this graph is called O-shaped grid graph denoted by $O(m,n,k,l)$. If $R(m,n)$ has

– exactly two border sides in common with $R(k,l)$ (see Fig. 1(c)), then we have an L-shaped grid graph denoted by $L(m,n,k,l)$.

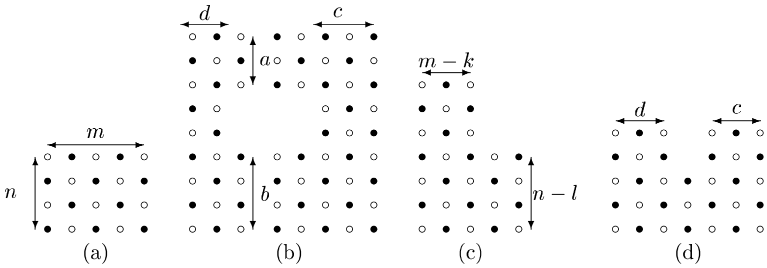

Fig. 1. (a) $R(5,4)$, (b) $O(8,9,3,2)$, (c) $L(5,7,2,3)$, and (d) $C(7,5,1,2)$.

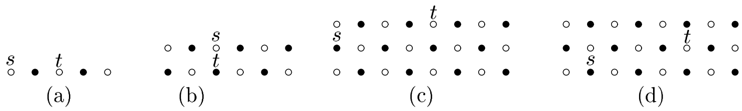

Fig. 2. The rectangular grid graphs in which there are no Hamiltonian path between s and t.

– exactly one border side in common with $R(k,l)$ (see Fig. 1(d)), then we have a C-shaped grid graph denoted by $C(m,n,k,l)$.

Throughout this paper in the figures, $(1,1)$ is the coordinates of the vertex in the upper left corner, except we explicitly change this assumption. Consider Fig. 1(b) and 1(d). Assume (x_1, y_1) is the coordinates of the vertex at the upper left corner of $R(k,l)$. Then $d = x_1 - 1$, $c = m - (d+k)$, $a = y_1 - 1$, and $b = n - (a+l)$.

We will refer to a grid graph $G(m,n)$ with two specified distinct vertices s and t as $(G(m,n),s,t)$. We say that $(G(m,n),s,t)$ is Hamiltonian if there is a Hamiltonian path between s and t in G. In the following by Hamiltonian (s,t)-path we mean a Hamiltonian path between s and t. Without loss of generality, we assume that $s_x \leq t_x$.

Definition 2.1. Suppose that $G = (V_1 \cup V_2, E)$ is a bipartite graph such that $|V_1| \geq |V_2|$ and the vertices of G are colored by two colors, black and white. All the vertices of V_1 are colored by the same color, the majority color, and the vertices of V_2 by the minority color. The Hamiltonian path problem (G,s,t) is *color-compatible* if

1. s and t are different-colored and G is even-sized ($|V_1| = |V_2|$), or
2. s and t have the majority color and G is odd-sized ($|V_1| = |V_2| + 1$).

Definition 2.2. Let G be a connected graph and V_1 be a subset of the vertex set $V(G)$. V_1 is a *vertex cut* of G if $G - V_1$ is disconnected. A vertex v of G is a *cut vertex* of G if $\{v\}$ is a vertex cut of G. For an example, in Fig. 2(b) $\{s,t\}$ is a vertex cut and in Fig. 2(a) t is a cut vertex.

An even-sized grid graph contains the same number of black and white vertices. Hence, the two end-vertices of any Hamiltonian path in the graph must have different colors. Similarly, in an odd-sized grid graph the number of vertices with the majority color is one more than the number of vertices with the minority color. Thus, the two end-vertices of any Hamiltonian path in such a graph must be the majority color. Hence, the color-compatibility of s and t is a necessary condition for a grid graph to be Hamiltonian.

Furthermore, Itai et al. [15] showed that if one of the following conditions holds, then $(R(m,n),s,t)$ is not Hamiltonian:

(F1) s or t is a cut vertex or $\{s,t\}$ is a vertex cut (Fig. 2(a) and 2(b)). Notice that, here, s or t is a cut vertex if $R(m,n)$ is a 1-rectangle and either s or t is not a corner vertex, and $\{s,t\}$ is a vertex cut if $R(m,n)$ is a 2-rectangle and $[(2 \leq s_x = t_x \leq m-1$ and $n=2)$ or $(2 \leq s_y = t_y \leq n-1$ and $m=2)]$.

(F2) All the cases that are isomorphic to the following case:
  1. m is even, $n=3$, and
  2. s is black, t is white, and
  3. $s_y = 2$ and $s_x < t_x$ (Fig. 2(c)) or $s_y \neq 2$ and $s_x < t_x - 1$ (Fig. 2(d)).

Definition 2.3. Let $G(m,n,k,l)$ be an L-shaped or C-shaped grid graph. A *separation* of a G is a partition of $G(m,n,k,l)$ into two disjoint grid subgraphs G_1 and G_2, i.e., $V(G) = V(G_1) \cup V(G_2)$, and $V(G_1) \cap V(G_2) = \emptyset$. G_1 and G_2 may be rectangular or L-shaped grid graphs. The three types of separations, vertical, horizontal and L-shaped separations are shown in Fig. 5.

In [19], authors showed that in addition to condition (F1) (as shown in Fig. 3(a) and 3(b)), whenever one of the following conditions is satisfied then $(L(m,n,k,l),s,t)$ has no Hamiltonian (s,t)-path.

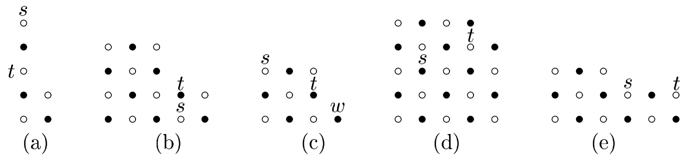

Fig. 3. Some L-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

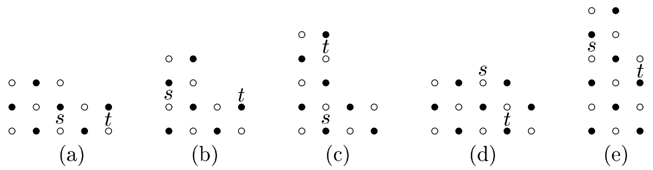

Fig. 4. Some L-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

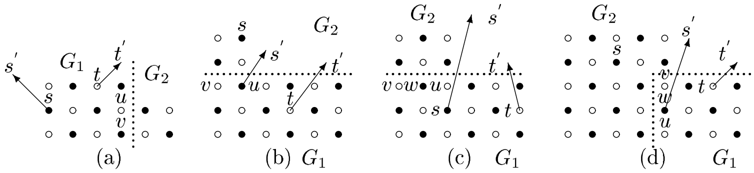

Fig. 5. Some L-shaped grid graphs in which there are no Hamiltonian (s,t)-path, where dotted lines indicate the separations; (a) a vertical separation, (b) and (c) a horizontal separation, and (d) an L-shaped separation.

(F3) Let G be a grid graph. $\exists w \in V(G)$, $degree(w) = 1$, $t \neq w$, and $s \neq w$ (Fig. 3(c)).
(F4) $L(m,n,1,1)$ is even-sized, $m-1 = \text{even} > 2$, $n-1 = \text{even} > 2$, $s = (m-1,2)$, and $t \neq (m-1,1)$ or $t \neq (m,2)$ (here the role of s and t can be swapped; i.e., $t = (m-1,2)$ and $s \neq (m-1,1)$) (Fig. 3(d)).
(F5) $L(m,n,k,l)$ is odd-sized, $n-l = 2$, $m-k = \text{odd} \geq 3$, and
 (i) $s_x, t_x > m-k$ (Fig. 3(e)); or
 (ii) $s = (m-k,n)$ and $t_x > m-k$ (Fig. 4(a)).
(F6) $L(m,n,k,l)$ is even-sized, $n-l = 2$, $m-k = 2$, and
 (i) $s = (1,n-l)$ and $t_x > 2$ (Fig. 4(b)); or
 (ii) $s = (2,n)$ and $t_y < l$ (here the role of s and t can be swapped; i.e., $t = (2,n)$ and $s_y \leq l$) (Fig. 4(c)).
(F7) $L(m,n,k,l)$ is even-sized and
 (i) $n = 3$, $l = 1$, $m-k = \text{even} > 2$, $s = (m-k-1,1)$, and $t = (m-k,3)$ (Fig. 4(d)); or
 (ii) $m = 3$, $k = 1$, and $n-l = \text{even} > 2$, $s = (1,l+1)$, and $t = (m,l+2)$ (Fig. 4(e)).
(F8) $L(m,n,k,l)$ is even-sized and $[(m-k = 2$ and $n-l > 2)$ or $(n-l = 2$ and $m-k > 2)]$. Let $\{G_1, G_2\}$ be a vertical or horizontal separation of $L(m,n,k,l)$ such that G_1 is a 3-rectangle grid graph, G_2 is a 2-rectangle grid graph, and exactly two vertices u and v are in G_1 that are connected to G_2. Let $s' = s$ and $t' = t$, if s' (or t') $\notin G_1$ then $s' = u$ (or $t' = u$). And (G_1, s', t') satisfies condition (F2) (Fig. 5(a) and 5(b)).
(F9) $L(m,n,k,l)$ is even-sized and $[(m-k = 3$ and $n-l \geq 3)$ or $(m-k > 3$ and $n-l = 3)]$. Let $\{G_1, G_2\}$ be a vertical, horizontal or L-shaped separation of $L(m,n,k,l)$ such that G_1 and G_2 are even-sized, G_1 is a 3-rectangle grid graph, and G_2 is
 (1) a rectangular grid graph (see Fig. 5(c)), or
 (2) an L-shaped grid graph, where $m \times n = \text{even} \times \text{odd}$, $k \times l = \text{odd} \times \text{even}$, $n-l = 3$, and $m-k \geq 5$. Here, $V(G_1) = \{m-k \leq x \leq m \textit{ and } l+1 \leq y \leq n\}$ and $G_2 = L(m,n,k,l) \backslash G_1$ (see Fig. 5(d)).
 Let exactly three vertices v, w and u are in G_1 that are connected to G_2. Let $s' = s$ and $t' = t$, if s' (or t') $\notin G_1$ then $s' = w$ (or $t' = w$). And (G_1, s', t') satisfies condition (F2).

Definition 2.4. [15,19] A rectangular or L-shaped Hamiltonian path problem $(R(m,n),s,t)$ or $(L(m,n,k,l),s,t)$ is *acceptable* if it is color compatible and

- $(R(m,n),s,t)$ does not satisfy conditions (F1) and (F2); or
- $(L(m,n,k,l),s,t)$ does not satisfy any of conditions (F1) and (F3)–(F9).

In [21] it is shown that in addition to conditions (F1) and (F3) (as depicted in Fig. 6(a)–6(c)) if one of the following conditions holds then $(C(m,n,k,l),s,t)$ has no Hamiltonian (s,t)-path.

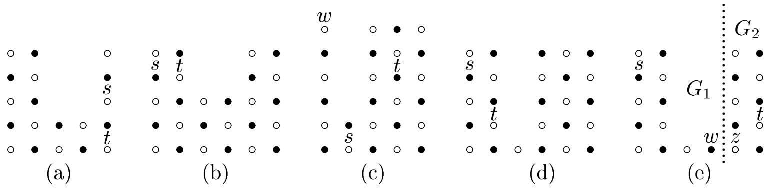

Fig. 6. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

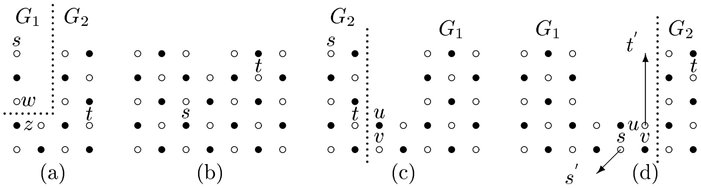

Fig. 7. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

(F10) $n-l=1$, $c,d>1$, and
(i) $s_x,t_x\le d$ or $s_x,t_x>d+k$ (Fig. 6(d)); or
(ii) Let $C(m,n,k,l)$ be even-sized, and let $\{G_1,G_2\}$ be a vertical separation of $C(m,n,k,l)$ such that $G_1=L(m',n,k,l)$, where $m'=d+k$, $G_2=R(m-m',n)$ (as shown Fig. 6(e)), and exactly a vertex w is in G_1 that is connected to G_2. Let $z\in G_2$ such that w and z are adjacent. And $s\in G_1$, $t\in G_2$, and (G_1,s,w) or (G_2,z,t) is not acceptable (Fig. 6(e)).

(F11) $n-l>1$ and $[(d=1,\ c>1$, and $s=(1,1))$ or $(d>1,\ c=1$, and $t=(m,1))]$. Let $\{G_1,G_2\}$ be L-shaped separation of $C(m,n,k,l)$ such that $G_1=R(m',l)$, $G_2=L(m,n,k',l)$, $m'=d$ if $d=1$; otherwise $m'=c$, and $k'=k+m'$. Let exactly a vertex w is in G_1 that is connected to G_2 and let $z\in G_2$ such that w and z are adjacent. And one of the following cases occurs:
(i) $d=1$, $t\in G_2$, and (G_2,z,t) is not acceptable (Fig. 7(a)); or
(ii) $c=1$, $s\in G_2$, and (G_2,s,z) is not acceptable.

(F12) $R(m,n)$ is odd $\times$ odd with white majority color and $R(k,l)$ is odd$\times$ odd with black majority color (Fig. 7(b)).

(F13) $n=$ odd, $n-l=2$, and $d,c>1$. Let $\{G_1,G_2\}$ be a vertical separation of $C(m,n,k,l)$ such that $G_1=L(m',n,k,l)$, $G_2=R(m-m',n)$, and $m'=d+k$ (or $G_1=L(m-m',n,k,l)$, $G_2=R(m',n)$, and $m'=d$). Let exactly two vertices u and v are in G_1 that are connected to G_2. And one of the following cases occurs:
(a) $C(m,n,k,l)$ is odd-sized and
(a_1) G_1 is odd-sized, G_2 is even-sized, and
(a_{11}) $s,t\in G_2$ (see Fig. 7(c)); or
(a_{12}) $s\in G_1$, $t\in G_2$, $s'=s$, $t'=u$ (or $t\in G_1$, $s\in G_2$, $t'=t$, $s'=u$), and (G_1,s',t') satisfies condition (F1) (i.e., $\{s',t'\}$ is a vertex cut) (see Fig. 7(d)).
(a_2) m is even, G_1 is even-sized, G_2 is odd-sized, $s\in G_1$, $t\in G_2$, $s'=s$, $t'=u$ (or $t\in G_1$, $s\in G_2$, $t'=t$, $s'=u$), and (G_1,s',t') satisfies condition (F6) or (F8) (see Fig. 8(a)).
(b) $C(m,n,k,l)$ is even-sized, $d=\text{odd}>1$, $c=\text{odd}>1$, and
(b_1) $s,t\in G_1$ (see Fig. 8(b)); or
(b_2) $s_x\le d$, $t_x>d+k$, $s'=s$, $t'=u$ (or $s'=u$, $t'=t$), and (G_1,s',t') is not acceptable (see Fig. 8(c)).

(F14) $n=$ odd, $n-l>2$, $[(d=\text{odd}>1$ and $c=2)$ or $(d=2$ and $c=\text{odd}>1)]$, and $[(C(m,n,k,l)$ is odd-sized) or ($m=$ even and $k\times l=\text{odd}\times\text{even})]$. Let $\{G_1,G_2\}$ be an L-shaped separation of $C(m,n,k,l)$ such that $G_1=L(m,n,k',l)$, where $k'=m-d$ or $k'=m-c$, $G_2=R(2,l)$ (see Fig. 8(d) and 9(a)), and exactly two vertices u and v be in G_1 that are connected to G_2. And one of the following cases occurs:
(a) $C(m,n,k,l)$ is odd-sized, $[(m=\text{even})$ or $(m=\text{odd}$ and $k=\text{even})]$, and
(a_1) $s,t\in G_2$ (see Fig. 8(d)); or
(a_2) $s=u$ and $t\in G_2$ (or $s\in G_2$ and $t=u$) (see Fig. 9(a)); or
(a_3) $m=$ odd, $l=$ odd, $s\in G_1$, $t\in G_2$, $s'=s$, $t'=u$ (or $t\in G_1$, $s\in G_2, t'=t$, $s'=u$) and (G_1,s',t') satisfies condition (F1) (that is, $\{s',t'\}$ is a vertex cut) (see Fig. 9(b)).
(b) $C(m,n,k,l)$ is even-sized, $s'=s$, $t'=t$, if s' (or t') $\notin G_1$ then $s'=u$ (or $t'=u$), and (G_1,s',t') satisfies condition (F9) (see Fig. 9(c)).

(F15) $C(m,n,k,l)$ is odd-sized, $m=$ even, $n=$ odd, $n-l=4$, and
(i) $d=\text{odd}>1$, $[(l=1$ and $c=\text{even}\ge 4)$ or $(s_y,t_y>l$ and $c=2)]$, and $s_x,t_x>d+k+1$ (Fig. 9(d)); or
(ii) $c=\text{odd}>1$, $[(l=1$ and $d=\text{even}\ge 4)$ or $(s_y,t_y>l$ and $d=2)]$, and $s_x,t_x<d$.

(F16) $C(m,n,k,l)$ is even-sized, $m\times n=\text{even}\times\text{odd}$, $c=\text{odd}>1$, $d=\text{odd}>1$, and $n-l=\text{odd}>1$. Assume that $\{G_1,G_2\}$ is a vertical separation of $C(m,n,k,l)$ such that $G_1=L(m',n,k',l)$, where $m'=d+1$ and $k'=m'-d$, $G_2=L(m-$

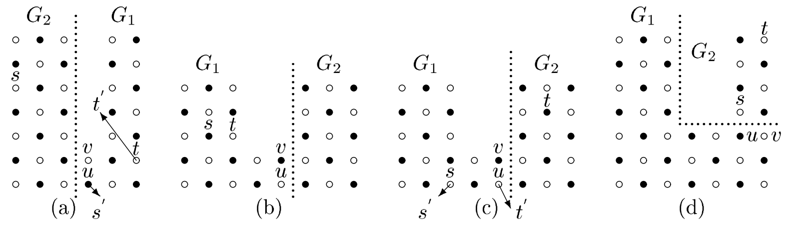

Fig. 8. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

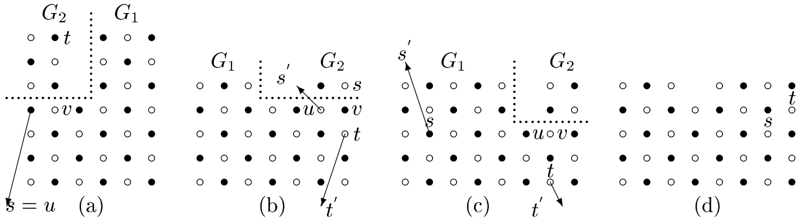

Fig. 9. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

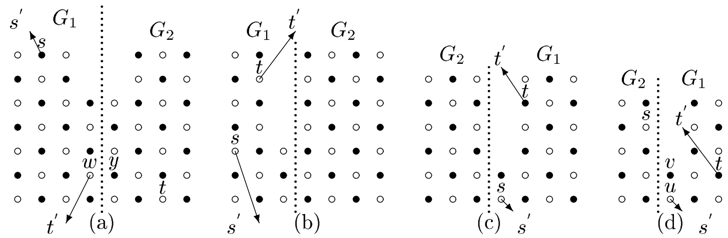

Fig. 10. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

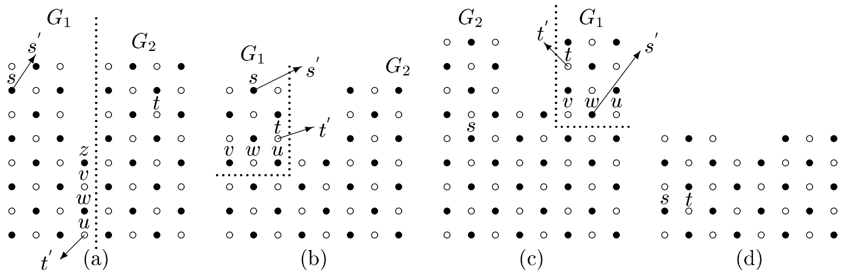

Fig. 11. Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

$m', n, k'', l)$, where $k'' = k - k'$ (see Fig. 10(a)), and at least three vertices u, w and v are in G_1 that are connected to G_2. Let $y \in G_2$ such that w and y are adjacent. And ($s' = s$ and $t' = w$, if $t \in G_1$ let $t' = t$) or ($s' = y$ and $t' = t$, if $s \in G_2$ let $s' = s$), and (G_1, s', t') or (G_2, s', t') satisfies condition (F9).

(F17) $n - l \geq 2$, $d, c > 1$ and $[(C(m,n,k,l)$ is odd-sized and $[(n = \text{even})$ or $(n = \text{odd}$ and $[(m = \text{even})$ or $(m = \text{odd}$ and $[k \times l = \text{odd} \times \text{even}$ or even $\times$ odd$])])])$ or $(C(m,n,k,l)$ is even-sized and $[(m \times n = \text{odd} \times \text{odd})$, $(n = \text{even})$, or $(m \times n = \text{even} \times \text{odd}$ and $[(c$ and d are even), $(c = \text{even} \geq 4$ and $d = \text{odd})$, or $(c = \text{odd}$ and $d = \text{even} \geq 4)])])]$. Let $\{G_1, G_2\}$ be a vertical separation of $C(m,n,k,l)$ such that $G_1 = L(m', n, k, l)$, $G_2 = R(m - m', n)$, $m' = d + k$ (or $G_1 = L(m - m', n, k, l)$, $G_2 = R(m', n)$, $m' = d$), G_2 is even-sized, and at least two vertices v and u are in G_1 which are connected to G_2. If $C(m,n,k,l)$ is even-sized, $k = 1$, and $n - l = \text{even} \geq 4$, then let $m' - k > 2$ (or $m - m' - k > 2$) in G_1. And $s' = s$, $t' = t$, if s' (or t') $\notin G_1$ then $s' = u$ (or $t' = u$), and (G_1, s', t') satisfies one of the conditions (F5), (F6), (F7), (F8), or (F9) (Fig. 10(b)–(d) and 11(a)).

(F18) $C(m,n,k,l)$ is even-sized, $n = \text{odd}$, $d = \text{odd} > 1$, $c = \text{odd} > 1$, $n - l = \text{even} \geq 4$, and one of the following cases occurs:

(a) Let $\{G_1, G_2\}$ be an L-shaped separation of $C(m,n,k,l)$ such that G_1 is an even-sized rectangular grid subgraph with $V(G_1) = \{1 \leq x \leq d$ (or $d + k + 1 \leq x \leq m)$ and $1 \leq y \leq l + 1\}$, G_2 is an even-sized solid grid subgraph (see Fig. 11(b) and 11(c)), and exactly three vertices v, w, and u are in G_1 that are connected to G_2. And $s' = s$, $t' = t$, if s' (or t') $\notin G_1$ then $s' = w$ (or $t' = w$), and (G_1, s', t') satisfies condition (F2); or

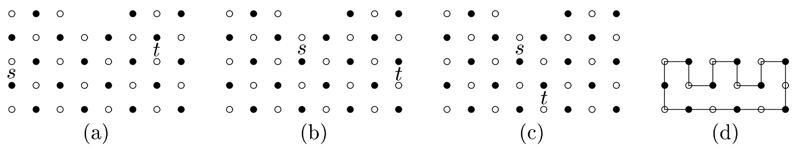

Fig. 12. (a)–(c) Some C-shaped grid graphs in which there are no Hamiltonian (s,t)-path, and (d) a Hamiltonian cycle in $R(6,3)$.

(b) $n-l=4$ and
- (b_1) $s_y, t_y > l+1$ and $[(d=3,\ s_x, t_x \le d,\ s=(1,n-1)$, and $t_x > s_x)$ or $(c=3,\ s_x, t_x > d+k,\ s_x < t_x$, and $t=(m,n-1))]$ (Fig. 11(d)); or
- (b_2) s is black and $[(s_x \le d$ and $t_x > d)$ or $(d+1 \le s_x \le d+k$ and $t_x > d+k)]$ (Fig. 12(a) and 12(b)); or
- (b_3) $d+1 \le s_x, t_x \le d+k$ and $[(t_x > s_x$ and s is black) or $(s_x = t_x,\ s_y$ (or t_y) $= l+2$, and t_y (or s_y) $= l+3)]$ (Fig. 12(c)).

Definition 2.5. [21] A C-shaped Hamiltonian path problem $(C(m,n,k,l),s,t)$ is *acceptable* if it is color compatible and $(C(m,n,k,l),s,t)$ does not satisfy any of conditions (F1), (F3), and (F10)–(F18).

