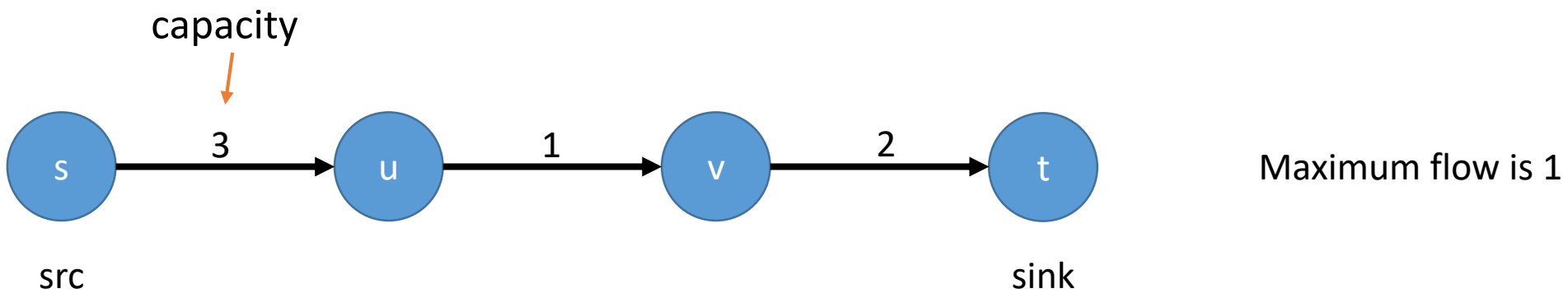
Push-Relabel Algorithm

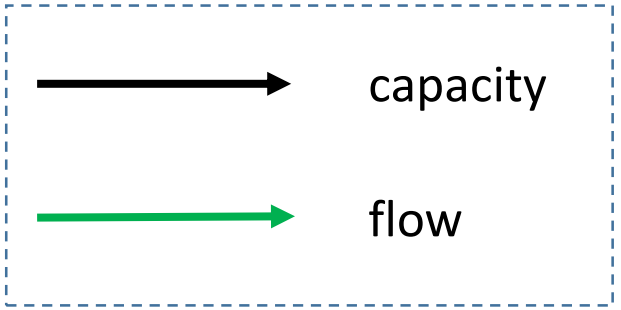
COMP5112/MSBD5009 assignments tutorial

Maximum Flow

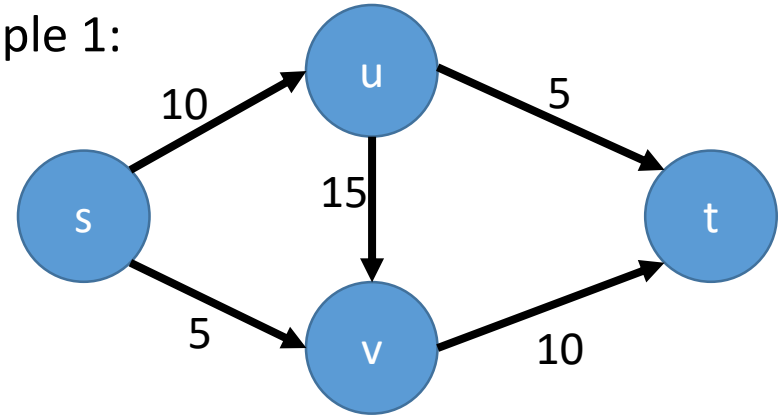
- A flow network is a directed graph, where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge.
- Maximum flow problems involve finding a feasible flow through a single-source (src) and a single-sink (sink) flow network that is maximum.

