Algorithm of the maximum likelihood estimation with Newton–Raphson iteration for single-population mortality models

The log-likelihood function of the **Poisson Lee–Carter model** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt}\left\{D_{xtj}\ (\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj}) - E^c_{xt}e^{\alpha_{xj}+\beta_{1,xj}\kappa_{1,tj}}\right\}$ + constant

$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj})$

The parameters θ are updated using the following iterative scheme:

$\widehat{\theta}^{\,(v+1)} = \widehat{\theta}^{\,(v)} - \frac{\partial \ell_j{}^{(v)}/\partial\theta}{\partial^2 \ell_j{}^{(v)}/\partial\theta^2}$ where $\ell_j{}^{(v)} = \ell_j{}^{(v)}(\widehat{\theta}^{\,(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters*

$\hat{\alpha}_{xj} = \frac{1}{n_Y}\sum_t ln\frac{D_{xtj}}{E^c_{xt}}$ where n_Y is the total number of calendar years

$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y+1)}{2}$ where n_Y is the total number of calendar years

$\hat{\beta}_{1,xj} = \frac{1}{n_A}$ where n_A is the total number of ages

2. *Update the parameter* $\hat{\alpha}_{xj}$

$$\hat{\alpha}^{(v+1)}_{xj} = \hat{\alpha}^{(v)}_{xj} + \frac{\sum_t\left(D_{xtj} - E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)}{\sum_t\left(E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj}\frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameter* $\hat{\kappa}_{1,tj}$

$$\hat{\kappa}^{(v+1)}_{1,tj} = \hat{\kappa}^{(v)}_{1,tj} + \frac{\sum_x\left(D_{xtj} - E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)\hat{\beta}^{(v)}_{1,xj}}{\sum_x\left(E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)\left(\hat{\beta}^{(v)}_{1,xj}\right)^2}$$

Adjust the parameter $\hat{\kappa}_{1,tj}$ in order to satisfy the constraint $\rightarrow \hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj}\,(\hat{\kappa}_{1,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj})$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameter* $\hat{\beta}_{1,xj}$

$$\hat{\beta}^{(v+1)}_{1,xj} = \hat{\beta}^{(v)}_{1,xj} + \frac{\sum_t\left(D_{xtj} - E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)\hat{\kappa}^{(v)}_{1,tj}}{\sum_t\left(E^c_{xt}e^{\hat{\alpha}^{(v)}_{xj}+\hat{\beta}^{(v)}_{1,xj}\hat{\kappa}^{(v)}_{1,tj}}\right)\left(\hat{\kappa}^{(v)}_{1,tj}\right)^2}$$

Adjust the parameter $\hat{\beta}_{1,xj}$ in order to satisfy the constraint $\rightarrow \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj}\frac{1}{\sum_x \hat{\beta}_{1,xj}}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)
In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **Binomial Lee–Carter model** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt}\left\{D_{xtj} ln\, q_{xtj} + \left(E_{xt}^0 - D_{xtj}\right) ln\left(1 - q_{xtj}\right)\right\}$ + constant
where $q_{xtj} = e^{\alpha_{xj}+\beta_{1,xj}\kappa_{1,tj}}/(1 + e^{\alpha_{xj}+\beta_{1,xj}\kappa_{1,tj}})$
$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj})$
The parameters θ are updated using the following iterative scheme:
$\widehat{\theta}^{(v+1)} = \widehat{\theta}^{(v)} - \frac{\partial \ell_j{}^{(v)}/\partial\theta}{\partial^2 \ell_j{}^{(v)}/\partial\theta^2}$ where $\ell_j{}^{(v)} = \ell_j{}^{(v)}(\widehat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters*

$\hat{\alpha}_{xj} = \frac{1}{n_Y}\sum_t ln \frac{D_{xtj}}{E_{xt}^0}$ where n_Y is the total number of calendar years
$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y+1)}{2}$ where n_Y is the total number of calendar years
$\hat{\beta}_{1,xj} = \frac{1}{n_A}$ where n_A is the total number of ages

