# Necessary and Sufficient Conditions for Hamiltonian based on Linear Diophantine Equation Systems with Cycle Vector

Guohun Zhu[1], Chunwei Song[2], Kaoru Hirota[3], Fangyan Dong[4], Yonghua Wu[1]
1) Guilin University of Electronic Technology 2) Beijing University
3) Tokyo Institute of Technology 4) Beijing University of Technology[4]
*Email: ccghzhu@guet.edu.cn wuyh@mails.guet.edu.cn*

***Abstract*—Two necessary and sufficient conditions are presented for Hamiltonian cycle problem in simple undirected graph using linear Diophantine equation systems with cycle vector. The first one is based on the incidence matrix and the second one is based on edge-adjacency matrix. It is proven that the solution set of the cycle vector correspond to the edges of Hamiltonian cycle in a given graph. Based on these result conditions, two necessary conditions for the Hamiltonian graph are given by determining the rank of the matrix.**

***Keywords-edge adjacency matrix; incidence matrix; Hamiltonian cycle; Linear Diophantine equation system;cycle vector; rank***

## I. INTRODUCTION

It's a well-known NP-complete Problem to determine Hamiltonian cycle existing or not[1,2]. Two survey of the Hamiltonian graph are presented in [1,10]. It can be identified that determining Hamiltonian existing or not has two cases: sufficient condition and necessary condition. All of methods have two limitations. The first limitation is that the sufficient conditions can not determine non-Hamiltonian graphs, and the necessary conditions can not verify Hamiltonian graphs. For example, non-Hamiltonian bipartite graph K3,4 could not be determined with the sufficient conditions proposed by Ore[4] ,Rahman[2],or Li[5]. The second limitation is that there has no edges which is composed a Hamiltonian cycle in any previously known conditions in [1,10].

In this paper the necessary and sufficient conditions for Hamiltonian cycle are formulated as two systems of linear equations $AX = B$, where one of coefficient matrix $A$ is incidence matrix, and another $A$ is edge-adjacency matrix.

## II. DEFINITION AND PROPERTIES

Throughout this paper is only consider the finite simple undirected graph $G = (V, E)$ which has no multi-edges and no self loops, the set of vertices and set of edges of $G$ is denoted by $V(G)$ and $E(G)$, respectively.

There are three matrix representations for graphs; the vertex-adjacency matrix, the incidence matrix and the edge-adjacency matrix. Let $n$ and $m$ denote the number of vertices and edges, respectively. Three matrices are defined as following.

**Definition 1.** [6~7] The incidence matrix $C$ of undirected graph $G$ is a two dimensional $n \times m$ table, each row respect one vertex, each column respect one edge, the $c_{ij}$ in $C$ are given by

$$c_{ij} = \begin{cases} 1, & \textit{if the jth edge is} \\ & \textit{incident on the ith vertex}; \\ 0, & \textit{otherwise}. \end{cases}$$

It is obvious that every column of an incidence matrix has exactly two 1 entries.

**Definition 2. [8]** The edge-adjacency matrix $F$ of undirected graph $G$ is a two dimensional $m \times m$ table, each row and column respect one edge, the $f_{ij}$ in $F$ are given by

$$f_{ij} = \begin{cases} 1, & \textit{if the ith edge share common} \\ & \textit{vertex with jth edge, and } i \neq j; \\ 0, & \textit{otherwise}. \end{cases}$$

Similar incidence matrix, let the cycle of undirected $G$ labeled the edge with {0, 1}, the cycle and the edges could be defined a relation by cycle matrix.

**Definition 3.** [7] The cycle matrix $L$ of undirected graph $G$ is a two dimensional $l \times m$ table, each row respect one cycle, each column respect one edge, the $l_{ij}$ of $L$ are given by

$$l_{ij} = \begin{cases} 1, & \textit{if the jth edge belong ith cycle}; \\ 0, & \textit{otherwise}. \end{cases}$$

The edge-adjacency matrix $F$ and the incidence matrix $C$ of $G$ has a relation.

**Theorem 2.1.** [7] The edge-adjacency matrix $F$ and the incidence matrix C of $G$ are related by

$$F = C^T C - 2I \quad (1)$$

Where $I$ is the identity matrix.

There is also a relation between the cycle matrix L and the incidence matrix **C**.

