HMM (1): Definition and two types of HMM

LING 570

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HMM

- Definition and properties of HMM
  - Two types of HMM

- Three basic questions in HMM

Definition of HMM

Hidden Markov Models

- There are *n* states $s_1, \ldots, s_n$ *in an HMM, and the states are connected.*

- The output symbols are produced by the states or edges in HMM.

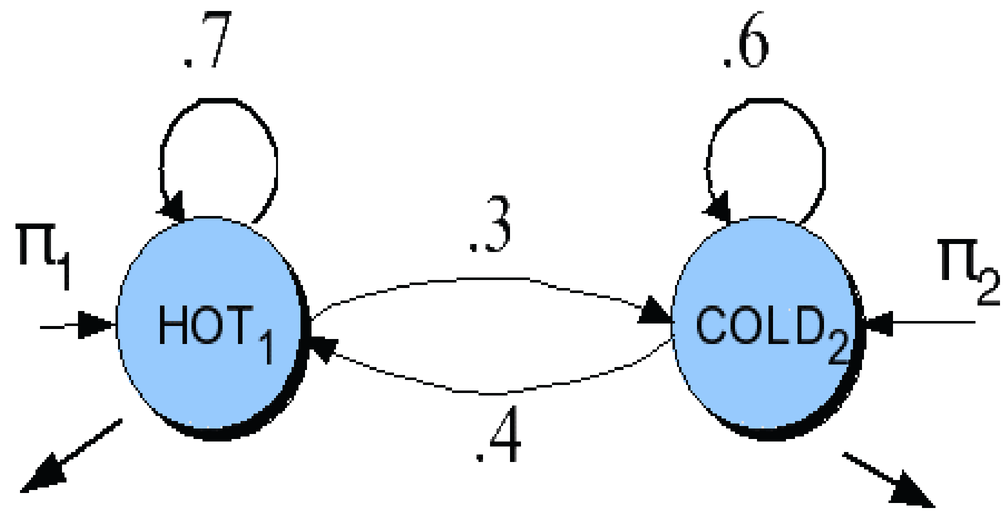
- An observation **O**=(o_1, …, o_T) is a sequence of output symbols.

- Given an observation, we want to recover the hidden state sequence.

- An example: POS tagging
  - States are POS tags (in a bigram POS tagger)
  - Output symbols are words
  - Given an observation (i.e., a sentence), we want to discover the tag sequence.

