① Algebra Review

- Know how to work with exponents (i.e. $\frac{2}{3}x^{-5/4} \Rightarrow \frac{2}{3\sqrt[4]{x^5}}$)
- Understand function notation

$f(x) = x^2 + 3x$
$f(2) = 2^2 + 3 \cdot 2 = 4 + 6 = 10$
$f(☆) = ☆^2 + 3☆$

Plug something in, get something out!

- Domain: all allowed "x" values
- Range: all possible "y" values

$\frac{\sqrt{x-5}}{x-2}$: $\sqrt{x-5} \Rightarrow x-5 \geq 0 \Rightarrow x \geq 5$ AND $(x-2) \Rightarrow x-2 \neq 0 \Rightarrow x \neq 2$

NO NEG. under square root.
Can't divide by zero.

Useful thing to know:
FACTOR DIFFERENCE* of 2 perfect SQUARES

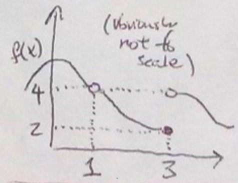
$(9-x^2) = (3-x)(3+x)$
$(a^2-b^2) = (a-b)(a+b)$

* DOESN'T WORK FOR ADDITION.

② Limits & Continuity

- A limit just means what value we're approaching.
- It only exists if the left-hand limit is the same as the right-hand limit

$\left(\text{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)\right)$



$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 4$
Therefore, $\lim_{x\to 1} f(x) = 4$.

HOWEVER...

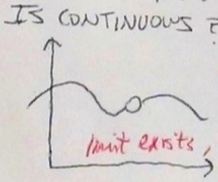
$\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to 3^+} f(x) = 4$ $2 \neq 4$ So, $\lim_{x\to 3} f(x) = \text{DNE}$ (DOES NOT EXIST)

- Continuity means the function ("y") value as we approach a certain point ("x" value) is equal to the actual value at that point.

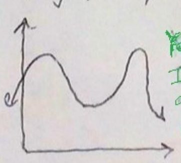
$f(a) = \lim_{x\to a} f(x)$

Pretty much, it's continuous if you can draw ~~the~~ function without lifting your pen.

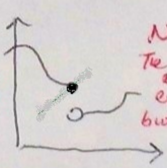
IS CONTINUOUS ??



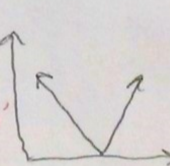
NO. ✗ limit exists, but actual value doesn't.



Yes. ✓ It's smooth and looks like camel humps.



NO. ✗ The function exists at every point, but NOT the limit.



Yes. ✓ This is an absolute value function by the way

Problem Solving:

Is $\frac{2}{x}$ continuous? Find the domain. Is it all real numbers? No, b/c $x \neq 0$. That's our point of discontinuity (how can we check if the function value is equal to the limit if it doesn't even exist right? (we can't)). There's actually an asymptote there.

The 4 cases of limits

ALWAYS PLUG IN FIRST!!

CASE 1:
$\lim_{x\to 2} x+3 = 2+3 = 5$

You just get a number.

CASE 2:
$\lim_{x\to 3} \frac{1}{x-3} = \frac{1}{0}$

∴ you can't divide by zero, so, it DNE.

One-sided limits can be $+\infty / -\infty$ though...

CASE 3:
$\lim_{x\to 2} \frac{x-2}{5} = \frac{0}{5} = 0$.

0 divided by any number is ALWAYS 0. Except for

CASE 4:
$\lim_{x\to -2} \frac{x^2+5x+6}{x^2+6x+8} = \frac{0}{0}$

THIS IS AN INDETERMINATE FORM.

DO MORE WORK!

$\lim_{x\to -2} \frac{(x+2)(x+3)}{(x+2)(x+4)} = \frac{1}{2}$

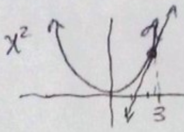
Now onto the next big topic... ⇒

③ DERIVATIVE

Here's the definition again: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ but don't memorize it or be intimidated. It's just slope, a rate of change, or more specifically,

SLOPE OF A TANGENT LINE

x^2



what's the slope of the tangent line at 3?

