

Algorithm of the maximum likelihood estimation with Newton–Raphson iteration for single-population mortality models

The log-likelihood function of the **Poisson Lee–Carter model** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xt} \{D_{xtj} (\alpha_{xj} + \beta_{1,xj} \kappa_{1,tj}) - E_{xt}^c e^{\alpha_{xj} + \beta_{1,xj} \kappa_{1,tj}}\} + \text{constant}$$

$$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j^{(v)}(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. *Get initial values for the parameters*

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^c} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\beta}_{1,xj} = \frac{1}{n_A} \text{ where } n_A \text{ is the total number of ages}$$

$$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y+1)}{2} \text{ where } n_Y \text{ is the total number of calendar years}$$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}})}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}})}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameter $\hat{\kappa}_{1,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) \hat{\beta}_{1,xj}^{(v)}}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) (\hat{\beta}_{1,xj}^{(v)})^2}$$

Adjust the parameter $\hat{\kappa}_{1,tj}$ in order to satisfy the constraint $\rightarrow \hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj})$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameter $\hat{\beta}_{1,xj}$*

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) (\hat{\kappa}_{1,tj}^{(v)})^2}$$

Adjust the parameter $\hat{\beta}_{1,xj}$ in order to satisfy the constraint $\rightarrow \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **Binomial Lee–Carter model** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xt} \{D_{xtj} \ln q_{xtj} + (E_{xt}^0 - D_{xtj}) \ln(1 - q_{xtj})\} + \text{constant}$$

where $q_{xtj} = e^{\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj}} / (1 + e^{\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj}})$

$$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. *Get initial values for the parameters*

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^0} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\beta}_{1,xj} = \frac{1}{n_A} \text{ where } n_A \text{ is the total number of ages}$$

$$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y + 1)}{2} \text{ where } n_Y \text{ is the total number of calendar years}$$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1) \right)}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1)^2 \right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameter $\hat{\kappa}_{1,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1) \right) \hat{\beta}_{1,xj}^{(v)}}{\sum_x \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1)^2 \right) \left(\hat{\beta}_{1,xj}^{(v)} \right)^2}$$

Adjust the parameter $\hat{\kappa}_{1,tj}$ in order to satisfy the constraint $\rightarrow \hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj})$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Update the parameter $\hat{\beta}_{1,xj}$*

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1) \right) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)}} + 1)^2 \right) \left(\hat{\kappa}_{1,tj}^{(v)} \right)^2}$$

Adjust the parameter $\hat{\beta}_{1,xj}$ in order to satisfy the constraint $\rightarrow \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 4)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **Poisson Lee–Carter model with two factors** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xtj} \{D_{xtj} (\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj} + \beta_{2,xj}\kappa_{2,tj}) - E_{xt}^c e^{\alpha_{xj} + \beta_{1,xj}\kappa_{1,tj} + \beta_{2,xj}\kappa_{2,tj}}\} + \text{constant}$$

$$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj}, \beta_{2,xj}, \kappa_{2,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. *Get initial values for the parameters*

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^c} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\beta}_{1,xj} = \frac{1}{n_A} \text{ where } n_A \text{ is the total number of ages}$$

$$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y+1)}{2} \text{ where } n_Y \text{ is the total number of calendar years}$$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}})}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}})}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) \hat{\beta}_{1,xj}^{(v)}}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) (\hat{\beta}_{1,xj}^{(v)})^2}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) \hat{\beta}_{2,xj}^{(v)}}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) (\hat{\beta}_{2,xj}^{(v)})^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$ in order to satisfy the constraints \rightarrow

$$\hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}) \text{ and } \hat{\kappa}_{2,tj} = \sum_x \hat{\beta}_{2,xj} (\hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj})$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. Update the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$

$$\hat{\beta}_{1,xj}^{(v+1)} = \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}}) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}}) (\hat{\kappa}_{1,tj}^{(v)})^2}$$

$$\hat{\beta}_{2,xj}^{(v+1)} = \hat{\beta}_{2,xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}}) \hat{\kappa}_{2,tj}^{(v)}}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}}) (\hat{\kappa}_{2,tj}^{(v)})^2}$$

