Synchronization in Chaotic Systems

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Certain subsystems of nonlinear, chaotic systems can be made to synchronize by linking them with common signals. The criterion for this is the sign of the sub-Lyapunov exponents. We apply these ideas to a real set of synchronizing chaotic circuits.

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Chaotic systems would seem to be dynamical systems that defy synchronization.[1] Two identical autonomous chaotic systems started at nearly the same initial points in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space. It is thus a practical impossibility to construct identical, chaotic, synchronized systems in the laboratory.

In this paper we describe the linking of two chaotic systems with a common signal or signals. We show that when the signs of the Lyapunov exponents for the subsystems are all negative the systems will synchronize. By synchronize we mean that the trajectories of one of the systems will converge to the same values as the other and they will remain in step with each other. The synchronization appears to be structurally stable.

We apply these ideas to several well-known systems[2] (e.g., Lorenz and Rössler) as well as the construction of a real set of chaotic synchronizing circuits.

The capability of synchronization is not obvious in nonlinear systems. We derive the results for flows (differential equations), but only a slight variation is needed to use them for iterated maps. Consider an autonomous n-dimensional dynamical system,

$$\dot{u} = f(u) . \tag{1}$$

Divide the system, arbitrarily, into two subsystems $[u = (v, w)]$,

$$\dot{v} = g(v, w), \quad \dot{w} = h(v, w) , \tag{2}$$

where $v = (u_1, \ldots, u_m)$, $g = (f_1(u), \ldots, f_m(u))$, $w = (u_{m+1}, \ldots, u_n)$, and $h = (f_{m+1}(u), \ldots, f_n(u))$.

Now create a new subsystem w' identical to the w system, substitute the set of variables v for the corresponding v' in the function h, and augment Eqs. (2) with this new system, giving

$$\dot{v} = g(v, w) , \quad \dot{w} = h(v, w) , \quad \dot{w}' = h(v, w') . \tag{3}$$

Examine the difference, $\Delta w = w' - w$. The subsystem components w and w' will synchronize only if $\Delta w \rightarrow 0$ as $t \rightarrow \infty$. In the infinitesimal limit this leads to the variational equations for the subsystem,

$$\dot{\xi} = D_w h(v(t), w(t)) \xi , \tag{4}$$

where $D_w h$ is the Jacobian of the w subsystem vector field with respect to w only. The behavior of Eq. (4) or its matrix version[9] depends on the Lyapunov exponents of the w subsystem. We refer to these as sub-Lyapunov exponents. We now have the following theorem: The subsystems w and w' will synchronize only if the sub-Lyapunov exponents are all negative.

The above theorem is a necessary, but not sufficient, condition for synchronization. It says nothing about the set of "initial conditions" in w' which will synchronize with w. We do not mention here any results regarding these sets of points. They are under investigation and will be reported elsewhere.

Taking a broader view, one can think of the $v = (v_1, \ldots, v_m)$ components as being driving variables and the $w' = (w'_{m+1}, \ldots, w'_n)$ as being responding variables. We take just such a view in our application to a chaotic electronic circuit, below.

It is natural to ask how the synchronization is affected by differences in parameters between the w and w' systems which would be found in real applications. Let μ be a vector of the parameters of the y subsystem and μ' of the w' subsystem, so that $h = h(v, w, \mu)$, for example. If the w subsystem were one dimensional, then for small Δw and small $\Delta \mu = \mu' - \mu$,

$$\Delta \dot{w} \approx h_w \Delta w + h_\mu \Delta \mu , \tag{5}$$

where h_w and h_μ are the derivatives of h. Roughly, if h_w and h_μ are nearly constant in time, the solution of this will follow the form

$$\Delta w(t) = \left[\Delta w(0) - \frac{h_\mu}{h_w} \right] e^{h_w t} + \frac{h_\mu}{h_w} . \tag{6}$$

If $h_w < 0$, the difference between w and w' will level off at some constant value. Although this is a simple one-dimensional approximation, it turns out to be the case for all systems we have investigated numerically, even when the differences in parameters are rather large ($\sim 10\%-20\%$).