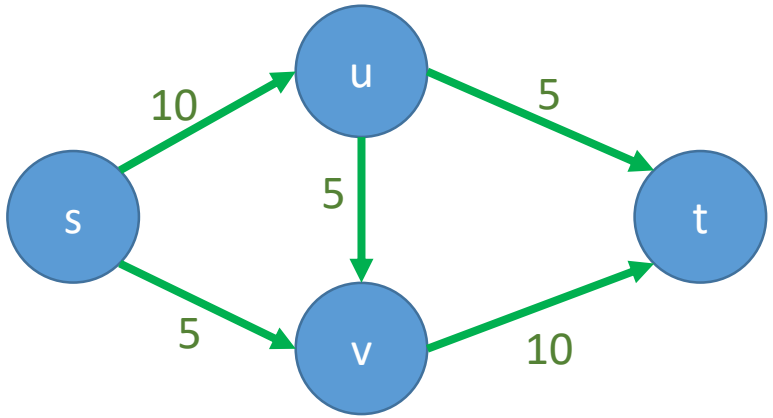


Maximum Flow Examples



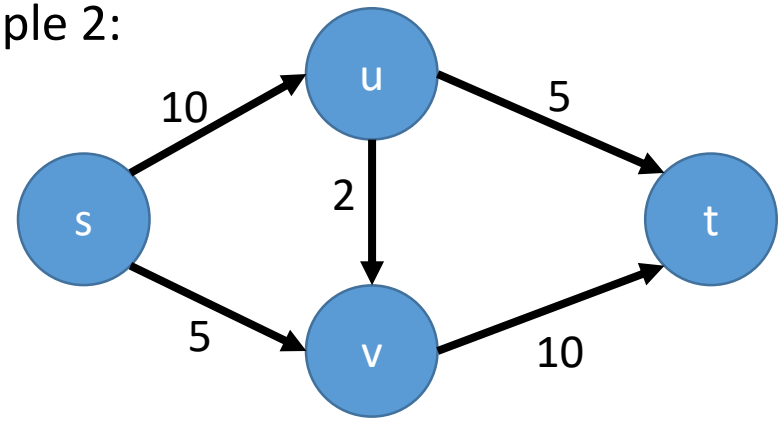
Example 1:

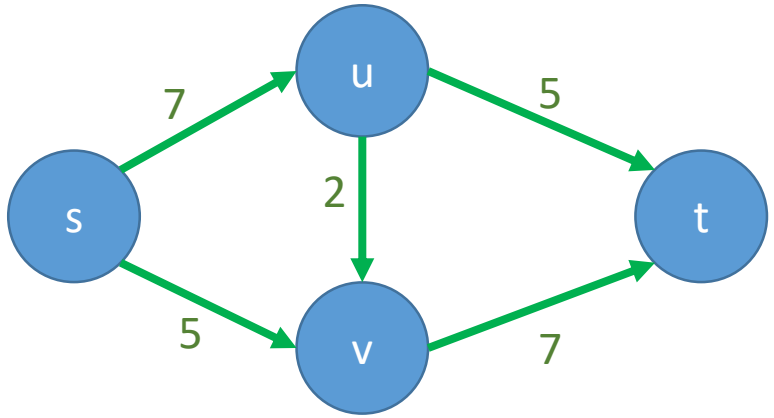




Maximum flow is 15

Example 2:



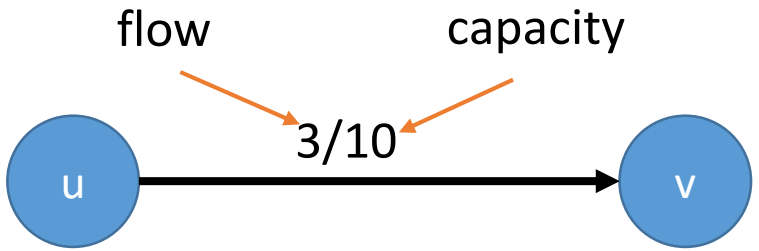


Maximum flow is 12

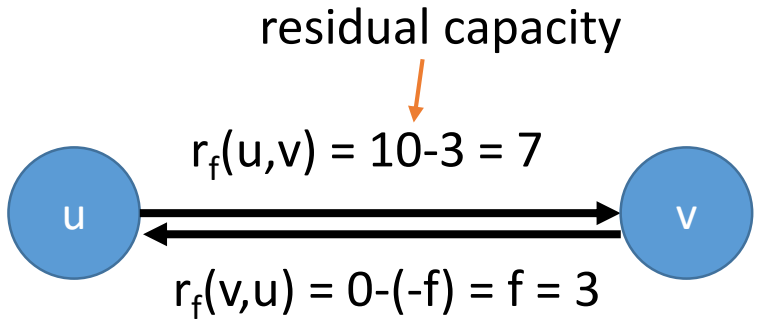
- Flow on an edge doesn't exceed the given capacity of the edge.
- Incoming flow is equal to outgoing flow for every vertex except s and t.