**Theorem 2.2.[7]** The cycle matrix L and the incidence matrix C of $G$ are related as

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$$LC^T \equiv 0 \pmod{2} \tag{2}$$

Let the vector $X^T$ be one of the row vector of cycle matrix and named as *cycle vector*, where $Xi \in \{0,1\}$, Xi=1 means the respect edge existed in the cycle, otherwise, it is not in the cycle. Since it is belong the cycle matrix, the cycle vector X and the incidence matrix C undirected graph G has a relation .

**Theorem 2.3.** The cycle vector X and the incidence matrix C of $G$ are related as

$$CX \equiv 0 \pmod{2} \tag{3}$$

The length of each cycle vector X in a graph is same, but the number of "1" and location of "1" is different. To distinguish the different cycle vector, this paper define a partial relation operator: $x \prec y \;\; iff \;\; (x = x \cap y) \wedge (x \neq y)$, and the equivalent relation is;

$x \prec y \;\; iff \;\; (y = x \cup y) \wedge (x \neq y)$. For example, let $x_1 = (1,0,1,1,0,0)^T$, $x_2 = (1,0,1,0,1,0)^T$, $y = (1,0,1,0,1,1)$, then $x_2 \prec y$, and x1 ,y do not satisfy the relation.

Let $E_m$ denote a row vector (1,1,1,1,….), which has m entries of "1" in the vector, let $E_m$ denote a column vector with all entries of "1", which is satisfied $E^m = E_m{}^T$.

It is obvious that the Hamiltonian cycle can also be viewed as a cycle vector, which is noted as Hamiltonian Vector in this paper. According the Hamiltonian cycle definition, this paper gives a necessary condition for the Hamiltonian vector.

**Theorem 2.4.** Giving an undirected graph G(V,E), |V|=n, |E|=m, A Hamiltonian cycle vector $X$ for G has following nature.

(1) $X^T E_m{}^T = E_m X = n$,

(2) $C_i X = 2$, where $1 \le i \le n$,

(3) $CX = \begin{pmatrix} 2 \\ \cdots \\ 2 \end{pmatrix} = 2E_n$

While $C_{nm}$ is the incidence matrix, $C_i$ is the $ith$ row vector of $C_{nm}$

**Proof:**

(1) since Hamiltonian cycle across each vertex exactly once, and |V|=n

$$X^T E_m{}^T = \sum_{i=1}^{m} X_i \Rightarrow X^T E_m{}^T = n, \quad \Rightarrow$$

$$(X^T E_m{}^T)^T = n \Rightarrow E_m{}^T X = n.$$

(2) since each vertex has two edges in Hamiltonian cycle, thus the set of $X \cap C_i$ has only two elements, and also $C_i X = \sum_{j=1}^{m} C_{ij} X_j$ ,for binary operator $1 \times 1 = 1 \cap 1,\ 0 \times 1 = 0 \cap 1,\ 0 \times 0 = 0 \cap 0$, so that $C_i X = 2$.

(3)according to (2)

$$CX = \begin{pmatrix} C_1 X \\ \cdots \cdots \\ C_n X \end{pmatrix} = \begin{pmatrix} 2 \\ \cdots \\ 2 \end{pmatrix} = 2E_n \qquad \Delta$$

III. TWO NECESSARY AND SUFFICIENT CONDITIONS

This section gives two necessary and sufficient conditions for determining and solving Hamiltonian cycle vector method based on the cycle vector.

**Theorem 3.1.** Giving incidence matrix $C_{nm}$ of a undirected graph G=(V,E) , |V|=n, |E|=m, $n \ge 1 \;\; and \;\; m \ge 1$, if and only if a cycle vector $X_m$ are satisfied

$$\begin{pmatrix} C \\ E_m \end{pmatrix} X = \begin{pmatrix} 2E^n \\ n \end{pmatrix} \tag{5}$$

And any $X' \prec X$ are not satisfied

$$CX' \equiv 0 \pmod{2} \tag{6}$$

Then G exists at least one Hamiltonian cycle, the $x_i = 1$ respects the edge $e_i$ existence in Hamiltonian cycle.