Theorem 2.1. *[15,19,21] Let $G(m,n)$ be a rectangular, L-shaped or C-shaped grid graph. $G(m,n)$ has a Hamiltonian (s,t)-path if and only if $(G(m,n),s,t)$ is acceptable.*

Theorem 2.2. *[15,19,21] In an acceptable Hamiltonian path problem $(R(m,n),s,t)$, $(L(m,n,k,l),s,t)$, and $(C(m,n,k,l),s,t)$, a Hamiltonian (s,t)-path can be found in linear time.*

The following lemmas state some results about the Hamiltonicity of even-sized rectangular, L-shaped, and C-shaped grid graphs.

Lemma 2.3. *[2] $R(m,n)$ has a Hamiltonian cycle if and only if it is even-sized and $m,n > 1$.*

Fig. 12(d) shows a Hamiltonian cycle for an even-sized rectangular grid graph, found by Lemma 2.3. This shows that for an even-sized rectangular grid graph R, we can always find a Hamiltonian cycle, such that it contains all the boundary edges, except of exactly one side of R which contains an even number of vertices.

Lemma 2.4. *[19] $L(m,n,k,l)$ has a Hamiltonian cycle if and only if it is even-sized, $m-k>1$, and $n-l>1$.*

Lemma 2.5. *[21] $C(m,n,k,l)$ has a Hamiltonian cycle if and only if it is even-sized, the number of vertices white is equal to the number of vertices black, $d,c>1$, and $[(n=\text{even}$ and $n-l\ge 2)$ or $(n=\text{odd}$ and $[(n-l\ge 2$ and $d=\text{even}$ or $c=\text{even})$ or $(n-l>2,\ d=\text{odd}$, and $c=\text{odd})])]$.*

In this paper, we consider even-sized O-shaped grid graphs. We will only consider the following cases, the other cases are isomorphic to these cases.

1. $a=1$ and $[(d>1$ and $[(b=c=1)$ or $(b>1)])$ or $(d=1$ and $s_y, t_y \le a+l)]$; or
2. $[(d=1)$ or $(c=1$ and $a,b,d>1)]$ and $a+1 \le s_y, t_y \le a+l$; or
3. $a,b,c,d>1$.

Based on the position of the vertices s and t in $O(m,n,k,l)$, the three cases can be considered: $(s_x, t_x \le d)$, $(s_x, t_x > d)$, or $(s_x \le d$ and $t_x > d)$. In this paper, we only consider cases $(s_x, t_x \le d)$ and $(s_x \le d$ and $t_x > d)$. Note that the case $s_x, t_x > d$ is isomorphic to the case $(s_x, t_x \le d)$ or $(s_x \le d$ and $t_x > d)$.

3. The necessary conditions

The purpose of this section is to obtain the necessary conditions for the existence of a Hamiltonian (s,t)-path in an O-shaped grid graph $O(m,n,k,l)$.

Definition 3.1. A *separation* of an O-shaped grid graph $O(m,n,k,l)$ is a partition of $O(m,n,k,l)$ into two (or three) disjoint grid subgraphs G_1, G_2, and G_3, that is, $V(O(m,n,k,l)) = V(G_1) \cup V(G_2) \cup V(G_3)$, and $V(G_1) \cap V(G_2) \cap V(G_3) = \emptyset$. G_1, G_2, and G_3 may be rectangular, L-shaped, C-shaped, or O-shaped grid graphs. There are eight types of separations for $O(m,n,k,l)$: vertical, horizontal, E-shaped, F-shaped, L-shaped, C-shaped, O-shaped, or T-shaped separations (as shown in Fig. 13 and 14).

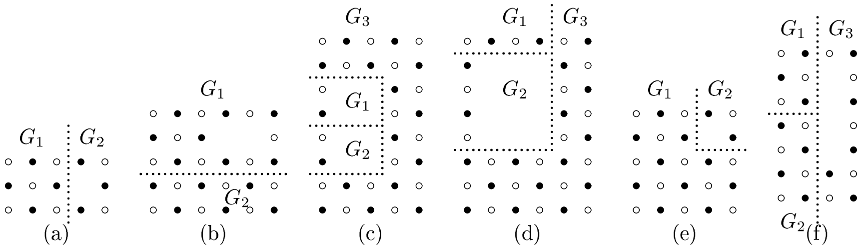

Fig. 13. (a) A vertical separation, (b) a horizontal separation, (c) an E-shaped separation, (d) an F-shaped separation, (e) an L-shaped separation type I, and (f) a L-shaped separation type II, where dotted lines indicate the separations.

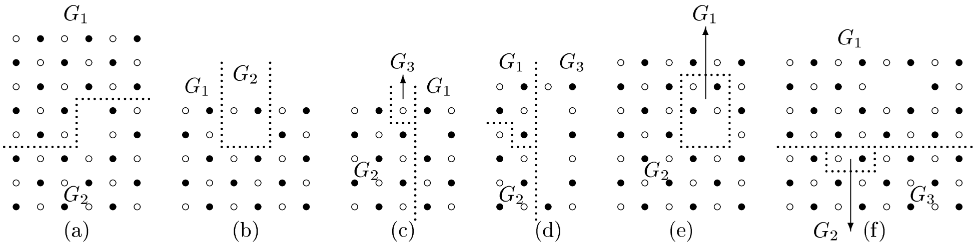

Fig. 14. (a) An L-shaped separation type III, (b) a C-shaped separation type I, (c) a C-shaped separation type II, (d) a C-shaped separation type III, (e) an O-shaped separation type I, and (f) an O-shaped separation type II, where dotted lines indicate the separations.

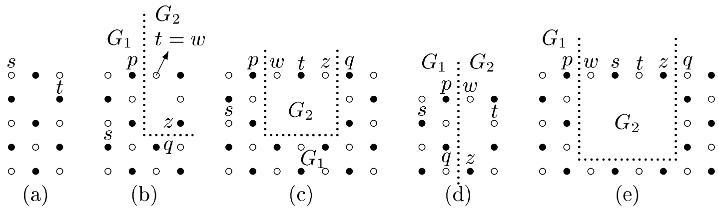

Fig. 15. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

Lemma 3.1. [19] *Let G be any grid graph. Let s and t be two given vertices of G such that (G,s,t) is color-compatible. If we can partition (G,s,t) into n subgraphs $G_1, G_2, \ldots, G_{n-1}, G_n$ such that $s,t \in G_n$ and in $V(G_1 \cup G_2 \cup \ldots \cup G_{n-1})$ the number of white and black vertices are equal, then (G_n,s,t) is color-compatible.*

Since colors of vertices of any Hamiltonian (s,t)-path alternate between black and white, it follows that the color-compatibility of s and t in $O(m,n,k,l)$ is necessary condition for $(O(m,n,k,l),s,t)$ to be Hamiltonian. Furthermore, in addition to condition (F1) (i.e., $\{s,t\}$ is a vertex cut; see Fig. 15(a)) if one of the following conditions holds, then $(O(m,n,k,l),s,t)$ has no Hamiltonian (s,t)-path.

(O1) Assume that $\{G_1,G_2\}$ is a vertical (L-shaped or C-shaped) separation of $O(m,n,k,l)$ such that G_2 is a path graph; i.e., G_2 is a subgraph of $O(m,n,k,l)$ which is a path and degree of its vertices in $O(m,n,k,l)$ is equal to 2, $|G_2|>1$, and $G_1 = O(m,n,k,l)\backslash G_2$. Let w and z be two end vertices of G_2 and let $p,q \in G_1$ such that w and p and q and z are adjacent. And one of the following cases occurs:
  - (a) $t=w$ (or z), $s \in G_1$, and $(G_1,s,q$ (or p)) is not acceptable (see Fig. 15(b)) (here the role of s and t can be swapped; i.e., $s=w$ (or z), $t \in G_1$, and $(G_1,t,q$ (or p)) is not acceptable); or
  - (b) t (or s) $\in G_2$, t (or s) $\neq w$, t (or s) $\neq z$, and s (or t) $\in G_1$ (see Fig. 15(c) and 15(d)).
  - (c) $s,t \in G_2$, s and t are adjacent, and (G_1,p,q) is not acceptable (see Fig. 15(e)).

(O2) $R(m,n)$ is odd $\times$ odd with white majority color and $R(k,l)$ is odd $\times$ odd with black majority color (Fig. 16(a)).

(O3) $a=b=1$ and one of the following cases holds:
  - (a) $c,d>1$, $s_x \leq d$, and $t_x > d+k$ (see Fig. 16(b)); or
  - (b) $n=$ even, $d=4$, s (or t) $=(2,1)$, and t (or s) $=(2,n)$ (see Fig. 16(c) and 16(d)); or
  - (c) $n=$ odd, $m=$ even, $d=$ even, and $s_x,t_x \leq d$ (see Fig. 17(a)); or
  - (d) $n=3$, $d=$ odd, $s_x,t_x \leq d$, s is black, and [($s_y=$ odd and $t_x > s_x+1$ (see Fig. 17(b))) or ($s_y=2$ and $t_x > s_x$ (see Fig. 17(c)))].

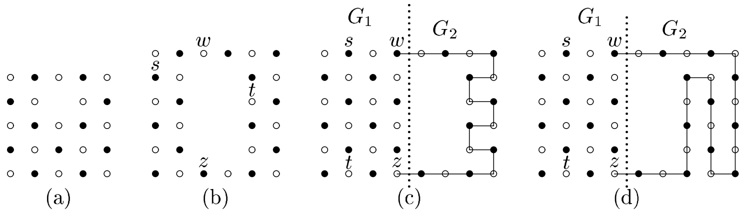

Fig. 16. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

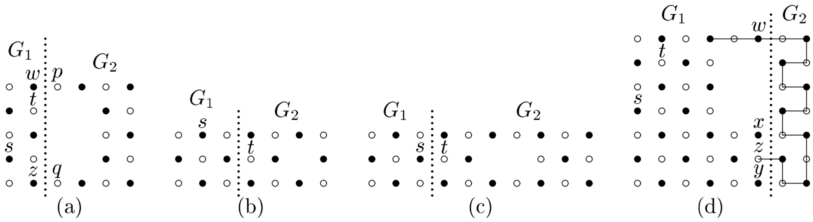

Fig. 17. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

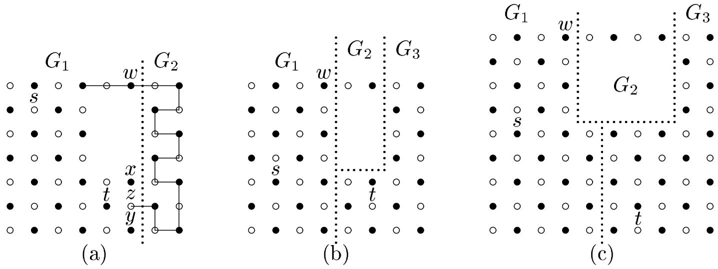

Fig. 18. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

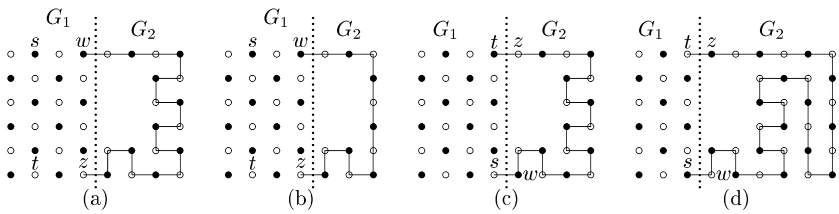

Fig. 19. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

(O4) $a=1$, $b=3$, $m=$ even, $n=$ odd, $d=$ even, and
  (a) $s_x, t_x \le d$ (see Fig. 17(d)); or
  (b) $s_x \le d$, s is white, and $d+1 \le t_x \le d+k-1$ (see Fig. 18(a)); or
  (c) $s_x \le d$, s is black, and $t_x > d$ (see Fig. 18(b)).
(O5) $n=$ odd, $m=$ even, $a=1$, $b=5$, $d=$ even, $s_x \le d$, s is black, and $t_x > d+1$ (see Fig. 18(c)).
(O6) $a=1$, $b=2$, and
  (a) $n=$ even, $d=4$, s (or t) $=(2,1)$, and t (or s) $=(2,n)$ (see Fig. 19(a) and 19(b)); or
  (b) $n=$ even, $d>2$, s (or t) $=(d,1)$, and t (or s) $=(d,n)$ (see Fig. 19(c) and 19(d)); or
  (c) $n=$ odd, $m=$ even, $d=$ even, and
    (c_1) $d>2$, $s=(d-1,n)$, and $t=(d,1,)$ (see Fig. 20(a)); or
    (c_2) t (or s) $=(d,n-1)$, s (or t) $\neq (d,n)$ and $[(d>2)$ or $(d=2$ and $s_y=t_y=n-1)]$ (see Fig. 20(b)).
  (d) $n=$ odd, $m=$ even, $d=2$, $k=1$, $c>1$, $s=(1,n-1)$, and $t=(d+1,1)$ (see Fig. 20(c)); or
  (e) $s_x \le d$, $t_x > d$, $w=(d,1)$, and [(s and w are same-colored and $d=$ even or $n=$ even (see Fig. 20(d))) or (s and w are different-colored, $n=$ odd, and $d=$ odd (see Fig. 21(a)))].
(O7) $a=1$, $d=2$, $[(n=$ even$)$ or $(n=$ odd and $[(b=$ even$)$, $(b=$ odd $>5)$, or $(b=5$ and $t_x \le d+1)])]$ and
  (a) $s_x, t_x \le d$, $s_y \neq t_y$, $a'_y \le a+l$, and a' is black, where $a'=s$ if $s_y<t_y$, otherwise $a'=t$ (see Fig. 21(b) and 21(c)); or
  (b) $b>2$, $s_x \le d$, $t_x > d$, $s_y \le a+l$, and s is black (see Fig. 21(d)); or

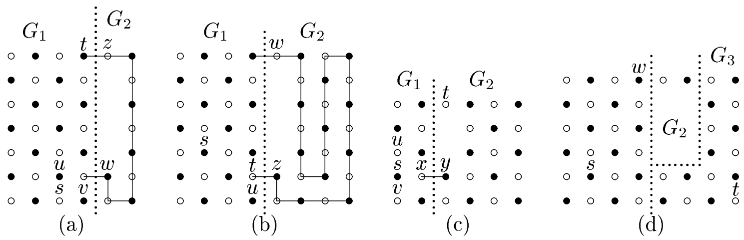

Fig. 20. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

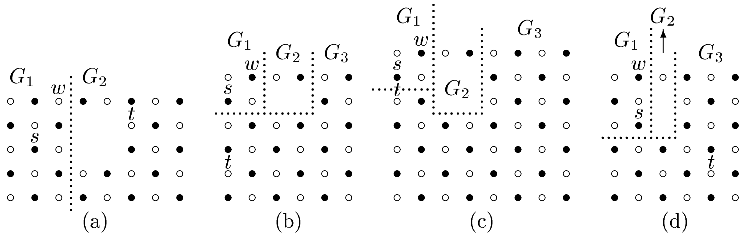

Fig. 21. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

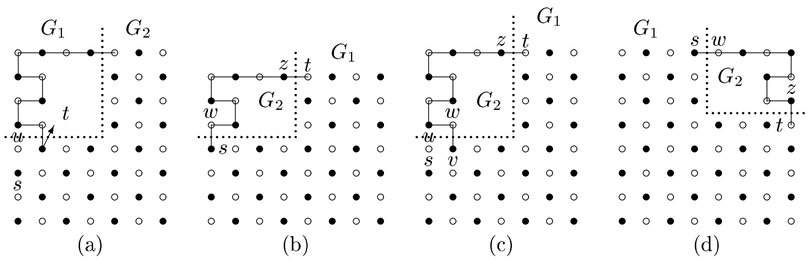

Fig. 22. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

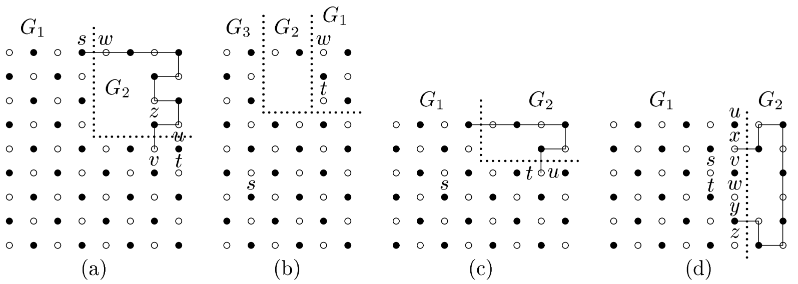

Fig. 23. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

(c) t (or s) $= (2, a+l+1)$, s (or t) $\neq (1, a+l+1)$, $[(s_x, t_x \leq d)$ or $(s_x \leq d$, $t_x > d$, and $b > 2)]$, and $[(n = \text{even}$ and $b = \text{even})$ or $(n = \text{odd}$ and $b = \text{odd})]$ (see Fig. 22(a)).
(d) $c > 1$, $b > 2$, $t = (d+k+1, 1)$, and $[(l = \text{even}$ and $s = (1, a+l+1)$ (see Fig. 22(b))) or $(l = \text{odd}$ and $s = (1, a+l+2)$ (see Fig. 22(c)))].

(O8) $a = 1$, $c = 2$, $d > 1$, $b > 2$, $s_x \leq d$, $t_x > d$, $[(n = \text{even})$ or $(n = \text{odd}$ and $[(b = \text{even})$ or $(b = \text{odd} > 5)])]$, and
(a) $d > 2$, $s = (d, 1)$, and $[(l = \text{even}$ and $t = (m, a+l+1)$ (see Fig. 22(d))) or $(l = \text{odd}$ and $t = (m, a+l+2)$ (see Fig. 23(a)))]; or
(b) $t_y \leq a+l$ and t and w are same-colored, where $w = (d+k+1, 1)$ (see Fig. 23(b)); or
(c) $t = (d+k+1, a+l+1)$ and $[(n = \text{even}$ and $b = \text{even})$ or $(n = \text{odd}$ and $b = \text{odd})]$ (see Fig. 23(c)).

(O9) $c = 1$, $d > 1$, $a+1 \leq s_y, t_y \leq a+l$, $a, b > 1$, and
(a) $a = b = 2$, $n = \text{even}$, s (or t) $= (d-1, a+1)$, t (or s) $= (d-1, a+l)$, and $[(l = 2$ and $[(m = \text{odd}$ and $b = \text{odd} > 5)$ or $(m = \text{even}$ and $d = \text{even} > 2$ (see Fig. 23(d)))]) or $(l > 2$, $m = \text{even}$, and $d = 4$ (see Fig. 24(a)))]; or

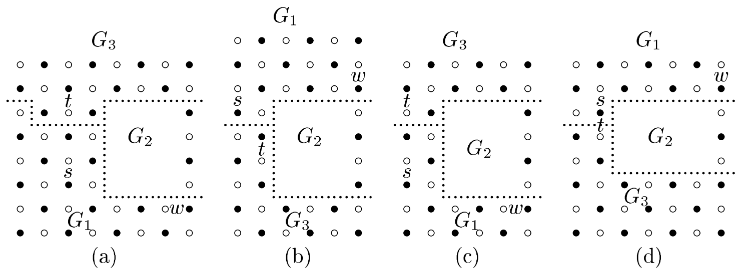

Fig. 24. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

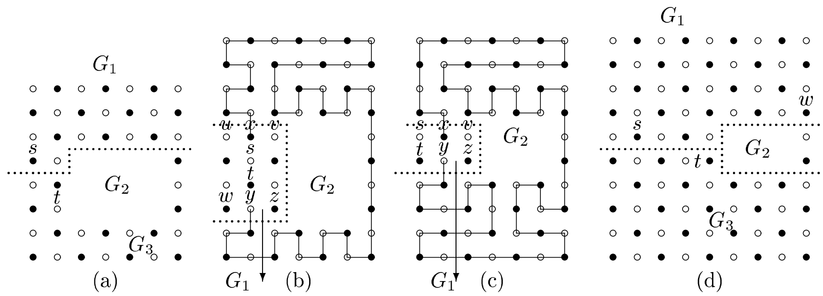

Fig. 25. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

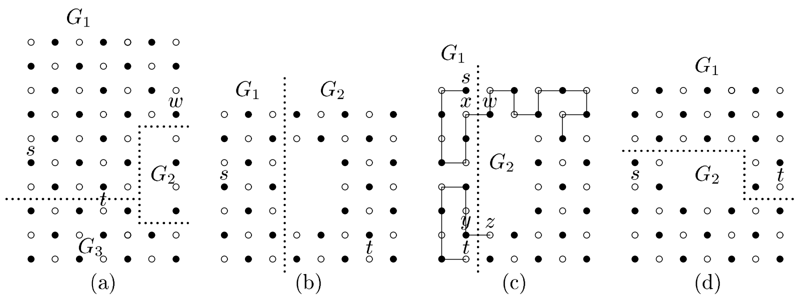

Fig. 26. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

(b) $d=2$, $l>1$, $s_y \neq t_y$, $s_x, t_x \leq d$, $w=(m,a)$, and

(b$_1$) a' and w are same-colored and [($m=$ even (see Fig. 24(b) and 24(c))) or ($m=$ odd and $a=$ even (see Fig. 24(d)))]; or

(b$_2$) a' and w are different-colored, $m=$ odd, $n=$ even, and $a=$ odd (see Fig. 25(a)).

Where $a'=s$ if $s_y<t_y$, otherwise $a'=t$.

(c) $m=$ odd, $n=$ even, $s_x, t_x \leq d$, $d=$ odd >1, $a=$ even, $b=$ even, and

(c$_1$) $d=3$, $l>2$, and [($s_y, t_y > a+1$), (s_y (or t_y) $=a+1$, t_y (or s_y) $=a+l$, and $s_x=t_x=2$), or ($s_y=t_y=a+1$)] (see Fig. 25(b)); or

(c$_2$) $d=3$, $l=2$, and [($s_x=t_x$) or ($s_y=t_y$)] (see Fig. 25(c)).

(c$_3$) $d=5$, s is black, and [($s_x=$ even and $t_y>s_y$ (see Fig. 25(d))) or ($s_x=$ odd and $t_y>s_y+1$ (see Fig. 26(a)))] (here the role of s and t can be swapped, i.e. [($t_x=$ even and $s_y>t_y$) or ($t_x=$ odd and $s_y>t_y+1$)]).

(O10) $a=b=2$, and

(a) $n=$ odd, $m=$ even, $d=$ odd >1, $c=$ odd >1, $s_x \leq d$, $t_x>d$, and s is black (see Fig. 26(b)); or

(b) $n=$ even, $m=$ odd, $d=2$, $c=3$, s (or t) $=(d,1)$, and t (or s) $=(d,n)$ (see Fig. 26(c)).

(O11) $d=c=2$, $m=$ odd, $n=$ even, $a=$ odd >1, $b=$ odd >1, $a+1 \leq s_y, t_y \leq a+l$, and

(a) $s_x \leq d$ and $t_x > d+k$ (see Fig. 26(d) and 27(a)); or

(b) $s_x, t_x \leq d$, $s_y \neq t_y$, and a' is black, where $a'=s$ if $s_y<t_y$, otherwise $a'=t$ (see Fig. 27(b)).

(O12) $d=c=3$, $m=$ odd, $n=$ even, $a=$ even, $b=$ even, $a+1 \leq s_y, t_y \leq a+l$, and

(a) $s_x, t_x \leq d$, s is black, and [($s_x=$ even and $t_y>s_y$ (see Fig. 27(c))) or ($s_x=$ odd and $t_y>s_y+1$ (see Fig. 27(d)))] (here the role of s and t can be swapped, i.e. t is black, and [($t_x=$ even and $s_y>t_y$) or ($t_x=$ odd and $s_y>t_y+1$)]); or

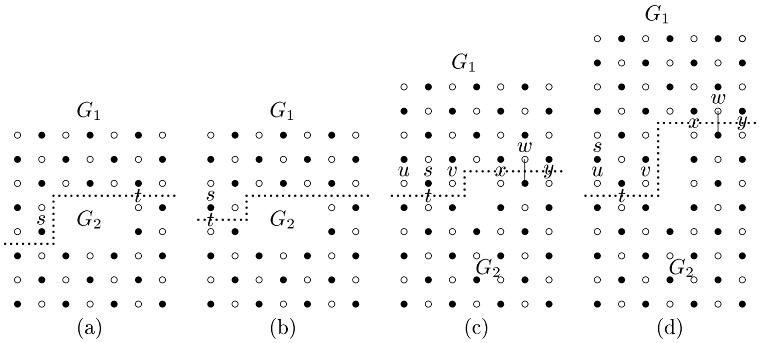

Fig. 27. Some O-shaped grid graphs in which there are no Hamiltonian (s,t)-path.

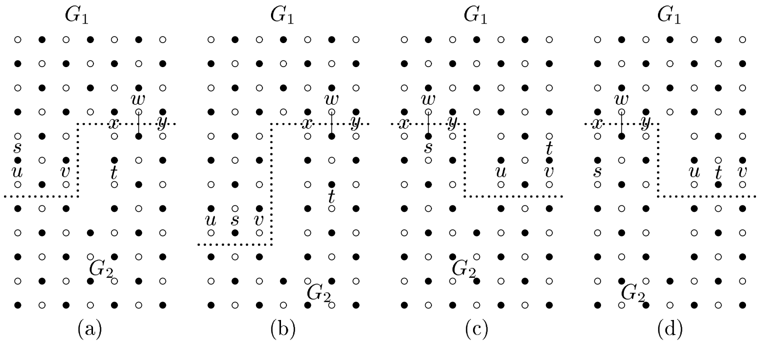

Fig. 28. Some O-shaped grid graphs in which there is no Hamiltonian (s,t)-path.