An example: Ice Cream-Weather HMM

$\pi = [.8, .2]$



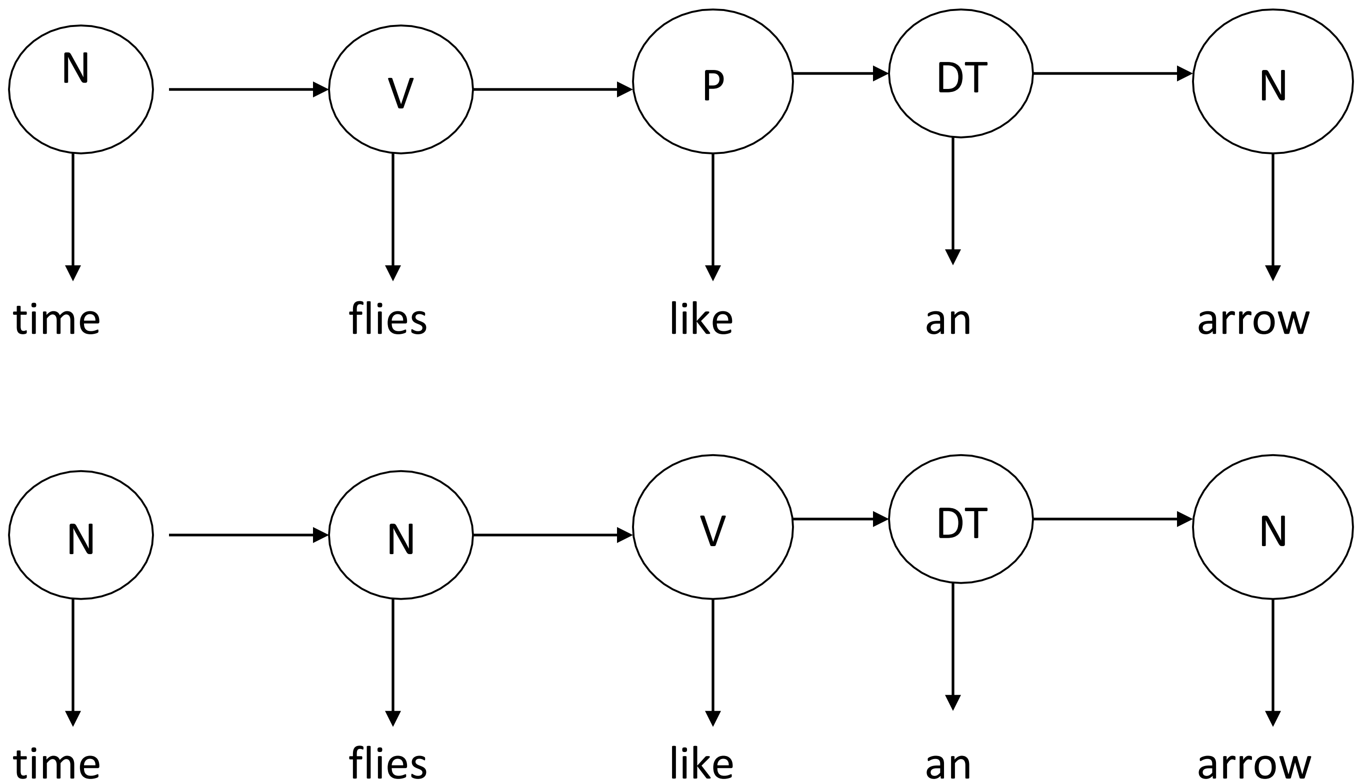
$$B_1: \quad \begin{bmatrix} P(1 \mid HOT) \\ P(2 \mid HOT) \\ P(3 \mid HOT) \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix}$$

$$B_2: \quad \begin{bmatrix} P(1 \mid COLD) \\ P(2 \mid COLD) \\ P(3 \mid COLD) \end{bmatrix} = \begin{bmatrix} .5 \\ .4 \\ .1 \end{bmatrix}$$

Highlights of HMM

- Can’t directly observe states
  - Those states are said to be “hidden”
    - Distinguished from observations: ice cream vs weather
- Allows us to capture ambiguity
- Characteristic of many real-world problems
  - POS Tagging:
    - Observe words, not the underlying tag sequence
  - Speech recognition:
    - Observe acoustic signal, not underlying word/meaning sequence
- Widely used sequence labeler

Same observation, different state sequences



HMM assumptions

- Independence assumptions:
  - Markov assumption: Finite history
    - Typically 1st order
    - $P(q_i \mid q_1 \ldots. q_{i-1}) = P(q_i \mid q_{i-1})$
  - Output independence:
    - Output probability depends only on state that produced it
    - $P(o_i \mid q_1 \ldots q_i \ldots q_T, o_1 \ldots o_i \ldots o_T) = P(o_i \mid q_i)$
- Typically:
  - Ergodic: Fully connected; every state transition to all others
  - Stationary: All probabilities are independent of time step

Two types of HMMs

- State-emission HMM (Moore machine):
  - The output symbol is produced by states:
    - By the from-state
    - By the to-state

- Arc-emission HMM (Mealy machine):
  - The output symbol is produce by the edges; i.e., by the (from-state, to-state) pairs.