The phenomena of synchronization is reminiscent of the "slaving principle" of Haken.[10] Haken applied his principle mostly to systems near singularities, like bifurcations, showing that the degrees of freedom of the system for which the eigenvalue of the linear part of the
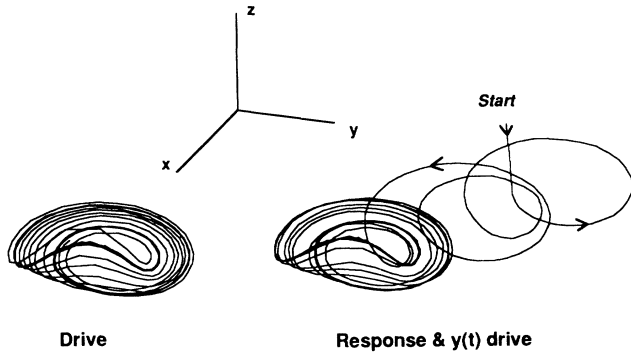
FIG. 1. The attractors for the Rössler drive system and the $(x'-z')$ response system and $y(t)$ drive variable.

vector field were ≥ 0 determined the behavior of all other variables associated with negative eigenvalues. Just as the Lyapunov exponent is the generalization of the Jacobian for stability studies, our use of the sub-Lyapunov exponents appears to be a generalization of concepts like Haken's slaving.

We have tested these ideas on several models, including several two-dimensional maps. Here we present the results for the Rössler[4] and Lorenz[3] attractors which are typical for all our systems.[2]

We found that in the Rössler system it was possible to use the y component to drive an (x',z') response Rössler system and attain synchronization with the (x,z) components of the driving system. Figure 1 shows three-dimensional views of the drive and response systems for a particular set of parameters in the chaotic regime. One can see that although the response system starts far away from the drive values it soon spirals into the same type of attractor where it remains in synchronization with the drive-system attractor. Table I shows the sub-Lyapunov exponents[11] of various configurations of drive and response for the Rössler system. Note that only the y drive configuration will synchronize.

Table I also shows the sub-Lyapunov exponents for the Lorenz system in the chaotic regime. In this case, synchronization will occur for either x or y driving. Figure 2(a) shows a plot of time versus log of the differences $y'-y$ and $z'-z$ for the Lorenz attractor. The convergences to synchronization are consistent with the values in Table I.

Figure 2(b) shows the results for the same situation, but with a slight change in the parameters of the response system. As expected from the simple one-dimensional argument above, the differences level off. The systems partially synchronize in that y' and z' stay within some neighborhood of y and z as they proceed around the attractor.

We have investigated all the above phenomena in other models[2] and have found similar results.

We used a modified version of an electronic chaotic circuit by Newcomb and Sathyan[6] to test these ideas on
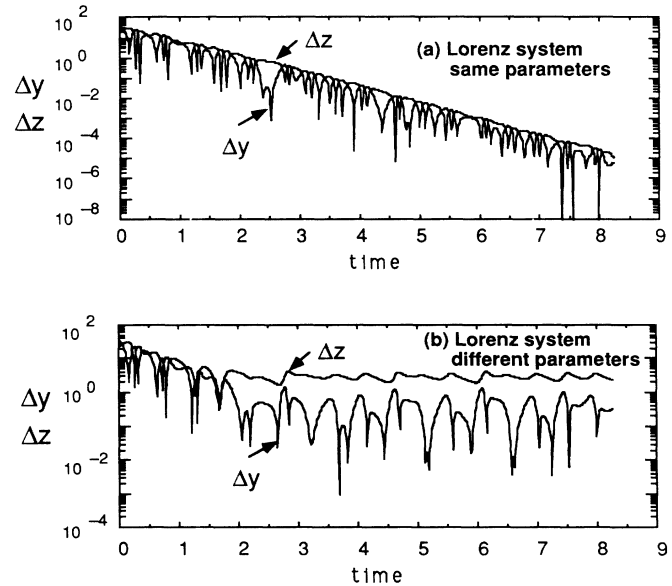

FIG. 2. The differences $y'-y$ and $z'-z$ between the response variables and their drive counterparts for the Lorenz system for (a) when parameters are the same for both systems and (b) when the parameters differ by 5%.

a real system. The drive circuit consists of an unstable second-degree oscillator coupled to a hysteritic circuit which continually shifts the center of the unstable focus causing the system to be reinjected into the region near one of two unstable focii. This keeps the motion bounded and chaotic in certain parameter regimes. This is a three-dimensional dynamical system. The response circuit was chosen to be a subcircuit in which the hysteritic circuitry was mostly cut off, so the drive signal came from a point just at the cutoff. The details of the circuits and these experiments will be given elsewhere.