Residual Graph

- ***Residual capacity*** $r_f(v, w)$ of a vertex pair is $c(v, w) – f(v, w)$.
- Residual graph $G_f = (V, E_f)$ where E_f is the set of ***residual edges*** (v,w) with $r_f(v, w) > 0$.
- E.g, consider two vertices *u* and *v*:



Original graph



Residual graph

Push-Relabel Algorithm

- Push-Relabel algorithm works on the ***residual graphs.***
- The intuition behind the push-relabel algorithm is that we consider edges as water pipes and nodes are joints:
  - The source is considered to be at the ***highest level*** and it send water to all adjacent nodes;
  - Once a node has excess water (inflow – outflow > 0), it ***pushes*** water to a smaller height node;
  - If water gets locally trapped at a vertex, the vertex is ***relabeled*** which means its height is increased.

Push-Relabel Algorithm

- Excess Flow of ***u*** = Total inflow to ***u*** – Total outflow from ***u***
- Three operations of Push-Relabel algorithm:

  1) ***Initialize Preflow*** : Initialize Flows and Heights

  2) ***While*** it is possible to perform a ***Push()*** or ***Relablel()*** on a vertex

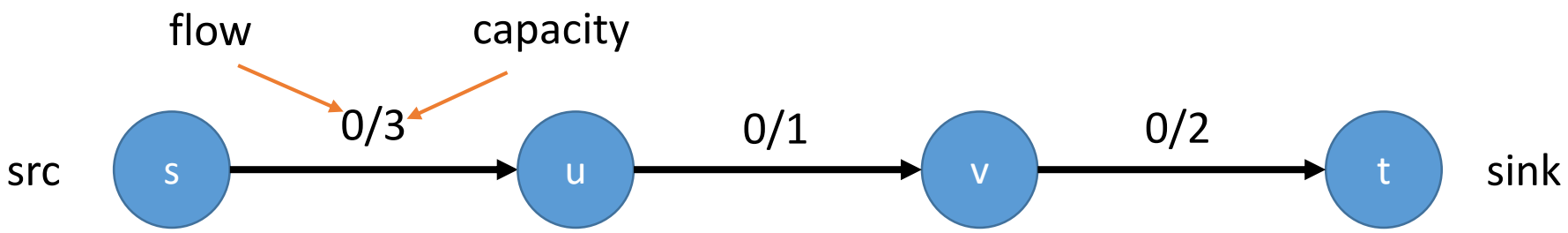
  // Or while there is a vertex that has excess flow

  Do Push() or Relabel()

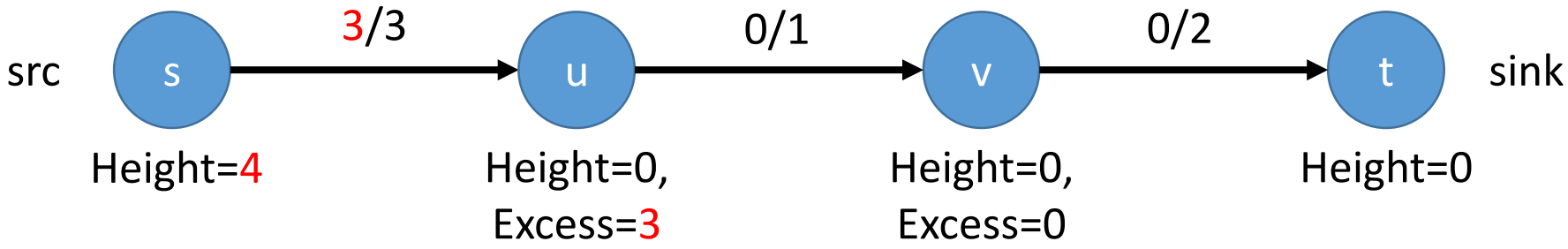
  // At this point all vertices have Excess Flow as 0 (Except source and sink)

  3) ***Return*** flow.