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t \left((D_{xtj}e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + D_{xtj})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)\right)}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)^2\right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj}\frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj}$
Calculate the log-likelihood function ℓ_j

3. *Update the parameter $\hat{\kappa}_{1,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left((D_{xtj}e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + D_{xtj})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)\right)\hat{\beta}_{1,xj}^{(v)}}{\sum_x \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)^2\right)\left(\hat{\beta}_{1,xj}^{(v)}\right)^2}$$

Adjust the parameter $\hat{\kappa}_{1,tj}$ in order to satisfy the constraint $\rightarrow \hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj}\,(\hat{\kappa}_{1,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj})$
Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameter $\hat{\beta}_{1,xj}$*

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t \left((D_{xtj}e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + D_{xtj})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)\right)\hat{\kappa}_{1,tj}^{(v)}}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}})/(e^{\hat{\alpha}_{xj}^{(v)}+\hat{\beta}_{1,xj}^{(v)}\hat{\kappa}_{1,tj}^{(v)}} + 1)^2\right)\left(\hat{\kappa}_{1,tj}^{(v)}\right)^2}$$

Adjust the parameter $\hat{\beta}_{1,xj}$ in order to satisfy the constraint → $\hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **Poisson Lee–Carter model with two factors** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt}\left\{D_{xtj}\,(\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj} + \beta_{2,xj}\kappa_{2,tj}) - E_{xt}^{c} e^{\alpha_{xj}+\beta_{1,xj}\kappa_{1,tj}+\beta_{2,xj}\kappa_{2,tj}}\right\}$ + constant

$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj}, \beta_{2,xj}, \kappa_{2,tj})$

The parameters θ are updated using the following iterative scheme:

$\widehat{\theta}^{(v+1)} = \widehat{\theta}^{(v)} - \frac{\partial \ell_j{}^{(v)}/\partial\theta}{\partial^2 \ell_j{}^{(v)}/\partial\theta^2}$ where $\ell_j{}^{(v)} = \ell_j{}^{(v)}(\widehat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters*

$\hat{\alpha}_{xj} = \frac{1}{n_Y}\sum_t ln \frac{D_{xtj}}{E_{xt}^{c}}$ where n_Y is the total number of calendar years

$$\left.\begin{aligned}\hat{\kappa}_{1,tj} &= (n_Y, \dots, 1) - \frac{(n_Y+1)}{2}\\ \hat{\kappa}_{2,tj} &= (n_Y, \dots, 1) - \frac{(n_Y+1)}{2}\end{aligned}\right\}\text{ where } n_Y \text{ is the total number of calendar years}$$

$$\left.\begin{aligned}\hat{\beta}_{1,xj} &= \frac{1}{n_A}\\ \hat{\beta}_{2,xj} &= \frac{1}{n_A}\end{aligned}\right\}\text{ where } n_A \text{ is the total number of ages}$$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t\left(D_{xtj} - E_{xt}^{c} e^{\widehat{\alpha}_{xj}^{(v)}+\widehat{\beta}_{1,xj}^{(v)}\widehat{\kappa}_{1,tj}^{(v)}+\widehat{\beta}_{2,xj}^{(v)}\widehat{\kappa}_{2,tj}^{(v)}}\right)}{\sum_t\left(E_{xt}^{c} e^{\widehat{\alpha}_{xj}^{(v)}+\widehat{\beta}_{1,xj}^{(v)}\widehat{\kappa}_{1,tj}^{(v)}+\widehat{\beta}_{2,xj}^{(v)}\widehat{\kappa}_{2,tj}^{(v)}}\right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj}\frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj}\frac{1}{n_Y}\sum_t \hat{\kappa}_{2,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x\left(D_{xtj} - E_{xt}^{c} e^{\widehat{\alpha}_{xj}^{(v)}+\widehat{\beta}_{1,xj}^{(v)}\widehat{\kappa}_{1,tj}^{(v)}+\widehat{\beta}_{2,xj}^{(v)}\widehat{\kappa}_{2,tj}^{(v)}}\right)\hat{\beta}_{1,xj}^{(v)}}{\sum_x\left(E_{xt}^{c} e^{\widehat{\alpha}_{xj}^{(v)}+\widehat{\beta}_{1,xj}^{(v)}\widehat{\kappa}_{1,tj}^{(v)}+\widehat{\beta}_{2,xj}^{(v)}\widehat{\kappa}_{2,tj}^{(v)}}\right)\left(\hat{\beta}_{1,xj}^{(v)}\right)^2}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \hat{\beta}_{2,xj}^{(v)}}{\sum_x \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \left(\hat{\beta}_{2,xj}^{(v)} \right)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$ in order to satisfy the constraints →
$\hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} \, (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj})$ and $\hat{\kappa}_{2,tj} = \sum_x \hat{\beta}_{2,xj} \, (\hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj})$
Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$*