The equations of motion for the model of the drive circuit can be written in terms of the above oscillator-hysterisis description (see Ref. 12 for a description of modeling hysteresis). These must be transformed so that

TABLE I. A listing of the various subsystems and driving components for the Lorenz and Rossler systems and their sub-Lyapunov exponents.

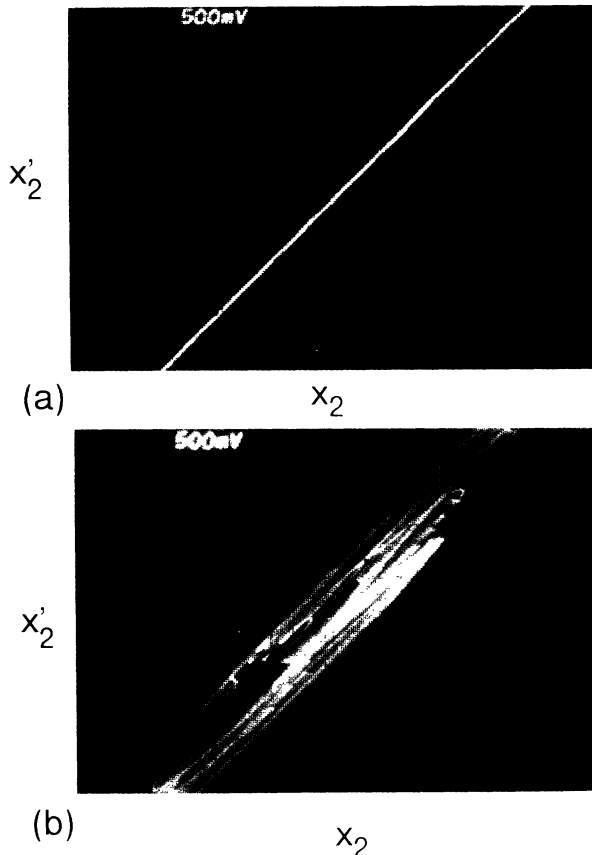| System	Drive	Response	Sub-Lyapunov exponents
Rössler	x	(y,z)	$(+0.2, -8.89)$
$a=0.2, b=0.2$	y	(x,z)	$(-0.056, -8.81)$
$c=9.0$	z	(x,y)	$(+0.1, +0.1)$
Lorenz	x	(y,z)	$(-1.81, -1.86)$
$\sigma=10, b=\frac{8}{3}$	y	(x,z)	$(-2.67, -9.99)$
$r=60.0$	z	(x,y)	$(+0.0108, -11.01)$

FIG. 3. Oscilloscope traces of the response voltage x_2' vs its drive counterpart voltage x_2 for (a) circuit parameters the same and (b) circuit parameters different by 50%.

the drive signal, x_3, is explicitly shown. This gives

$$\dot{x}_1 = x_2 + \gamma x_1 + c(ax_3 - \beta x_1) \, ,$$

$$\dot{x}_2 = -\omega_2 x_1 - \delta_2 x_2 \, ,$$

$$\epsilon \dot{x}_3 = a^{-1}\{[1 - (ax_3 - \beta x_1)^2](sx_1 - r + ax_3 - \beta x_1)$$

$$- \delta_3 ax_3 - \beta x_1 - \beta x_2 - \beta \gamma x_1 - \beta c(ax_3 - \beta x_1)\} \, . \tag{7}$$

The equations for x_1 and x_2 model the response circuit as well. For the chaotic regime the circuit settings dictate that $\gamma = 0.2$, $c = 2.2$, $a = 6.6$, $\beta = 7.9$, $\delta_2 = 0.01$, $\omega_2 = 10$, $s = 1.667$, and $r = 0.0$. The sub-Lyapunov exponents can be calculated directly since the Jacobian for Eqs. (7) is a constant in the x_1 and x_2 variables. The exponents are -16.587 and -0.603, implying synchronization will occur.