$\frac{d}{dx}(x^2) = 2x$

$2(3) = \boxed{6}$

Important notation (don't be fooled!)

given $y = 2x+3$ or $f(x) = 2x+3$

they could say find $\frac{dy}{dx}$ or y' or $f'(x)$ ← 1st Derivative

$\frac{d^2y}{dx^2}$ or y'' or $f''(x)$ ← 2nd Derivative

It all means the same thing! (within each row at least)

IMPORTANT RULES:

Product Rule:
$f(x)g'(x) + g(x)f'(x)$

Quotient Rule (BOTTOM FIRST)
if $\frac{g(x)}{h(x)}$ ... $\frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

Chain Rule
$(●)^5 \Rightarrow 5(●)^4 \ast$ DERIVATIVE OF ●
DON'T FORGET!

⚠ Warning:
$x \ln(x)$ is the product of x and $\ln(x)$ NOT $x \ln$ times x. This is nonsense.

ALSO... $e^{●} \xrightarrow{\text{Derivative}} e^{●} \cdot$ Derivative of ●

$\ln(●) \xrightarrow{\text{Derivative}} \frac{1}{●} \cdot$ Derivative of ●

ALSO: Bacon and Eggs.

$\log_B A = E$ can be changed to $B^E = A$.

Note "ln" just means "$\log_e$". So what's $\ln 1$?

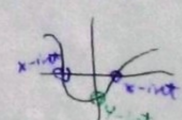
$\ln 1 = ?$ which means $e^? = 1$

ANYTHING RAISED TO THE ZERO POWER = 1. So...

$\ln 1 = 0$.

④ GRAPHING!

X-intercepts occur when $Y = 0$
Y-intercepts occur when $X = 0$, just plug in and solve!



Horizontal Asymptotes (HA)

If power on bottom is greater...

$\frac{3x+1}{2x^2-2}$ as $x \to \infty$, the denominator is growing MUCH FASTER than the top, so we're getting closer and closer to ZERO $(y=0)$

If power on top is greater...

$\frac{2x^2+1}{5x}$ when $x \to \infty$, the whole thing just shoots up to infinity and doesn't "level out" at a certain value. So, no HA.

If they're the same...

$\frac{5x+4}{2x-3}$

Just take coefficients $y = 5/2$.

(VA) Vertical Asymptotes (where the denominator is zero).

$y = \frac{x^2+3x+1}{4x^2-9}$

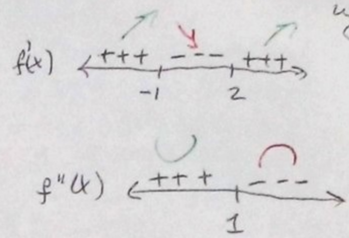
$4x^2 - 9 = 0$
$4x^2 = 9$
$x^2 = 9/4$

$x = \pm 3/2$
So 2 VA:
$x = -3/2$ AND $x = 3/2$

BEWARE! If you're given something like $\frac{x(x-2)}{x^2}$ the bottom doesn't actually have the higher power. Multiply out the top and you'll see...

$f'(x) = 0$	Critical Points	Possible max/mins (not for sure though!)
$f''(x) = 0$	Inflection Points	Places where concavity changes.

then make a sign chart (include the asymptotes)



which means plug back into the original equation to find the other value (if asked for it)

& Plug in test values for each interval.

Finally, GRAPH IT!

5) INTEGRALS (ANTIDIFFERENTIATION)

So with a derivative we usually do nx^{n-1}. Now, just go backwards, so $\frac{x^{n+1}}{n+1}$.

Example: $\int\left(\sqrt[3]{x} + \frac{2}{x}\right)dx = \int\left(x^{1/3} + \frac{2}{x}\right)dx = \frac{3}{4}x^{4/3} + 2\ln x + C$ Remember this!

Oh, and if you have numbers (definite integral) just plug in.

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$$

ALMOST FORGOT. If they ask for ABSOLUTE max/min, just plug in your candidates and pick out the point that gives the "highest" value. Don't forget to plug in/test the endpoints they give you too.