Adjust the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$ in order to satisfy the constraints $\rightarrow \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$ and

$$\hat{\beta}_{2,xj} = \hat{\beta}_{2,xj} \frac{1}{\sum_x \hat{\beta}_{2,xj}}$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. Check convergence: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$$\Delta \ell_j = \ell_j - \ell_j^{updated} \text{ (} \ell_j^{updated} \text{ is the updated log-likelihood from step 4)}$$

In case of nonconvergence ($\Delta \ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **Binomial Lee–Carter model with two factors** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xtj} \{D_{xtj} \ln q_{xtj} + (E_{xt}^0 - D_{xtj}) \ln(1 - q_{xtj})\} + \text{constant}$$

$$\text{where } q_{xtj} = e^{\alpha_{xj} + \beta_{1,xj} \kappa_{1,tj} + \beta_{2,xj} \kappa_{2,tj}} / (1 + e^{\alpha_{xj} + \beta_{1,xj} \kappa_{1,tj} + \beta_{2,xj} \kappa_{2,tj}})$$

$$\theta = (\alpha_{xj}, \beta_{1,xj}, \kappa_{1,tj}, \beta_{2,xj}, \kappa_{2,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. Get initial values for the parameters

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^0} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\beta}_{1,xj} = \frac{1}{n_A} \text{ where } n_A \text{ is the total number of ages}$$

$$\hat{\kappa}_{1,tj} = (n_Y, \dots, 1) - \frac{(n_Y + 1)}{2} \text{ where } n_Y \text{ is the total number of calendar years}$$

2. Update the parameter $\hat{\alpha}_{xj}$

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right)}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right)}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \hat{\beta}_{1,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj} + \hat{\beta}_{2,xj} \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj}$

Calculate the log-likelihood function ℓ_j

3. Update the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$

$$\begin{aligned} \hat{\kappa}_{1,tj}^{(v+1)} &= \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\beta}_{1,xj}^{(v)}}{\sum_x \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\beta}_{1,xj}^{(v)} \right)^2} \\ \hat{\kappa}_{2,tj}^{(v+1)} &= \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\beta}_{2,xj}^{(v)}}{\sum_x \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\beta}_{2,xj}^{(v)} \right)^2} \end{aligned}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$ in order to satisfy the constraints \rightarrow

$$\hat{\kappa}_{1,tj} = \sum_x \hat{\beta}_{1,xj} (\hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}) \text{ and } \hat{\kappa}_{2,tj} = \sum_x \hat{\beta}_{2,xj} (\hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj})$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. Update the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$

$$\begin{aligned} \hat{\beta}_{1,xj}^{(v+1)} &= \hat{\beta}_{1,xj}^{(v)} + \frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\kappa}_{1,tj}^{(v)}}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\kappa}_{1,tj}^{(v)} \right)^2} \\ \hat{\beta}_{2,xj}^{(v+1)} &= \hat{\beta}_{2,xj}^{(v)} + \frac{\sum_t \left((D_{xtj} e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1) \right) \hat{\kappa}_{2,tj}^{(v)}}{\sum_t \left((E_{xt}^0 e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\alpha}_{xj}^{(v)} + \hat{\beta}_{1,xj}^{(v)} \hat{\kappa}_{1,tj}^{(v)} + \hat{\beta}_{2,xj}^{(v)} \hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) \left(\hat{\kappa}_{2,tj}^{(v)} \right)^2} \end{aligned}$$

Adjust the parameters $\hat{\beta}_{1,xj}$ and $\hat{\beta}_{2,xj}$ in order to satisfy the constraints $\rightarrow \hat{\beta}_{1,xj} = \hat{\beta}_{1,xj} \frac{1}{\sum_x \hat{\beta}_{1,xj}}$ and

$$\hat{\beta}_{2,xj} = \hat{\beta}_{2,xj} \frac{1}{\sum_x \hat{\beta}_{2,xj}}$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