(b) $s_x \le d$, $t_x > d+k$, and
  (b_1) s is black, $t_y > a+1$, and $a+2 \le s_y \le a+l-1$ (see Fig. 28(a) and 28(b)); or
  (b_2) s is white, $s_y > a+1$, and $a+2 \le t_y \le a+l-1$ (see Fig. 28(c) and 28(d)).

Definition 3.2. An O-shaped Hamiltonian path problem $(O(m,n,k,l), s, t)$ is *acceptable* if it is color-compatible and $(O(m,n,k,l), s, t)$ does not satisfy any of conditions (F1) and (O1)–(O12).

Theorem 3.2. *Assume that $O(m,n,k,l)$ is a rectangular grid graph with a hole and s and t are two distinct vertices of it. If $(O(m,n,k,l), s, t)$ is Hamiltonian, then $(O(m,n,k,l), s, t)$ is acceptable.*

Proof. Assume that $(O(m,n,k,l), s, t)$ is not acceptable, then we show that $(O(m,n,k,l), s, t)$ has no Hamiltonian (s,t)-path. It is obvious that if $(O(m,n,k,l), s, t)$ is not color-compatible then $(O(m,n,k,l), s, t)$ has no Hamiltonian (s,t)-path. Therefore, without loss of generality, suppose $(O(m,n,k,l), s, t)$ is color-compatible. We will show that if one of the conditions (F1) and (O1)–(O12) occurs, then $(O(m,n,k,l), s, t)$ has no Hamiltonian (s,t)-path.

Condition (F1): It is obvious that if $\{s,t\}$ is a vertex cut, then any path between s and t can only pass through one component; see Fig. 15(a).

Condition (O1): (a) Consider Fig. 15(b). The Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through all the vertices of G_1, leaves G_1 at q, then enters to G_2 at z and should pass through all the vertices of G_2 and ends at t. It is clear that if (G_1, s, q) is not acceptable, then by Theorem 2.1, (G_1, s, q) has no Hamiltonian path. Hence, $(O(m,n,k,l), s, t)$ has no Hamiltonian path. (b) Consider Fig. 15(c) and 15(d). It is easy to see that there is no Hamiltonian (s,t)-path in $O(m,n,k,l)$ containing both of the vertices w and z. (c) Consider Fig. 15(e). The Hamiltonian path P of $O(m,n,k,l)$ starts from s and passes through some vertices G_2, leaves G_2 at w, enters to G_1 at p and should pass through all the vertices of G_1, reenters to G_2 at z, then passes through all the remaining vertices of G_2 and ends at t. Obviously if (G_1, q, p) is not acceptable, then by Theorem 2.1, (G_1, q, p) has no Hamiltonian path. Hence, $(O(m,n,k,l), s, t)$ has no Hamiltonian path.

Condition (O2): Consider Fig. 16(a). Notice that $O(m,n,k,l)$ is even-sized and the number of vertices with white color is two more than the number of vertices with black color. Since $O(m,n,k,l)$ is even-sized and colors of vertices of any path must alternate between black and white, it is clear that two vertices with white color remain out of the path, and hence $O(m,n,k,l)$ has no Hamiltonian (s,t)-path.

Condition (O3): (a) Consider Fig. 16(b). Since $t \ne w$ and $t \ne z$, thus w or z could not be in any path from s to t, except when $t = w$ or $t = z$. (b) In this case, we partition $O(m,n,k,l)$ into two components G_1 and G_2 by a vertical separation as shown in Fig. 16(c) and 16(d). To have a Hamiltonian path in $O(m,n,k,l)$, the path should pass through vertices w

and z. It is easy to see that the path should start from s go to w then z, and finally reach to t. So in G_1, we have two disjoint subpaths. To have these two disjoint subpaths in G_1, we should partition G_1 into two odd-sized components. It is easy to see that at least one of these components has a vertex of degree one. So, that component satisfies condition (F3) and does not have a Hamiltonian path. (c) Consider Fig. 17(a). The Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through some vertices of G_1, leave G_1 at w (or z), enter to G_2 and leave it after visiting the vertices of G_2 by q (or p), then reenter to G_1 and passe through all the remaining vertices of G_1 and end at t. Since G_2 is even-sized and p and q are same-colored, (G_2,q,p) is not acceptable. Thus by Theorem 2.1, (G_2,q,p) does not have any Hamiltonian (p,q)-path. Hence, $O(m,n,k,l)$ has no Hamiltonian (s,t)-path. (d) We partition $O(m,n,k,l)$ into two parts G_1 and G_2 as shown in Fig. 17(b) and 17(c). Clearly, G_1 is an odd-sized grid subgraph with white majority color. It is easy to see, to cover G_1 by some paths we need at least two paths. These paths should have three white end vertices at the common border of G_1 and G_2. But there are less than three white vertices in this common border. So, there is not any Hamiltonian path in $O(m,n,k,l)$.

Condition (O4): (a) and (b) Consider Fig. 17(d) and 18(a). Since G_2 is even-sized, $a=1$, and $b=3$, it is clear that the Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through some vertices of G_1, leaves G_1 at w (or z), enters G_2, then should pass through all the vertices of G_2, leave G_2, reenter G_1 at z (or w), then pass through the remaining vertices of G_1 and end at t. We can easily see that one of the two vertices x or y remains out of path. (c) We partition $O(m,n,k,l)$ into three parts G_1, G_2, and G_3 as shown in Fig. 18(b). Here, G_1 is an even-sized grid graph. It is easy to see, to cover G_1 by some paths that pass through black vertices s and w we need at least two paths. These paths should have two white end vertices at the common border of G_1 with G_2 and G_3. But there are less than two white vertices in this common border. Therefore, there is not any Hamiltonian path in $O(m,n,k,l)$.

Condition (O5): We partition $O(m,n,k,l)$ into three parts G_1, G_2, and G_3 as shown in Fig. 18(c). Clearly, G_1 is an odd-sized grid graph with white majority color. It is easy to see, to cover G_1 by some paths that pass through black vertices s and w, we need at least three paths. These paths should have four white end vertices at the common border of G_1 with G_2 and G_3. But there are less than four white vertices in this common border. Thus, there is not any Hamiltonian path in $O(m,n,k,l)$.

Condition (O6): (a) Consider Fig. 19(a) and 19(b). The proof is similar to the proof of condition (O3) (case (b)). (b) and (c) Consider Fig. 19(c), 19(d), 20(a), and 20(b). Since $a=1$, $b=2$, and G_2 is even-sized, the Hamiltonian path P of $O(m,n,k,l)$ that starts from s must enter G_2 at w, passes through all the vertices of G_2, leaves G_2 at z, then enters G_1 and ends at t. Consider Fig. 19(c) and 19(d). We can easily see that there is no Hamiltonian (s,t)-path in $O(m,n,k,l)$ containing all of the vertices of G_1. Consider Fig. 20(a). It is easy to see that one of the two vertices u or v remains out of the path. Now, consider Fig. 20(b). Since $s \neq u$, thus u could not be in any path from s to t. (d) Consider Fig. 20(c). Since $t=(d+1,1)$ and $b=2$, the Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through all the vertices of G_1, leave G_1 at x, enter to G_2 at y and pass through all the vertices of it and end at t. Clearly, one of the vertices u or v remains out of the path. (e) Consider Fig. 20(d) and 21(a). Let $d=$ even or $n=$ even (resp. $d=$ odd and $n=$ odd), then by the same method as in the proof condition (O4) (case (c)) (resp. (O3) (case (d))), we derive that $(O(m,n,k,l),s,t)$ has no Hamiltonian path.

Condition (O7): (a) and (b) Consider Fig. 21(b)–21(d). By the same argument as in the proof condition (O4) (case (c)), we derive that $(O(m,n,k,l),s,t)$ has no Hamiltonian path. (c) Consider Fig. 22(a). Since $s \neq u$, thus u could not be in any path from s to t, except when $s=u$. (d) Consider Fig. 22(b) and 22(c). The proof is the same as the proof condition (O6) (case (b) and (c)).

Condition (O8): The proof is similar to condition (O7), see Fig. 22(d) and 23(a)–23(c).

Condition (O9): (a) First, let $l=2$. Consider Fig. 23(d). Since $a=b=2$, $c=1$, and G_2 is even-sized, the Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through some vertices of G_1, leaves G_1 at x (or y), enters G_2 and should pass through all the vertices of G_2, leaves G_2, then reenters G_1 at y (or x) and passes through all the remaining vertices of G_1 and ends at t. We can easily see that one of the four vertices u, v, w, and z remains out of the path. Now, let $l>2$. Consider Fig. 24(a). By the same argument as in the proof of condition (O5), we derive that $(O(m,n,k,l),s,t)$ has no Hamiltonian path. (b) Suppose that ($m=$ even) or ($m=$ odd and $a=$ even). Consider Fig. 24(b)–24(d). By the same argument as in the proof condition (O4) (case (c)), we derive that $(O(m,n,k,l),s,t)$ has no Hamiltonian path. Now, suppose that $m=$ odd and $a=$ odd. Consider Fig. 25(a). Then the proof is the same as the proof of condition (O3) (case (d)). (c_1) and (c_2) Consider Fig. 25(b) and 25(c). The proof is similar to case (a), where $l=2$. Notice that, here, $d=3$, $c=1$, and G_2 is even-sized. (c_3) Consider Fig. 25(d) and 26(a). Then the proof is similar to the proof of condition (O5).

Condition (O10): (a) Consider Fig. 26(b). The proof is similar to the proof of condition (O3) (case (d)). (b) Consider Fig. 26(c). Since G_2 is even-sized and $a=b=2$, the Hamiltonian path P of $O(m,n,k,l)$ that starts from s should pass through some vertices of G_1, leaves G_1 at x, enters to G_2 at w and should pass through all the vertices of it, leaves G_2 at z, then reenters to G_1 at y and passes through all the remaining vertices and ends at t. A simple check shows that (G_2,w,z) is not acceptable. Thus, by Theorem 2.1, it has no Hamiltonian (w,z)-path. Hence, $(O(m,n,k,l),s,t)$ has no Hamiltonian (s,t)-path.

Condition (O11): Consider Fig. 26(d), 27(a), and 27(b). The proof is the same as the proof of condition (O3) (case (d)).

Condition (O12): We partition $O(m,n,k,l)$ into two parts G_1 and G_2 as shown in Fig. 27(c), 27(d), and 28. Note that G_1 is an odd-sized grid graph with white majority color. It is easy to see, to cover G_1 by some paths, we need exactly two subpaths P_1 and P_2. P_1 from s (or t) to u, v or w and P_2 from v to w, u to w, or u to v. We can easily see that one of the vertices x or y remains out of the path. □

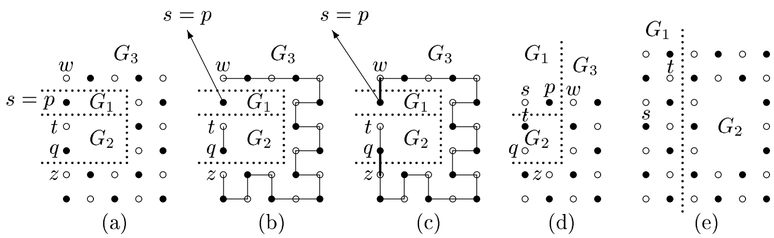

Fig. 29. (a) An E-shaped separation of $O(m,n,k,l)$, (b) Hamiltonian paths in (G_1,s,p), (G_2,q,t) and (G_3,w,z), (c) a Hamiltonian (s,t)-path in $O(m,n,k,l)$, (d) an F-shaped separation of $O(m,n,k,l)$, and (e) a vertical separation of $O(m,n,k,l)$.

4. The algorithm

In this section, we present a linear-time algorithm for finding a Hamiltonian (s,t)-path in a rectangular grid graph with a hole. Throughout this section, we assume $(O(m,n,k,l),s,t)$ is acceptable. Our algorithm is given in Algorithm 4.1.

Algorithm 4.1 The Hamiltonian path algorithm HamPath$(O(m,n,k,l),s,t)$.

Input: An acceptable Hamiltonian path problem $(O(m,n,k,l),s,t)$
Output: A Hamiltonian (s,t)-path of $O(m,n,k,l)$
Step 1. By vertical, horizontal, E-shaped, F-shaped, L-shaped, C-shaped, or O-shaped separation, partitions $O(n,m,k,l)$ into at most three disjoint grid subgraphs G_1 to G_3.
Step 2. Find a Hamiltonian path or cycle in each subgraph.
Step 3. Combine the Hamiltonian paths or cycles for constructing a Hamiltonian (s,t)-path.

In the following, we describe the steps of the algorithm in detail by Lemmas 4.1–4.4.

Definition 4.1. A separation is *acceptable* if all of its components are acceptable.

Definition 4.2. Two non-incident edges (u_1,v_1) and (u_2,v_2) are *parallel*, if u_1 (resp. v_1) is adjacent to u_2, and v_1 (resp. u_1) is adjacent to v_2.

Lemma 4.1. *Assume $(O(m,n,k,l),s,t)$ is an acceptable Hamiltonian path problem with $d=$ odd. Let $s_x,t_x \le d$. Then there is an acceptable separation for $(O(m,n,k,l),s,t)$ and it has a Hamiltonian path.*

Proof. There are two possible cases, $d=1$ and $d>1$. In two cases, first we prove that $(O(m,n,k,l),s,t)$ has an acceptable separation. Then we show that $(O(m,n,k,l),s,t)$ has a Hamiltonian path.

Case 1. $d=1$. Notice that, in this case, s and t are adjacent. If s and t are not adjacent, then $(O(m,n,k,l),s,t)$ satisfies condition (F1). For this case, we consider the following subcases.

Subcase 1.1. $a+1 \le s_y,t_y \le a+l$. Assume that $s_y<t_y$. Let $\{G_1,G_2,G_3\}$ be an E-shaped separation of $O(m,n,k,l)$ such that $V(G_1)=\{x=1, a+1 \le y \le s_y\}$, $V(G_2)=\{x=1, s_y+1 \le y \le a+l\}$, and $G_3=O(m,n,k,l)\backslash(G_1\cup G_2)$. Let $s,p\in G_1$, $q,t\in G_2$, $w,z\in G_3$, q and z are adjacent, w and p are adjacent, $p=(1,a+1)$, and $q=(1,a+l)$. Consider Fig. 29(a). In this case, G_1 and G_2 are one-rectangles and G_3 is a C-shaped grid graph. Notice that, here, if $|G_1|=1$ (resp. $|G_2|=1$), then $p=s$ (resp. $q=t$). Consider (G_1,s,p) and (G_2,q,t). Since G_1 and G_2 are one-rectangles and s and p and q and t are two end vertices of G_1 and G_2, respectively, it is clear that (G_1,s,p) and (G_2,q,t) are acceptable. Now, consider (G_3,w,z). It is obvious that if (G_3,w,z) is not acceptable, then $(O(m,n,k,l),s,t)$ satisfies condition (O1), a contradiction. Therefore, it follows that (G_3,w,z) is acceptable.

Now, we show that $(O(m,n,k,l),s,t)$ has a Hamiltonian path. Since (G_1,s,p), (G_2,q,t), and (G_3,w,z) are acceptable, by Theorem 2.1, they have a Hamiltonian path. So, we construct a Hamiltonian path in (G_1,s,p) and (G_2,q,t) by the algorithm in [2] and in (G_3,w,z) by the algorithm in [21] (as shown in Fig. 29(b)). Finally, a Hamiltonian path for $(O(m,n,k,l),s,t)$ can be constructed by connecting the vertices q and z, and w and p. The full construction of a Hamiltonian (s,t)-path in $O(m,n,k,l)$ is illustrated in Fig. 29(c). In this case, if $t_y<s_y$, then the role of p and q can be swapped (that is, $s,p\in G_2$ and $q,t\in G_1$).

Subcase 1.2. $s_y,t_y \le a+1$. This case is similar to Subcase 1.1 (here we have an F-shaped separation), where $V(G_1)=\{1\le x\le d+k, y=1\}$, $V(G_2)=\{x=1, s_y+1\le y\le a+l\}$, $p=(d+k,1)$, and $q=(1,a+l)$ (as shown Fig. 29(d)). Notice that, here, G_3 is an L-shaped grid graph and it has a Hamiltonian path by the algorithm in [19].

Case 2. $d>1$ and $[(n=\text{odd})$ or $(n=\text{even}$ and $[(m=\text{even})$ or $(m=\text{odd}$ and $[(a=\text{odd}$ or $b=\text{odd})$, $(a=\text{even}$, $b=\text{even}$, and $c=\text{even})$, or $(a=\text{even}$, $b=\text{even}$, and $c=\text{odd}>3)])])]$. For this case, we consider the following subcases.

Subcase 2.1. $s_x,t_x \le d-1$ and $[(d>3)$ or $(d=3$ and $[(s_y\ne t_y)$ or $(s_y=t_y=1$ or $n)])]$. Let $\{G_1,G_2\}$ be a vertical separation of $O(m,n,k,l)$ such that $G_1=R(m',n)$, $G_2=O(m-m',n,k,l)$, $m'=d-1$, and $s,t\in G_1$ (see Fig. 29(e)). Since $m'=\text{even}$, G_1 is even-sized. Moreover, since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1,

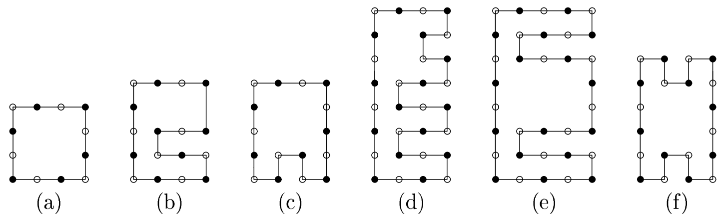

Fig. 30. A Hamiltonian cycle in an even-sized $O(m,n,k,l)$: (a) $a=b=d=c=1$, (b) $a=d=c=1$ and $b=\text{odd}>1$, (c) $a=d=c=1$ and $b=\text{even}$, (d) $d=a=1$, $b=\text{odd}$ and $l=\text{even}$, (e) $d=c=1$, $a=\text{odd}>1$, and $b=\text{odd}>1$, and (f) $d=c=1$, $m=\text{even}$, and $a=\text{even}$ or $b=\text{even}$.

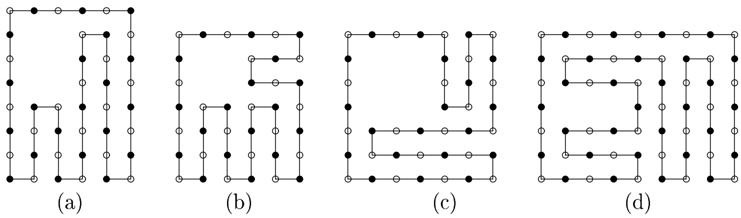

Fig. 31. A Hamiltonian cycle in an even-sized $O(m,n,k,l)$: (a) $a=d=1$, $c>1$, $b=\text{even}$, $m=\text{even}$, and $n=\text{even}$, (b) $a=d=1$, $c>1$, $b=\text{even}$, and $l=\text{even}$, (c) $a=d=1$, $b,c>1$, $m=\text{odd}$, and $n=\text{odd}$, and (d) $d=1$, $a,c,b>1$, $m=\text{odd}$, and $n=\text{odd}$.

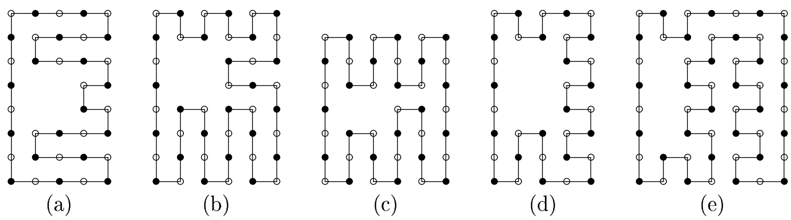

Fig. 32. A Hamiltonian cycle in an even-sized $O(m,n,k,l)$, where $d=1$, $a,b,c>1$: (a) $a=\text{odd}$, $b=\text{odd}$, and $l=\text{even}$, (b) [$a=\text{even}$ or $b=\text{even}$], $m=\text{even}$, and $l=\text{even}$, (c) $m=\text{even}$ and $l=\text{odd}$, (d) [$a=\text{even}$ or $b=\text{even}$], $k=\text{even}$, $m=\text{odd}$, and $n=\text{even}$, and (e) $a=\text{even}$, $b=\text{even}$, $k=\text{odd}$, $m=\text{odd}$, and $n=\text{even}$.

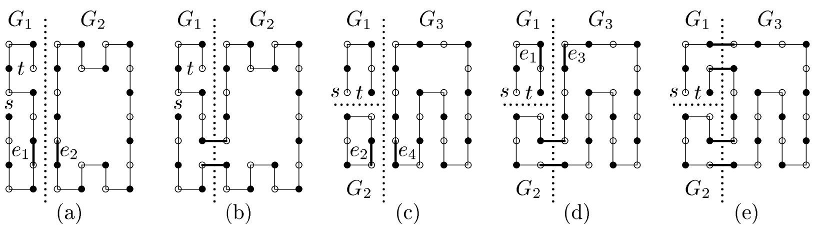

Fig. 33. (a) A Hamiltonian path in G_1 and a Hamiltonian cycle in G_2, (b) combining a Hamiltonian cycle in G_2 and path in G_1, (c) a Hamiltonian path in G_1 and Hamiltonian cycles in G_2 and G_3, (d) combining Hamiltonian cycles in G_2 and G_3, and (e) combining a Hamiltonian cycle and path.

(G_1,s,t) is color-compatible. Now, we show that (G_1,s,t) is not in conditions (F1) and (F2). The condition (F1) holds if $m'=2$ and $2\le s_y=t_y\le n-1$. This is impossible, because of in this case $s_y=t_y=1$ or n. Thus, (G_1,s,t) is not in condition (F1). The condition (F2) occurs, when $n=3$. Clearly, if (G_1,s,t) satisfies condition (F2), then $(O(m,n,k,l),s,t)$ satisfies condition (O3), a contradiction. Therefore, it follows that (G_1,s,t) is not in condition (F2). Hence, (G_1,s,t) is acceptable.

Now, we show that $(O(m,n,k,l),s,t)$ has a Hamiltonian path. Since (G_1,s,t) is acceptable, we construct a Hamiltonian path in (G_1,s,t) by the algorithm in [2]. Moreover, since G_2 is even-sized, it has a Hamiltonian cycle. The pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32 (the isomorphic cases are omitted). Then by combining Hamiltonian cycle and path using two parallel edges e_1 and e_2 (Fig. 33(a)), a Hamiltonian path for $(O(m,n,k,l),s,t)$ is obtained; see Fig. 33(b). Now, we describe combining a Hamiltonian (s,t)-path in G_1 with the constructed cycle in G_2. Any Hamiltonian path P in G_1 contains all the vertices of G_1. Therefore, P should contain a boundary edge of G_1 that has a parallel edge in G_2. Moreover, since $n\ge 3$, thus there exists at least one edge for combining Hamiltonian cycle in G_2 and path in G_1.

