LING 473: Day 5

START THE RECORDING

Bayes Theorem

Announcements

- I will not be physically here August 8 & 10
- Lectures will be made available right before I go to sleep in Oslo
  - So, something like 2:30-3:00pm here. I’ll send out an email.
  - This means that Assignment 2 will have to be reviewed on August 10.
- There is a grader for the class
  - He’ll be helping me with grades, but I’m the final arbiter.
  - Let me know if you have a question

Projects Generally

- Read instructions carefully
- Modeling language data vs full linguistic analysis
  - Your requested implmentation may not be fully linguistically correct (that's ok!)
- Must run on Patas with all requested files turned in
- Make sure to log out of Patas when you're done by typing "logout"
  - You may encounter strange state problems if you are disconnected without logging out

Writing assignment

- Due September 5th , 2017

  http://courses.washington.edu/ling473/writing-assignment.html
- Short Critical review of a paper from the computational linguistics literature
- Formatted according to ACL-2017 guidelines
  - http://acl2017.org/calls/papers/
- Any published journal or peer-reviewed paper on a comp. ling. topic is acceptable

Review: Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B)P(B)$$

$$P(A \cap B) = P(A|B)P(B)$$

joint probability = conditional probability × marginal probability
(or "prior" probability)

Independent random variables

Random variables A and B are independent *iff*

$$P(A \cap B) = P(A)P(B)$$

Recall conditional probability:

$$P(A \cap B) = P(A|B)P(B)$$

This means that, if A and B are independent,

$$P(A|B) = P(A)$$

$P(A \cap B)$ $P(A, B)$ $P(AB)$
Reminder: these are three notations for the same thing:
the joint probability of *A* and *B*. That is, that both events occur in a single trial

Conditional independence

A and *B* are independent *iff*

$$P(A \cap B) = P(A)P(B)$$

A and *B* are conditionally independent given *K* *iff*

$$P(A \cap B|K) = P(A|K)P(B|K)$$

Just as with conditional probability, *K* constrains the sample space. Conditional independence means that *A* and *B* are independent if we know that *K* has occurred.

Conditional independence

A and *B* are conditionally independent given *K* *iff*

$$P(A \cap B|K) = P(A|K)P(B|K)$$

Given that *K* has occurred, knowing that *B* has occurred gives us no additional information about the probability of *A* (and vice-versa)

Q: Does this imply that *A* and *B* are independent?

A: No. *A* and *B* could be either independent or dependent in the absence of knowledge about *K*

Conditional independence

$$P(A \cap B|K) = P(A|K)P(B|K)$$

Two events (*A* and *B*) are conditionally independent given a third event (*K*) if their probabilities conditioned on K are independent. The following will also be true:

$$P(A|B \cap K) = P(A|K)$$
$$P(B|A \cap K) = P(B|K)$$

Chain rule

- This can be extended $P(A \cap B) = P(A|B)P(B)$

$$P(A \cap B \cap C \cap D)$$

$$= \overbrace{P(A|B \cap C \cap D)P(B \cap C \cap D)}$$

$$= P(A|B \cap C \cap D)\overbrace{P(B|C \cap D)P(C \cap D)}$$

$$= P(A|B \cap C \cap D)P(B|C \cap D)\overbrace{P(C|D)P(D)}$$

etc... This is called the chain rule

$$P(AB) = P(A|B)P(B)$$

$$P(ABCDE)$$

$$= P(A|BCDE)P(BCDE)$$

$$= P(A|BCDE)P(B|CDE)P(CDE)$$

$$= P(A|BCDE)P(B|CDE)P(C|DE)P(DE)$$

$$= P(A|BCDE)P(B|CDE)P(C|DE)P(D|E)P(E)$$

Chain rule

$$P(X_1 = x_1, \dots, X_n = x_n) =$$
$$P(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_1 = x_1) \times P(X_{n-1} = x_{n-1}, \dots, X_1 = x_1)$$

$$P(A, B, C, D) = P(A \cap B \cap C \cap D)$$
$$= P(A|B, C, D)P(B|C, D)P(C|D)P(D)$$

(The “given” notation ‘|’ has lowest precedence)

Chain Rule and Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$P(A|B)P(B) = P(A \cap B)$$

$$P(B|A)P(A) = P(B \cap A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes Theorem

Rev. Thomas Bayes (1701-1761)

Conditional probability of B given A "likelihood"

marginal or prior probability of A

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Conditional probability of A given B "posterior" probability

marginal or prior probability of B

Bayes Theorem

- Relates hypothesis to observation (evidence or prior knowledge)

hypothesis

observation

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Bayes Theorem

- Expresses one probability in terms of another
- P(A|B) depends on B, but also P(A) and P(B) in the general population.
- When might Bayes theorem be useful?