The circuit itself runs in the realm of a few kHz. We find that the response synchronizes with the drive within about 2 ms which is consistent with the above sub-Lyapunov exponents whose units are inverse milliseconds. Figure 3 shows oscilloscope traces of the variable x_2 versus its response counterpart x_2' for the syn-

chronizing circuits for two different parameter values. The parameter varied was a resistor in the response circuit which effectively changed α and β. In Fig. 3(b) $\alpha = 9.9$ and $\beta = 10.4$. The values for the driving circuit remained unchanged. This shows changes ($\sim 50\%$) of the circuit parameters effect synchronization greatly. Even though the sub-Lyapunov exponents in the latter cases both remain negative, synchronization is degraded.

At this point much more remains to be done (theoretically and experimentally) on synchronizing systems. All of the systems studied so far have been low dimensional with one positive Lyapunov exponent. Can synchronization be accomplished in the case of two or more positive exponents, but with only one drive? Can one predict which components will synchronize based on the structure of the center, unstable, and stable manifolds? Despite these and other open questions, we would like to offer some speculations.

The ability to design synchronizing systems in nonlinear and, especially, chaotic systems may open interesting opportunities for applications of chaos to communications, exploiting the unique features of chaotic signals. One now has the capability of having two remote systems with many internal signals behaving chaotically yet still synchronized with each other through the one linking drive signal.

Recent interesting results[13,14] suggest the possibility of extending the synchronization concept to that of a metaphor for some neural processes. Freeman has suggested that one should view the brain response as an attractor. The process of synchronization can be viewed as a response system that "knows" what state (attractor) to go to when driven (stimulated) by a particular signal. It would be interesting to see whether this dynamical view could supplant the more "fixed-point" view of neural nets.[15,16]

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[2]References to "all systems" in this paper include the Lorenz (Ref. 3), Rössler (Ref. 4), scroll (Ref. 5), Newcomb hysteresis (Ref. 6), three-mode spin system (Ref. 7), and laser emulation (Ref. 8) systems. We hope to report on these results in the future.

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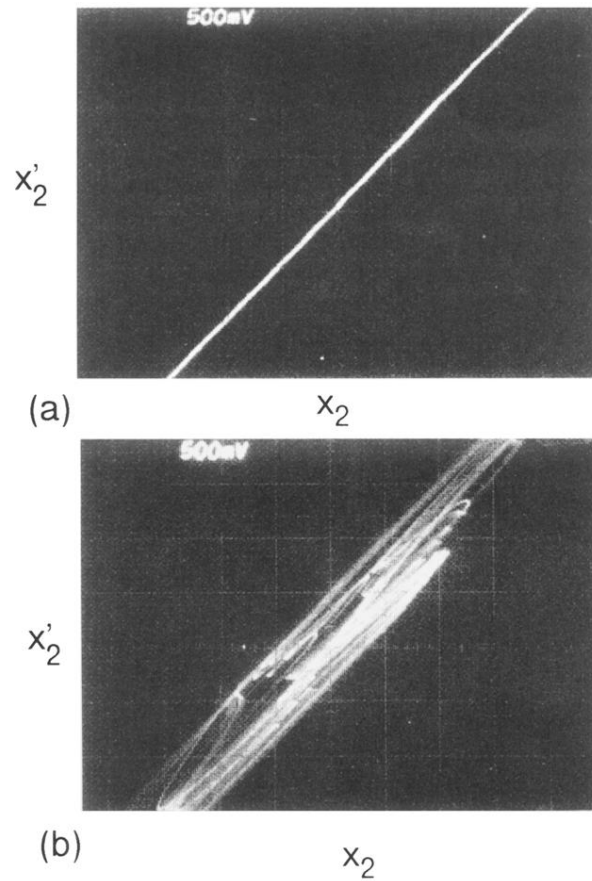
x_2'

x_2

(a)

x_2'

x_2

(b)

FIG. 3. Oscilloscope traces of the response voltage x_2' vs its drive counterpart voltage x_2 for (a) circuit parameters the same and (b) circuit parameters different by 50%.