<u>Subcase 2.2.</u> $d=3$, $s_x,t_x\le d-1$, and $2\le s_y=t_y\le n-1$. Notice that, in this case, $n>3$. If $n=3$, then $(O(m,n,k,l),s,t)$ satisfies condition (O3). Let $\{G_1,G_2,G_3\}$ be an L-shaped separation (type II) of $O(m,n,k,l)$ such that $G_1=R(m',n')$, $G_2=R(m',n-n')$, $G_3=O(m,n,k,l)\backslash(G_1\cup G_2)$, $m'=d-1$, $n'=s_y$ if $s_y\ne n-1$; otherwise $n'=s_y-1$ (see Fig. 33(c)). In this case, $s,t\in G_1$ if $s_y\ne n-1$; otherwise $s,t\in G_2$. Since $m'=d-1$, thus G_1 and G_2 are even-sized. Additionally, since $O(m,n,k,l)$

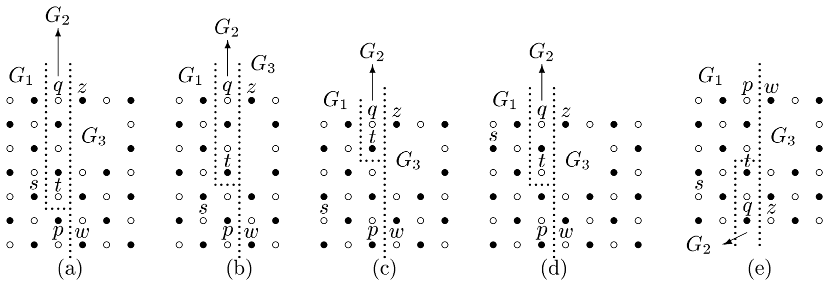

Fig. 34. A C-shaped separation (type II) of $O(m, n, k, l)$.

is even-sized, we conclude that G_3 is even-sized. Assume that $s, t \in G_1$. For the case $s, t \in G_2$, the proof is similar to the case $s, t \in G_1$. By Lemma 3.1, (G_1, s, t) is color-compatible. Since $s_y = t_y = n'$, a simple check shows that (G_1, s, t) is not in conditions (F1) and (F2). Hence, (G_1, s, t) is acceptable. In the following, we show that $(O(m, n, k, l), s, t)$ has a Hamiltonian path. Since (G_1, s, t) is acceptable, by Theorem 2.1, it has a Hamiltonian path. So, we construct a Hamiltonian path in (G_1, s, t) by the algorithm in [2]. Also since G_2 and G_3 are even-sized, we construct Hamiltonian cycles in G_2 and G_3 by Lemma 2.3 and by the pattern given in Fig. 30–32, respectively. Let C be a Hamiltonian cycle in G_3. First, we combine Hamiltonian cycles in G_2 and G_3 using two parallel edges and we obtain a large Hamiltonian cycle C_1 (Fig. 33(d)). Then by combining Hamiltonian cycle C_1 and the Hamiltonian path in G_1 using two parallel edges (as shown in Fig. 33(d)), a Hamiltonian path for $(O(m, n, k, l), s, t)$ is obtained; see Fig. 33(e). Consider Fig. 33(c) and 33(d). Let $u_1 = (m', 1)$, $v_1 = (m' + 1, 1)$, $u_2 = (m', n)$, and $v_2 = (m' + 1, n)$. Since $s \neq u_1$ (or u_2) and $t \neq u_1$ (or u_2), thus in any Hamiltonian path or cycle in G_1 or G_2 edges e_1 and e_2 must be exist. Also since $degree(v_1) = degree(v_2) = 2$, in any Hamiltonian cycle of G_3 edges e_3 and e_4 must be exist. Therefore, there exists at least one edge for combining Hamiltonian cycles in G_2 and G_3 and the Hamiltonian path in G_1.

Subcase 2.3. $s_x \leq d - 1$ and $t_x = d$. For this case, we consider the following subcases.

Subcase 2.3.1. $t_y \leq n - 1$ and $[(d > 3)$ or $(d = 3$ and $[(s \neq (1, n - 1))$ or $(s = (1, n - 1)$ and $t_y \leq n - 3)])]$. This case is similar to Subcase 1.1 (notice that here, we have a C-shaped separation (type II)), where $V(G_1) = \{1 \leq x \leq d - 1, 1 \leq y \leq t_y$ and $1 \leq x \leq d, t_y + 1 \leq y \leq n\}$, $V(G_2) = \{x = d, 1 \leq y \leq t_y\}$, $p = (d, n)$, and $q = (d, 1)$; as shown in Fig. 34(a)–(d). Here, $G_2 = R(m', n')$, where $m' = 1$ and $n' = t_y$, and $G_1 = L(m', n, k', l')$, where $m' = d$, $k' = 1$, $l' = t_y$. Notice that if $|G_2| = 1$, then $q = t$. A simple check shows that (G_1, s, p), (G_2, q, t), and (G_3, w, z) are color-compatible. Consider (G_2, q, t). Since G_2 is a one-rectangle and q and t are two corner vertices of it, thus (G_2, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. Consider (G_1, s, p). The condition (F1) holds, if $n - l' = 1$ and $s = (d - 1, n)$. Clearly, this case does not occur. The condition (F3) holds, if $n - l' = 1$ and $p \neq (d, n)$. This is impossible, because of $p = (d, n)$. The condition (F4) holds, if $n =$ odd, $t = (d, 1)$, and $s = (d - 1, 2)$. It is obvious that this case can not occur. The condition (F5) occur, when $m' - k' =$ odd. This is impossible, because of $m' - k' =$ even. The condition (F6) holds, if $t_y = n - 2$, $m' - k' = 2$, $n - l' = 2$, and $s = (1, n - 1)$. By assumption, this case does not occur. The condition (F7) holds, if $d = 3$, $n - l' > 2$, $s = (1, t_y + 1)$, and $p = (d, s_y + 1)$. This is impossible, because of $p = (d, n)$. The condition (F8) holds, if $d = 3$ and $p_y < s_y$. This case does not occur, because of $p = (d, n)$. The condition (F9) holds, if $m' - k' = 3$ or $n - l' = 3$. This case does not occur, because of in this case $m' - k' =$ even and $n - l' =$ even. Thus, (G_1, s, p) is acceptable. Now, consider (G_3, w, z). Since $w_x = z_x = d + 1$, $w_y = n$, and $z_y = a$, a simple check shows that (G_3, w, z) is not in conditions (F1), (F3), and (F10)–(F18), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1.

Subcase 2.3.2. $(d = 3$, $s = (1, n - 1)$, and $t_y = n - 2)$ or $(t = (d, n))$. This case is similar to Subcase 2.3.1, where $V(G_1) = \{1 \leq x \leq d, 1 \leq y \leq t_y - 1$ and $1 \leq x \leq d - 1, t_y \leq y \leq n\}$, $V(G_2) = \{x = d, t_y \leq y \leq n\}$, $p = (d, 1)$, and $q = (d, n)$; as depicted in Fig. 34(e).

Subcase 2.4. $s_x = t_x = d$. For this case, we consider the following subcases.

Subcase 2.4.1. $s_y, t_y \leq n - 1$. This case is similar to Subcase 2.3.1. Notice that, in this case, if $s_y < t_y$, then the role of q and p can be swapped; i.e., $s, p \in G_2$ and $q, t \in G_1$.

Subcase 2.4.2. $2 \leq s_y$ (or t_y) $\leq n - 1$ and t (or s) $= (d, n)$. This case is similar to Subcase 2.3.2. Notice that, in this case, if $s = (d, n)$, then the role of q and p can be swapped; i.e., $s, p \in G_2$ and $q, t \in G_1$.

Subcase 2.4.3. s (or t) $= (d, 1)$ and t (or s) $= (d, n)$. Notice that, in this case, $n =$ even. Consider the following subcases.

Subcase 2.4.3.1. $b > 2$. Let $t = (d, 1)$ and $s = (d, n)$. This case is similar to Subcase 2.3.1, where $p = (d, n - 2)$ and $q = (d, 1)$. Notice that if $s = (d, 1)$ and $t = (d, n)$, then the role of q and p can be swapped; i.e., $s, p \in G_2$ and $q, t \in G_1$.

Subcase 2.4.3.2. $b = 2$. In this case, $a > 1$. If $a = 1$, then $(O(m, n, k, l), s, t)$ satisfies condition (O6) (case (b)). This case is similar to Subcase 2.1, where $m' = d$. In this case, $G_1 = R(m', n)$ and G_2 is a C-shaped grid graph. Consider Fig. 35(a). It is clear that (G_1, s, t) is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1. Since $a > 1$ and $b = 2$, thus there exists at least one edge for combining Hamiltonian cycle in G_2 and path in G_1. Notice that, by the algorithm in [2], the Hamiltonian path in G_1 always contains the edge e_1 or e_2; see Fig. 35(b).

Case 3. $d > 1$, $m =$ odd, $n =$ even, $a =$ even, $b =$ even, and $c =$ odd ≤ 3. In this case, $l =$ even. Since $a =$ even, $l =$ even, and $b =$ even, it follows that $n \geq 6$. The subcases are considered as follows.

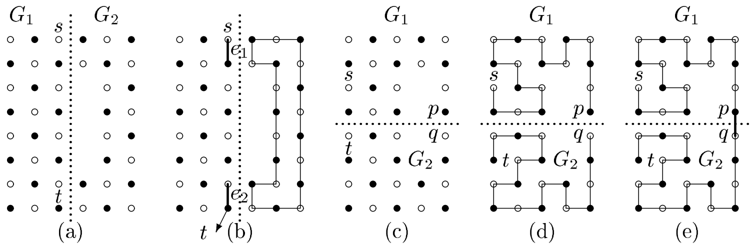

Fig. 35. (a) and (b) A vertical separation of $O(m,n,k,l)$, (c) a horizontal separation of $O(m,n,k,l)$, (d) Hamiltonian paths in (G_1,s,p) and (G_2,q,t), and (e) a Hamiltonian (s,t)-path in $O(m,n,k,l)$.

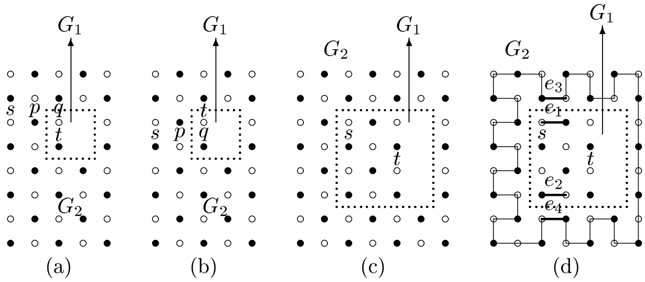

Fig. 36. An O-shaped separation of $O(m,n,k,l)$.

Subcase 3.1. $c=1$.

Subcase 3.1.1. $d=3$.

Subcase 3.1.1.1. $l>2$. In this case, s_y (or t_y) $=a+1$, t_y (or s_y) $=a+l$, $s_x=$ odd, and $t_x=$ odd. If $(s_y,t_y>a+1)$, $(s_y=t_y=a+1)$, or (s_y (or t_y) $=a+1$, t_y (or s_y) $=a+l$, and $s_x=t_x=2$), then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (c_1)). Let $s_y=a+1$ and $t_y=a+l$. Assume that $\{G_1,G_2\}$ is a horizontal separation of $O(m,n,k,l)$ such that $G_1=C(m,n',k,l')$, $G_2=C(m,n-n',k,l'')$, $n'=a+2$, $l'=2$, and $l''=l-l'$ (Fig. 35(c)). Let $s,p\in G_1$, $q,t\in G_2$, q and p are adjacent, and $p=(m,n')$. Since $s_x=$ odd and $s_y=a+1=$ odd, thus s is white and t is black. Since $a=$ even, thus $n'=$ even, and since $l=2$, we conclude that G_1 is even-sized. Additionally, since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. Since $m=$ odd and $n'=$ even, thus p is black and q is white. Clearly, (G_1,s,p) and (G_2,q,t) are color-compatible. Since $c=1$, it suffices to prove that (G_1,s,p) and (G_2,q,t) are not in conditions (F1) and (F10). Since $s_x,t_x\le d$ and $q_x=p_x=m$, a simple check shows that (G_1,s,p) and (G_2,q,t) are not in condition (F1). The condition (F10) holds, if ($s_x=$ even, s is white, and $s_y>a+1$) or ($t_x=$ even, t is black, and $t_y<a+l$). This is impossible, because of $s_y=a+1$ and $t_y=a+l$. Thus, (G_1,s,p) and (G_2,q,t) are not in condition (F1) and (F10), and hence it is acceptable.

It remains to show that $(O(m,n,k,l),s,t)$ has a Hamiltonian path. Since (G_1,s,p) and (G_2,q,t) are acceptable, by Theorem 2.1, they have a Hamiltonian path. So, we construct Hamiltonian paths in (G_1,s,p) and (G_2,q,t) by the algorithm in [21]; as illustrated in Fig. 35(d). Finally, a Hamiltonian path for $(O(m,n,k,l),s,t)$ can be constructed by connecting two vertices q and p; as shown in Fig. 35(e).

Subcase 3.1.1.2. $l=2$. Notice that, here, $s_x\ne t_x$ and $s_y\ne t_y$. If $(s_x=t_x)$ or $(s_y=t_y)$, then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (c_2)). Let $\{G_1,G_2\}$ be an O-shaped separation of $O(m,n,k,l)$ such that $V(G_1)=\{x_1\le x\le x_2,\ y_1\le y\le y_2\}$, $x_1=x_2=d$, $y_1=a+1$, $y_2=a+l$, and $G_2=O(m,n,k,l)\backslash G_1$ (as illustrated in Fig. 36(a) and 36(b)). Let $s,p\in G_2$, $q,t\in G_1$, q and p are adjacent, and $p=(d-1,s_y)$. Consider Fig. 36(a) and 36(b). In this case, $G_1=R(m',n')$; where $m'=1$ and $n'=l$. Since $l=$ even, thus G_1 is even-sized. Moreover since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. A simple check shows that (G_2,s,p) and (G_1,q,t) are acceptable. In this case, (G_2,s,p) is in Case 1 of Lemma 4.2. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1.

Subcase 3.1.2. $d\ge 5$.

Subcase 3.1.2.1. $s_x,t_x\le d-2$. This case is similar to Subcase 2.1, where $m'=d-2$. Since $n=$ even, thus G_1 is even-sized and since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_1,s,t) is color-compatible. Since $n\ge 6$ and $m'=$ odd ≥ 3, it is enough to show that (G_1,s,t) is not in condition (F2). If (G_1,s,t) satisfies condition (F2), then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (c_3)), a contradiction. Therefore, it follows that (G_1,s,t) is not in condition (F2), and hence it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1. Since $n\ge 6$, thus there exists at least one edge for combining Hamiltonian cycle and path. Notice that, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 32(b).

Subcase 3.1.2.2. $[(s_x=d-2$ and $t_x>d-2)$ or $(s_x,t_x>d-2)]$ and $[(d=5)$ or $(d>5$ and $[(l>2)$ or $(l=2$ and s (or t) $\ne (d-1,a+1)$ or t (or s) $=(d-1,a+l))])]$. This case is similar to Subcase 3.1.1.2, where $x_1=3$, $x_2=d$, and $s,t\in G_1$; as shown in Fig. 36(c). From Subcase 3.1.1.2, we know that G_1 and G_2 are even-sized. By Lemma 3.1, (G_1,s,t) is color-compatible. The condition (F1) holds, if $l=2$, $s_x=t_x=d-1$, and $d=5$. Clearly, in this case, $(O(m,n,k,l),s,t)$ satisfies condition (O9)

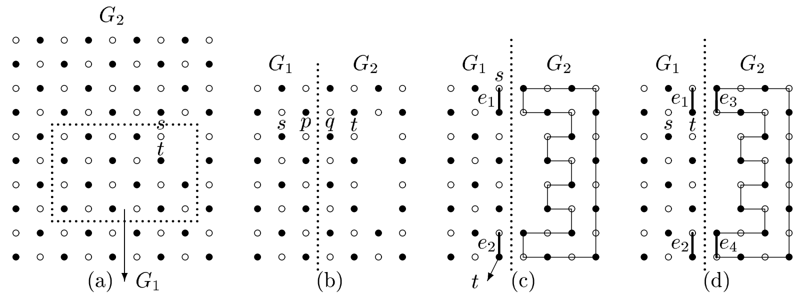

Fig. 37. (a) An O-shaped separation of $O(m,n,k,l)$, (b), (c), and (d) a vertical separation of $O(m,n,k,l)$.

(case (c_3)), a contradiction. Therefore, it follows that (G_1,s,t) is not in condition (F1). (G_1,s,t) is not in condition (F2), the proof is the same as Subcase 3.1.2.1. Thus, (G_1,s,t) is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1. Notice that, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 36(d). Since $t_x > d-2$, we can always find a Hamiltonian path in G_1 such that it contains an edge e_1 or e_2 as illustrated in Fig. 36(d). Thus clearly, we can combine a Hamiltonian cycle and a Hamiltonian path using two parallel edges e_1 and e_3 or e_2 and e_4.

Subcase 3.1.2.3. $d>5$, $l=2$, and $s_x=t_x=d-1$. Notice that, in this case, $a>2$ or $b>2$. If $a=b=2$, then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (a)). Let $a\ge 2$ and $b>2$. By symmetry, the result follows, if $a>2$ and $b=2$. This case is similar to Subcase 3.1.1.2, where $V(G_1)=\{3\le x\le d, a+1\le y\le a+l+2$ and $d+1\le x\le d+k, a+l+1\le y\le a+l+2\}$. Consider Fig. 37(a). A simple check shows that (G_1,s,t) is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.2.2.

Subcase 3.1.2.4. $s_x<d-2$ and $t_x>d-2$.

Subcase 3.1.2.4.1. $d>5$. This case is similar to Subcase 3.1.2.1, where $s,p\in G_1$, $q,t\in G_2$, q and p are adjacent, and

$$p=\begin{cases}(m',n-1); & \text{if } s \text{ is black}\\ (m',a); & \text{otherwise}\end{cases}$$

It follows from Subcase 3.1.2.1 that G_1 and G_2 are even-sized. A simple check shows that (G_1,s,p) and (G_2,q,t) are color-compatible. Consider (G_1,s,p). (G_1,s,p) is not in condition (F1) and (F2), the proof is the same as Subcase 3.1.2.1. Hence, (G_1,s,p) is acceptable. Now, consider (G_2,q,t). Since $d=2$, $q_x,t_x\le d$, and $c=1$, it suffices to show that (G_2,q,t) is not in conditions (O6) (case (e)) and (O9) (case (b)). The condition (O6) holds, if ($q_y\le a$ and q is black) or ($q_y>a+l$ and q is white). A simple check shows that this case does not occur. The condition (O9) holds, if $q_y<t_y$, and q is black. Clearly, this is impossible, because of in this case $q_y>t_y$. Hence, (G_2,q,t) is acceptable. In this case, (G_2,q,t) is isomorphic to Subcase 3.2.1 of Lemma 4.3. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1.

Subcase 3.1.2.4.2. $d=5$. This case is similar to Subcase 3.1.2.2, where $s,p\in G_2$, $q,t\in G_1$, q and p are adjacent, and

$$p=\begin{cases}(2,s_y); & \text{if } s_x=1\\ (2,s_y+1); & \text{if } s_x=2 \text{ and } s \text{ and } w \text{ are different-colored}\\ (2,s_y-1); & \text{if } s_x=2,\ s \text{ and } w \text{ are same-colored, and } s_y>a+1\end{cases}$$

where $w=(m,a)$. From Subcase 3.1.2.2, we know that G_1 and G_2 are even-sized. A simple check shows that (G_1,q,t) and (G_2,s,p) are color-compatible. Consider (G_2,s,p). Since $d=2$ and $s_x,p_x\le d$, it is enough to show that (G_2,s,p) is not in condition (O9) (case (b)). The condition (O9) holds, if $p_y<s_y$ and p and w are same-colored. This is impossible, because of in this case p and w are different-colored. Hence, (G_2,s,p) is acceptable. Now, consider (G_1,q,t). The condition (F1) holds, if $l=2$ and $t_x=q_x=4$. This case does not occur, because $q_x=3$. The condition (F2) occurs, when $m'=3$ and (i) t is white, and $t_y>q_y+1$; Clearly, in this case $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (c_3)), a contradiction. Therefore, it follows that (G_1,q,t) is not in condition (F2). (ii) t is black and [($t_x=$ even and $q_y>t_y$) or ($t_x=$ odd and $q_y>t_y+1$)]. It is obvious that in this case $q_y=t_y$ if $t_x=$ even and $q_y=t_y+1$ if $t_x=$ odd. Thus (G_1,q,t) is not in condition (F2). Hence, it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1.

Now, let s and w be same-colored, $s_x=2$, and $s_y=a+1$. The only case that occurs is $s_y=t_y$. Because if $t_y>a+1$, then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (c_3)). This case is similar to Subcase 3.1.2.4.1, where $p=(3,a+1)$. Consider Fig. 37(b). Clearly, (G_1,q,t) and (G_2,s,p) are acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1.

Subcase 3.2. $c=3$. This case is similar to Subcase 3.1.2.1, where $m'=d$. In this case, $G_1=R(m',n)$ and G_2 is a C-shaped grid graph. From the proof of Subcase 3.1.2.1, we know that G_1 and G_2 are even-sized. By Lemma 3.1, (G_1,s,t) is color-compatible. Since $n\ge 6$ and $m'\ge 3$, it is sufficient to show that (G_1,s,t) is not in condition (F2). The condition (F2) holds, if $a+1\le s_y,t_y\le a+l$, $m'=3$, s (or t) is black and [(s_x (or t_x) = odd and $t_y>s_y+1$ (or $s_y>t_y+1$)) or (s_x (or t_x) = even and $t_y>s_y$ (or $s_y>t_y$))]. It is clear that in this case $(O(m,n,k,l),s,t)$ satisfies condition (O12), a contradiction. Therefore, it follows that (G_1,s,t) is not in condition (F2), and hence it is acceptable. Notice that, in this case, G_2 is an even-sized

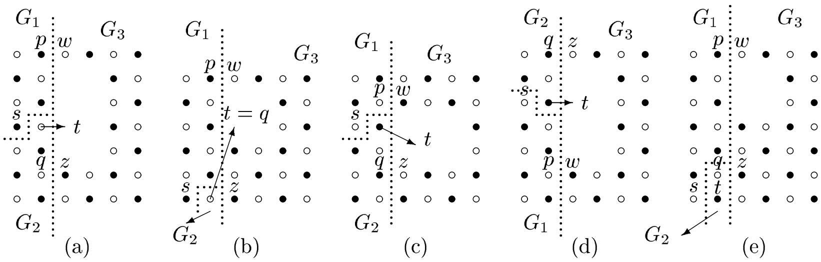

Fig. 38. A separation of $O(m,n,k,l)$.

C-shaped grid graph and $m-m' = $ even, thus it has a Hamiltonian cycle by Lemma 2.5. Let s (or t) $= (d,1)$ and t (or s) $= (d,n)$ (as shown in Fig. 37(c)), then the Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.4.3.2. Now, suppose that s (or t) $\neq (d,1)$ or t (or s) $\neq (d,n)$ (as shown in Fig. 37(d)), then the Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1. Let $u_1 = (m',1)$, $v_1 = (m'+1,1)$, $u_2 = (m',n)$, and $v_2 = (m'+1,n)$. Since s (or t) $\neq (d,1)$ or t (or s) $\neq (d,n)$, thus in any Hamiltonian path in G_1 and e_1 or e_2 must be exist. Also since $degree(v_1) = degree(v_2) = 2$, then in any Hamiltonian cycle in G_2 edges e_3 and e_4 must be exist. Hence, we can combine the path of G_1 with the cycle of G_2 using two parallel edges e_1 and e_3 or e_2 and e_4. □

Lemma 4.2. *Assume $(O(m,n,k,l),s,t)$ is an acceptable Hamiltonian path problem with $d = $ even. Let $s_x, t_x \leq d$. Then there is an acceptable separation for $(O(m,n,k,l),s,t)$ and it has a Hamiltonian path.*

Proof. Consider the following cases. In each of these cases, first we will prove that $(O(m,n,k,l),s,t)$ has an acceptable separation, then we show that it has a Hamiltonian path.

Case 1. $d = 2$ and $[(a = 1)$ or $(c = 1$, $m = $ even, $a = $ even, and $b = $ even$)]$. Since $d = 2$ and $a,b,c,k,l \geq 1$, we have $m \geq 4$ and $n \geq 3$. Notice that, in this case, $m = $ even or $n = $ even. Consider the following subcases.

Subcase 1.1. $(a = 1$, $s_y = t_y = $ even, and $[(n = $ even$)$ or $(n = $ odd and $s_y = t_y \neq n-1)])$ or $(c = 1$ and $s_y = t_y = $ odd$)$. This case is similar to Subcase 1.1 of Lemma 4.1 (note that, here, we have a C-shaped separation (type II or III)), where $V(G_1) = \{1 \leq x \leq d, 1 \leq y \leq s_y - 1$ and $x = 1, y = s_y\}$, $V(G_2) = \{1 \leq x \leq d, t_y + 1 \leq y \leq n$ and $x = d, y = t_y\}$,

$$p = \begin{cases} (d,1); & \text{if } a = 1 \\ (d,a); & \text{otherwise} \end{cases} \quad \text{and} \quad q = \begin{cases} (d,n-1); & \text{if } (a = 1 \text{ and } n = \text{odd}) \text{ or } (c = 1) \\ (d,n); & \text{otherwise} \end{cases}$$

Notice that, in this case, if $|G_1| = 1$ (resp. $|G_2| = 1$), then $s = p$ (resp. $q = t$). Consider Fig. 38(a)–(c). Here, G_1 and G_2 are odd-sized L-shaped grid graphs, where $2 \leq s_y, t_y \leq n-1$, or G_1 is an odd-sized L-shaped grid graph and G_2 is an odd-sized rectangular grid graph, where $s_y = t_y = n$. Note that, in this case, G_3 is a C-shaped grid graph. Since $s_y = $ even (resp. $s_y = $ odd) and $s_y = t_y$, it follows that s is black (resp. white) and t is white (resp. black). Additionally, since $d = $ even, a simple check shows that p is black and q is white (resp. p is white and q is black). Hence, (G_1,s,p), (G_2,q,t), and (G_3,w,z) are color-compatible. Consider (G_1,s,p), (G_2,q,t). We can easily see that (G_1,s,p) and (G_2,q,t) are not in condition (F1)–(F9). Hence, they are acceptable.

Consider (G_3,w,z). Since $a = 1$ or $c = 1$, it is enough to show that (G_3,w,z) is not in conditions (F1), (F3), (F10), and (F11). Since $w_x = z_x = d+1$, $w_y \leq a$ and $z_y > a+l$, it is clear that (G_3,w,z) is not in conditions (F1), (F3), and (F10). The condition (F11) holds, if $n = $ odd and $b = 3$. In this case, $(O(m,n,k,l),s,t)$ satisfies condition (O4), a contradiction. Therefore, (G_3,w,z) is not in condition (F11). Thus, it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 1.1 of Lemma 4.1.