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

  – Medical test vs condition! P(condition|test) is hard to know, but P(test|condition) is easier!

Bayes Theorem

		actual condition A	
		yes $P(A)$	no
test result B	positive $P(B)$	true positive	false positive
	negative	false negative	true negative

- We can empirically discover P(B|A=*true*) given a population of people with a condition. Same for P(B|A=*false*).
- We can use a sample population to get P(B=*true*) and P(B=*false*).
- We can use other sources to estimate *P(A)* in the general population.
- Now we have enough to generate P(A|B), the probability of an actual condition given a test result.

Example 1

A gambler has two coins in his pocket, one fair coin and one two-headed one.

a. He selects one at random and flips it. It comes up heads. What is the probability that is the fair coin?

Example 1

- Assuming an equal chance of picking from the pocket:

  $P(F) = P(F^C) = \frac{1}{2}$

- Probability of obtaining heads from the fair coin:

  $P(H|F) = \frac{1}{2}$

- Probability of obtaining heads from the two-headed coin:

  $P(H|F^C) = 1$

Example 1

- Overall prior probability of flipping heads:

$$P(H) = P(H|F)P(F) + P(H|F^C)P(F^C)$$
$$= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{3}{4}$$

- Probability of having the fair coin given heads:

$$P(F|H) = \frac{P(H|F)P(F)}{P(H)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$$

Example 1

b. He now flips the same coin a second time, and it again comes up heads. What is the probability that it is the fair coin?

Example 1

- Probability of two heads given the fair coin

  $P(H,H|F) = \frac{1}{4}$ (it is one of 4 outcomes)

- Probability of two heads given the double-headed coin

  $P(H,H|F^C) = 1$

- Overall probability of two heads:

  $P(H,H|F)P(F) + P(H,H|F^C)P(F^C)$

  $= \frac{1}{4} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{5}{8}$

Example 1

- Probability of having selected the fair coin given the observation {H,H}

$$P(F|H,H) = \frac{P(H,H|F)P(F)}{P(H)} = \frac{\frac{1}{4}\times\frac{1}{2}}{\frac{5}{8}} = \frac{1}{5}$$

Example 1

c. Suppose he flips the coin a third time, and it comes up tails. What is the probability that it is the fair coin?

Example 1

- 1.0
- The double headed coin can’t come up tails.

Example 2

The Monty Hall problem

- There are 3 doors: A, B, & C. One of them has a prize behind it.
- You choose door A. The host knows where the prize is and reveals that door B does not have the prize. The host asks if you want to switch.
- Should you switch?

Example 2

- Original chance of choosing the prize: $\frac{1}{3}$
- Event B: {door B is revealed}
- Random variable Z: { door with the prize }
- $P(Z = a|B) = \frac{P(B|Z = a)P(Z=a)}{P(B)} = \frac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$
- $P(Z = b|B) = \frac{P(B|Z = b)P(Z=b)}{P(B)} = \frac{0\times\frac{1}{3}}{\frac{1}{2}} = 0$
- $P(Z = c|B) = \frac{P(B|Z = c)P(Z=c)}{P(B)} = \frac{1\times\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