**Proof:**

- **Necessity**

According theorem 2.4, the Hamiltonian vector X is satisfied

$$CX = 2E^n \tag{7}$$

Since X is cycle vector, $x_i \in \{0,1\}$

$$E_m X = n \tag{8}$$

The equations (5) could be combined by (7) (8),

Suppose there exists a $X' \prec X$ which is satisfied equation (6) and $|X'| = n_1$. According to theorem 2.3, $X'$ is a cycle vector, the remind edges set $Y' = X - X'$ and $|Y'| = n - n_1$. since

$$CY' = CX - CX' \equiv 0 \pmod{2} \tag{9}$$

Then $Y'$ is also a sub cycle. Since $|X| = |X'| + |Y'| - |X' \cap Y'| = n$, so $Y' \cap X' = \theta$, it is deduce that $X$ has two sub cycle and no Hamiltonian cycle. So the suppose is false.

- **Sufficient**

since $CX = 2E_n \Rightarrow CX \equiv 0 \pmod{2}$, According to Diophantine equation solution existing condition, which exits the integral solution, and according theorem 2.3 and $X_i \in \{0,1\}$, the X is a cycle vector. Since

$$E_m X = n \Rightarrow |X| = n, \qquad (10)$$

It deduces that the cycle vector $X$ has two cases.

case 1) vector $X$ consists of only one cycle, in this case, $X$ represents the Hamiltonian Cycle.

case 2) vector $X$ consist of two or more cycle,.

In this case, Since the $X' \prec X$ is not satisfied equation(6), which means if the number of selected edges are less than n, no sub cycle existing, so vector X can only combined as 1 cycle, each vertex has only two edges, it is satisfied the definition of Hamiltonian. Δ

Now let the vector $X$ be a minimal solution in theorem 3.1 in following section.

**Theorem 3.2.** Giving the edge-adjacency matrix $F_{mm}$ of an undirected graph G=(V,E), |V|=n, |E|=m, if and only if a cycle vector $X$, $xi \in \{0,1\}$ is satisfied

$$\begin{pmatrix} F \\ E_m \end{pmatrix} X = \begin{pmatrix} K^m \\ n \end{pmatrix} \qquad (11)$$

And any $X' \prec X$ are not satisfied

$$FX' \equiv 0 \pmod 2$$

Where the value of column vector $k_i = 2$ when $x_i = 1$, otherwise $k_i = 4$.

**Proof :**

- **Necessity**

According to theorem 3.1,

$$C^T \begin{pmatrix} C \\ E_m \end{pmatrix} X = C^T \begin{pmatrix} 2E^m \\ n \end{pmatrix} \qquad (12)$$

So $$(C^T C)X = 2C^T E^n \qquad (13)$$

Since each column of $C$ has two 1, then each row of $C^T$ has two 1.

$$2C^T E^n = 4E^m,$$

According to theorem 2.1

$$(C^T C)X = (F + 2I)X = (4E^m) \qquad (14)$$

$$FX = (4E^m) - (2I)X \qquad (15)$$

Let $X_i = 1$, then $2X_i = 2 \Rightarrow FX_i = 2$

Let $X_i = 0$, then $2X_i = 0 \Rightarrow FX_i = 4$

So column vector $K$ is a value when $x_i = 1$ then $k_i = 2$, otherwise $k_i = 4$.

- **Sufficient**

Let the $e_i$ labeled with $X_i$, all of the $X_i = 1$ is represented as a edge list $P = (e_1, e_2, e_3 \ldots, e_n)$.

Since $$FX = \sum_{j=1}^{m} f_{ij} e_j$$

where

$$f_{ij} e_j = \begin{cases} 1, & if <ei,ej> and\ e_j \in P \\ 0, & otherwise \end{cases}$$

According to equation (15), when $e_j \in P$, then $F_i\ X = 2$ otherwise $F_i\ X = 4$. This imply that if $e_j \in P$, $e_j$ share common vertex exactly only two times, so $P$ is a cycle set. And let $X'$ be a sub set of $P$, which is not satisfied

$$FX' \equiv 0 \pmod 2$$

And according equations (10),

$$|P| = |X| = n$$

So $P$ is only a Hamiltonian cycle and length of $P$ is value of $n$. Δ

Seeing from theorem 3.1 and theorem 3.2, the solution of Hamiltonian problem changes to the solution of Diophantine problem.

## IV. Two Necessary Conditions Based on Rank of Matrix

It is obvious that the systems of linear equations in theorem 3.1 and theorem 3.2 is a sub set of 0-1 programming [3], so the complexity of the solving the system of linear equation is also NP-complete.