Subcase 1.2. $(a = 1$, $s_y = t_y = $ odd, and $[(n = $ even$)$ or $(n = $ odd and $s_y, t_y \neq n)])$ or $(c = 1$ and $s_y = t_y = $ even$)$. This case is similar to Subcase 1.1 (see Fig. 38(d)), where $V(G_1) = \{1 \leq x \leq d, s_y + 1 \leq y \leq n$ and $x = 1, y = s_y\}$, $V(G_2) = \{1 \leq x \leq d, 1 \leq y \leq t_y - 1$ and $x = d, y = t_y\}$,

$$q = \begin{cases} (d,1); & \text{if } a = 1 \\ (d,a); & \text{otherwise} \end{cases} \quad \text{and} \quad p = \begin{cases} (d,n-1); & \text{if } (a = 1 \text{ and } n = \text{odd}) \text{ or } (c = 1) \\ (d,n); & \text{otherwise} \end{cases}$$

Subcase 1.3. $a = 1$, $n = $ odd, and $n - l \leq s_y = t_y \leq n$.

Subcase 1.3.1. $s_y = t_y = n$. This case is similar to Subcase 1.1, where $V(G_1) = \{1 \leq x \leq d, 1 \leq y \leq n-2$ and $x = 1, n-1 \leq y \leq n\}$ and $V(G_2) = \{x = 2, n-1 \leq y \leq n\}$ (see Fig. 38(e)). Note that, in this case, G_1 and G_2 are even-sized.

Subcase 1.3.2. $s_y = t_y = n-1$. Notice that, in this case, $b = $ even > 2 or $b = $ odd ≥ 5. If $b \leq 2$ (resp. $b = 3$), then $(O(m,n,k,l),s,t)$ satisfies condition (O3) or (O6) (case (c$_2$)) (resp. (O4) (case (a))). This case is similar to Subcase 2.2 of Lemma 4.1 (note that, here, we have a C-shaped separation (type II)), where $G_3 = O(m,n',k,l)$, $n' = n-3$, $V(G_2) = \{1 \leq x \leq 2, y = n-2\}$, $G_1 = O(m,n,k,l) \backslash (G_3 \cup G_2)$, and $s,t \in G_1$; as shown in Fig. 39(a). It is easy to see that (G_1,s,t) is acceptable. Now, we show that $(O(m,n,k,l),s,t)$ has a Hamiltonian path. In this case, G_2 is a one-rectangle, where $|G_2| = 2$. Let two

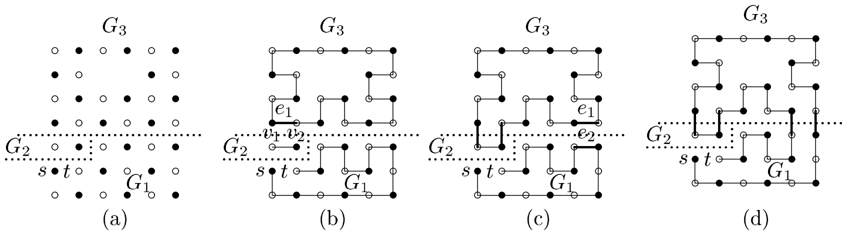

Fig. 39. (a) A C-shaped separation (type II) of $O(m,n,k,l)$, (b) a Hamiltonian path in G_1 and a Hamiltonian cycle in G_3, (c) combining a Hamiltonian cycle in G_3 and an edge (v_1, v_2), and (d) combining a Hamiltonian cycle C_1 and a Hamiltonian path in G_1.

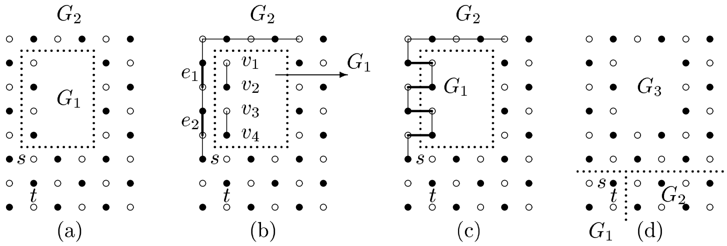

Fig. 40. (a) An O-shaped separation (type II) of $O(m,n,k,l)$, (b) a Hamiltonian subpath in G_1, (c) combining Hamiltonian path in G_2 and edge (v_1, v_2) and (v_3, v_4), and (d) an L-shaped separation (type II) of $O(m,n,k,l)$.

vertices $v_1, v_2 \in G_2$ and C be a Hamiltonian cycle in G_3. The pattern for constructing a Hamiltonian cycle in G_3 is shown in Fig. 30–32. Consider Fig. 39(b). There exists an edge e_1 such that $e_1 \in C$ is on the boundary of G_3 facing G_2. Thus, by merging (v_1, v_2) to this edge, we obtain a large Hamiltonian cycle C_1 (Fig. 39(c)). Finally, by combining a Hamiltonian cycle C_1 and path in G_1 using two parallel edges e_1 and e_2 (Fig. 39(c)), a Hamiltonian path for $(O(m,n,k,l), s, t)$ is obtained; see Fig. 39(d). Since $m - d \geq 2$, thus there exists at least one edge for combining a Hamiltonian cycle C_1 and path in G_1.

<u>Subcase 1.4.</u> $s_y \neq t_y$. Let $y' = s$ if $s_y < t_y$, otherwise $y' = t$ and let $x = (d, 1)$ if $a = 1$; otherwise $x = (m, a)$.

<u>Subcase 1.4.1.</u> x and y' are different-colored and $[(n = \text{even})$ or $(n = \text{odd}$ and $y'_y < n-1)]$. Let $s_y < t_y$. This case is similar to Subcase 2.2 of Lemma 4.1, where $m' = d$ and $n' = max(s_y, t_y) - 2$ if $n = \text{odd}$ and $max(s_y, t_y) = n$; otherwise $n' = max(s_y, t_y) - 1$. In this case, p and q are defined similar to Subcase 1.1. Since $d = \text{even}$, thus G_1 and G_2 are even-sized. Moreover, since $O(m,n,k,l)$ is even-sized, we conclude that G_3 is even-sized. A simple check shows that (G_1, s, p), (G_2, q, t), and (G_3, w, z) are color-compatible. Consider (G_1, s, p) and (G_2, q, t). The condition (F1) holds, if $s_y = p_y = a$, $a > 1$, and $t_y > a + 1$. This is impossible, because of in this case $a + 1 \leq s_y, t_y \leq a + l$. The condition (F2) holds, if $s_y = p_y = a$, $a = 2$, and $t_y = a + 2$. Since $a + 1 \leq s_y, t_y \leq a + l$, this case does not occur. Thus, (G_1, s, p) and (G_2, q, t) are not in conditions (F1) and (F2), and hence they are acceptable. Now, consider (G_3, w, z). (G_3, w, z) is not in conditions (F1), (F3), and (F10)–(F18), the proof is the same as Subcase 1.1. Thus, (G_3, w, z) is acceptable. The Hamiltonian path in $(O(m,n,k,l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.1. If $t_y < s_y$, then the role of q and p can be swapped; i.e., $s, p \in G_2$ and $q, t \in G_1$.

<u>Subcase 1.4.2.</u> (x and y' are same-colored) or (x and y' are different-colored, $n = \text{odd}$, and $y'_y \geq n - 1$). First note that, here, $a = 1$, $b \geq 2$, and $s_y, t_y > a + l$. If $y_y \leq a + l$ or $b = 1$, then $(O(m,n,k,l), s, t)$ satisfies condition (O7) or (O3).

<u>Subcase 1.4.2.1.</u> ($n = \text{even}$ and $b = \text{odd}$) or ($n = \text{odd}$ and $b = \text{even}$). In this case, $l = \text{even}$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $s, t \in G_2$; as shown in Fig. 40(a). Here, $G_2 = O(m,n,k',l)$, where $k' = k + 1$. It follows from Subcase 3.1.1.2 that G_1 and G_2 are even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $s_y, t_y > a + l$, $s_x, t_x \leq d$, $b \geq 2$, $a = 1$, and $d = 1$ (in G_2), it is enough to show that (G_2, s, t) is not in condition (O7) (case (a)). The condition (O7) holds, if $b = 2$ and $s_y = t_y$. This is impossible, because of $s_y \neq t_y$. Thus (G_2, s, t) is not in condition (O7), and hence it is acceptable. Assume that $(1, 1)$ is the coordinates of vertex in the bottom left corner in $O(m,n,k,l)$, then $(O(m,n,k,l), s, t)$ is in Lemma 4.1, Subcase 1.1, Case 2 or 3. Now, we show that $(O(m,n,k,l), s, t)$ has a Hamiltonian path. Since (G_2, s, t) is acceptable, thus by Lemma 4.1, Subcase 1.1, Case 2 or 3, it has a Hamiltonian path. Let $|G_1| = 4$ and let four vertices v_1, v_2, v_3 and v_4 be in G_1. Let P be a Hamiltonian path in G_2. In this case, we can always construct a Hamiltonian path P in G_2 that contains a subpath P_1, as shown in Fig. 40(b). Clearly, there exist two edges e_1 and e_2 such that $e_1, e_2 \in P_1$ are on boundary of G_2 facing G_1. By merging (v_1, v_2) and (v_3, v_4) to these edges, we obtain a Hamiltonian path for $(O(m,n,k,l), s, t)$, as illustrated in Fig. 40(c). When $|G_1| = 2$ or $|G_1| > 4$, a similar to the case $|G_1| = 4$, the result follows.

<u>Subcase 1.4.2.2.</u> ($n = \text{even}$ and $b = \text{even}$) or ($n = \text{odd}$ and $b = \text{odd}$). In this case, $b = \text{even} > 2$ or $b = \text{odd} \geq 5$. If $b \leq 3$, then $(O(m,n,k,l), s, t)$ satisfies condition (O4) or (O7) (case (c)). There are two possible cases $(s_y, t_y > a + l + 2)$ or (s_y (or t_y) $\leq a + l + 2$ and t_y (or s_y) $> a + l + 2$). Notice that, if $s_y, t_y \leq a + l + 2$, then $(O(m,n,k,l), s, t)$ satisfies condition (O7) (case (c)).

<u>Subcase 1.4.2.2.1.</u> $s_y, t_y > a + l + 2$. This case is similar to Subcase 2.2 of Lemma 4.1, where $G_3 = O(m, n', k, l)$, $G_1 = R(m', n - n')$, $G_2 = R(m - m', n - n')$, $n' = a + l + 2$, $m' = d$; as shown in Fig. 40(d). Since $d = \text{even}$, thus G_1 is even-sized. Since $n' = \text{even}$, it follows that G_3 is even-sized. Additionally, since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized.

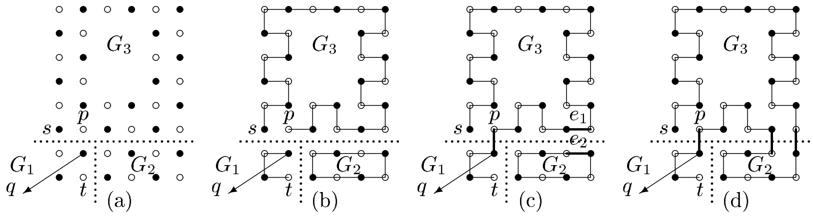

Fig. 41. (a) An L-shaped separation (type II) of $O(m,n,k,l)$, (b) Hamiltonian paths in G_1 and G_3 and a Hamiltonian cycle in G_2, (c) combining two Hamiltonian paths by two vertices p and q, and (d) combining a Hamiltonian cycle in G_2 and the Hamiltonian path P_1.

By Lemma 3.1, (G_1,s,t) is color-compatible. Since G_1 is even×even, $s_y \neq t_y$, and $d=2$, a simple check shows that (G_1,s,t) is not in conditions (F1) and (F2). Hence, it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.2 of Lemma 4.1. Notice that the pattern for constructing a Hamiltonian cycle in G_3 is shown in Fig. 30–32.

Subcase 1.4.2.2.2. s_y (or t_y) $\leq a+l+2$ and t_y (or s_y) $> a+l+2$. The only case that occurs is s (or t) $=(1,a+l+2)$ (note that if s (or t) $=(d,a+l+1)$, then $(O(m,n,k,l),s,t)$ satisfies condition (O7) (case (c))). Let $s=(1,a+l+2)$. This case is similar to Subcase 1.4.2.2.1, where $s,p\in G_3$, $q,t\in G_1$, q and p are adjacent, and $p=(d,a+l+2)$ (as shown in Fig. 41(a)). It follows from Subcase 1.4.2.2.1 that G_1, G_2, and G_3 are even-sized. A simple check shows that (G_3,s,p) and (G_1,q,t) are color-compatible. Consider (G_3,s,p). Since $s_y=p_y=a+l+2$, a simple check shows that (G_3,s,p) is not in conditions (F1) and (O1)–(O12), and hence it is acceptable. Now, consider (G_1,q,t). (G_1,q,t) is not in condition (F1) and (F2), the proof is the same as Subcase 1.4.2.2.1. Hence it is acceptable. In this case, (G_3,s,p) is in Subcase 1.1. Since (G_1,q,t) and (G_3,s,p) is acceptable, by Theorem 2.1 and Subcase 1.1, they have Hamiltonian paths. So, we construct Hamiltonian paths in (G_1,q,t) and (G_3,s,p) by the algorithm in [2] and Subcase 1.1, respectively (see Fig. 41(b)). Then we merge two Hamiltonian paths in G_3 and G_1 by connecting two vertices q and p, this path called P_1 (see Fig. 41(c)). Moreover, since G_2 is an even-sized rectangular grid graph, it has a Hamiltonian cycle by Lemma 2.3. Finally, we combine Hamiltonian cycle in G_2 with P_1 using two parallel edges e_1 and e_2; as depicted in Fig. 41(d). If $t=(1,a+l+2)$, then the role of p and q can be swapped; i.e., $s,p\in G_1$ and $q,t\in G_3$.

Case 2. $d>2$ and $[(a=1)$ or $(c=1,\ m=\text{even},\ a=\text{even},$ and $b=\text{even})]$.

Subcase 2.1. $s_x,t_x\leq d-2$.

Subcase 2.1.1. $[(n=\text{odd})$ or $(n=\text{even}$ and s (or t) $\neq (d-2,1)$ or t (or s) $\neq (d-2,n))]$ and $[(d>4)$ or $(d=4$ and $[(s_y\neq t_y)$ or $(s_y=t_y=1$ or $n)])]$. This case is similar to Subcase 2.1 of Lemma 4.1, where $m'=d-2$. Notice that, here, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32.

Subcase 2.1.2. $d=4$ and $2\leq s_y=t_y\leq n-1$. This case is similar to Subcase 2.2 of Lemma 4.1, where $m'=d-2$. Notice that, here, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32.

Subcase 2.1.3. $n=\text{even}$, s (or t) $=(d-2,1)$, and t (or s) $=(d-2,n)$.

Subcase 2.1.3.1. $b>2$. Let $s=(d-2,1)$ and $t=(d-2,n)$. This case is similar to Subcase 3.1.1.1 of Lemma 4.1, where $G_1=O(m,n',k,l)$, $n'=n-2$, $G_2=R(m,n-n')$, and $p=(d,n')$. Since $n=\text{even}$, we have $n'=\text{even}$ and $n-n'=\text{even}$. Therefore, G_1 and G_2 are even-sized. Since $d=\text{even}$ and $n'=\text{even}$ implies that p is white. Clearly, (G_1,s,p) and (G_2,q,t) are color-compatible. Consider (G_2,q,t). Since $q_x\neq t_x$, $m\geq 5$, and $n-n'=\text{even}$, a simple check shows that (G_2,q,t) is not in conditions (F1) and (F2). Hence, it is acceptable. Now, consider (G_1,s,p). Since $d\geq 4$ and $s_x,p_x\leq d$, it is enough to show that (G_1,s,p) is not in condition (O3) (case (b)) and (O6) (case (a)). The conditions (O3) and (O6) occur, when $d=4$, $s=(d-2,1)$, $p=(d-2,n')$, and $b\leq 2$. This is impossible, because of $p=(d,n')$. Hence, it is acceptable. In this case, (G_1,s,p) is in Subcase 2.3. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1. Here, if $t=(d-2,1)$ and $s=(d-2,n)$, then the role of p and q can be swapped; i.e., $s,p\in G_2$ and $q,t\in G_1$.

Subcase 2.1.3.2. $b\leq 2$. In this case $d\geq 6$. If $d=4$, then $(O(m,n,k,l),s,t)$ satisfies condition (O3) (case (b)) or (O4) (case (a)). This case is similar to Subcase 2.1 of Lemma 4.1. Since $n=\text{even}$, thus G_1 is even-sized. Moreover, since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. Since $d\geq 6$, we have $m'\geq 5$. By Lemma 3.1, (G_1,s,t) is color-compatible. Since $m'\geq 5$ and $n\geq 4$, it is clear that (G_1,s,t) is not in condition (F1) and (F2), and hence it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Here, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32.

Subcase 2.2. $s_x,t_x>d-2$. Let $y'=s$ if $s_y<t_y$, otherwise $y'=t$ and let $x=(d,1)$ if $a=1$; otherwise $x=(m,a)$.

Subcase 2.2.1. $(s_y=t_y)$ or $(s_y\neq t_y$, x and y' are different-colored). This case is similar to Subcase 2.1.1, where $s,t\in G_2$. Since $m'=\text{even}$, thus G_1 is even-sized and since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_2,s,t) is color-compatible. Since $d=2$ (in G_2) and $s_x,t_x\leq d$, it is enough to show that (G_2,q,t) is not in conditions (O7) (case (a)) and (O9) (case (b)). The conditions (O7) and (O9) hold, if $s_y\neq t_y$ and x and y' are same-colored. Clearly, this is impossible. Thus, (G_2,s,t) is not in conditions (O7) and (O9), and hence it is acceptable. In this case, (G_2,s,t) is in Case 1. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Notice that, here, G_1 is an even-sized rectangular grid graph and by Lemma 2.3 it has a Hamiltonian cycle.

Subcase 2.2.2. $s_y\neq t_y$ and x and y' are same-colored.

Subcase 2.2.2.1. s (or t) $=(d,1)$ or s (or t) $=(d,n)$. In this case $n\geq 4$. If $n=3$, then $(O(m,n,k,l),s,t)$ satisfies condition (O3) (case (c)). Suppose t (or s) $=(d,1)$. By symmetry, the result follows, t (or s) $=(d,n)$. Assume that $t=(d,1)$. This

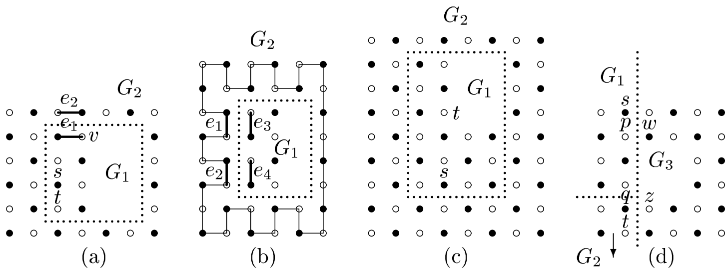

Fig. 42. (a)–(c) An O-shaped separation of $O(m, n, k, l)$, and (d) an L-shaped separation (type II) of $O(m, n, k, l)$.

case is similar to Subcase 2.3.1 of Lemma 4.1, where $q = (d, 1)$ and $p = (d, n)$ if $n =$ even; otherwise $p = (d, n-1)$. Since $d =$ even > 2, $k' \times l' = 1$, and $O(m, n, k, l)$ is even-sized, it follows that G_1 and G_2 are odd-sized and G_3 is even-sized. A simple check shows that (G_1, s, p), (G_2, q, t), and (G_3, w, z) are color-compatible. Consider (G_1, s, p). Since $n - l' \geq 3$ and $m' - k' \geq 3$, we can easily see that (G_1, s, p) is not in conditions (F3) and (F5). The condition (F1) holds, if $n =$ odd, $b = 2$, and $s = (d-1, n)$. Obviously, in this case, $(O(m, n, k, l), s, t)$ satisfies condition (O6) (case (c)), a contradiction. Therefore, it follows that (G_1, s, p) is not in condition (F1). Hence, (G_1, s, p) is acceptable. Consider (G_2, q, t). Since $q = t$, it is obvious that (G_2, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_3, w, z). (G_3, w, z) is not in conditions (F1), (F3), and (F10)–(F18), the proof is the same as Subcase 1.1. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 1.1 of Lemma 4.1. Notice that if $s = (d, 1)$, then the role of q and p can be swapped; i.e., $s, p \in G_2$ and $q, t \in G_1$.

Subcase 2.2.2.2. t (or s) $= (d, 1)$ and s (or t) $= (d, n)$. In this case, $n =$ even. Note that, in this case, $b > 2$. If $b \leq 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (F1) or (O6) (case (b)). This case is similar to Subcase 2.2.2.1, where $p = (d, n-2)$.

Subcase 2.2.2.3. t (or s) $\neq (d, 1)$, s (or t) $\neq (d, n)$, and [($b =$ odd and $n =$ even), ($b =$ even and $n =$ odd), or ($c = 1$ and $a, b > 1$)].

Subcase 2.2.2.3.1. $c = 1$.

Subcase 2.2.2.3.1.1. $s_y, t_y \leq a + l$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1 = d - 1$, $x_2 = d$, and $s, t \in G_1$. In this case, $G_1 = R(m', n')$, where $m' = 2$ and $n' = l$. Since $m' =$ even, thus G_1 is even-sized. Additionally, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_1, s, t) are color-compatible. Since $s_y \neq t_y$, thus (G_1, s, t) is not in condition (F1). The condition (F2) holds, if $l = 3$ and $s_y = t_y = a + 2$. This is impossible, because of $s_y \neq t_y$. Therefore, (G_1, s, t) is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Let $a = 1$ and let $v = (d, a+1)$. Since $degree(v) = 2$ (in G_2), then in any Hamiltonian path in G_1 always there exists an edge e_1; as shown in Fig. 42(a). Therefore for combining Hamiltonian cycle in G_2 and path in G_1, there exists always edges e_1 and e_2. Now, let $a, b > 1$ and consider Fig. 42(b). There always exists edges e_1 and e_3 or e_2 and e_4 for combining Hamiltonian cycle in G_2 and path in G_1. Notice that, here, the pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32.

Subcase 2.2.2.3.1.2. $s_y, t_y > a + l$. Assume that $(1, 1)$ is the coordinates of vertex in the bottom right corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Lemma 4.1, where $b =$ odd, or in Case 1, Subcase 2.3, or Subcase 2.2.1, where $b =$ even.

Subcase 2.2.2.3.1.3. s_y (or t_y) $\leq a + l$ and t_y (or s_y) $> a + l$. In this case, $b > 1$. Assume that $(1, 1)$ is the coordinates of vertex in the bottom right corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Lemma 4.3.

Subcase 2.2.2.3.2. $c > 1$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1 = d + k + 1$, $x_2 = m - 1$, and $s, t \in G_2$. From the proof of Subcase 3.1.1.2 of Lemma 4.1, we know that G_1 and G_2 are even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $b > 1$ and $s_x, t_x \leq d$, it suffices to prove that (G_2, s, t) is not in condition (O7). If (G_2, s, t) satisfies condition (O7), then $(O(m, n, k, l), s, t)$ satisfies condition (O6), a contradiction. Therefore, it follows that (G_2, s, t) is not in condition (O7), and hence it is acceptable. In this case, (G_2, s, t) is in Subcase 2.2.2.3.1. Let $m', n' > 1$, then the Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Now, let G_1 is a one-rectangle, then the Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 1.4.2.1.

Subcase 2.2.2.4. $a = 1$, t (or s) $\neq (d, 1)$, s (or t) $\neq (d, n)$, and [($b =$ even and $n =$ even) or ($b =$ odd and $n =$ odd)]. In this case, $l =$ odd and $k =$ even. Notice that, for $n =$ odd, we have $b =$ odd ≥ 5. If $n =$ odd and $b \leq 3$, then $(O(m, n, k, l), s, t)$ satisfies conditions (O3) and (O4).

Subcase 2.2.2.4.1. $b = 2$. In this case, $n =$ even. The only case that occurs is ($s_y, t_y \leq a + l$) or (s_y (or t_y) $\leq a + l$ and t_y (or s_y) $> a + l$). If $s_y, t_y > a + l$, then $(O(m, n, k, l), s, t)$ satisfies condition (O7) (case (c)).

Subcase 2.2.2.4.1.1. $s_y, t_y \leq a + l$. This case is similar to Subcase 2.2.2.3.1.1.

Subcase 2.2.2.4.1.2. s_y (or t_y) $\leq a + l$ and t_y (or s_y) $> a + l$. Let $t_y \leq a + l$ and $s_y > a + l$. Since $b = 2$ and $s \neq (d, n)$, the only case that occurs is $s = (d-1, n-1)$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1 = d - 1$ and $p = (d, n-1)$. A simple check shows that (G_1, q, t) and (G_2, s, p) is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.2 of Lemma 4.1.