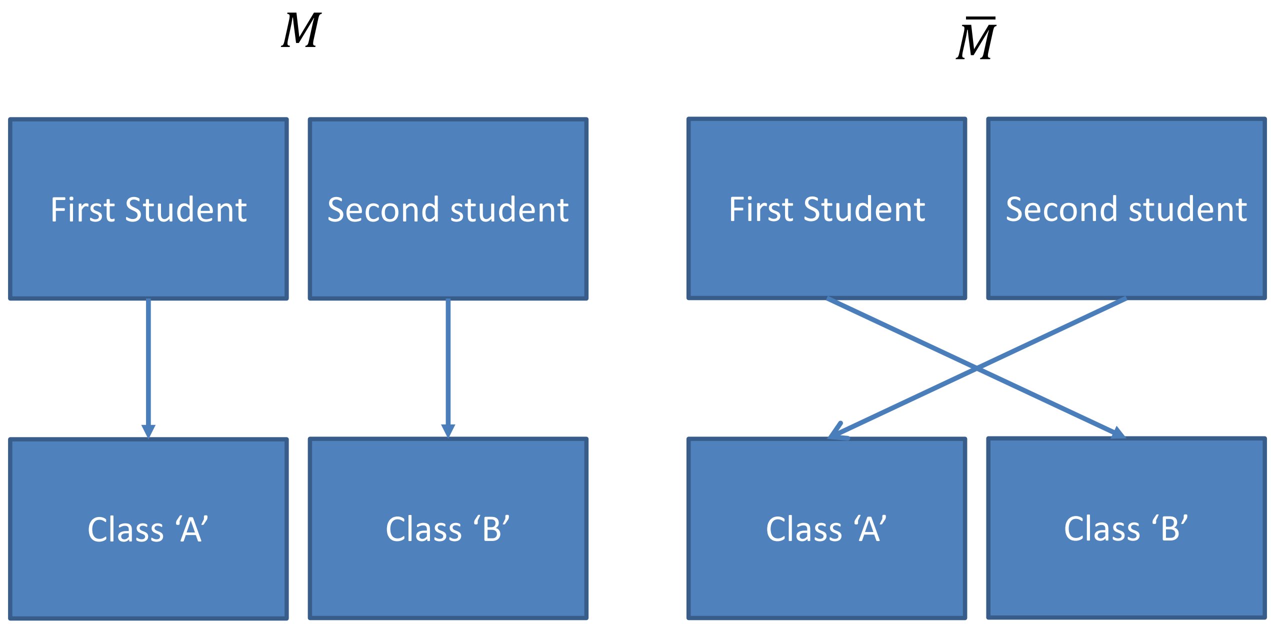
Example 2

- Assuming the host always selects at random when he can, it is always better to switch your choice.

Example 3

- Class A has 15 PhD students, 10 CLMS students, and 5 students from other majors.
- Class B has 5 PhD students, 10 CLMS students, and 15 students from other majors.
- One student from each class is chosen at random. The first is a CLMS student and the second is from another major. What is the probability the CLMS student is from Class A?

2 possibilities for the actual matchup



Example 3

- Prior probabilities:

$$P(M) = 0.5$$
$$P(\bar{M}) = 1 - P(M) = 0.5$$

(either match-up is equally likely)

We observe the sequence: (CLMS, other)

Probability of seeing this given M:

$$P(\,(\text{CLMS, other})\,|M) = \frac{\overset{\text{\# CLMS in ‘A’}}{10}}{30} \times \frac{\overset{\text{\# other in ‘B’}}{15}}{30} = \frac{1}{6}$$

Probability of seeing this given $\bar{M}$:

$$P(\,(\text{CLMS, other})\,|\bar{M}) = \frac{\overset{\text{\# CLMS in ‘B’}}{10}}{30} \times \frac{\overset{\text{\# other in ‘A’}}{5}}{30} = \frac{1}{18}$$

Example 3

- Overall prior probability of seeing (CLMS, other):

$$P(M) \times P(\text{ (CLMS, other) } | M) + P(\bar{M}) \times P(\text{ (CLMS, other) } | \bar{M})$$

$$= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{18}$$

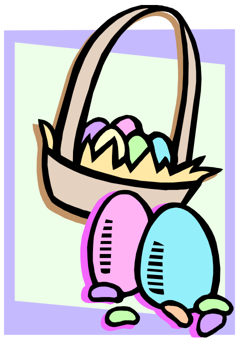
$$\mathrm{P}((\text{CLMS, other})) = \frac{1}{9}$$

Example 3

- $P(M|(CLMS, other)) = \frac{P((CLMS,other)|M)\times P(M)}{P\ ((CLMS,other))}$
- $= \frac{\frac{1}{6}\times\frac{1}{2}}{\frac{1}{9}} = \frac{3}{4}$

Example 4

A basket contains many small plastic eggs, some painted red and some are painted blue.



40% of the eggs in the bin contain pearls

30% of eggs containing pearls are painted blue, and 10% of eggs containing nothing are painted blue.

What is the probability that a blue egg contains a pearl?