5. Check convergence: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$$\Delta \ell_j = \ell_j - \ell_j^{updated} \text{ (} \ell_j^{updated} \text{ is the updated log-likelihood from step 4)}$$

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **CBD model** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xt} \{D_{xtj} \ln q_{xtj} + (E_{xt}^0 - D_{xtj}) \ln(1 - q_{xtj})\} + \text{constant}$$

$$\text{where } q_{xtj} = e^{\kappa_{1,tj} + (x - \bar{x})\kappa_{2,tj}} / (1 + e^{\kappa_{1,tj} + (x - \bar{x})\kappa_{2,tj}})$$

$$\theta = (\kappa_{1,tj}, \kappa_{2,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. Get initial values for the parameters and fix the values of the parametric coefficient $\beta_{2,xj}$

$$\hat{\kappa}_{1,tj} = 0$$

$$\hat{\kappa}_{2,tj} = 0$$

$$\beta_{2,xj} = (x - \bar{x})$$

2. Update the parameter $\hat{\kappa}_{1,tj}$

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x \left((D_{xtj} e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1) \right)}{\sum_x \left((E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right)}$$

Calculate the log-likelihood function ℓ_j

3. Update the parameter $\hat{\kappa}_{2,tj}$

$$\begin{aligned} \hat{\kappa}_{2,tj}^{(v+1)} &= \hat{\kappa}_{2,tj}^{(v)} + \\ &+ \frac{\sum_x \left((D_{xtj} e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} - E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + D_{xtj}) / (e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1) \right) (x - \bar{x})}{\sum_x \left((E_{xt}^0 e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}}) / (e^{\hat{\kappa}_{1,tj}^{(v)} + (x - \bar{x})\hat{\kappa}_{2,tj}^{(v)}} + 1)^2 \right) (x - \bar{x})^2} \end{aligned}$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. Check convergence: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$$\Delta\ell_j = \ell_j - \ell_j^{updated} \text{ (}\ell_j^{updated} \text{ is the updated log-likelihood from step 3)}$$

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **reduced Plat model with three factors** can be formulated as follows:

$$\begin{aligned}\ell_j(\theta) &= \\ &= \sum_{xt} \{D_{xtj} (\alpha_{xj} + \kappa_{1,tj} + (\bar{x} - x)\kappa_{2,tj} + (\bar{x} - x)^+ \kappa_{3,tj}) - E_{xt}^c e^{\alpha_{xj} + \kappa_{1,tj} + (\bar{x} - x)\kappa_{2,tj} + (\bar{x} - x)^+ \kappa_{3,tj}}\} + \\ &+ \text{constant} \\ \theta &= (\alpha_{xj}, \kappa_{1,tj}, \kappa_{2,tj}, \kappa_{3,tj})\end{aligned}$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. Get initial values for the parameters and fix the values of the parametric coefficients $\beta_{2,xj}$ and $\beta_{3,xj}$

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^c} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\kappa}_{1,tj} = 0$$

$$\hat{\kappa}_{2,tj} = 0$$

$$\hat{\kappa}_{3,tj} = 0$$

$$\beta_{2,xj} = (\bar{x} - x)$$

$$\beta_{3,xj} = (\bar{x} - x)^+$$

2. Update the parameter $\hat{\alpha}_{xj}$

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}})}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}})}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj} + (\bar{x} - x) \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj} + (\bar{x} - x)^+ \frac{1}{n_Y} \sum_t \hat{\kappa}_{3,tj}$