Subcase 2.2.2.4.2. $b > 2$.

Subcase 2.2.2.4.2.1. $s_y, t_y > a + l + 2$. This case is similar to Subcase 1.4.2.2.1.

Subcase 2.2.2.4.2.2. s_y (or t_y) $\leq a + l + 2$ and t_y (or s_y) $> a + l + 2$. This case is similar to Subcase 1.4.2.2.2.

Subcase 2.2.2.4.2.3. $s_y, t_y \le a+l+2$. This case is similar to Subcase 3.1.1.2, where $x_1 = d-1$, $x_2 = d+k$, $y_1 = a+1$, $y_2 = a+l+3$, and $s, t \in G_1$. Consider Fig. 42(c). In this case, $G_1 = L(m', n', k, l)$, where $m' = k+2$ and $n' = l+3$. It is obvious that G_1 and G_2 are even-sized. By Lemma 3.1, (G_1, s, t) is color-compatible. Here, $m' - k = 2$ and $n' - l = 3$. Since $s_x, t_x \le m' - k$ and $s_y \ne t_y$, a simple check shows that (G_1, s, t) is not in conditions (F1) and (F3)–(F9), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.2.2.3.1.1. Notice that, here, G_1 is an L-shaped grid graph, and hence we construct a Hamiltonian path in (G_1, s, t) by the algorithm in [19]. The pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 30–32.

Subcase 2.3. $s_x \le d-2$ and $t_x > d-2$. Notice that, for $n =$ odd, we have $b =$ odd ≥ 5. If $n =$ odd and $b \le 3$, then $(O(m, n, k, l), s, t)$ satisfies conditions (O3) and (O4). This case is similar to Subcase 2.1 of Lemma 4.1, where $m' = d-2$, $s, p \in G_1$, $q, t \in G_2$, q and p are adjacent, and

$$p = \begin{cases} (m', 1); & \text{if } a = 1 \text{ and } s \text{ is white} \\ (m', n); & \text{if } a = 1,\ n = \text{even},\ \text{and } s \text{ is black} \\ (m', a); & \text{if } c = 1,\ \text{and } s \text{ is black} \\ (m', n-1); & \text{if } (c = 1,\ \text{and } s \text{ is white}) \text{ or} \\ & (a = 1,\ n = \text{odd},\ s \text{ is black},\ \text{and } [(d > 4) \text{ or } (d = 4 \text{ and } s \ne (1, n-1))]) \end{cases}$$

Since $d =$ even, thus G_1 is even-sized. Additionally since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_1, s, p). The condition (F1) holds, if $m' = 2$ and $[(s_y = p_y = a)$ or $(s_y = p_y = n-1)]$. Clearly, this is impossible. The condition (F2) occurs, when $n = 3$ and $s_y = p_y = n-1$. Clearly, in this case $(O(m, n, k, l), s, t)$ satisfies condition (O3), a contradiction. Therefore, (G_1, s, p) is not in condition (F2). Hence, it is acceptable. Now, consider (G_2, q, t). Since $t_x, q_x \le d$, $d = 2$, it is sufficient to show that (G_2, q, t) is not in conditions (O6), (O7) (case (a) and (c)) and (O9) (case (b)). The conditions (O7) and (O9) hold, if $q_y < t_y$ and q is black (where $a = 1$) or q is white (where $c = 1$). It is obvious that this case can not occur. The condition (O6) occurs, when $n =$ odd, $b = 2$, and $t = (d, n-1)$. Clearly, if (G_2, q, t) satisfies condition (O6), then $(O(m, n, k, l), s, t)$ satisfies condition (O6), a contradiction. Therefore, it follows that (G_2, q, t) is not in condition (O6). Thus, (G_2, q, t) is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Now, assume that $n =$ odd, $d = 4$, $s = (1, n-1)$, $a = 1$. Since $b > 3$, $a = 1$, and $l \ge 1$, it follows that $n \ge 5$. This case is similar to Subcase 1.4.2.2.2, where $G_2 = R(m', n')$, $G_3 = R(m', n-n')$, $G_1 = O(m-m', n, k, l)$, $m' = d-2$, $n' = n-2$, and $p = (d-2, n-1)$. Since $d =$ even, we have $m' =$ even. Hence, G_2 and G_3 are even-sized. Also, since $O(m, n, k, l)$ is even-sized, we conclude that G_1 is even-sized. Since $m' =$ even and $n-1 =$ odd, thus p is white. A simple check shows that (G_3, s, p) and (G_1, q, t) are color-compatible. Consider (G_3, s, p). It is easy to see that (G_3, s, p) is not in conditions (F1) and (F2). Hence, it is acceptable. Now, consider (G_1, q, t). Since $n \ge 5$, $d = 2$, and $q_x, t_x \le d$, it suffices to prove that (G_1, q, t) is not in conditions (O6) and (O7). The condition (O6) holds, if $b = 2$ and $t = (d, n-1)$. Obviously, in this case, $(O(m, n, k, l), s, t)$ satisfies condition (O6), a contradiction. Therefore, (G_1, q, t) is not in condition (O6). The condition (O7) holds, if $q_y < t_y$ and $q_y \le a+l$. This is impossible, because of $q_y = n-1$. Thus, (G_1, q, t) is acceptable. In this case, (G_1, q, t) is in Case 1. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 1.4.2.2.2.

Case 3. $c = 1$ and $[(m =$ odd$)$ or $(m =$ even and $a =$ odd or $b =$ odd$)]$. Note that, in this case, $a+1 \le s_y, t_y \le a+l$ and $a, b > 1$. For the case $m =$ even, let $a =$ odd and $b \ge 2$. By symmetry, the result follows, if $a = 2$ and $b =$ odd. This case is similar to Subcase 3.1.1.2, where $x_1 = d+1$, $x_2 = d+k$, $y_1 = 2$, $y_2 = a$, and $s, t \in G_2$. In this case, $G_1 = R(m', n')$, where $m' = k$ and $n' = a-1$. Let $m =$ odd (resp. $m =$ even), then $k =$ even (resp. $n' =$ even). Since $k =$ even or $n' =$ even, thus G_1 is even-sized. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $a = 1$ (in G_2), $n \ge 4$, $d =$ even, and $s_y, t_y \le a+l$, it is enough to show that (G_2, s, t) is not in condition (O7). It is clear that if (G_2, s, t) satisfies condition (O7), then $(O(m, n, k, l), s, t)$ satisfies condition (O9), a contradiction. Therefore, it follows that (G_2, s, t) is not in condition (O7), and hence it is acceptable. In this case (G_2, s, t) is in Case 1 or 2. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.2.2.3.2.

Case 4. $a, b, c, d > 1$. Since $a, b, c, d > 1$ and $k, l \ge 1$, we have $m, n \ge 5$.

Subcase 4.1. $[(d > 2)$ or $(d = 2$ and $[(s_y \ne t_y)$ or $(s_y = t_y = 1$ or $n)])]$ and $[(n =$ odd$)$ or $(n =$ even and $[(a =$ even, $b =$ even, and s (or t) $\ne (d, 1)$ or t (or s) $\ne (d, n))$ or $(a =$ odd or $b =$ odd and $[(m =$ even$)$ or $(m =$ odd and $c > 2)])])]$. This case is similar to Subcase 2.1 of Lemma 4.1, where $m' = d$ and $s, t \in G_1$. In this case, $G_1 = R(m', n)$ and G_2 is a C-shaped grid graph. Since $d =$ even, thus G_1 is even-sized. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_1, s, t) is color-compatible. Since $n \ge 5$ and $m' =$ even, thus (G_1, s, t) is not in condition (F2). The condition (F1) holds, if $d = 2$ and $2 \le s_y = t_y \le n-1$. This is impossible, because in this case $s_y \ne t_y$. Therefore, (G_1, s, t) is not in condition (F1). Hence, it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Consider G_2. Let $m =$ even, then $m - m' =$ even. Therefore, by Lemma 2.5, it has a Hamiltonian cycle. Now, let $m =$ odd, then $m - m' =$ odd. Since $(c > 2)$ or $(c =$ even, $a =$ even, and $b =$ even$)$, by Lemma 2.5, it has a Hamiltonian cycle. Also since $a, b > 1$, thus there exists at least one edge for combining Hamiltonian cycle in G_2 and path in G_1.

Subcase 4.2. $d = 2$, $2 \le s_y = t_y \le n-1$, and $[(n =$ odd$)$ or $(n =$ even and $[(a =$ even, $b =$ even$)$ or $(a =$ odd or $b =$ odd and $[(m =$ even$)$ or $(m =$ odd and $c > 2)])])]$. This case is similar to Subcase 2.2 of Lemma 4.1, where $m' = d$. Notice that, in this case, G_3 is an even-sized C-shaped grid graph and by Lemma 2.5 it has a Hamiltonian cycle.

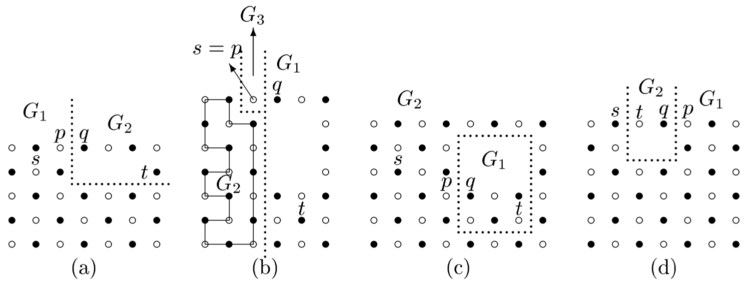

Fig. 43. (a) An L-shaped separation (type I) of $O(m,n,k,l)$, (b) an O-shaped separation of $O(m,n,k,l)$, and (c) and (d) a C-shaped separation of $O(m,n,k,l)$.

Subcase 4.3. $n=$ even, $b=$ even, $a=$ even, s (or t) $=(d,1)$, and t (or s) $=(d,n)$. Assume that $s=(d,1)$ and $t=(d,n)$. This case is similar to Subcase 2.2 of Lemma 4.1, where $m'=d$, $n'=a+l$, $s,p\in G_1$, $q,t\in G_2$, and $w,z\in G_3$ (as shown in Fig. 42(d)). Assume that p and w and q and z are adjacent, $q=(d,n-1)$, and $p=(d,a)$. Since $n=$ even and $b=$ even, we have $n'=$ even. Since $n-n'=$ even and $d=$ even, thus G_1 and G_2 are even-sized. Moreover since $O(m,n,k,l)$ is even-sized, we conclude that G_3 is even-sized. A simple check shows that (G_1,s,p), (G_2,q,t), and (G_3,w,z) are color-compatible. Since $q_x=p_x=s_x=t_x=d$, it is easy to see that (G_1,s,p) and (G_2,q,t) are not in conditions (F1) and (F2), and hence they are acceptable. Now, consider (G_3,w,z). Since $m-m'=$ even, $n=$ even, $a=$ even, $b=$ even, $c\geq 2$, $w=(d+1,a)$, and $z=(d+1,n-1)$, a simple check shows that (G_3,w,z) is not in conditions (F1), (F3) and (F9)–(F18). Hence, it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 1.1 of Lemma 4.1. Notice that, if $s=(d,n)$ and $t=(d,1)$, then the role of q and p can be swapped; i.e., $s,p\in G_2$ and $q,t\in G_1$.

Subcase 4.4. $m=$ odd, $n=$ even, $c=2$, and $a=$ odd or $b=$ odd. In this case, $a=$ odd and $b=$ odd.

Subcase 4.4.1. $s_y,t_y\leq a$ or $s_y,t_y>a+l$. This case is isomorphic to the case $m=$ even and $n=$ odd.

Subcase 4.4.2. (s_y (or t_y) $\leq a$ and t_y (or s_y) $>a$), or (s_y (or t_y) $>a+l$ and $a+1\leq t_y$ (or s_y) $\leq a+l$). This case is isomorphic to Lemma 4.4.

Subcase 4.4.3. $a+1\leq s_y,t_y\leq a+l$. This case is similar to Subcase 3.1.1.1 of Lemma 4.1, where $n'=a-1$ and $s,t\in G_2$. In this case, $G_1=R(m,n')$ and $G_2=O(m,n-n',k,l)$. Since $a=$ odd, thus G_1 is even-sized. Moreover since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_2,s,t) is color-compatible. Since $d\geq 2$, $s_x,t_x\leq d$, $a=1$ (in G_2), and $b=$ odd >1, it is enough to show that (G_2,s,t) is not in condition (O7) (case (a)). It is obvious that if (G_2,s,t) satisfies condition (O7), then $(O(m,n,k,l),s,t)$ satisfies condition (O9) (case (b)) or (O11) (case (b)), a contradiction. Therefore, it follows that (G_2,s,t) is not in condition (O7), and hence it is acceptable. In this case, (G_2,s,t) is in Case 1 or 2. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. Since $m\geq 5$, thus there exists at least one edge for combining Hamiltonian cycle in G_1 and path in G_2. □

Lemma 4.3. *Assume that $(O(m,n,k,l),s,t)$ is an acceptable Hamiltonian path problem with $(a=1)$ or $(a,b>1$ and $c=1)$. Let $s_x\leq d$ and $t_x>d$. Then there is an acceptable separation for $(O(m,n,k,l),s,t)$ and it has a Hamiltonian path.*

Proof. For all the following cases, we prove that $(C(m,n,k,l),s,t)$ has an acceptable separation and show that it has a Hamiltonian path.

Case 1. $d>1$ and $a=b=c=1$. In this case, $t=(d+1,1)$ or $t=(d+1,n)$. If $t_x>d+1$, then $(O(m,n,k,l),s,t)$ satisfies condition (O1). This case is similar to Subcase 2.1 of Lemma 4.1, where $m'=d$, $s,p\in G_1$, $q,t\in G_2$, q and p are adjacent, and $p=(d,1)$ if $t=(d+1,n)$; otherwise $p=(d,n)$. In this case, $G_1=R(m',n)$ and G_2 is a C-shaped grid graph. Consider (G_2,q,t). Here, G_2 is a path subgraph such that q and t are two end vertices of it. It is obvious that (G_2,q,t) is acceptable. Now, consider (G_1,s,p). Clearly, if (G_1,s,p) is not acceptable, then $(O(m,n,k,l),s,t)$ satisfies condition (O1) (case (a)), a contradiction. Therefore, it follows that (G_1,s,p) is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1. Note that, here, G_1 is a rectangular grid graph and by the algorithm in [2] it has a Hamiltonian (s,p)-path.

Case 2. $d,b>1$ and $a=c=1$. Consider the following subcases.

Subcase 2.1. $t_y\leq a+l$. In this case, $t=(d+1,1)$ or $t=(m,a+l)$. If $t_x>d+1$ and $t_y<a+l$, then $(O(m,n,k,l),s,t)$ satisfies condition (O1). Let $\{G_1,G_2\}$ be an L-shaped separation (type I) of $O(m,n,k,l)$ such that $G_1=L(m,n,k',l')$, where $k'=m-d$ and $l'=n-b$, and $G_2=L(m',n',k,l)$, where $m'=k'$ and $n'=l'$ (see Fig. 43(a)). Assume that $s,p\in G_1$, $q,t\in G_2$, q and p are adjacent, and $p=(d,1)$ if $t=(m,a+l)$; otherwise $p=(m,a+l+1)$. (G_2,q,t) and (G_1,s,p) are acceptable, the proof is the same as Case 1. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1. Notice that, in this case, G_1 and G_2 are an L-shaped grid graph, and by the algorithm in [21] they have a Hamiltonian path.

Subcase 2.2. $t_y>a+l$. Let $w=(d,1)$. Since $a=c=1$, $k,l\geq 1$, $d,b>1$, we have $m,n\geq 4$.

Subcase 2.2.1. (s and w are different-colored and [($n=$ even) or ($d=$ even)]) or (s and w are same-colored, $n=$ odd, and $d=$ odd).

Subcase 2.2.1.1. $s\neq(d,1)$ and $t\neq(m,a+l+1)$. This case is similar to Case 1, where $p=(d,1)$. Let $n=$ even or $d=$ even, then G_1 and G_2 are even-sized. Now, let $n=$ odd and $d=$ odd, then G_1 is an odd-sized rectangular grid subgraph with the white majority color and G_2 is an odd-sized C-shaped grid subgraph with the black majority color. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_1, s, p). The condition (F1) holds, if $m' = 2$ and $2 \leq s_y = p_y \leq n-1$. This is impossible, because of $p_y = 1$. The condition (F2) occurs, when $d = 3$, $n =$ even, s is black, and $p_y > s_y$. This case does not occur, because of $p_y = 1$. Therefore, (G_1, s, p) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_2, q, t). Since $a = 1$, it is enough to show that (G_2, q, t) is not in conditions (F1), (F3), and (F11). The condition (F1) holds, if ($c > 1$, $t_y = 1$, and $t_x \leq d+k+1$) or ($c = 1$ and $t_y \leq a+l$). We can easily see that this case can not occur. The condition (F3) holds, if $q_x > d+1$. This is impossible, because of $q_x = d+1$. The condition (F11) holds, if $n =$ odd, $m =$ even, $d =$ even, $b = 3$, $t_x < d+k$, and t is black. Clearly in this case, $(O(m,n,k,l), s, t)$ satisfies condition (O4) (case (b)), a contradiction. Therefore, it follows that (G_2, q, t) does not satisfy condition (F11). Thus, (G_2, q, t) is acceptable. The Hamiltonian path in $(O(m,n,k,l), s, t)$ is obtained similar to Case 1.

Subcase 2.2.1.2. $s = (d, 1)$. In this case, $t \neq (m, a+l+1)$. If $t = (m, a+l+1)$, then $(O(m,n,k,l), s, t)$ satisfies condition (F1). In this case, $n =$ odd and $d =$ odd. This case is similar to Subcase 1.4.2.2.2 of Lemma 4.2 (notice that, here, we have a C-shaped separation (type II)), where $V(G_3) = \{x = d, y = 1\}$, $V(G_2) = \{1 \leq x \leq d-1, y = 1$ and $1 \leq x \leq d, 2 \leq y \leq n\}$, and $G_1 = O(m,n,k,l) \backslash (G_3 \cup G_2)$; as shown in Fig. 43(b). A simple check shows that (G_3, s, p) and (G_1, q, t) are acceptable. The Hamiltonian path in $(O(m,n,k,l), s, t)$ is obtained similar to Subcase 1.4.2.2.2 of Lemma 4.2. Notice that, in this case, G_2 is an even-sized L-shaped grid graph, and by Lemma 2.4 it has a Hamiltonian cycle. The pattern for constructing a Hamiltonian cycle in G_2 is shown in Fig. 43(b). Moreover, since $n - l = b > 1$, thus there exists at least one edge for combining Hamiltonian cycle in G_2 and a Hamiltonian path P_1.

Subcase 2.2.1.3. $t = (m, a+l+1)$. In this case, $s \neq (d, 1)$. If $s = (d, 1)$, then $(O(m,n,k,l), s, t)$ satisfies condition (F1). In this case, $b =$ odd and [($m =$ even and $d =$ odd) or ($m =$ odd and $d =$ even)]. First let $s_y > a+l$, then $(O(m,n,k,l), s, t)$ is isomorphic to Lemma 4.1. Now, let $s_y \leq a+l$. Assume that $(1, 1)$ is the coordinates of vertex in the top right corner in $O(m,n,k,l)$, then by the same argument as in the proof of Subcase 3.2.2.1.2.1 or 3.2.2.1.2.2 the Hamiltonian path is obtained.

Subcase 2.2.2. (s and w are same-colored and [($n =$ even) or ($d =$ even)]) or (s and w are different-colored, $n =$ odd, and $d =$ odd). Notice that, here, $b > 2$. If $b = 2$, then $(O(m,n,k,l), s, t)$ satisfies condition (O6) (case (e)). Since $b > 2$, $a = 1$, and $l \geq 1$, it follows that $n \geq 4$.

Subcase 2.2.2.1. $s_y > a+l$. Assume that $(1, 1)$ is the coordinates of vertex in the bottom right corner in $O(m,n,k,l)$, then $(O(m,n,k,l), s, t)$ is in Case 2 of Lemma 4.2, where $b =$ even, or in Case 2 or 3 of Lemma 4.1, where $b =$ odd.

Subcase 2.2.2.2. $s_y \leq a+l$. Notice that, here, $d > 2$. If $d = 2$, then $(O(m,n,k,l), s, t)$ satisfies condition (O7). Since $d > 2$, $k \geq 1$, and $c = 1$, we have $m \geq 5$.

Subcase 2.2.2.2.1. ($b =$ odd and $t_y > a+l+1$) or ($b =$ even and $t_y > a+l+2$). This case is similar to Subcase 3.1.1.1 of Lemma 4.1, where $G_1 = O(m, n', k, l)$, $G_2 = R(m, n-n')$, $n' = a+l+1$ if $b =$ odd; otherwise $n' = a+l+2$, and

$$p = \begin{cases} (1, n'); & \text{if } (n = \text{even and } d = \text{odd}) \text{ or } (n = \text{odd}) \\ (2, n'); & \text{if } n = \text{even},\ d = \text{even},\ \text{and } s = (d, 1) \\ (d, n'); & \text{otherwise} \end{cases}$$

Let $n =$ odd (resp. $n =$ even) and $b =$ odd. Since $a = 1$, we conclude that $l =$ odd (resp. $l =$ even). Similarly, let $n =$ odd (resp. $n =$ even) and $b =$ even. Since $a = 1$, we conclude that $l =$ even (resp. $l =$ odd). Therefore, $n' =$ odd (resp. $n' =$ even) and $n - n' =$ even. Hence, G_1 and G_2 are even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_2, q, t). Since $m \geq 5$ and $n - n' =$ even, it is enough to show that (G_2, q, t) is not in condition (F1). The condition (F1) holds, if $n - n' = 2$ and $t_x = q_x$. This is impossible, because of $q_x \leq d$ and $t_x > d$. Thus, (G_2, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_1, s, p). Since $d > 2$, $s_x, p_x \leq d$, $a = 1$, and $b \leq 2$, it suffices to prove that (G_1, s, p) is not in conditions (O3) (cases (b)–(d)), (O6) (cases (a)–(c)). The condition (O3) (case (b)) and (O6) (case (a)) hold, if $n =$ even, $d = 4$, and $s_x = p_x = 2$. This is impossible, because of in this case $p = (d, n')$. The condition (O3) (case (c)) occurs, when $n =$ odd, $d =$ even, and $b = 1$. It is easy to see that this case does not occur. The condition (O3) (case (d)) holds, if $n' = 3$, $d =$ odd, and $p_x = d$. This is impossible, because of in this case $p_x = 1$. The condition (O6) (case (b)) holds, if $n =$ even, $b = 2$, $s = (d, 1)$, and $p = (d, n')$. This is impossible, because of $p_x < d$. The condition (O6) (case (c)) holds, if $d =$ even, $n =$ odd, $b = 2$, $s = (d, 1)$, and $p = (d-1, n')$. This case does not occur, because of $p = (1, n')$. Therefore, (G_1, s, p) is not in conditions (O3) (cases (b)–(d)), (O6) (cases (a)–(c)), and hence it is acceptable. In this case, (G_1, s, p) is in Case 2 or 3 of Lemma 4.1, where $d =$ odd, or in Case 2 of Lemma 4.2, where $d =$ even. The Hamiltonian path in $(O(m,n,k,l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 2.2.2.2.2. $t \neq (m, a+l+1)$ and [($b =$ odd and $t_y = a+l+1$) or ($b =$ even and $t_y \leq a+l+2$)]. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1 = d+1$, $x_2 = d+k$, $y_1 = a+l+1$, $y_2 = a+l+2$, and

$$p = \begin{cases} (d, a+l+1); & \text{if } s \text{ and } z \text{ are different-colored} \\ (d, a+l+2); & \text{otherwise} \end{cases}$$

where $z = (d, a+l+1)$. Consider Fig. 43(c). In this case, $G_1 = R(m', n')$, where $m' = k$ and $n' = 2$, and $G_2 = O(m,n,k,l')$, where $l' = l+2$. Since $n' = 2$, thus G_1 is even-sized. Moreover, since $O(m,n,k,l)$ is even-sized, we conclude that G_2 is even-sized. A simple check shows that (G_2, s, p) and (G_1, q, t) are color-compatible. Consider (G_1, q, t). The condition (F1) holds, $m' > 2$ and $d+2 \leq q_x = t_x \leq d+k-1$. Since $q_x = d+1$, clearly (G_1, q, t) is not in condition (F1). The condition (F2) holds, if $m' = 3$ and $q_x = t_x = d+2$. This is impossible, because of $q_x = d+1$. Thus, (G_1, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_2, s, p). Since $d > 2$, $n \geq 4$, $a = 1$, $b \geq 1$, $s_x, p_x \leq d$, and $a + l' - 1 \leq p_y \leq a + l'$, a simple check shows that (G_2, s, p) is not in conditions (F1) and (O1)–(O12). Hence (G_2, s, p) is acceptable. In this case, (G_2, s, p) is in Case 2 or 3 of Lemma 4.1, where $d =$ odd, or in Case 2 of Lemma 4.2, where $d =$ even. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 2.2.2.2.3. $t = (m, a + l + 1)$. In this case, $s \neq (d, 1)$. If $s = (d, 1)$, then $(O(m, n, k, l), s, t)$ satisfies condition (F1). This case is the same as Subcase 2.2.1.3.