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \left(\hat{\kappa}_{1,tj}^{(v)} \right)^2}$$

$$\hat{\beta}_{2,xj}^{(v+1)} = \hat{\beta}_{2,xj}^{(v)} + \frac{\sum_t \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \hat{\kappa}_{2,tj}^{(v)}}{\sum_t \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} \right) \left(\hat{\kappa}_{2,tj}^{(v)} \right)^2}$$

Adjust the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$ in order to satisfy the constraints → $\hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$ and $\hat{\beta}_{2,xj} = \hat{\beta}_{2,xj} \frac{1}{\sum_x \hat{\beta}_{2,xj}}$
Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)
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The log-likelihood function of the **Binomial Lee–Carter model with two factors** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt} \{ D_{xtj} ln \, q_{xtj} + \left(E_{xt}^0 - D_{xtj} \right) ln\left(1 - q_{xtj} \right) \} +$ constant
where $q_{xtj} = e^{\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj} + \beta_{2,xj}\kappa_{2,tj}} / (1 + e^{\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj} + \beta_{2,xj}\kappa_{2,tj}})$
$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj}, \beta_{2,xj}, \kappa_{2,tj})$
The parameters θ are updated using the following iterative scheme:
$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial\theta}{\partial^2 \ell_j^{(v)} / \partial\theta^2}$ where $\ell_j^{(v)} = \ell_j^{(v)}(\hat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters*

$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t ln \frac{D_{xtj}}{E_{xt}^0}$ where n_Y is the total number of calendar years

$$\left.\begin{aligned}\hat{\kappa}_{1,tj} &= (n_Y, \dots, 1) - \frac{(n_Y+1)}{2} \\ \hat{\kappa}_{2,tj} &= (n_Y, \dots, 1) - \frac{(n_Y+1)}{2}\end{aligned}\right\} \text{where } n_Y \text{ is the total number of calendar years}$$

$$\left.\begin{aligned}\hat{\beta}_{1,xj} &= \frac{1}{n_A} \\ \hat{\beta}_{2,xj} &= \frac{1}{n_A}\end{aligned}\right\} \text{where } n_A \text{ is the total number of ages}$$

2. *Update the parameter* $\hat{\alpha}_{xj}$

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} +$$

$$+\frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right)}{\sum_t \left((E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \to \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameters* $\hat{\kappa}_{1,tj}$ *and* $\hat{\kappa}_{2,tj}$

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} +$$

$$+\frac{\sum_x \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\beta}_{1,xj}^{(v)}}{\sum_x \left((E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\beta}_{1,xj}^{(v)} \right)^2}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} +$$

$$+\frac{\sum_x \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\beta}_{2,xj}^{(v)}}{\sum_x \left((E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\beta}_{2,xj}^{(v)} \right)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{1,tj}$ in order to satisfy the constraints $\to$
$\hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} \, (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj})$ and $\hat{\kappa}_{2,tj} = \sum_x \hat{\beta}_{2,xj} \, (\hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj})$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameters* $\hat{\beta}_{1,xj}$ *and* $\hat{\beta}_{2,xj}$