Example 4

- $P(pearl|blue) = \frac{P(blue|pearl)P(pearl)}{P(blue)}$
- $P(pearl|blue) = \frac{0.3\times0.4}{P(blue)}$
- $P(blue) = P(blue|pearl)P(pearl) + P\left(blue\middle|\overline{pearl}\right)P\left(\overline{pearl}\right)$
- $P(blue) = 0.3\times0.4 + 0.1\times0.6 = 0.18$
- $P(pearl|blue) = \frac{0.12}{0.18} = \frac{2}{3}$

Earthquakes & Burglaries

- You own a house in California with an alarm system. If your alarm goes off, one of your neighbors will call you.
- The alarm could go off because of an earthquake, or a burglary.

- P(burglary) = 0.001
- P(earthquake) = 0.002

Earthquakes & Burglaries

- P(alarm, burglary, earthquake) =

Burglary	**Earthquake**	**P(Alarm|B,E)**
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

Earthquakes & Burglaries

- The alarm is going off. What is the most likely reason for it?

- $P(burglary|alarm) = \frac{P(alarm|burglary)P(burglary)}{P(alarm)}$
- $P(earthquake|alarm) = \frac{P(alarm|earthquake)P(earthquake)}{P(alarm)}$
- $P(nothing|alarm) = \frac{P(alarm|nothing)P(nothing)}{P(alarm)}$

Earthquakes & Burglaries

- $P(A) = P(A|B,E)P(B)P(E) + P(A|B,\bar{E})P(B)P(\bar{E}) + P(A|\bar{B},E)P(\bar{B})P(E) + P(A|\bar{B},\bar{E})P(\bar{B})P(\bar{E})$
- $= 0.95\times0.001\times0.002 + 0.94\times0.001\times0.998 + 0.29\times0.9999\times0.002 + 0.001\times0.999\times0.998$
- $P(A) = 0.002516964$

Earthquakes & Burglaries

- $P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{(P(A|B,\bar{E})P(\bar{E})+P(A|B,E)P(E))P(B)}{P(A)}$
- $\frac{(0.94\times 0.998+0.95\times 0.002)(0.001)}{0.002516964} = 0.3734737$

- $P(E|A) = \frac{P(A|E)P(E)}{P(A)} = \frac{(P(A|E,\bar{B})P(\bar{B})+P(A|E,B)P(B))P(E)}{P(A)}$
- $\frac{(0.29\times 0.999+0.95\times 0.001)(0.002)}{0.002516964} = 0.2309607$

Earthquakes & Burglaries

- $P(\bar{E}, \bar{B}\ |A) = \frac{P(A|\bar{E},\bar{B})P(\bar{E})P(\bar{B})}{P(A)}$
- $\frac{0.001\times 0.992\ \times 0.999}{0.002516964} = 0.3937314$

- It's almost a toss-up between nothing and a burglary. Ask your neighbor if they can see anyone (and if the ground is shaking).

Review: Derivation of Bayes Theorem

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$P(A|B)P(B) = P(A \cap B) \qquad P(B|A)P(A) = P(B \cap A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes Theorem

Rev. Thomas Bayes (1701-1761)

Conditional probability of B given A "likelihood"

marginal or prior probability of A

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Conditional probability of A given B "posterior" probability

marginal or prior probability of B

Bayes Theorem

- Expresses one conditional probability in terms of its inverse
- $P(A|B)$ depends not only on B, but also on $P(A)$ and $P(B)$ in the general population

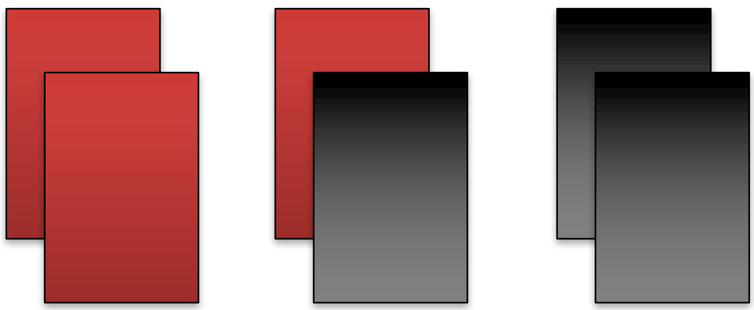
		actual condition A	
		yes $P(A)$	no
test result B	positive $P(B)$	true positive	false positive
	negative	false negative	true negative

Recipe for Bayes Theorem

- What you need:
  1. The probability of actually satisfying the criteria (regardless of 2)
  2. The probability of testing positive for the criteria (regardless of 1)
  3. And either:
     a. the probability of testing positive given the criteria is satisfied
     b. the probability of satisfying the criteria given the test is positive