Case 3. $d, b, c > 1$ and $a = 1$. Suppose that $w = (d, 1)$.

Subcase 3.1. $t_y = 1$ and $t_x \leq d + k$. In this case, $t = (d + 1, 1)$ or $t = (d + k, 1)$. Notice that if $d + 2 \leq t_x \leq d + k - 1$ and $k > 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O1). Let $\{G_1, G_2\}$ be a C-shaped separation (type I) of $O(m, n, k, l)$ such that $G_2 = R(m', n')$, where $m' = k$ and $n' = 1$, and $G_1 = C(m, n, k, l')$, where $l' = n - b$ (as shown in Fig. 43(d)). Assume that $s, p \in G_1$, $q, t \in G_2$, q and p are adjacent, and

$$p = \begin{cases} (d, 1); & \text{if } (k > 1 \text{ and } t = (d + k, 1)) \text{ or } (k = 1, m = \text{even}, n = \text{odd}, b = 2,\ d = \text{odd}, \\ & s_x \geq d \text{ and } s_y > a + l) \\ (d + k + 1, 1); & \text{otherwise} \end{cases}$$

(G_2, q, t) is acceptable, the proof is the same as Case 1. Now, consider (G_1, s, p). First, assume that $k > 1$. Clearly, if (G_1, s, p) is not acceptable, then $(O(m, n, k, l), s, t)$ satisfies condition (O1) (case (a)), a contradiction. Therefore, it follows that (G_1, s, p) is acceptable. Now, suppose $k = 1$, then $|G_1| = 1$. Hence, G_1 and G_2 are odd-sized. Notice that, here, $q = t$. Assume that t is black (resp. white), then p is white (resp. black). A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Since $b, c, d > 1$ and G_1 is odd-sized, it is enough to show that (G_1, s, p) is not in conditions (F1), (F13), (F14), (F15), and (F17). The conditions (F1), (F14), and (F15) hold, if $s_x, p_x > d + k$. Clearly, this is impossible. The conditions (F13) and (F17) holds, if $d = b = 2$, $n =$ odd, and $s = (1, n - 1)$. It is obvious that, in this case, $(O(m, n, k, l), s, t)$ satisfies condition (O6) (case (d)), a contradiction. Thus, it follows that (G_1, s, p) is not in conditions (F13) and (F17). Hence, it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2. $(t_x > d + k)$ or $(t_x \leq d + k$ and $t_y > a + l)$.

Subcase 3.2.1. (s and w are different-colored and [$(n =$ even$)$ or $(d =$ even$)$]) or (s and w are same-colored, $n =$ odd, and $d =$ odd).

Subcase 3.2.1.1. $s \neq (d, 1)$ and $t \neq (d + k + 1, 1)$. This case is similar to Subcase 2.2.1.1.

Subcase 3.2.1.2. $s = (d, 1)$. In this case, $t \neq (d + k + 1, 1)$. If $t = (d + k + 1, 1)$, then $(O(m, n, k, l), s, t)$ satisfies condition (F1). This case is similar to Subcase 2.2.1.2.

Subcase 3.2.1.3. $t = (d + k + 1, 1)$. In this case, $s \neq (d, 1)$. If $s = (d, 1)$, then $(O(m, n, k, l), s, t)$ satisfies condition (F1). Assume that $(1, 1)$ is the coordinates of vertex in the top right corner in $O(m, n, k, l)$, then by the same argument as in the proof of Subcases 3.2.1.2, 3.2.2.1, or 3.2.2.2 the Hamiltonian path is obtained.

Subcase 3.2.2. (s and w are same-colored and [$(n =$ even$)$ or $(d =$ even$)$]) or (s and w are different-colored, $n =$ odd, and $d =$ odd). Notice that, here, $b > 2$. If $b = 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O6) (case (d)). Since $b > 2$, $l \geq 1$, and $a = 1$, it follows that $n \geq 4$.

Subcase 3.2.2.1. $(l =$ even$)$ or $(l =$ odd and $c =$ odd$)$.

Subcase 3.2.2.1.1. $t_y > a + l$ and $s \neq (d, 1)$ or $t \neq (m, a + l + 1)$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1 = d + k + 1$, $x_2 = m - 1$, and $s, t \in G_2$. In this case, $G_1 = R(m', n')$, where $m' = c - 1$ and $n' = l$, and $G_2 = O(m, n, k', l)$, where $k' = k + m'$. Let $l =$ even (resp. $c =$ odd), then $n' =$ even (resp. $m' =$ even). Thus, G_1 is even-sized. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $s_x \leq d$, $t_x > d$, $d \geq 2$, $b > 2$, and $a = c = 1$ (in G_2), it is enough to show that (G_2, s, t) is not in conditions (F1), (O4), and (O7). The condition (F1) holds, if $s = (d, 1)$ and $t = (m, a + l + 1)$. This is impossible, because we assume that $t \neq (m, a + l + 1)$. It is obvious that if (G_2, s, t) satisfies conditions (O4) and (O7), then $(O(mn, k, l), s, t)$ satisfies conditions (O4) and (O7), a contradiction. Therefore, it follows that, (G_2, s, t) is not in condition (O4) and (O7). Hence it is acceptable. In this case, (G_2, s, t) is in Case 2. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.2.2.3.2 of Lemma 4.2.

Subcase 3.2.2.1.2. $(t_y \leq a + l$ and $t \neq (d + k + 1, 1))$ or $(s = (d, 1)$ and $t = (m, a + l + 1))$. In this case, $c > 2$. If $c = 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O8).

Subcase 3.2.2.1.2.1. $(c =$ odd and $t_x > d + k + 1)$ or $(c =$ even and $t_x > d + k + 2)$. This case is similar to Case 1, where $G_1 = O(m', n, k, l)$, $G_2 = R(m - m', n)$, $m' = d + k + 1$ if $c =$ odd; otherwise $m' = d + k + 2$, and

$$p = \begin{cases} (m', n); & \text{if } s \text{ and } z \text{ are different-colored} \\ (m', n - 1); & \text{otherwise} \end{cases}$$

where $z = (m', n)$. Let $m =$ even (resp. $m =$ odd) and $c =$ odd, then $d + k =$ odd (resp. $d + k =$ even). Similarly let $m =$ even (resp. $m =$ odd) and $c =$ even, then $d + k =$ even (resp. $d + k =$ odd). Hence, $m' =$ even (resp. $m' =$ odd) and $m - m' =$ even. Therefore, G_1 and G_2 are even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_2, q, t). Since $m - m' =$ even, $n \geq 4$, $t_y \leq a + l + 1$, and $q_y \geq n - 1$, clearly (G_2, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_1, s, p). Since $s_x \leq d$, $p_x > d + k$, $a = 1$, $c \leq 2$, $b > 2$, and $p_y \geq n - 1$, it is enough to

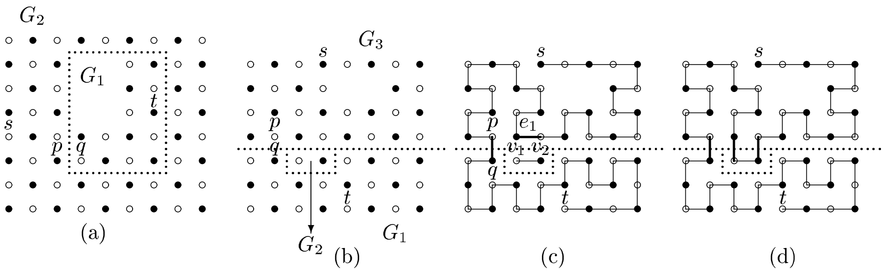

Fig. 44. (a) and (b) An O-shaped separation of $O(m,n,k,l)$, (c) Hamiltonian paths in (G_3,s,p) and (G_1,q,t), and (d) a Hamiltonian (s,t)-path in $O(m,n,k,l)$.

show that (G_1,s,p) is not in conditions (O4) and (O7). (G_1,s,p) is not in conditions (O4) and (O7), the proof is the same as Subcase 3.2.2.1.1. Hence, (G_1,s,p) is acceptable. In this case, (G_1,s,p) is in Case 2 or Subcase 3.2.2.1.1. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.1.2.2. ($c=$ odd and $t_x=d+k+1$) or ($c=$ even and $t_x\le d+k+2$). This case is similar to Subcase 3.1.1.2, where $x_1=d+k+1$, $x_2=d+k+2$, and

$$p=\begin{cases}(d+k+2,a+l+1); & \text{if } s \text{ and } z \text{ are different-colored}\\ (d+k+1,a+l+1); & \text{otherwise}\end{cases}$$

where $z=(d+k+2,a+l+1)$. In this case, $G_1=R(m',n')$, where $m'=2$ and $n'=l$. It is obvious that G_1 and G_2 are even-sized. A simple check shows that (G_1,q,t) and (G_2,s,p) are color-compatible. Consider (G_1,q,t). The condition (F1) holds, if $l>2$ and $a+2\le q_y=t_y\le a+l-1$. This is impossible, because of $q_y=a+l$. The condition (F2) holds, if $l=3$ and $q_y=t_y=a+l-1$. Since $q_y=a+l$, thus this case does not occur. Hence, (G_1,q,t) is acceptable. Now, consider (G_2,s,p). Since $s_x\le d$, $d+k+1\le p_x\le d+k+2$, $a=1$, $b>2$, and $c\ge 1$, a simple check shows that (G_2,s,p) is not in conditions (F1), (O1)–(O12). Therefore, (G_2,s,p) is acceptable. In this case, (G_2,s,p) is in Case 2 or Subcase 3.2.2.1.1. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.1.3. $t=(d+k+1,1)$. Assume that $(1,1)$ is the coordinates of vertex in the top right corner in $O(m,n,k,l)$, then $(O(m,n,k,l),s,t)$ is in Subcase 3.2.2.1.1 or 3.2.2.1.2, where ($l=$ even) or ($l=$ odd and $d=$ odd), or in Subcase 3.2.2.2, where $l=$ odd and $d=$ even.

Subcase 3.2.2.2. $l=$ odd and $c=$ even. In this case, $k=$ even. Since $d>1$, $c=$ even, $k=$ even, $a=1$, $l=$ odd, and $b>2$, we have $m\ge 6$ and $n\ge 5$.

Subcase 3.2.2.2.1. ($n=$ even) or ($n=$ odd and $b>5$). In this case, $k=$ even and $d=$ even if $m=$ even; otherwise $d=$ odd. Let $n=$ odd (resp. $n=$ even), then $n\ge 9$ (resp. $n\ge 6$).

Subcase 3.2.2.2.1.1. $s_y,t_y>a+l+2$. This case is similar to Subcase 3.1.1.1 of Lemma 4.1, where $G_1=O(m,n',k,l)$, $G_2=R(m,n-n')$, $n'=a+l+2$, and $s,t\in G_2$. Since $a=1$ and $l=$ odd, we conclude that $n'=$ even. Let $n=$ odd (resp. $n=$ even), then $n-n'=$ odd ≥ 5 (resp. $n-n'=$ even). Since $m=$ even or $n-n'=$ even, it follows that G_2 is even-sized and since $O(m,n,k,l)$ is even-sized, we conclude that G_1 is even-sized. By Lemma 3.1, (G_2,s,t) is color-compatible. Since $s_x\le d$, $t_x>d$, $n-n'>1$, $n-n'\ne 3$, and $m\ge 6$, it is clear that (G_2,s,t) does not satisfies conditions (F1) and (F2), and hence it is acceptable. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 2.1 of Lemma 4.1. In this case, the pattern for constructing a Hamiltonian cycle in G_1 is shown in Fig. 32(b). Since $m\ge 6$, there exists at least one edge for combining Hamiltonian cycle and path.

Subcase 3.2.2.2.1.2. $t_y\le a+l+2$ and $t_x\le d+k$. This case is similar to Subcase 2.2.2.2.2. Consider (G_2,s,p). Since $n\ge 6$, $d\ge 2$, $s_x,p_x\le d$, and $p_y\le a+l'$, it suffices to prove that (G_2,s,p) is not in conditions (O7) (case (a)). The condition (O7) holds, if $d=2$ and $s_y<a+l+2$. Obviously in this case, $(O(m,n,k,l),s,t)$ satisfies conditions (O7), a contradiction. Therefore, it follows that (G_2,s,p) does not satisfy conditions (O7). Hence, (G_2,s,p) is acceptable.

Subcase 3.2.2.2.1.3. $s_y\le a+l+2$ and $t_y>a+l+2$. Here, $d>2$. If $d=2$, then $(O(m,n,k,l),s,t)$ satisfy conditions (O7). This case is similar to Subcase 3.2.2.2.1.1, where $s,p\in G_1$, $q,t\in G_2$, q and p are adjacent, and $p=(d,n')$ if $s\ne(d,1)$; otherwise $p=(1,n')$. It follows from the proof of Subcase 3.2.2.2.1.1 that G_1 and G_2 are even-sized. It is obvious that, (G_1,s,p) and (G_2,q,t) are color-compatible. (G_1,s,p) does not satisfies conditions (F1), (O1)–(O12), the proof is the same as Subcase 2.2.2.2.1. Thus, it is acceptable. (G_2,q,t) is not in conditions (F1) and (F2), the proof is the same as Subcase 3.2.2.2.1.1. Hence, it is acceptable. In this case, (G_1,s,p) is in Case 2 of Lemma 4.2, where $d=$ even, or in Case 2 or 3 of Lemma 4.1, where $d=$ odd. The Hamiltonian path in $(O(m,n,k,l),s,t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.2.1.4. $t_y\le a+l+2$ and $t_x>d+k$. In this case, $c>2$. If $c=2$, then $(O(m,n,k,l),s,t)$ satisfies condition (O8).

Subcase 3.2.2.2.1.4.1. $t_x>d+k+2$. This case is similar to Subcase 3.2.2.1.2.1.

Subcase 3.2.2.2.1.4.2. $t_x\le d+k+2$ and $t\ne(d+k+1,1)$. This case is similar to Subcase 3.1.1.2 of Lemma 4.1, where $x_1=d+1$, $x_2=d+k+2$, $y_1=a+1$, $y_2=a+l+2$, and $p=(d,a+l+2)$ (see Fig. 44(a)). In this case, $G_1=L(m',n',k,l)$, where $m'=k+2$, and $n'=l+2$, and $G_2=O(m,n,k,',l')$, where $k'=m'$ and $l'=n'$. Assume that $m=$ even (resp. $m=$ odd). Since $c=$ even and $k=$ even, thus $d=$ even (resp. $d=$ odd) and $k'=$ even. Also, since $l=$ odd, we have $l'=$ odd. Since

$k =$ even (resp. $l =$ odd), it follows that $m' - k =$ even (resp. $n' - l =$ even). Obviously, G_1 and G_2 are even-sized. A simple check shows that (G_2, s, p) and (G_1, q, t) are color-compatible. Consider (G_1, q, t). Since $m' - k = 2$ and $n' - l = 2$, it suffices to prove that (G_1, q, t) does not satisfies conditions (F1) and (F6). Since $q = (d + 1, a + l + 2)$, a simple check shows that (G_1, q, t) is not in condition (F1). The condition (F6) holds, if $m' - k = n' - l = 2$ and $t = (d + k + 2, a + l + 1)$. This is impossible, because in this case s and t are same-colored. Therefore, (G_1, q, t) is not in conditions (F1) and (F6), and hence it is acceptable. Now, consider (G_2, s, p). Since $d, b \geq 2$, $a = 1$, $s_x, p_x \leq d$, and $p_y = a + l'$, it sufficient to show that (G_2, s, p) is not in conditions (O7). The condition (O7) occur, when $d = 2$ and $s_y < a + l'$. It is obvious that, in this case, $(O(m, n, k, l), s, t)$ satisfies condition (O7), a contradiction. Thus, it follows that (G_2, s, p) is not in condition (O7), and hence it is acceptable. In this case, (G_2, s, p) is in Case 2 of Lemma 4.2, where $d =$ even, or in Case 2 or 3 of Lemma 4.1, where $d =$ odd. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.2.1.4.3. $t = (d + k + 1, 1)$. Notice that, in this case, $s \neq (d, 1)$. If $s = (d, 1)$, then $(O(m, n, k, l), s, t)$ satisfies condition (F1). Assume that $(1, 1)$ is the coordinates of vertex in the top right corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Subcase 3.2.2.2, where $l =$ odd and $d =$ even, or in Subcase 3.2.2.1, where $l =$ odd and $d =$ odd.

Subcase 3.2.2.2.2. $m =$ even, $n =$ odd and $b \leq 5$. In this case, $d =$ even and $b =$ odd. The only case that occur is $b = 5$ and $t_x = d + 1$. If $(b = 3)$ or $(b = 5$ and $t_x > d)$, then $(O(m, n, k, l), s, t)$ satisfies condition (O4) or (O5). Since $b = 5$, $a = 1$, $l =$ odd, then $n \geq 7$.

Subcase 3.2.2.2.2.1. $t_y \leq a + l + 3$. This case is similar to Subcase 2.2.2.2.2., where $y_2 = a + l + 3$. In this case, $G_1 = R(m', n')$, where $m' = k$ and $n' = 3$. Since $k =$ even, thus G_1 is even-sized. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. Clearly, (G_1, q, t) and (G_2, s, p) are color-compatible. Since $n \geq 7$, $a = 1$, $n' = 3$, and $l =$ odd, it follows that $b = 2$ (in G_2). Consider (G_1, q, t). Since $q_x = t_x = d + 1$, it is obvious that (G_1, q, t) is not in conditions (F1) and (F2), and hence it is acceptable. (G_2, s, p) is not in conditions (F1) and (O1)–(O12), the proof is the same as Subcase 3.2.2.2.1.4.2. Thus, (G_2, s, p) is acceptable. In this case, (G_2, s, p) is in Case 2 of Lemma 4.2. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.2.2.2. $t = (d + 1, n)$ and $s_y > a + l + 2$.

Subcase 3.2.2.2.2.2.1. $d \leq 4$ and $s = (1, n - 1)$. This case is similar to Subcase 1.3.2 of Lemma 4.2. Notice that, in this case, $|G_2| = 2$ or $|G_2| = 4$. When $|G_2| = 4$, a similar to the case $|G_2| = 2$, the result follows. In this case, the pattern for constructing a Hamiltonian cycle in G_1 is shown in Fig. 32(b).

Subcase 3.2.2.2.2.2.2. $s_x = d$ and $d \geq 2$. This case is similar to Subcase 3.2.2.2.1.1. From the proof of Subcase 3.2.2.2.1.1, we know that G_1 and G_2 are even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $b = 5$ and $n' = a + l + 2$, we have $n - n' = 3$. Since $s_x = d$ and $t_x = d + 1$, we can easily see that (G_2, s, t) is not in conditions (F1) and (F2), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.2.2.2.1.1. In this case, the pattern for constructing a Hamiltonian cycle in G_1 is shown in Fig. 32(b).

Subcase 3.2.2.2.2.2.3. $s_x \leq d - 2$ and $[(d = 4$ and $s \neq (1, n - 1))$ or $(d > 4)]$. This case is similar to Subcase 2.3 of Lemma 4.2, where $m' = d - 2$ and $p = (m', n - 1)$. It follows from Subcase 2.3 of Lemma 4.2 that G_1 and G_2 are even-sized. Since $m' =$ even and $n - 1 =$ even, it follows that p is white. Therefore, (G_1, s, p) and (G_2, q, t) are color-compatible. Since $n \geq 7$ and $m' =$ even, it is enough to show that (G_1, s, p) is not in condition (F1). The condition (F1) holds, if $d = 4$ and $s = (1, n - 1)$. This is impossible, because of $s \neq (1, n - 1)$. Thus (G_1, s, p) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_2, q, t). Since $n =$ odd, $b = 5$, $q_y, t_y > a + l$, and $a = 1$, it suffices to prove that (G_2, q, t) is not in condition (O7). Since $t_x = d + 1$ and $q = (m' + 1, n - 1)$, it is clear that (G_2, q, t) is not in condition (O7), and hence it is acceptable. In this case, (G_2, q, t) is in Subcase 3.2.2.2.2.2.1. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.2.2.3. $t = (d + 1, n)$ and $s_y \leq a + l + 2$.

Subcase 3.2.2.2.2.3.1. $s \neq (d, 1)$. In this case, $d > 2$. If $d = 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O7). This case is similar to subcase 3.2.2.2.1.3. Since $a = 1$ and $l =$ odd, it follows that $n' =$ even. It follows from the proof of Subcase 3.2.2.2.1.3, G_1 and G_2 are even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. (G_2, q, t) does not satisfies conditions (F1) and (F2), the proof is the same as Subcase 3.2.2.2.2.2.2, and hence it is acceptable. Now, consider (G_1, s, p). Since $d > 2$, $n' =$ even, and $b = 2$, it is enough to show that (G_1, s, p) is not in condition (O6) (case (b)). The condition (O6) holds, if $s = (d, 1)$. This is impossible, because of $s \neq (d, 1)$. Thus, (G_1, s, p) does not satisfies condition (O6), and hence it is acceptable. In this case, (G_1, s, p) is in Case 2 of Lemma 4.2. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1.

Subcase 3.2.2.2.2.3.2. $s = (d, 1)$. This case is similar to Subcase 1.4.2.2.2 of Lemma 4.2 (note that, in this case, we have an O-shaped separation (type II)), where $G_3 = O(m, n', k, l)$, $n' = a + l + 2$, $V(G_2) = \{d - 1 \leq x \leq d, y = a + l + 3\}$, $G_1 = O(m, n, k, l) \backslash (G_2 \cup G_3)$, and $p = (d - 2, n')$. Consider Fig. 44(b). A simple check shows that (G_1, s, p) and (G_2, q, t) are acceptable. In this case, (G_3, s, p) is in Case 2 of Lemma 4.2. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 1.4.2.2.2 of Lemma 4.2. In this case, G_1 is a C-shaped grid graph, thus we construct a Hamiltonian (q, t)-path in (G_1, q, t) by the algorithm in [21]; see Fig. 44(c). Also, G_2 is a one-rectangle, where $|G_2| = 2$. Let $v_1, v_2 \in G_2$. Clearly, by Case 2 of Lemma 4.2, there exists an edge e_1 for combining a Hamiltonian path and an edge (v_1, v_2) (as depicted in Fig. 44(d)).

Case 4. $a, b, d > 1$ and $c = 1$. In this case $t = (m, a + 1)$ or $t = (m, a + l)$. Notice that if $a + 2 \leq t_y \leq a + l - 1$, then $(O(m, n, k, l), s, t)$ satisfies condition (O1). Assume that $(1, 1)$ is the coordinates of vertex in the bottom right corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Subcase 3.1.

Case 5. $a = b = 1$ and $c, d > 1$. In this case $t = (d + 1, 1)$ or $t = (d + 1, n)$. If $t_x > d + 1$, then $(O(m, n, k, l), s, t)$ satisfies condition (O1). This case is similar to Subcase 2.2.1, where $p = (d, 1)$ if $t = (d + 1, n)$; otherwise $p = (d, n)$.

Case 6. $d = 1$. In this case, $(a, b, c > 1)$ or $(a = 1$ and $b > 1)$. If $(a, b > 1$ and $c = 1)$ or $(a = b = 1$ and $c \geq 1)$, then $O(m, n, k, l)$ satisfies conditions (F1) and (O1). Suppose $(1, 1)$ is the coordinates of vertex in the bottom left corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Cases 1–3, Case 2 of Lemma 4.2, or Case 2 or 3 of Lemma 4.1. □

Lemma 4.4. *Assume $(O(m, n, k, l), s, t)$ is an acceptable Hamiltonian path problem with $a, b, c, d > 1$. Let $s_x \leq d$ and $t_x > d$. Then there is an acceptable separation for $(O(m, n, k, l), s, t)$ and it has a Hamiltonian path.*

Proof. Here, we consider the following cases.

1. $s_y, t_y \leq a$ or $s_y, t_y > a + l$. This case is isomorphic to Case 2 or 3 of Lemma 4.1 or Case 5 of Lemma 4.2.
2. $(s_y \leq a + l$ and $t_y > a)$ or $(s_y > a$ and $t_y \leq a + l)$. In this case, we only consider the first case; i.e., $s_y \leq a + l$ and $t_y > a$. By symmetry, the result follows, if $s_y > a$ and $t_y \leq a + l$.

In the following we only consider that case $s_y \leq a + l$ and $t_y > a$. Consider the following cases. In each of these cases, first we will prove that $(O(m, n, k, l), s, t)$ has an acceptable separation, then we show that it has a Hamiltonian path. Since $a, b, c, d \geq 2$ and $k, l \geq 1$, it follows that $m, n \geq 5$.

