In [159…
```
import os
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.axes_grid1.inset_locator import zoomed_inset_axes
from mpl_toolkits.axes_grid1.inset_locator import mark_inset
```

In [160…
```
fs = 20 # fontsize
lw = 2  # line width
width = 15 # figure width
height = 8 # figure height
plt.rcParams["axes.formatter.useoffset"] = False # No tick label offset
plt.rcParams.update({'font.size': fs-4}) # Standard font size
```

In [161…
```
path = "Data\co2_mm_mlo.csv"
t, co2, co2_av = np.loadtxt(path, skiprows = 53,delimiter = ',',comments='#',useco
```

Given: Time series of measurements of concentration $\vec{c} = (c_1, c_2, \ldots, c_m)$ According to time $\vec{t} = (t_1, t_2, \ldots, t_)$.

Forward model: Polynome of order 4

$$c(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3 + a_4 \cdot t^4 = \sum_{i=0}^{4} a_i \cdot t^i \tag{1}$$

Matrix notation:

$$c(\vec{t}) = K\vec{a} \tag{2}$$

with $\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ and $K = \begin{pmatrix} t_1^0 & t_1 & t_1^2 & t_1^3 & t_1^4 \\ t_2^0 & t_2 & t_2^2 & t_2^3 & t_2^4 \\ t_3^0 & t_3 & t_3^2 & t_3^3 & t_3^4 \\ \ldots & & \ldots & & \ldots \end{pmatrix}$

In [162…
```
def forward_model(timeseries, n):

    K_T= np.zeros([n,len(timeseries)])
    for j in range(n):
        K_T[j]= timeseries**j

    K_model=np.transpose(K_T)

    return K_model
```

In [163…
```
orderp1 = 5
K = forward_model(t, orderp1)
a_fit, r, rank, sv = np.linalg.lstsq(K,co2, rcond=1)
c_fit= np.matmul(K,a_fit)

print("K is an (mxn) matrix with m= "+ str(len(t))+ " and n=" + str(orderp1)+ ". Th
```

```
K is an (mxn) matrix with m= 776 and n=5. The rank of K is 3.
```

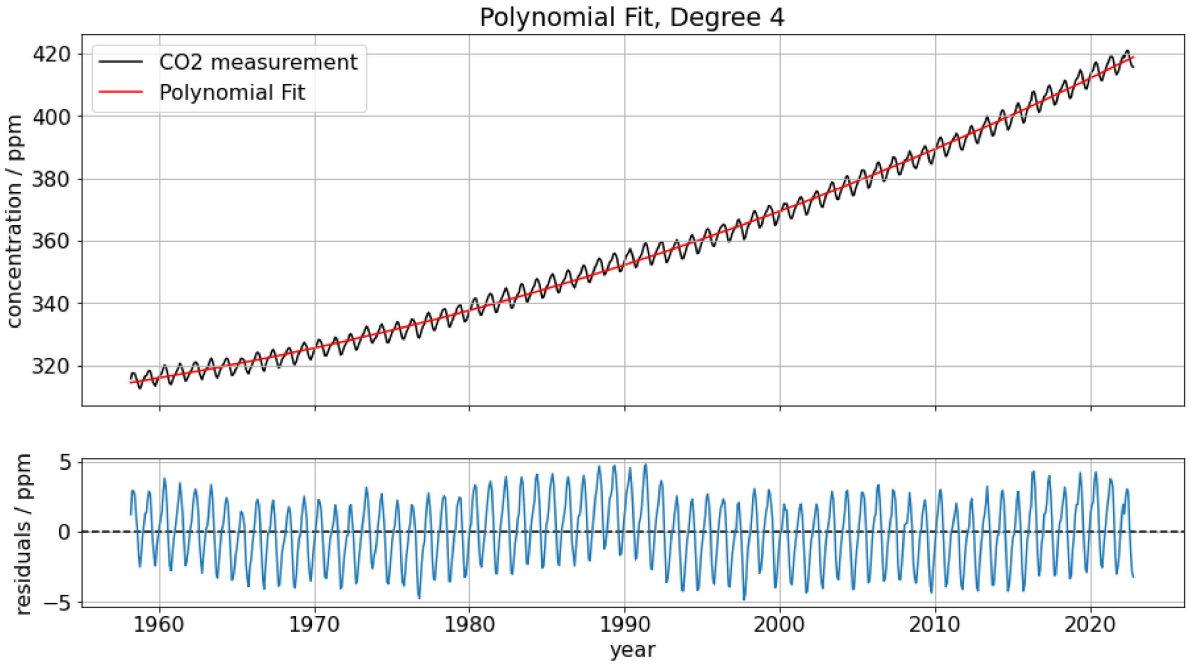
As K is no square matrix, there is no inverse of K, thus there is no exact solution of the problem.