Bayes theorem

- 3 cards:



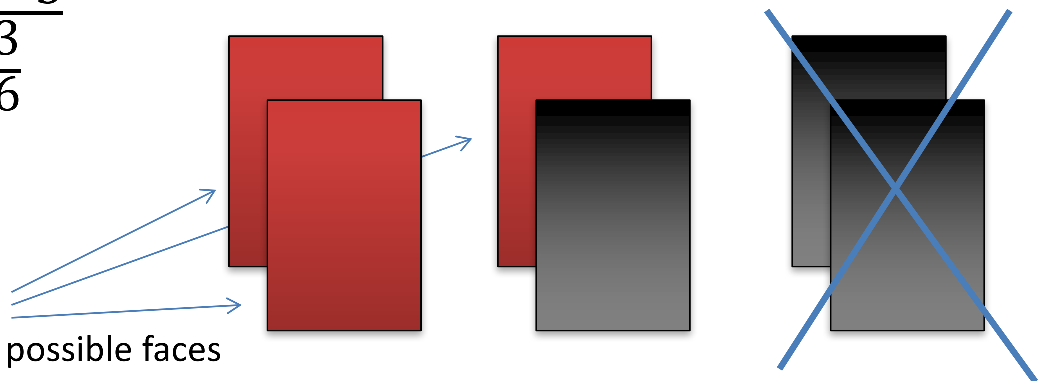
- We select a card at random and note that one side is red. What is the chance that it's the red-red card?

Bayes theorem

$$P(\,(red,red)|R) = \frac{P(R|(red,red)P(red,red)}{P(R)}$$

$$= \frac{1 \times \frac{1}{3}}{\frac{3}{6}}$$

$$= \frac{2}{3}$$



For each of the 3 possible faces that you could be looking at, how many of them have red on its *other* side?

Probability distributions

Assuming that a random variable exhibits a fixed, characteristic probability distribution, e.g.

$$\Omega = \{\, a, b, c \,\}$$

$$P(X = x) = \begin{cases} {}^{1}\!/_{3}\,, if\ x = \{a\}; \\ {}^{1}\!/_{3}\,, if\ x = \{b\}; \\ {}^{1}\!/_{3}\,, if\ x = \{c\}; \end{cases}$$

allows us justify our intuition about events from last week:

$$A = \{a\}$$

$$A^C = \{\, b, c \,\}$$

$$P(A) = \frac{|A|}{|\Omega|}$$

$$P(A^C) = \frac{|A^C|}{|\Omega|}$$

$$P(A) + P(A^C) = \frac{|A|}{|\Omega|} + \frac{|A^C|}{|\Omega|} = \frac{|A| + |A^C|}{|\Omega|} = \frac{|\Omega|}{|\Omega|} = 1$$

Probability distributions

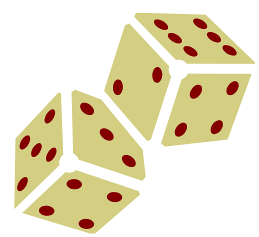
- A random variable's probability distribution encapsulates both:
  - a characteristic type of "spread" or "shape" (distribution)
    - uniform
    - normal
    - etc.
  - the scaling and normalization factors that map between probabilities [0.0, 1.0] and the range of measurement values

This is why the capital letter subscript is (supposed to be) used: $P_X(X = x)$

Uniform distribution

- Dividing the probability mass evenly between the values of a discrete random variable creates a uniform distribution

$$a = 1, b = 6$$



the mean μ is the average value

$$\mu = \frac{a+b}{2} = 3.5$$

Non-uniform distribution

(the, cat, in, the, hat)

$$X = \{ \textit{the word which is selected} \}$$

$$P_X(X = the) = 0.4$$
$$P_X(X = cat) = 0.2$$
$$P_X(X = in) = 0.2$$
$$P_X(X = hat) = 0.2$$

$$Y = \{ \textit{the number of times } \text{X=the} \textit{ in } 3 \textit{ trials, with replacement} \}$$

$$P_Y(Y = 0) = .6 \times .6 \times .6 = .216$$
$$P_Y(Y = 3) = .4 \times .4 \times .4 = .064$$
$$P_Y(Y = 1) = .4 \times .6 \times .6 \times \binom{3}{1} = .432$$
$$P_Y(Y = 2) = .4 \times .4 \times .6 \times \binom{3}{1} = .288$$
$$P_Y(Y \geq 2) = .352$$