State-emission HMM

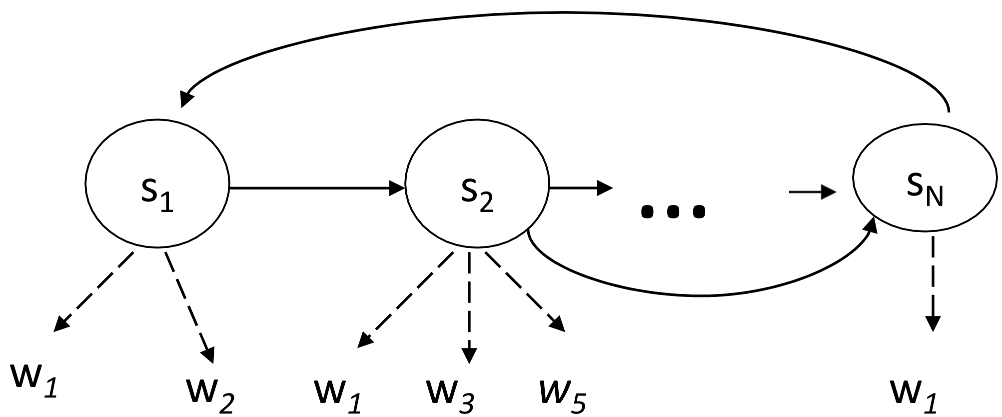
Definition of state-emission HMM

- A HMM is a tuple (S, Σ, Π, A, B) :
  - A set of states $S=\{s_1, s_2, \ldots, s_N\}$.
  - A set of output symbols $\Sigma=\{w_1, \ldots, w_M\}$.
  - Initial state probabilities $\Pi = \{\pi_i\}$
  - Transition prob: $A=\{a_{ij}\}$
  - Emission prob: $B=\{b_{jk}\}$

- We use s_i and w_k to refer to what is in an HMM structure.

- We use X_i and O_i to refer to what is in a particular HMM path and its output

An example: the HMM structure



Two kinds of parameters:

- Transition probability: $P(s_j \mid s_i)$
- Emission probability: $P(w_k \mid s_i)$

➔ # of Parameters: $O(NM+N^2)$

Constraints in a state-emission HMM

$$\sum_{i=1}^{N} \pi_i = 1$$

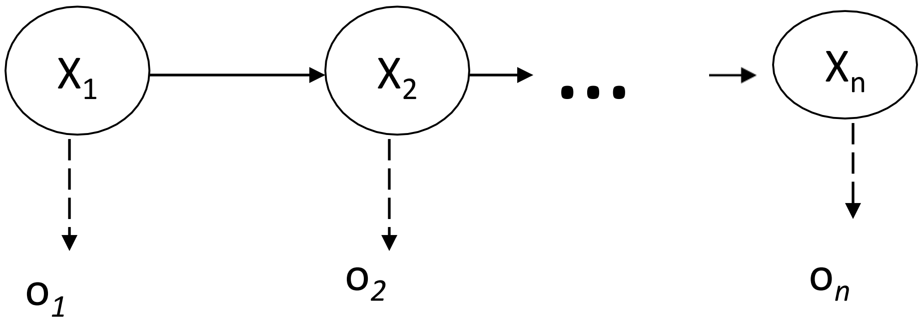
$$\forall i \quad \sum_{j=1}^{N} a_{ij} = 1$$