In [164…
```
fig, (ax, res) = plt.subplots(2, 1, sharex=True, gridspec_kw={"height_ratios": [5,
ax.plot(t,co2,label="CO2 measurement",color="black",zorder=1)
ax.plot(t,c_fit,label="Polynomial Fit",color="red",zorder=1)

res.errorbar( t, co2 - c_fit, capsize=3)

res.axhline(0, linestyle="--", color="black")
res.grid()
ax.grid()
res.set_xlabel("year")
res.set_ylabel('residuals / ppm')
ax.set_ylabel("concentration / ppm")
ax.set_title("Polynomial Fit, Degree 4")
ax.legend()
```

Out[164]:
```
<matplotlib.legend.Legend at 0x20384bc8820>
```



Residuals show systematic deviation from fit curve (Oscillation around zero). --> polynomial fit does not describe underlying measurement very well --> Oscillating term should be introduced.

Advanced forward model: polynome of order 4, overlaid with fourier series of order 4

$$c(t) = \sum_{i=0}^{4} a_i \cdot t^i + b_i \cos(2\pi \cdot i \cdot t) + c_i \sin(2\pi \cdot i \cdot t) \tag{3}$$

Matrix notation:

$$\vec{c(t)} = K\vec{a} \tag{4}$$

$$\text{with } \vec{a} = \begin{pmatrix} a_0 \\ \dots \\ a_4 \\ b_0 \\ \dots \\ b_4 \\ c_0 \\ \dots \\ c_4 \end{pmatrix} \text{ and}$$

$$K = \begin{pmatrix} t_1^0 & \dots & t_1^4 & \cos(2\pi \cdot 0 \cdot t_1) & \dots & \cos(2\pi \cdot 0 \cdot t_1) & \sin(2\pi \cdot 0 \cdot t_1) & \dots & \sin(\\ t_2^0 & \dots & t_2^4 & \cos(2\pi \cdot 0 \cdot t_2) & \dots & \cos(2\pi \cdot 4 \cdot t_2) & \sin(2\pi \cdot 0 \cdot t_2) & \dots & \sin(\\ t_3^0 & \dots & t_3^4 & \cos(2\pi \cdot 0 \cdot t_3) & \dots & \cos(2\pi \cdot 4 \cdot t_3) & \sin(2\pi \cdot 0 \cdot t_3) & \dots & \sin(\\ & \dots & & & \dots & & & \dots & \end{pmatrix}$$

In [165…

```
def advanced_forward_model(timeseries, order_poly, order_fourier):

    K_T= np.zeros([2*order_fourier+order_poly,len(timeseries)])

    for j in range(order_poly):
        K_T[j]= timeseries**j
    for j in range(order_fourier):
        K_T[j+order_poly]= np.cos(2*np.pi*j*timeseries)
        K_T[j+order_poly+order_fourier]= np.sin(2*np.pi*j*timeseries)

    K=np.transpose(K_T)

    return K
```

In [166…

```
n_poly = 5
n_fourier= 5
K_2 = advanced_forward_model(t, n_poly, n_fourier)

x_fit_2, r_2, rank_2, sv_2 = np.linalg.lstsq(K_2,co2, rcond=1)
y_fit_2= np.matmul(K_2,x_fit_2)

print("K is an (mxn) matrix with m= "+ str(len(t))+ " and n=" + str(5)+ ". The ranl
```

```
K is an (mxn) matrix with m= 776 and n=5. The rank of K is 3.
```

In [168…

```
fig, (ax, res) = plt.subplots(2, 1, sharex=True, gridspec_kw={"height_ratios": [5,

axins = zoomed_inset_axes(ax, 2.5, loc=4)
axins.plot(t,co2,color="black" )
axins.plot(t,y_fit,color="red" )
axins.plot(t,y_fit_2,color="green" )
x1, x2, y1, y2 = 1960, 1970, 310, 330 # specify the limits
axins.set_xlim(x1, x2) # apply the x-limits
axins.set_ylim(y1, y2)
mark_inset(ax, axins, loc1=2, loc2=4, fc="none", ec="0.05")

ax.plot(t,co2,label="CO2 measurement",color="black",zorder=1)
ax.plot(t,y_fit,label="Polynomial Fit",color="red",zorder=1)
ax.plot(t,y_fit_2,label="Fourier Fit",color="green",zorder=1)

res.errorbar( t, co2 - y_fit,label="Polynomial Residuals",color="red", capsize=3)
res.errorbar( t, co2 - y_fit_2,label="Fourier Residuals",color="green", capsize=3)
```

```
res.axhline(0, linestyle="--", color="black")
res.grid()
ax.grid()
res.set_xlabel("year")
res.set_ylabel('residuals / ppm')
ax.set_ylabel("concentration / ppm")
ax.set_title("Polynomial-Fourier Fit, Degree 4")
ax.legend(loc=2)
```

Out[168]: <matplotlib.legend.Legend at 0x2038522ca90>