MSBD 5004 Mathematical Methods for Data Analysis
Homework 3

Due date: 27 March, 9pm, Wednesday

1. Let V be a Hilbert space. Let S_1 and S_2 be two hyperplanes in V defined by

$$S_1 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_1, \boldsymbol{x} \rangle = b_1\}, \quad S_2 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_2, \boldsymbol{x} \rangle = b_2\}.$$

Let $\boldsymbol{y} \in V$ be given. We consider the projection of $\boldsymbol{y}$ onto $S_1 \cap S_2$, i.e., the solution of

$$\min_{\boldsymbol{x} \in S_1 \cap S_2} \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{1}$$

   (a) Prove that $S_1 \cap S_2$ is a plane, i.e., if $\boldsymbol{x}, \boldsymbol{z} \in S_1 \cap S_2$, then $(1+t)\boldsymbol{z} - t\boldsymbol{x} \in S_1 \cap S_2$ for any $t \in \mathbb{R}$.

   (b) Prove that $\boldsymbol{z}$ is a solution of (1) if and only if $\boldsymbol{z} \in S_1 \cap S_2$ and

$$\langle \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x} \rangle = 0, \quad \forall \boldsymbol{x} \in S_1 \cap S_2. \tag{2}$$

   (c) Find an explicit solution of (1).

   (d) Prove the solution found in part (c) is unique.

2. Let $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^N$ be given with $\boldsymbol{x}_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}$. Assume $N < n$, and $\boldsymbol{x}_i$, $i = 1, 2, \ldots, N$, are linearly independent. Consider the ridge regression

$$\min_{\boldsymbol{a} \in \mathbb{R}^n} \sum_{i=1}^N \left(\langle \boldsymbol{a}, \boldsymbol{x}_i \rangle - y_i\right)^2 + \lambda \|\boldsymbol{a}\|_2^2,$$

where $\lambda \in \mathbb{R}$ is a regularization parameter, and we set the bias $b = 0$ for simplicity.

   (a) Prove that the solution must be in the form of $\boldsymbol{a} = \sum_{i=1}^N c_i \boldsymbol{x}_i$ for some $\boldsymbol{c} = [c_1, c_2, \ldots, c_N]^T \in \mathbb{R}^N$. (*Hint: Similar to the proof of the representer theorem.*)

   (b) Re-express the minimization in terms of $\boldsymbol{c} \in \mathbb{R}^N$, which has fewer unknowns than the original formulation.

3. Find the Frechet derivative of the following functions $f : V \mapsto \mathbb{R}$, where V is a Hilbert space.

   (a) $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|$ for a given $\boldsymbol{a} \in V$, where $\boldsymbol{x} \neq \boldsymbol{a}$.

   (b) $f(\boldsymbol{x}) = \|2\boldsymbol{x} - \boldsymbol{a}\|^2$ for a given $\boldsymbol{a} \in V$.

   (c) $f(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$, where $\boldsymbol{x} \neq \boldsymbol{0}$.

   (d) $V = \mathbb{R}^n$ and $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{x_i^2 + c}$ for some $c > 0$.