Calculate the log-likelihood function ℓ_j

3. Update the parameters $\hat{\kappa}_{1,tj}$, $\hat{\kappa}_{2,tj}$ and $\hat{\kappa}_{3,tj}$

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}})}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}})}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}}) (\bar{x} - x)}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}}) (\bar{x} - x)^2}$$

$$\hat{\kappa}_{3,tj}^{(v+1)} = \hat{\kappa}_{3,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}}) (\bar{x} - x)^+}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)} + (\bar{x} - x)^+ \hat{\kappa}_{3,tj}^{(v)}}) ((\bar{x} - x)^+)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$, $\hat{\kappa}_{2,tj}$ and $\hat{\kappa}_{3,tj}$ in order to satisfy the constraints \rightarrow

$$\hat{\kappa}_{1,tj} = \hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}, \hat{\kappa}_{2,tj} = \hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj} \text{ and } \hat{\kappa}_{3,tj} = \hat{\kappa}_{3,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{3,tj}$$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta\ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 3)

In case of nonconvergence ($\Delta\ell_j > 10^{-6}$), select different initial values for the parameters.

The log-likelihood function of the **reduced Plat model with two factors** can be formulated as follows:

$$\ell_j(\theta) = \sum_{xt} \{D_{xtj} (\alpha_{xj} + \kappa_{1,tj} + (\bar{x} - x)\kappa_{2,tj}) - E_{xt}^c e^{\alpha_{xj} + \kappa_{1,tj} + (\bar{x} - x)\kappa_{2,tj}}\} + \text{constant}$$

$$\theta = (\alpha_{xj}, \kappa_{1,tj}, \kappa_{2,tj})$$

The parameters θ are updated using the following iterative scheme:

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\partial \ell_j^{(v)} / \partial \theta}{\partial^2 \ell_j^{(v)} / \partial \theta^2} \text{ where } \ell_j^{(v)} = \ell_j(\hat{\theta}^{(v)})$$

The iterative updating scheme

1. *Get initial values for the parameters and fix the values of the parametric coefficient $\beta_{2,xj}$*

$$\hat{\alpha}_{xj} = \frac{1}{n_Y} \sum_t \ln \frac{D_{xtj}}{E_{xt}^c} \text{ where } n_Y \text{ is the total number of calendar years}$$

$$\hat{\kappa}_{1,tj} = 0$$

$$\hat{\kappa}_{2,tj} = 0$$

$$\beta_{2,xj} = (\bar{x} - x)$$

2. *Update the parameter $\hat{\alpha}_{xj}$*

$$\hat{\alpha}_{xj}^{(v+1)} = \hat{\alpha}_{xj}^{(v)} + \frac{\sum_t (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}})}{\sum_t (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}})}$$

Adjust the parameter $\hat{\alpha}_{xj} \rightarrow \hat{\alpha}_{xj} = \hat{\alpha}_{xj} + \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj} + (\bar{x} - x) \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj}$

Calculate the log-likelihood function ℓ_j

3. *Update the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$*

$$\hat{\kappa}_{1,tj}^{(v+1)} = \hat{\kappa}_{1,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}})}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}})}$$

$$\hat{\kappa}_{2,tj}^{(v+1)} = \hat{\kappa}_{2,tj}^{(v)} + \frac{\sum_x (D_{xtj} - E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}}) (\bar{x} - x)}{\sum_x (E_{xt}^c e^{\hat{\alpha}_{xj}^{(v)} + \hat{\kappa}_{1,tj}^{(v)} + (\bar{x} - x)\hat{\kappa}_{2,tj}^{(v)}}) (\bar{x} - x)^2}$$

Adjust the parameters $\hat{\kappa}_{1,tj}$ and $\hat{\kappa}_{2,tj}$ in order to satisfy the constraints $\rightarrow \hat{\kappa}_{1,tj} = \hat{\kappa}_{1,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{1,tj}$
and $\hat{\kappa}_{2,tj} = \hat{\kappa}_{2,tj} - \frac{1}{n_Y} \sum_t \hat{\kappa}_{2,tj}$

Recalculate the log-likelihood function: compute $\ell_j^{updated}$

4. *Check convergence*: the difference between the log-likelihood and the updated log-likelihood is less than 10^{-6}

$\Delta \ell_j = \ell_j - \ell_j^{updated}$ ($\ell_j^{updated}$ is the updated log-likelihood from step 3)

In case of nonconvergence ($\Delta \ell_j > 10^{-6}$), select different initial values for the parameters.