Extending The Thue-Morse Sequence

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***Abstract*—In this paper, we discuss various ways to extend the Thue-Morse Sequence [3] when used as a fair-share sequence. Included are N definitions of the original sequence, M extensions to n players, and proofs of equality for all definitions. In the appendix are several complexity analyses for both space and time of each definition.**

***Index Terms*—component, formatting, style, styling, insert**

I. Introduction

II. The Original Sequence

A. Definition 1 - Parity of Hamming Weight

This definition appears in [1, 5]

The Hamming Weight, as typically defined, is the digit sum of a binary number. In other words, it is a count of the high bits in a given number. A common way to generate the Thue-Morse Sequence is to take the parity of the Hamming Weight for each natural number. We can use that as follows:

$$\begin{aligned} p(0) &= 0 \\ p(n) &= n + p\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \pmod 2 \end{aligned} \tag{1}$$

Under this definition, you can construct the Thue-Morse Sequence using the following, starting at 0:

$$T_{2,1}(n) = p(n) \tag{2}$$

The subscript indicates that we are using 2 players (writing in base 2) and that we are using the first definition laid out in this paper. Note that when we extend to n players, the T function will get a second parameter for the number of players, so it will look like $T_{n,d}(x, s)$, where s is the size of the player pool.

B. Definition 2 - Invert and Extend

This definition is more natural to think about as extending a tuple that contains the sequence. We will give a recurrence relation below, but to build an intuition we will work in this framework first.

Let $t(n)$ be the first 2^n elements of the Thue-Morse Sequence. Given this, we can define:

$$\text{inv}(\mathbf{x}) = \begin{cases} 0, & \text{if } x_i = 1 \\ 1, & \text{if } x_i = 0 \end{cases} \quad \text{for } \mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \tag{3}$$

$$\begin{aligned} t(0) &= \langle 0 \rangle \\ t(n) &= t(n-1) \cdot \text{inv}(t(n-1)) \end{aligned} \tag{4}$$

Given the above, we can define a recurrence relation that will give us individual elements. It will be less efficient to compute, but will allow proofs of equivalence to be easier.

$$\begin{aligned} T_{2,2}(0) &= 0 \\ T_{2,2}(n) &= T_{2,2}\left(n - 2^{\lfloor \log_2(n) \rfloor}\right) + 1 \pmod 2 \end{aligned} \tag{5}$$

C. Definition 3 - Substitute and Flatten

This definition appears in [4, 5]

D. Definition 4 - Recursion

This definition appears in [4]

$$\begin{aligned} T_{2,4}(0) &= 0 \\ T_{2,4}(2n) &= T_{2,4}(n) \\ T_{2,4}(2n+1) &= 1 - T_{2,4}(n) \pmod 2 \end{aligned} \tag{6}$$

E. Definition 5 - Highest Bit Difference

This definition appears in [2]

The text below is from Wiki and needs to be entirely rewritten. I was able to derive the formula on my own from translating their code. This method leads to a fast method for computing the Thue–Morse sequence: start with t0 = 0, and then, for each n, find the highest-order bit in the binary representation of n that is different from the same bit in the representation of n - 1. If this bit is at an even index, tn differs from tn - 1, and otherwise it is the same as tn - 1.

$$\begin{aligned} T_{2,5}(0) &= 0 \\ T_{2,5}(n) &= \begin{matrix} \lfloor \log_2(n \oplus (n-1)) \rfloor \\ + T_{2,5}(n-1) + 1 \end{matrix} \pmod 2 \end{aligned} \tag{7}$$

F. Summary

III. The Extensions

A. Definition 1 - Modular Digit Sums

To extend definition 1 from 2 to n players, we must first map our concept of parity to base n. We can do this by taking the parity equation defined above and replacing the 2s with n, as follows:

$$
\begin{aligned}
p_n(0) &= 0 \\
p_n(x) &= x + p_n\left(\left\lfloor \frac{x}{n} \right\rfloor\right) \pmod{n}
\end{aligned} \tag{8}
$$

Under this definition, you can construct the Thue-Morse Sequence using the following, starting at 0:

$$
T_{n,1}(x, s) = p_s(x) \tag{9}
$$

1) Proof of Equivalence with Original Definition 1:

B. Definition 2 - Increment and Extend

In the original version of this definition, we inverted the elements. In base 2, this is the same thing as adding 1 (mod 2). Given that, let $t(x, n)$ be the first n^x elements of the Extended Thue-Morse Sequence.

$$
\mathrm{inc}(\mathbf{x}, n) = \begin{matrix} x_i + 1 \pmod{n} \\ \text{for } \mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \end{matrix} \tag{10}
$$

$$
\begin{aligned}
t(0, n) &= \langle 0 \rangle \\
t(1, n) &= \langle 0, 1, \ldots, n-1 \rangle \\
t(x, n) &= t(x-1, n) \cdot \mathrm{inc}(t(x-1, n), n)
\end{aligned} \tag{11}
$$

Given the above, we can define a recurrence relation that will give us individual elements. It will be less efficient to compute, but will allow proofs of equivalence to be easier.

$$
\begin{aligned}
T_{n,2}(0, s) &= 0 \\
T_{n,2}(x, s) &= T_{n,2}\left(x - s^{\lfloor \log_s(x) \rfloor}\right) + 1 \pmod{s}
\end{aligned} \tag{12}
$$

1) Proof of Equivalence with Original Definition 2:

C. Definition 3 - Substitute and Flatten

1) Proof of Equivalence with Original Definition 3:

D. Definition 4 - Recursion

1) Proof of Equivalence with Original Definition 4:

E. Definition 5 - Highest Digit Difference

Note: This is speculative, and testing needs to be done

1) Proof of Equivalence with Original Definition 5:

F. Summary

IV. Proving Equivalence Between Extensions

A. Correlating Definition 1 and Definition 2

B. Correlating Definition 1 and Definition 3

C. Correlating Definition 1 and Definition 4

D. Summary

V. Proving Persistence of Original Properties

VI. Acknowledgment

The preferred spelling of the word "acknowledgment" in America is without an "e" after the "g". Avoid the stilted expression "one of us (R. B. G.) thanks …". Instead, try "R. B. G. thanks…". Put sponsor acknowledgments in the unnumbered footnote on the first page.

VII. Appendix

A. Complexity of Original Definition 1

1) Time Complexity:

2) Space Complexity:

B. Complexity of Original Definition 2

1) Time Complexity:

2) Space Complexity:

C. Complexity of Original Definition 3

1) Time Complexity:

2) Space Complexity:

D. Complexity of Original Definition 4

1) Time Complexity:

2) Space Complexity:

E. Complexity of Extension Definition 1

1) Time Complexity:

2) Space Complexity:

F. Complexity of Extension Definition 2

1) Time Complexity:

2) Space Complexity:

G. Complexity of Extension Definition 3

1) Time Complexity:

2) Space Complexity:

H. Complexity of Extension Definition 4

1) Time Complexity:

2) Space Complexity:

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