$$\forall i \quad \sum_{k=1}^{M} b_{ik} = 1$$

$\Rightarrow$ For any integer n and any HMM

$$\sum_{|O|=n} P(O \mid HMM) = 1$$

Output symbols are generated by the from-states

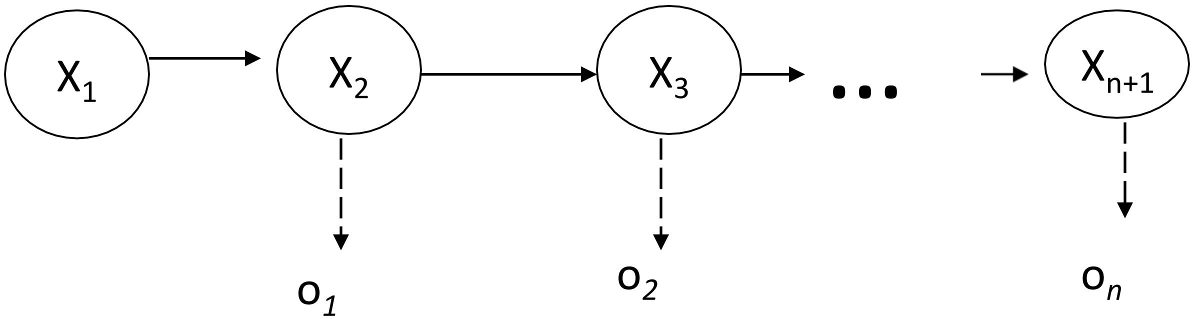


- State sequence: $X_{1,n}$
- Output sequence: $O_{1,n}$

$$P(O_{1,n}, X_{1,n}) = \pi(x_1)\ \left(\prod_{i=1}^{n-1} P(x_{i+1} \mid x_i)\right) \left(\prod_{i=1}^{n} P(o_i \mid x_i)\right)$$

$$P(O_{1,n}) = \sum_{X_{1,n}} P(O_{1,n}, X_{1,n})$$

Output symbols are generated by the to-states



- State sequence: $X_{1,n+1}$
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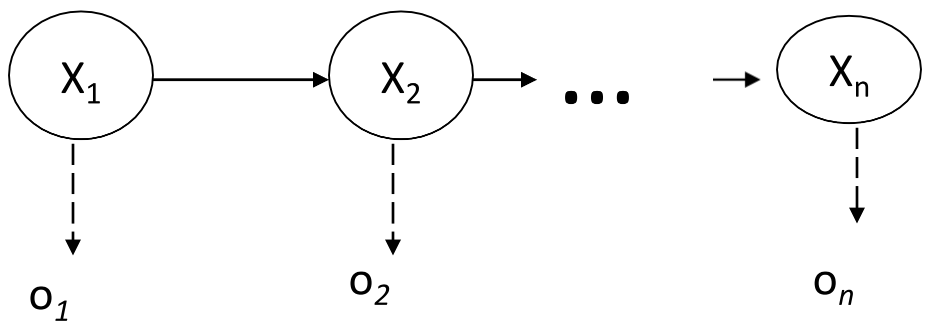
$$P(O_{1,n}, X_{1,n+1}) = \pi(x_1)\prod_{i=1}^{n}(P(x_{i+1} \mid x_i)P(o_i \mid x_{i+1}))$$

$$P(O_{1,n}) = \sum_{X_{1,n+1}} P(O_{1,n}, X_{1,n+1})$$

A path in a state-emission HMM

Output symbols are produced by the from-states:



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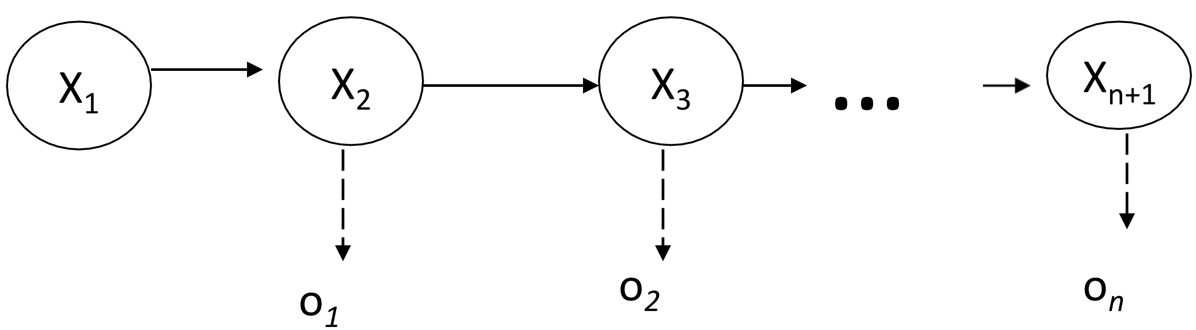


Properties of HMM

- Markov assumption (Limited horizon):

$$P(X_{t+1} | X_1, X_2, \ldots X_t) = P(X_{t+1} | X_t)$$

- Stationary distribution (Time invariance): the probabilities do not change over time:

$$P(X_{t+1} | X_t) = P(X_{t+1+m} | X_{t+m})$$

- The states are hidden because we know the structure of the machine (i.e., S and Σ), but we don't know which state sequences generate a particular output.