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} +$$

$$+\frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t \left((E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\kappa}_{1,tj}^{(v)} \right)^2}$$

$$\hat{\beta}_{2,xj}^{(v+1)} = \hat{\beta}_{2,xj}^{(v)} +$$

$$+\frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\kappa}_{2,tj}^{(v)}}{\sum_t \left((E_{xt}^{0} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\kappa}_{2,tj}^{(v)} \right)^2}$$

Adjust the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$ in order to satisfy the constraints $\to \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$ and
$\hat{\beta}_{2,xj} = \hat{\beta}_{2,xj} \frac{1}{\sum_x \hat{\beta}_{2,xj}}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)
In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **CBD model** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt}\left\{D_{xtj} ln\, q_{xtj} + \left(E_{xt}^0 - D_{xtj}\right) ln\left(1 - q_{xtj}\right)\right\}$ + constant
where $q_{xtj} = e^{\kappa_{1,tj}+(x-\bar{x})\kappa_{2,tj}}/(1 + e^{\kappa_{1,tj}+(x-\bar{x})\kappa_{2,tj}})$
$\theta = (\kappa_{1,tj}, \kappa_{2,tj})$
The parameters θ are updated using the following iterative scheme:
$\widehat{\theta}^{(v+1)} = \widehat{\theta}^{(v)} - \frac{\partial \ell_j{}^{(v)}/\partial\theta}{\partial^2 \ell_j{}^{(v)}/\partial\theta^2}$ where $\ell_j{}^{(v)} = \ell_j{}^{(v)}(\widehat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters and fix the values of the parametric coefficient* $\beta_{2,xj}$

$\hat{\kappa}_{1,tj} = 0$
$\hat{\kappa}_{2,tj} = 0$
$\beta_{2,xj} = (x - \bar{x})$

2. *Update the parameter* $\hat{\kappa}_{1,tj}$

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + D_{xtj})/(e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)\right)}{\sum_x \left((E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}})/(e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)^2\right)}$$

Calculate the log-likelihood function ℓ_j

3. *Update the parameter* $\hat{\kappa}_{2,tj}$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + D_{xtj})/(e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)\right)(x-\bar{x})}{\sum_x \left((E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}})/(e^{\hat{\kappa}_{1,tj}^{(v)}+(x-\bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)^2\right)(x-\bar{x})^2}$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 3)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **reduced Plat model with three factors** can be formulated as follows:

$\ell_j(\theta) =$
$= \sum_{xt}\left\{D_{xtj}\left(\alpha_{xj} + \kappa_{1,tj} + (\bar{x}-x)\kappa_{2,tj} + (\bar{x}-x)^+\kappa_{3,tj}\right) - E_{xt}^c e^{\alpha_{xj} + \kappa_{1,tj} + (\bar{x}-x)\kappa_{2,tj} + (\bar{x}-x)^+\kappa_{3,tj}}\right\} +$
$+$ constant
$\theta = (\alpha_{xj}, \kappa_{1,tj}, \kappa_{2,tj}, \kappa_{3,tj})$
The parameters θ are updated using the following iterative scheme:
$\widehat{\theta}^{(v+1)} = \widehat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)}/\partial\theta}{\partial^2 \ell_j^{(v)}/\partial\theta^2}$ where $\ell_j^{(v)} = \ell_j^{(v)}(\widehat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters and fix the values of the parametric coefficients $\beta_{2,xj}$ and $\beta_{3,xj}$*