Case 1. $(n = \text{even}$ and $m = \text{even})$ or $(n = \text{odd}$ and $d = \text{even})$. Suppose that $w = (d, 1)$. This case is the same as Case 1 of Lemma 4.3, where

$$p = \begin{cases} (d,1); & \text{if } s \text{ and } w \text{ are different-colored, } [(c>2) \text{ or } (c=2 \text{ and } [(b>2) \text{ or} \\ & (b=2 \text{ and } t \neq (m-1,n))])], \textit{ and } [(n=\text{even and } [(k=\text{even}) \text{ or } (k=\text{odd and} \\ & [(d=\text{even and } b=\text{odd}) \text{ or } (d=\text{odd and } [(c>2) \text{ or } (c=2 \text{ and } t \neq (m,a+l+1))])])]) \text{ or} \\ & (n=\text{odd and } b=\text{even})] \\ (d,n-1); & \text{if } (n=\text{even}, \ s \text{ and } w \text{ are different-colored}, \ k=\text{odd}, b=\text{even}, \ \text{and } d=\text{even}) \text{ or} \\ & (n=\text{odd}, s \text{ and } w \text{ are same-colored}, [(b=\text{even}) \text{ or } (b=\text{odd and } c=\text{even})]), \ \text{and} \\ & [(c>2) \text{ or } (c=2 \text{ and } [(b>2) \text{ or } (b=2 \text{ and } t \neq (m,n-1))])] \\ (d,n); & \text{if } [(c>2) \text{ or } (c=2 \text{ and } [(b>2) \text{ or } (b=2 \text{ and } t \neq (m,n-1))])] \text{ and} \\ & [(n=\text{odd}, s \text{ and } w \text{ are different-colored}, \ \text{and } b=\text{odd}) \text{ or} \\ & (n=\text{even}, s \text{ and } w \text{ are same-colored}, \ \text{and } [(k=\text{even}) \text{ or} \\ & (k=\text{odd and } [(d=\text{even and } b=\text{odd}) \text{ or } (d=\text{odd and } [(b=\text{even}) \text{ and} \\ & (b=\text{odd and } c>2)])])])] \\ (d,a) & \text{if } s \text{ and } w \text{ are same-colored, } [(d>2) \text{ or } (d=2 \text{ and } s \neq (1,a))], \ \text{and} \\ & [(n=\text{even}, k=\text{odd}, d=\text{even}, \ \text{and } b=\text{even}) \text{ or } (n=\text{odd}, b=\text{odd}, \ \text{and } c=\text{odd})] \end{cases}$$

Since $d = \text{even}$ or $n = \text{even}$, it follows that G_1 is even-sized. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is even-sized. Clearly, (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_1, s, p). The condition (F1) holds, if $d = 2$ and $2 \leq s_y = p_y \leq n - 1$. The only case that occurs is $s_y = p_y = a$. By our assumption this case does not occur. The condition (F2) holds, if $d = 3$, $n = \text{even}$, s is black, and $p_y \geq n - 1$. This is impossible, because of $p = (d, 1)$. Therefore, (G_1, s, p) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_2, q, t). Since G_2 is even-sized and $a, b, c > 1$, it is enough to show that (G_2, q, t) does not satisfy conditions (F1), (F13) (case (b)), (F14) (case (b)), (F16), (F17), and (F18). The condition (F1) holds, if $b = 2$, $q_x = t_x > d + 1$, and $q_y > a + l$. This is impossible, because in this case $q = (d + 1, n)$ or $q = (d + 1, n - 1)$. The condition (F13) holds, if $m - m' = \text{odd}$, $n = \text{even}$, $a = \text{odd}$, $b = \text{odd}$, $c = 2$, and

(i) $t = (m, a + l + 1)$. The only case that occurs is $n = \text{even}$ and $k = \text{odd}$. A simple check shows that this case does not occur because we assume that in this case $t \neq (m, a + l + 1)$.
(ii) t is black and $q_y \leq a$. This is impossible, because of in this case s is white and $p = (d, n)$.

Thus, (G_2, q, t) does not satisfies condition (F13). The condition (F14) holds, if $m - m' = \text{odd}$, $a = 2$, and $(b = 3$, $d = \text{even}$, t is white, and $q_y \leq a)$, $(b = 3$, $d = \text{odd}$, t is black, and $q_y \leq a)$, or $(c = 3$, t and z are same-colored, where $z = (m - 1, a + 1)$, and $q_y > a + l)$. A simple check shows that these cases does not occur, hence (G_2, q, t) is not in condition (F14). The condition (F16) holds, if $m - m' = \text{odd}$, $n = \text{even}$, $a = \text{odd}$, $b = \text{odd}$, $c = \text{odd}$, t and z are same-colored, where $z = (m - 1, a + 1)$, and $q_y \leq a$. This is impossible, because s and w are same-colored and $q = (d, n)$. Therefore, (G_2, q, t) does not match with condition (F16). The condition (F17) holds, if

(a) $b = c = 2$ and $[(q_y \leq a$, and $t = (m - 1, n))$ or $(q_y > a + l$ and $t = (m, n - 1))]$. Clearly, this case does not occur, because we assume that $t \neq (m - 1, n)$ or $t \neq (m, n - 1)$.

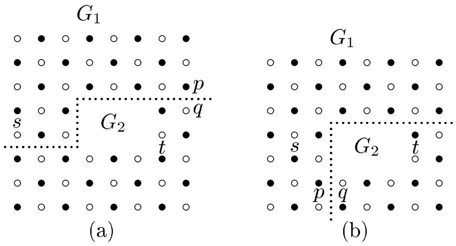

Fig. 45. An L-shaped separation of $O(m, n, k, l)$.

(b) $m - m' = 3$, $b > 2$, $q = (d + 1, a + l + 2)$, and $t = (m, a + l + 1)$. This is impossible, because of $q_y \leq a$ or $q_y \geq n - 1$.
(c) $[m - m' = \text{even}$ or $a = \text{even}]$ and
 - $c = 3$, t and z are same-colored, and $q_y > a + l$, where $z = (m - 1, a + 1)$. This is impossible, because of $q_y \leq a$; or
 - $b = 3$, t and z are same-colored, and $q_y \leq a$, where $z = (d + 1, n - 1)$. This is impossible, because of $q_y > a + l$.

Hence, (G_2, q, t) is not in condition (F17). The condition (F18) holds, if $m - m' = \text{odd}$, $n = \text{even}$, $c = \text{even} \geq 4$, $a = \text{odd}$, $b = \text{odd}$, t and z are same-colored, and $q_y \leq a$; where $z = (d + 1, n - 1)$. This case does not occur because $q = (d + 1, n)$. Therefore, (G_2, q, t) does not satisfies conditions (F1), (F13) (case (b)), (F14) (case (b)), (F16), (F17), and (F18), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Case 1 of Lemma 4.3. Now, let ($c = b = 2$ and $[t = (m - 1, n)$ or $t = (m, n - 1)]$), ($n = \text{even}$, $c = 2$, $d = \text{odd}$, $k = \text{odd}$, and $t = (m, a + l + 1)$), ($d = 2$ and $s = (1, a)$), or ($n = \text{even}$, $k = \text{odd}$, $d = \text{odd}$, $b = \text{odd}$, and $c = 2$). Consider the following subcases.

Subcase 1.1. ($c = b = 2$ and $[t = (m - 1, n)$ or $t = (m, n - 1)]$) or ($n = \text{even}$, $c = 2$, $d = \text{odd}$, $k = \text{odd}$, and $t = (m, a + l + 1)$). This case is similar to Subcase 3.1.1.1 of Lemma 4.1, where $G_1 = C(m, n', k, l)$, $n' = a + l$, $G_2 = R(m, n - n')$, and $p = (1, n')$. Since $m = \text{even}$ or $b = 2$, thus G_2 is even-sized. Additionally, since $O(m, n, k, l)$ is even-sized, we conclude that G_1 is even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_2, q, t). Since $q_x \leq d$ and $t_x > d$, it is obvious that (G_2, q, t) is not in condition (F1). The condition (F2) holds, if $n - n' = 3$, $m = \text{even}$, and t is black. This is impossible, because of in this case $t = (m, a + l + 1)$ and it is white. Hence, (G_2, q, t) is acceptable. Now, consider (G_1, s, p). Since $s_x, p_x \leq d$ and $a, c, d \geq 2$, it is enough to show that (G_1, s, p) is not in conditions (F1), (F13), (F14), (F16), (F17), and (F18). The condition (F1) holds, if $d = 2$ and $2 \leq s_y = p_y \leq a + l - 1$. This is impossible because of $p = (1, n')$. The conditions (F13), (F16), and (F18) hold, if $m = \text{even}$, $n' = \text{odd}$, $d = \text{odd}$, $c = \text{odd}$, and $[(s_x \leq d$ and $p_x > d)$ or $(s_x, p_x \leq d$, $d = 3$, and $p_y < n')]$. It is clear that this case does not occur. The conditions (F14) and (F17) hold, if $d = 3$, s and z are same-colored, and $p_y < s_y$; where $z = (2, n')$. This is impossible, because of $p = (1, n')$. Therefore, (G_1, s, p) is not in conditions (F1), (F13), (F14), (F16), (F17), and (F18), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 3.1.1.1 of Lemma 4.1. Notice that, in this case, G_2 is a rectangular grid graph and it has a Hamiltonian path by the algorithm in [2].

Subcase 1.2. $d = 2$ and $s = (1, a)$. In this case, $m = \text{even}$ and $c = \text{odd}$. This case is similar to Subcase 1.1, where $G_1 = R(m, n')$, $G_2 = C(m, n - n', k, l)$, $n' = a$, and $p = (m, a)$.

Subcase 1.3. $m = \text{even}$, $n = \text{even}$, $k = \text{odd}$, $b = \text{odd}$, and $c = 2$. In this case, $d = \text{odd}$ and s is white. Let $\{G_1, G_2\}$ be an L-shaped separation (type II) of $O(m, n, k, l)$ such that $V(G_1) = \{1 \leq x \leq m, 1 \leq y \leq a$ and $1 \leq x \leq d, a + 1 \leq y \leq a + l\}$ and $G_2 = O(m, n, k, l) \backslash G_1$ (see Fig. 45(a)). Let $s, p \in G_1$, $q, t \in G_2$, q and p are adjacent, and $p = (m, a)$. A simple check shows that (G_1, s, p) and (G_2, q, t) are acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.1 of Lemma 4.3.

Case 2. $n = \text{even}$ and $m = \text{odd}$.

Subcase 2.1. ($s_y \leq a$ and $t_y > a$) or ($a + 1 \leq s_y \leq a + l$ and $t_y > a + l$). This case is isomorphic to the case $m = \text{even}$ and $n = \text{odd}$.

Subcase 2.2. $a + 1 \leq s_y, t_y \leq a + l$.

Subcase 2.2.1. ($k = \text{even}$) or ($k = \text{odd}$ and $[(d = \text{odd}$ and $b = \text{odd})$ or $(d = \text{even}$ and $[(b = \text{even})$ or $(b = \text{odd}$ and $c > 2)])]$). Let $w = (d, 1)$. This case is similar to Case 1, where $p = (d, 1)$ if s and w are different-colored; otherwise $p = (d, n)$.

Subcase 2.2.2. $k = \text{odd}$, $d = \text{even}$, $b = \text{odd}$, and $c = 2$. In this case, $d > 2$. If $d = 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O11) (case (b)). Suppose $(1, 1)$ is the coordinates of vertex in the top right corner in $O(m, n, k, l)$, then $(O(m, n, k, l), s, t)$ is in Subcase 2.2.1.

Subcase 2.2.3. $k = \text{odd}$, $d = \text{odd}$, and $b = \text{even}$. Note that, here, ($d \geq 5$ and $c \geq 3$), ($d \geq 3$ and $c \geq 5$), or ($d = c = 3$ and $[(s$ is white and $[(t_y = a + l)$ or $(s_y = a + 1)])$ or $(s$ is black and $[(s_y = a + l)$ or $(t_y = a + 1)])])$. If $d = c = 3$ and $[(s$ is white, $s_y > a + 1$, and $a + 2 \leq t_y \leq a + l - 1)$ or $(s$ is black, $t_y > a + 1$, and $a + 2 \leq s_y \leq a + l - 1)]$, then $(O(m, n, k, l), s, t)$ satisfies condition (O12) (case (b)). Assume that s is white. This case is similar to Subcase 2.2.1, where $p = (d, n)$ if $(c \geq 5)$ or ($d = c = 3$ and $t_y = a + l$); otherwise $p = (d, a)$. For the case s is black, then let $(1, 1)$ is the coordinates of vertex in the top right corner in $O(m, n, k, l)$.

Case 3. $n = \text{odd}$ and $d = \text{odd}$.

Subcase 3.1. s is white. This case is similar to Case 1, where $p = (d, 1)$ if $s \neq (d, 1)$. Since $d = \text{odd}$ and $n = \text{odd}$, it follows that G_1 is odd-sized with the with majority color. Moreover, since $O(m, n, k, l)$ is even-sized, we conclude that G_2 is odd-sized with the black majority color. It is obvious that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_1, s, p). Since $d = \text{odd} \geq 3$ and $n \geq 5$, (G_1, s, p) is not in conditions (F1) and (F2), and hence it is acceptable. Now, consider (G_2, q, t). Since $q = (d + 1, 1)$, $b, c, a > 1$, and $t_y > a$, a simple check shows that (G_2, q, t) does not satisfies conditions (F1), (F3), and (F10)–(F18), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Case 1. Now, let $s = (d, 1)$. This case is similar to Subcase 2.2.1.2 of Lemma 4.3.

Subcase 3.2. s is black.

Subcase 3.2.1. $b = \text{odd}$. This case is similar to Subcase 1.3.

Subcase 3.2.2. $b = \text{even}$ and $a = \text{odd}$. Let $\{G_1, G_2\}$ be an L-shaped separation (type I) of $O(m, n, k, l)$ such that $G_2 = L(m', n', k, l)$, $m' = m - d$, $n' = n - a$, $G_1 = O(m, n, k, l) \backslash G_2$, as depicted Fig. 45(b). Assume that $s, p \in G_1$, $q, t \in G_2$, q and p are adjacent, and $p = (d, n)$ if $(c > 2)$ or $(c = 2$ and $[(b > 2)$ or $(b = 2$ and $t \neq (m, n-1))])$. Since $n = \text{odd}$, $m = \text{even}$, $d = \text{odd}$, and $a = \text{odd}$, we have $m' = \text{odd}$ and $n' = \text{even}$. Clearly, G_1 and G_2 are even-sized. A simple check shows that (G_1, s, p) and (G_2, q, t) are color-compatible. Consider (G_1, s, p). Since $m, n \geq 6$ and $d, a \geq 3$, it is enough to show that (G_1, s, p) is not in condition (F9). The condition (F9) holds, if $p_x > d$ and $d = 3$. Since $p = (d, n)$, thus (G_1, s, p) is not in condition (F9), and hence it is acceptable. Now, consider (G_2, q, t). Since $b = \text{even}$ and $c \geq 2$, it suffices to prove that (G_2, q, t) is not in conditions (F1), (F6), (F7), (F8), and (F9). The condition (F1) holds, if $(b = 2$ and $d + 2 \leq q_x = t_x \leq m - 1)$ or $(c = 2$ and $a + 2 \leq q_y = t_y \leq n - 1)$. Since $q = (d + 1, n)$, clearly (G_2, q, t) does not satisfy condition (F1). The condition (F6) holds, if $c = b = 2$ and $t = (m, n - 1)$. By our assumption, this case does not occur. The conditions (F8) and (F9) holds, if $c = 3$ and t is black. This is impossible, because t is white. The condition (F7) holds, if $m' = 3$, $b > 2$, and $q = (d + 1, a + l + 2)$. Since $q = (d + 1, n)$, thus (G_2, q, t) is not in condition (F7). Therefore, (G_2, q, t) is not in conditions (F1), (F6), (F7), (F8), and (F9), and hence it is acceptable. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Case 1. Now, let $b = c = 2$ and $t = (m, n - 1)$. This case is similar to Subcase 1.1.

Subcase 3.2.3. $b = \text{even}$ and $a = \text{even}$. Notice that, here, $(a \geq 2$ and $b > 2)$ or $(b = 2$ and $a > 2)$. If $a = b = 2$, then $(O(m, n, k, l), s, t)$ satisfies condition (O10) (case (a)).

Subcase 3.2.3.1. $b > 2$ and $a \geq 2$. This case is similar to Case 3 of Lemma 4.2. From the proof of Case 3 of Lemma 4.2, we know that G_1 and G_2 are even-sized. By Lemma 3.1, (G_2, s, t) is color-compatible. Since $n \geq 5$, $a = 1$ (in G_2), $d = \text{odd} > 1$, $c = \text{odd} > 1$, and $b \geq 3$, it is enough to show that (G_2, s, t) is not in conditions (O3)–(O8). A simple check shows that (G_2, s, t) does not satisfy conditions (O3)–(O8), and hence it is acceptable. In this case, (G_2, s, t) is in Case 3 of Lemma 4.3. The Hamiltonian path in $(O(m, n, k, l), s, t)$ is obtained similar to Subcase 2.2.2.3.2 of Lemma 4.2.

Subcase 3.2.3.2. $b = 2$ and $a > 2$.

Subcase 3.2.3.2.1. $t_x > d + k$. This case is isomorphic to Subcase 3.2.3.1.

Subcase 3.2.3.2.2.. $t_x \leq d + k$. This case is similar to Subcase 1.1. □

Theorem 4.5. *The cases that are mentioned in Lemmas 4.1–4.4 include all the possible cases that may occur in problem $(O(m, n, k, l), s, t)$.*

Proof. We enumerate all the possible cases in the following.

Case 1. $s_x, t_x \leq d$.

Subcase 1.1. $d = \text{odd}$.

Subcase 1.1.1. $d = 1$. $(O(m, n, k, l), s, t)$ is in Case 1 of Lemma 4.1.

Subcase 1.1.2. $d > 1$.

Subcase 1.1.2.1. $[(n = \text{odd})$ or $(n = \text{even}$ and $[(m = \text{even})$ or $(m = \text{odd}$ and $[(a = \text{odd}$ or $b = \text{odd})$, $(a = \text{even}$, $b = \text{even}$, and $c = \text{even})$, or $(a = \text{even}$, $b = \text{even}$, and $c = \text{odd} > 3)])])]$.

Subcase 1.1.2.1.1. $s_x, t_x \leq d - 1$. $(O(m, n, k, l), s, t)$ is in Subcase 2.1 or 2.2 of Lemma 4.1.

Subcase 1.1.2.1.2. $s_x = t_x = d$. $(O(m, n, k, l), s, t)$ is in Subcase 2.4 of Lemma 4.1.

Subcase 1.1.2.1.3. $s_x \leq d - 1$ and $t_x = d$. $(O(m, n, k, l), s, t)$ is in Subcase 2.3 of Lemma 4.1.

Subcase 1.1.2.2. $m = \text{odd}$, $n = \text{even}$, $a = \text{even}$, $b = \text{even}$, and $c = \text{odd} \leq 3$.

Subcase 1.1.2.2.1. $c = 1$. $(O(m, n, k, l), s, t)$ is in Subcase 3.1 of Lemma 4.1.

Subcase 1.1.2.2.2. $c = 3$. $(O(m, n, k, l), s, t)$ is in Subcase 3.2 of Lemma 4.1.

Subcase 1.2. $d = \text{even}$.

Subcase 1.2.1. $a = 1$.

Subcase 1.2.1.1. $d = 2$.

Subcase 1.2.1.1.1. $s_y = t_y$. $(O(m, n, k, l), s, t)$ is in Subcases 1.1–1.3 of Lemma 4.2.

Subcase 1.2.1.1.2 $s_y \neq t_y$. $(O(m, n, k, l), s, t)$ is in Subcase 1.4 of Lemma 4.2.

Subcase 1.2.1.2. $d > 2$.

Subcase 1.2.1.2.1. $s_x, t_x \leq d - 2$. $(O(m, n, k, l), s, t)$ is in Subcase 2.1 of Lemma 4.2.

Subcase 1.2.1.2.2. $s_x, t_x > d - 2$. $(O(m, n, k, l), s, t)$ is in Subcase 2.2 of Lemma 4.2.

Subcase 1.2.1.2.3. $s_x \leq d - 2$ and $t_x > d - 2$. $(O(m, n, k, l), s, t)$ is in Subcase 2.3 of Lemma 4.2.

Subcase 1.2.2. $c = 1$. In this case, $a + 1 \le s_y, t_y \le a + l$.

Subcase 1.2.2.1. $m =$ even.

Subcase 1.2.2.1.1. $a =$ even and $b =$ even.

Subcase 1.2.2.1.1.1. $d = 2$. This case is the same as Subcase 1.2.1.1.

Subcase 1.2.2.1.1.2. $d > 2$. This case is the same as Subcase 1.2.1.2.

Subcase 1.2.2.1.2. $a =$ odd or $b =$ odd. $(O(m, n, k, l), s, t)$ is in Case 3 of Lemma 4.2.

Subcase 1.2.2.2. $m =$ odd. $(O(m, n, k, l), s, t)$ is in Case 3 of Lemma 4.2.

Subcase 1.2.3. $a, b, c > 1$. $(O(m, n, k, l), s, t)$ is in Case 4 of Lemma 4.2.

Case 2. $s_x \le d$ and $t_x > d$.

Subcase 2.1. $d = 1$. $(O(m, n, k, l), s, t)$ is in Case 6 of Lemma 4.3.

Subcase 2.2. $d > 1$ and $a = 1$.

Subcase 2.2.1. $b = c = 1$. $(O(m, n, k, l), s, t)$ is in Case 1 of Lemma 4.3.

Subcase 2.2.2. $c = 1$ and $b > 1$.

Subcase 2.2.2.1. $t_y \le a + l$. $(O(m, n, k, l), s, t)$ is in Subcase 2.1 of Lemma 4.3.

Subcase 2.2.2.2. $t_y > a + l$. $(O(m, n, k, l), s, t)$ is in Subcase 2.2 of Lemma 4.3.

Subcase 2.2.3. $b, c > 1$.

Subcase 2.2.3.1. $t_y = 1$ and $t_x \le d + k$. $(O(m, n, k, l), s, t)$ is in Subcase 3.1 of Lemma 4.3.

Subcase 2.2.3.2. $(t_x > d + k)$ or $(t_x \le d + k$ and $t_y > a + l)$. $(O(m, n, k, l), s, t)$ is in Subcase 3.2 of Lemma 4.3.

Subcase 2.2.4. $b = 1$ and $c > 1$. $(O(m, n, k, l), s, t)$ is in Case 5 of Lemma 4.3.

Subcase 2.3. $d, a, b > 1$ and $c = 1$. $(O(m, n, k, l), s, t)$ is in Case 4 of Lemma 4.3.

Subcase 2.4. $a, b, c, d > 1$.

Subcase 2.4.1. $s_y, t_y > a + l$ or $s_y, t_y \le a$. $(O(m, n, k, l), s, t)$ is isomorphic to Subcase 1.1.2 or 1.2.3.

Subcase 2.4.2. $(s_y \le a + l$ and $t_y > a)$ or $(s_y > a$ and $t_y \le a + l)$. Let $s_y \le a + l$ and $t_y > a$. By symmetry, the result follows, if $s_y > a$ and $t_y \le a + l$.

Subcase 2.4.2.1. $(n =$ even and $m =$ even$)$ or $(n =$ odd and $d =$ even$)$. $(O(m, n, k, l), s, t)$ is in Case 1 of Lemma 4.4.

Subcase 2.4.2.2. $n =$ even and $m =$ odd. $(O(m, n, k, l), s, t)$ is in Case 2 of Lemma 4.4.

Subcase 2.4.2.3. $n =$ odd and $d =$ odd. $(O(m, n, k, l), s, t)$ is in Case 3 of Lemma 4.4.

All possible cases are exhausted, and the proof of Theorem 4.5 is completed. □

By Theorem 3.2 and Lemmas 4.1–4.4, we have the following result:

Theorem 4.6. *$O(m, n, k, l)$ has a Hamiltonian path between s and t if and only if $(O(m, n, k, l), s, t)$ is acceptable.*

Theorem 4.7 summarizes our results.

Theorem 4.7. *In an acceptable Hamiltonian path problem $(O(m, n, k, l), s, t)$, a Hamiltonian path between s and t can be found in linear time.*

Proof. Consider the pseudo-code of our algorithm in Algorithm 4.1. Step 1 does only a constant number of partitioning by Lemmas 4.1–4.4, which is done in constant time. Since $O(m, n, k, l)$ divides into some rectangular grid subgraphs and Hamiltonian paths and cycles in these grid subgraphs can be found in linear time by Theorem 2.2, thus Step 2 can be done in linear time. Step 3 which combines Hamiltonian paths and cycles by Lemmas 4.1–4.4 requires only constant time. Therefore, in total, our algorithm has linear time complexity, i.e. linear in the number of vertices of $O(m, n, k, l)$. □

5. Conclusion

We gave necessary conditions for the existence of Hamiltonian (s, t)-paths in rectangular grid graphs with a rectangular hole. Then we present a linear-time algorithm for finding a Hamiltonian (s, t)-path in these graphs. The proposed algorithm considers the holes, which helps us to apply it to more realistic applications. For example, in the robotic applications it is now possible to consider obstacles, or in the picturesque maze generation the image may have some holes.

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