Arc-emission HMM

Definition of arc-emission HMM

- A HMM is a tuple (S, Σ, Π, A, B):
  - A set of states $S=\{s_1, s_2, \ldots, s_N\}$
  - A set of output symbols $\Sigma=\{w_1, \ldots, w_M\}$

  - Initial state probabilities $\Pi = \{\pi_i\}$
  - Transition prob: $A=\{ a_{ij} \}$
  - Emission prob: $B=\{ b_{ijk} \}$

Constraints in an arc-emission HMM

$$\sum_{i=1}^{N} \pi_i = 1$$

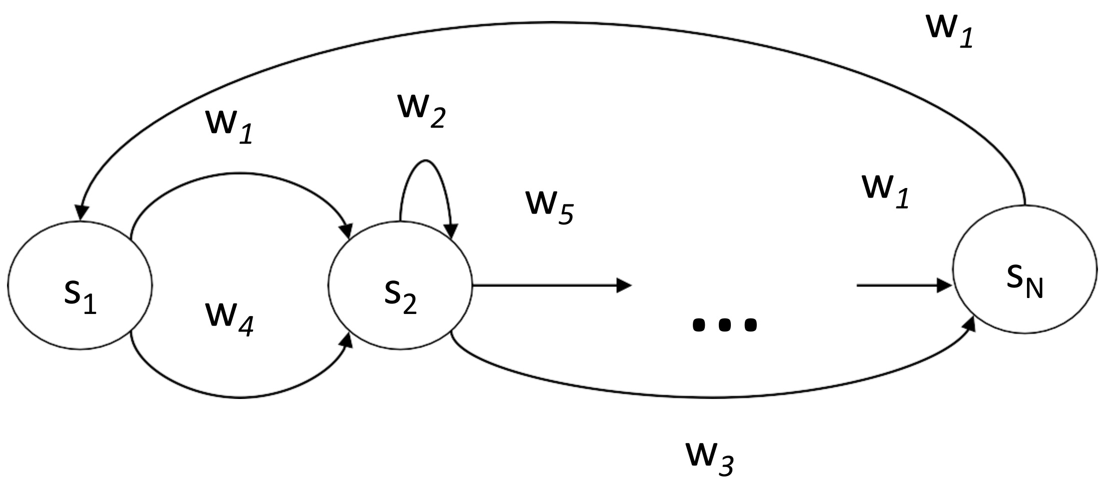
$$\forall i \quad \sum_{j=1}^{N} a_{ij} = 1$$

$$\forall i, j \quad \sum_{k=1}^{M} b_{ijk} = 1$$

For any integer n and any HMM

$$\sum_{|O|=n} P(O \mid HMM) = 1$$

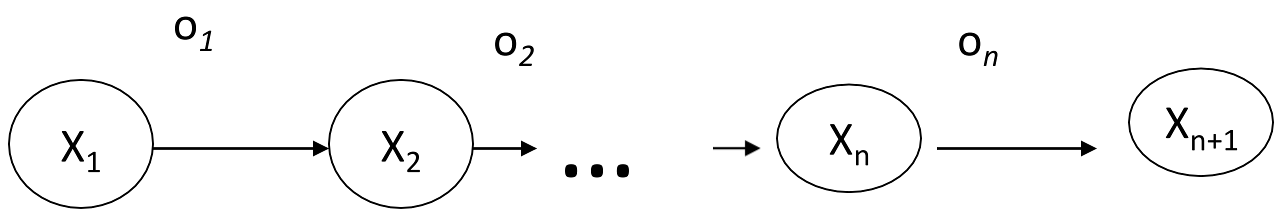
An example: HMM structure



Same kinds of parameters but the emission probabilities depend on both states: $P(w_k \mid s_i, s_j)$

➔ # of Parameters: $O(N^2M + N^2)$.