Considering the solution conditions for system of linear equations, the rank of coefficient matrix must equal rank of the argument matrix. So it can deduce necessary conditions determining by rank of matrix for Hamiltonian graphs.

**Corollary 4.1.** given the incidence matrix $C_{nm}$, of a connected undirected graph G=(V,E) $n \geq 1, m \geq 1$.

(1) If $r(C) \neq r(C, 2E^n)$, there does not exist Hamiltonian cycle.

(2) If the graph has a Hamiltonian Cycle, the incidence matrix must be satisfied

$$r(C) = r\begin{pmatrix} C \\ E_n \end{pmatrix} = r(C, 2E^n) = r\begin{pmatrix} C & 2E^n \\ E_m & n \end{pmatrix}$$

It is also easy to obtain a corollary from theorem 3.2.

**Corollary 4.2.** given the edge-adjacency matrix $F_{mm}$ of a connected undirected graph G=(V,E) $n \geq 1, m \geq 1$. It is satisfied

$$r(F) = r\begin{pmatrix} F \\ E_m \end{pmatrix} \qquad (16)$$

$$r(CF) = r\begin{pmatrix} CF & P \\ E_m & n \end{pmatrix} \qquad (17)$$

Where each column of vector $P^n$ and $P_i = 4\deg(V_i) - 4$,

**Proof :**

According theorem 3.2, it imply the equation (16).

Since Hamiltonian graph is satisfied.

$$FX = (4E^m) - (2I)X \tag{18}$$

$$CFX = 4CE^m - 2CX = 4CE^m - 4E^n \tag{19}$$

Since $CE^m$ equal the degree of each vertex

$$CFX = 4CE^m - 4E^n = 4\deg(V) - 4$$

According the solution condition for linear system of equations,

$$r(CF) = r(CF \quad P)$$

Since $E_m X = n$

$$r(CF) = r\begin{pmatrix} CF & P \\ E_m & n \end{pmatrix} \qquad \Delta$$

Based on corollary 4.1，it is easy to determining some non Hamiltonian graph, here gives three examples

Example (1) when n=1,n=2， simple graph has no Hamiltonian cycle.

Example (2) bipartite graph $K_{2,3}$， $K_{3,4}$ which has $r(C) \neq r\begin{pmatrix} C & 2E^n \\ E_m & n \end{pmatrix}$, so its have no Hamiltonian cycle.

Example (3) Cycle graph and Start graph has same edge-adjacency matrix $F_{mm}$, which are shown in figure 1 and figure 2.

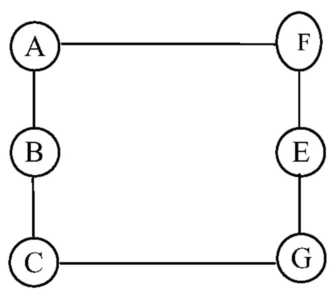


**Figure 1. Cycle graph $C_6$**

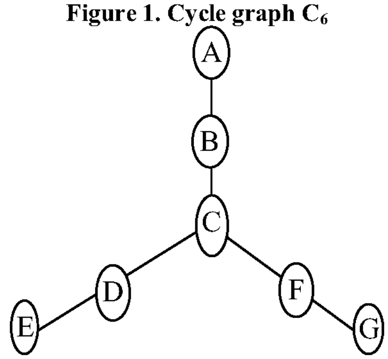


**Figure 2. Start fraph $S_7$**

Since incidence matrix of cycle graph is square matrix, the graph in figure.1 could obtain a solution by solving the equations in theorem 31. the second graph $r(C) \neq r\begin{pmatrix} C & 2E^n \\ E_m & n \end{pmatrix}$, it has no Hamiltonian cycle.

Example (4) In 1983, Horton find there is a 54 vector，3-regular， bipartite non Hamiltonian graph [9]， where $r(C) = 53$, and $r(C, 2E^n) = 54$， according corollary 4.1, it has no Hamiltonian existing.

## V. CONCLUSION

Having seen the theorem 3.1 and theorem 3.2, it is obvious that the necessary and sufficient conditions on Hamiltonian cycle solution overcome the weakness of conventional sufficient conditions which could not give the edges information, required connected and preceding.

With the corollary in section Ⅳ, this paper give two rank of matrix method to determining the non Hamiltonian graph method, which can be solved in polynomial time.

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