$\hat{\alpha}_{xj} = \frac{1}{n_Y}\sum_t ln\frac{D_{xtj}}{E_{xt}^c}$ where n_Y is the total number of calendar years
$\hat{\kappa}_{1,tj} = 0$
$\hat{\kappa}_{2,tj} = 0$
$\hat{\kappa}_{3,tj} = 0$
$\beta_{2,xj} = (\bar{x}-x)$
$\beta_{3,xj} = (\bar{x}-x)^+$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t\left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)}{\sum_t\left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj} + (\bar{x}-x)\frac{1}{n_Y}\sum_t \hat{\kappa}_{2,tj} + (\bar{x}-x)^+\frac{1}{n_Y}\sum_t \hat{\kappa}_{3,tj}$
Calculate the log-likelihood function ℓ_j

3. *Update the parameters $\hat{\kappa}_{1,tj}$, $\hat{\kappa}_{2,tj}$ and $\hat{\kappa}_{3,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x\left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)}{\sum_x\left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x\left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)(\bar{x}-x)}{\sum_x\left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+\hat{\kappa}_{3,tj}^{(v)}}\right)(\bar{x}-x)^2}$$

$$\hat{\kappa}_{3,tj}^{(v+1)} = \hat{\kappa}_{3,tj}^{(v)} + \frac{\sum_x \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+ \hat{\kappa}_{3,tj}^{(v)}} \right) (\bar{x} - x)^+}{\sum_x \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x}-x)^+ \hat{\kappa}_{3,tj}^{(v)}} \right) ((\bar{x} - x)^+)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$, $\hat{\kappa}_{2,tj}$ and $\hat{\kappa}_{3,tj}$ in order to satisfy the constraints →
$\hat{\kappa}_{1,tj} = \hat{\kappa}_{1,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj}$, $\hat{\kappa}_{2,tj} = \hat{\kappa}_{2,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{2,tj}$ and $\hat{\kappa}_{3,tj} = \hat{\kappa}_{3,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{3,tj}$
Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 3)
In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **reduced Plat model with two factors** can be formulated as follows:

$\ell_j(\theta) = \sum_{xt}\left\{D_{xtj}\ (\alpha_{xj} + \kappa_{1,tj} + (\bar{x} - x)\kappa_{2,tj}) - E_{xt}^c e^{\alpha_{xj} + \kappa_{1,tj} + (\bar{x}-x)\kappa_{2,tj}}\right\}$ + constant
$\theta = (\alpha_{xj}, \kappa_{1,tj}, \kappa_{2,tj})$
The parameters θ are updated using the following iterative scheme:
$\widehat{\theta}^{(v+1)} = \widehat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)}/\partial\theta}{\partial^2 \ell_j^{(v)}/\partial\theta^2}$ where $\ell_j^{(v)} = \ell_j^{(v)}(\widehat{\theta}^{(v)})$

<u>The iterative updating scheme</u>

1. *Get initial values for the parameters and fix the values of the parametric coefficient* $\beta_{2,xj}$

$\hat{\alpha}_{xj} = \frac{1}{n_Y}\sum_t ln\frac{D_{xtj}}{E_{xt}^c}$ where n_Y is the total number of calendar years
$\hat{\kappa}_{1,tj} = 0$
$\hat{\kappa}_{2,tj} = 0$
$\beta_{2,xj} = (\bar{x} - x)$

2. *Update the parameter* $\hat{\alpha}_{xj}$

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}} \right)}{\sum_t \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}} \right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \to \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj} + (\bar{x} - x)\frac{1}{n_Y}\sum_t \hat{\kappa}_{2,tj}$
Calculate the log-likelihood function ℓ_j

3. *Update the parameters* $\hat{\kappa}_{1,tj}$ *and* $\hat{\kappa}_{2,tj}$

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}}\right)}{\sum_x \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}}\right)}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x \left(D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}}\right)(\bar{x} - x)}{\sum_x \left(E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x}-x)\hat{\kappa}_{2,tj}^{(v)}}\right)(\bar{x} - x)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$ in order to satisfy the constraints → $\hat{\kappa}_{1,tj} = \hat{\kappa}_{1,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj} = \hat{\kappa}_{2,tj} - \frac{1}{n_Y}\sum_t \hat{\kappa}_{2,tj}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 3)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.