A path in an arc emission HMM



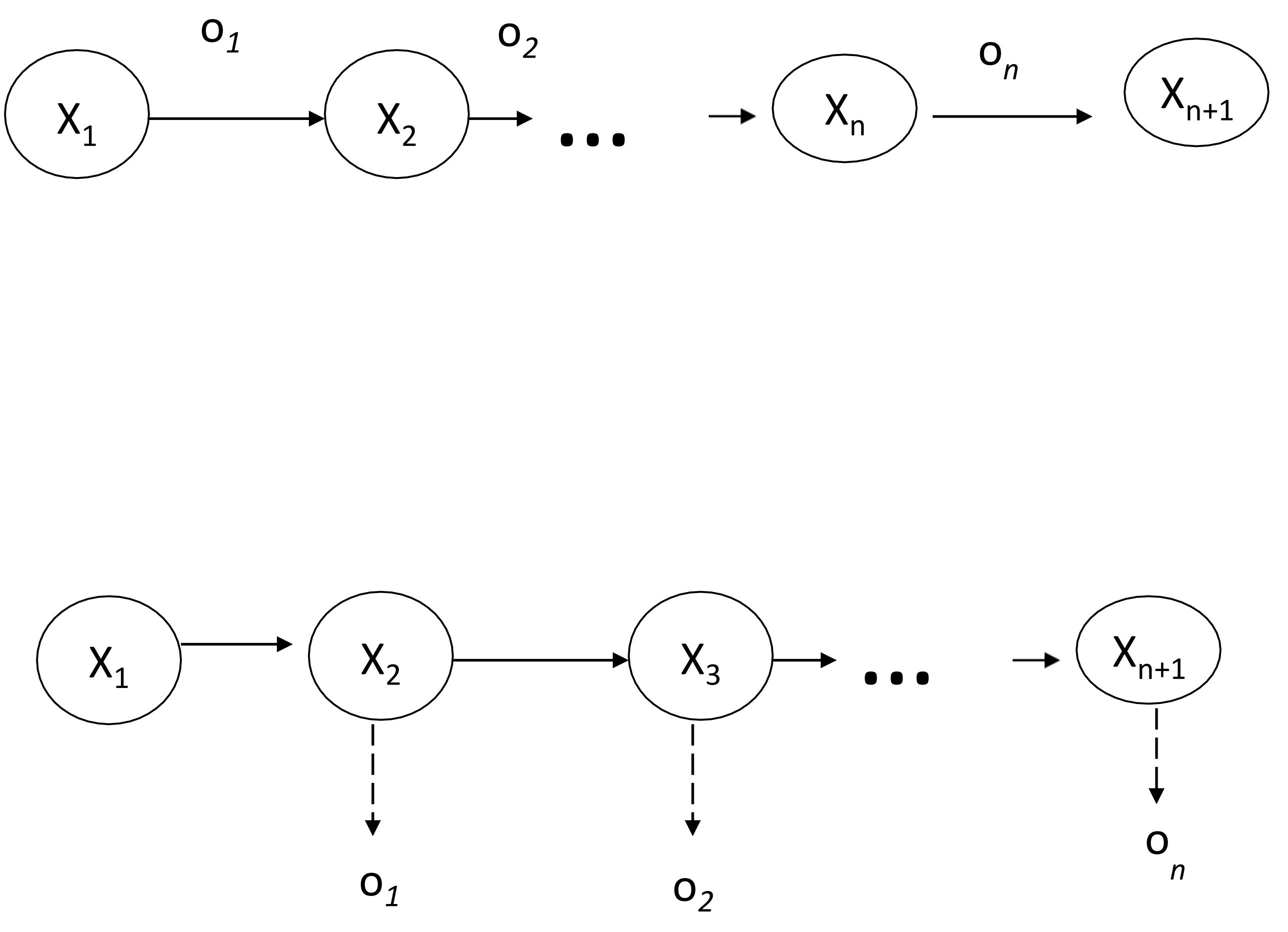
State sequence: $X_{1,n+1}$

Output sequence: $O_{1,n}$

$$P(O_{1,n}, X_{1,n+1}) = \pi(x_1) \prod_{i=1}^{n} P(x_{i+1} \mid x_i) P(o_i \mid x_i, x_{i+1})$$

$$P(O_{1,n}) = \sum_{X_{1,n+1}} P(O_{1,n}, X_{1,n+1})$$

Arc-emission vs. state-emission



Are the two types of HMMs equivalent?

- For each state-emission HMM_1, there is an arc-emission HMM_2, such that for any sequence O, $P(O|HMM_1)=P(O|HMM_2)$.

- The reverse is also true.

- How to prove that?

Applications of HMM

- N-gram POS tagging
  - Bigram tagger: o_i is a word, and s_i is a POS tag.
- Other tagging problems:
  - Word segmentation
  - Chunking
  - NE tagging
  - Punctuation predication
  - …
- Other applications: ASR, ….

Additional slides

PFA recap

Formal definition of PFA

A PFA is $(Q, \Sigma, I, F, \delta, P)$

- Q: a finite set of N states
- Σ: a finite set of input symbols
- I: Q ➔ R^+ (initial-state probabilities)
- F: Q ➔ R^+ (final-state **probabilities**)
- $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$: the transition relation between states.
