Homework 3

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Problem 1

We want to find a closed-form solution to the inhomogeneous recurrence relation:

$$X_{n+1} = X_n + 2X_{n-1} + 4^n + n$$

First consider the homogeneous part of the recurrence:

$$X_{n+1} - X_n - 2X_{n-1} = 0$$

and its characteristic equation:

$$r^2 - r - 2 = 0$$

Obviously solution to the homogeneous part of the recurrence takes the form:

$$h_n = \alpha 2^n + \beta(-1)^n$$

Now assume a solution to the inhomogeneous recurrence takes the form:

$$a_n = A4^n + Bn + C$$

Plug the solution into the relation gives us a system of equations:

$$\begin{cases} 4A = A + \frac{1}{2}A + 1 \\ B = 3B + 1 \\ B + C = 3C - 2B \end{cases}$$

Solving it gives us:

$$a_n = \frac{2}{5}4^n - \frac{1}{2}n - \frac{3}{4}$$

Then general solution to the recurrence should be:

$$X_n = a_n + h_n = \frac{2}{5}4^n - \frac{1}{2}n - \frac{3}{4} + \alpha 2^n + \beta(-1)^n$$

Plug in initial vales of X and solve for α, β:

$$\begin{cases} X_0 = \frac{2}{5} - \frac{3}{4} + \alpha + \beta = 0 \\ X_1 = \frac{2\times 4}{5} - \frac{1}{2} - \frac{3}{4} + 2\alpha - \beta = 1 \end{cases}$$

$\alpha = 0, \beta = \frac{7}{20}$, and finally we have the solution:

$$X_n = \frac{2}{5}4^n - \frac{1}{2}n + \frac{7}{20}(-1)^n - \frac{3}{4}$$

Problem 2

We want to show that $T(n) = \Theta(n)$. To begin with we try to set up an upper-bound. Let's assume $T(n) \leq 2n - c$, which gives us:

$$T(n) = 2T(\frac{n}{2}) + \frac{1}{4}2^{\frac{2T(n/2)}{n}} \leq 2(\frac{2n}{2} - c) + \frac{1}{4}2^{2-\frac{2c}{n}} = 2n - 2c + \frac{1}{2^{\frac{2c}{n}}}$$

We want to show:

$$2n - 2c + \frac{1}{2^{\frac{2c}{n}}} \leq 2n - c$$

which is equivalently:

$$\frac{1}{2^{\frac{2c}{n}}} \leq c$$

which is true because for $c \geq \frac{1}{2}, n > 0$:

$$\frac{1}{2^{\frac{2c}{n}}} \leq \frac{1}{2} \leq c$$

Now we need to show the base case to complete the inductive proof. Let $n = 1, c = \frac{1}{2}$ and we have:

$$T(1) = 1 \leq 2 \times 1 - \frac{1}{2} = 1.5$$

This shows $T(n) = O(n)$. Now we proceed to prove the lower-bound by assuming $T(n) \geq n$:

$$T(n) = 2T(\frac{n}{2}) + \frac{1}{4}2^{\frac{2T(n/2)}{n}} \geq 2\frac{n}{2} + \frac{1}{4}2^{\frac{2n/2}{n}} = n + \frac{1}{2} > n$$

together with the base case $T(1) = 1 \geq 1$ gives us: $T(n) = \Omega(n)$, which proves $T(n) = \Theta(n)$