The U-Substitution (use this when the derivative of an embedded function is spotted elsewhere... or if the problem just looks complicated pretty much).

$$\int 2x e^{2+x^2}\,dx \qquad \text{substitute} \qquad \int e^u\,du = e^u + C \qquad \text{plug back in} \qquad e^{2+x^2} + C.$$

$u = 2 + x^2$

$du = 2x\,dx$

Random thoughts:

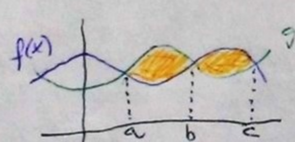
$\frac{x^2}{x^3}$ is just $\frac{1}{x}$ and $\frac{x^3}{x^2}$ is just x. You know, exponent rules.

and if you see $\frac{x^2+x^3}{x}$ it may help you to break up fractions such as $\frac{x^2}{x} + \frac{x^3}{x}$.

6) APPLICATIONS

Areas under curves / between (Recall: TOP − BOTTOM)

If you're not given the limits of integration set the appropriate given equations equal to each other to get the points where they intersect.



area from a to c is?

$$\int_a^b [g(x) - f(x)]\,dx + \int_b^c [f(x) - g(x)]\,dx$$

Average Value Problems

It's like normal averages. Sum and Divide.

$$\frac{1}{B-A}\int_a^b f(x)\,dx$$

example: x^2 from $[1, 6]$

$$\frac{1}{6-1}\int_1^6 x^2\,dx = \frac{1}{5}\left[\frac{x^3}{3}\right]_1^6 = \text{Some number.}$$

(I should've chosen a smaller number... oh well.)

DIFF EQs

Lets say you're given $xy' = 5$. Find ~~f~~ given that $f(1) = 2$.

* ALWAYS MOVE VARIABLES ONTO THEIR OWN SIDES! *
AND REWRITE y' as $\frac{dy}{dx}$.

$$x \frac{dy}{dx} = 5$$

$$\int dy = \int \frac{5}{x}\, dx$$

Integrate both sides:

$$y = 5\ln x + C$$

Plug in to find C:

$$2 = 5\ln 1 + C$$

$$2 = 5(0) + C$$

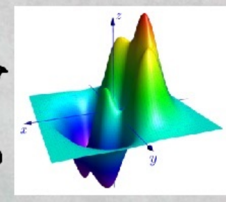
So $2 = C$.

So

$$\boxed{y = 5\ln x + 2}$$

(7) 3D STUFF

Functions can take in more than 1 parameter and you end up with weird stuff like $f(x,y)$ with graphs like



Anyway, this means you can take partial derivatives (just take derivatives normally and treat the other variable as a constant).

NOTATION f_x is the same as $\frac{\partial f}{\partial x}$

f_y is the same as $\frac{\partial f}{\partial y}$

To find critical points just solve the system of equations where $f_x = 0$ and $f_y = 0$

Also memorize: $D = f_{xx} f_{yy} - [f_{xy}]^2$

1. $D > 0$ & $f_{xx} > 0$ relative min
2. $D > 0$ & $f_{xx} < 0$ relative max
3. $D < 0$ Saddle point
4. $D = 0$ we don't know squat.

(8) Optimization

* OR $F(x) = f(x,y) - \lambda g(x,y)$

So, normally we just set some function's derivative $= 0$. But we can also use Lagrange Multipliers for multivariable functions.

You will always have 2 parts: the function you are trying to maximize or minimi[ze] $f(x,y)$ such as $f(x,y) = x^2 + 2y$ and a "constraint" $g(x,y)$ or condition they give you such as $2x + y = 10$

$g(x,y) = \boxed{2x + y - 10 = 0}$

So, $f_x = 2x$ $\quad f_y = 2$

$g_x = 2$ $\quad g_y = 1$

Just set up a system of equations and solve.

$$\begin{cases} f_x = g_x \lambda \\ f_y = g_y \lambda \\ g(x,y) \end{cases}$$

f_x: partial derivative w/respect to x of the function we're trying to max/minimize

g_x: Partial derivative of the constraint function.

Include this constraint equation when solving (otherwise you won't have enough equations)

Don't forget to multiply by lambda (λ)!

* OR just

$F_x = 0$

$F_y = 0$

$F_z = 0$

$F_\lambda = 0$

obviously, I can't go into great detail for each topic, but do some practice problems and it should (hopefully) make sense.

That's It! Good luck on your finals!