- P: $\delta \rightarrow R^+$ (transition probabilities)

Constraints on function:

$$\sum_{q \in Q} I(q) = 1$$

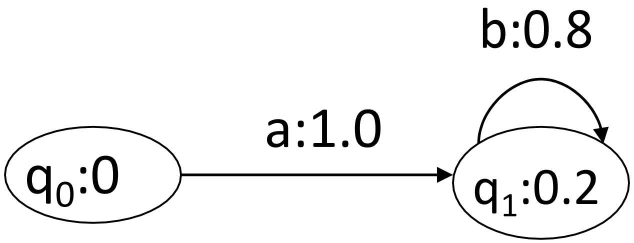
$$\forall q \in Q \quad F(q) + \sum_{\substack{a \in \Sigma \cup \{\varepsilon\} \\ q' \in Q}} P(q, a, q') = 1$$

Probability of a string:

$$P(w_{1,n}, q_{1,n+1}) = I(q_1) * F(q_{n+1}) * \prod_{i=1}^{n} p(q_i, w_i, q_{i+1})$$

$$P(w_{1,n}) = \sum_{q_{1,n+1}} P(w_{1,n}, q_{1,n+1})$$

An example of PFA



$F(q_0)=0$

$F(q_1)=0.2$

$I(q_0)=1.0$

$I(q_1)=0.0$

$$P(ab^n)=I(q_0)*P(q_0,ab^n,q_1)*F(q_1)$$
$$=1.0 \;*\; 1.0*0.8^n \;\; *0.2$$

$$\sum_x P(x) = \sum_{n=0}^{\infty} P(ab^n) = 0.2*\sum_{n=0}^{\infty} 0.8^n = 0.2*\frac{0.8^0}{1-0.8} = 1$$

PFA vs. Arc-emission HMM

A PFA is $(Q, \Sigma, I, F, \delta, P)$

- Q: a finite set of N states
- Σ: a finite set of input symbols
- I: Q ➔ R^+ (initial-state probabilities)
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A HMM is a tuple $(S, \Sigma, \Pi, A, :B)$

- A set of states $S=\{s_1, s_2, \ldots, s_N\}$
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