Push-Relabel Algorithm Running Example



Preflow



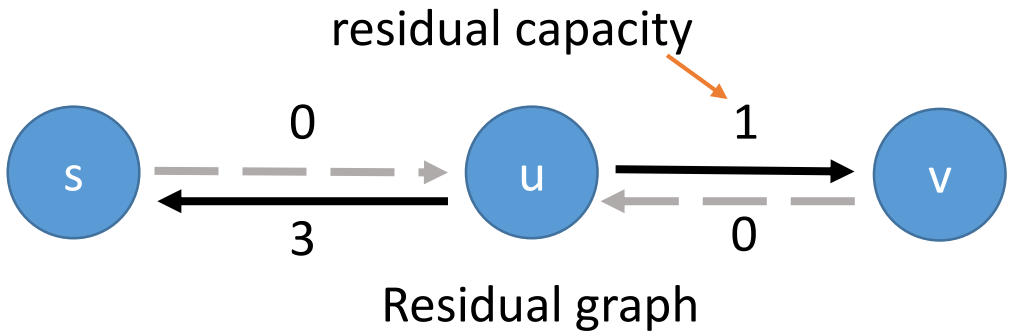
Initial Labeling:

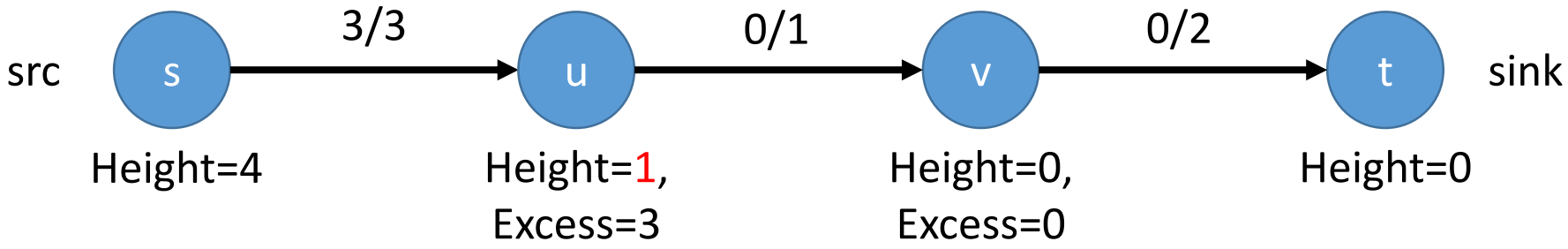
- Initial height of *src* = Number of vertices = 4
- Initial height of other vertices = 0

Initial Flow:

- Flow from *src* to its adjacents is equal to capacities of edges

Relabel ***u***

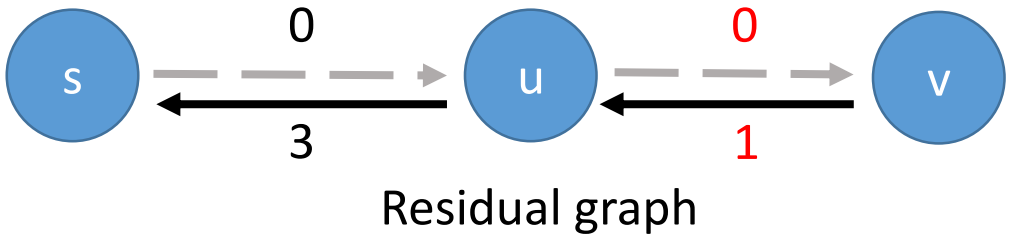


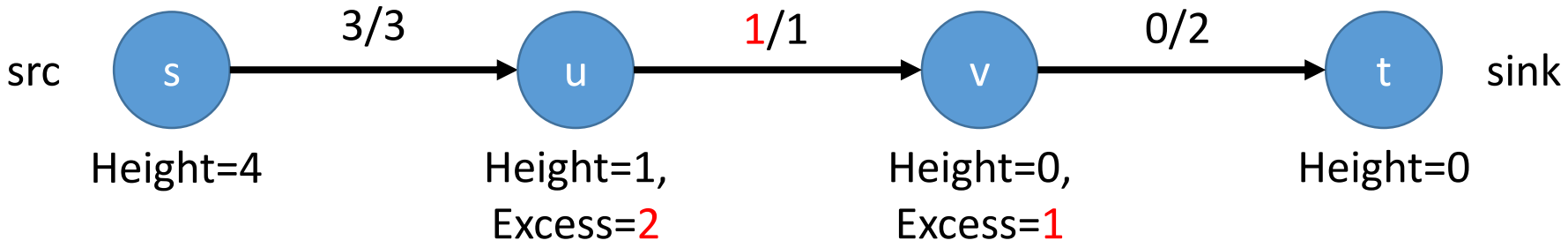


- The height of vertex ***u*** is relabeled (height becomes to 1) since it has excess flow and there is no adjacent with smaller height.

- The new height of ***u*** is equal to minimum of heights of its ***adjacent vertex on the residual graph*** ($r_f(v, w) > 0$) plus 1.

Push from ***u***

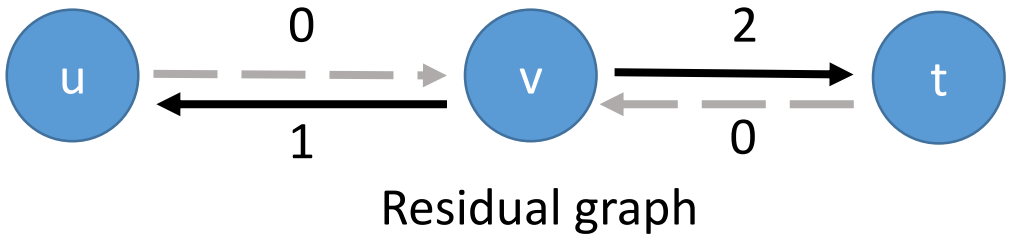


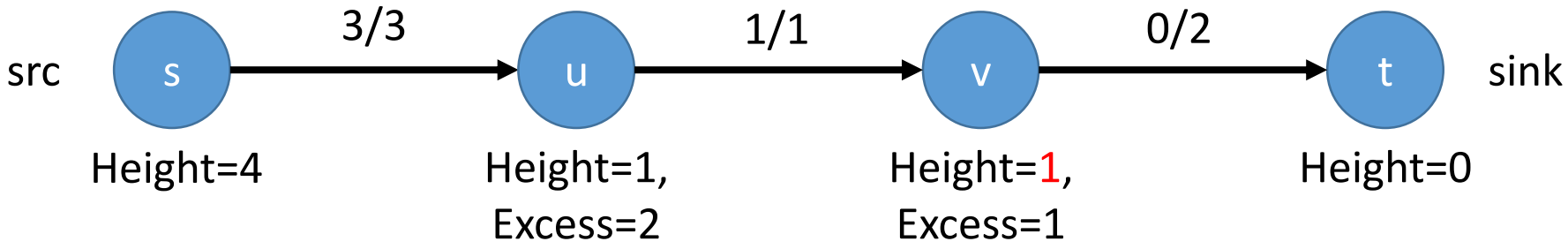


- The vertex ***u*** has excess flow and there is an adjacent with lower height, so ***push*** happens.

- Excess flow of vertex ***u*** is (3 – 0 = 3) and the capacity of edge(***u***,***v***) is 1; thus, he amount of pushed flow is 1 (minimum of excess flow and the edge capacity).

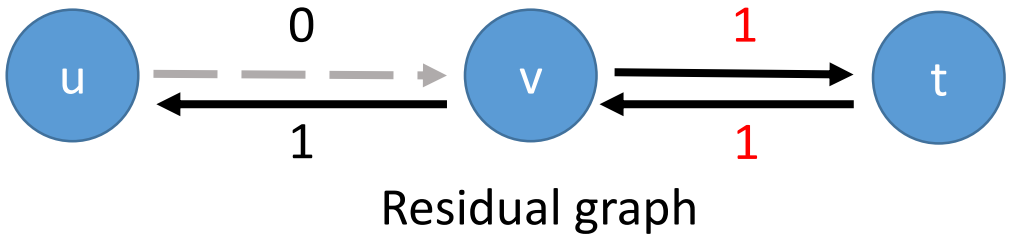
Relabel ***v***

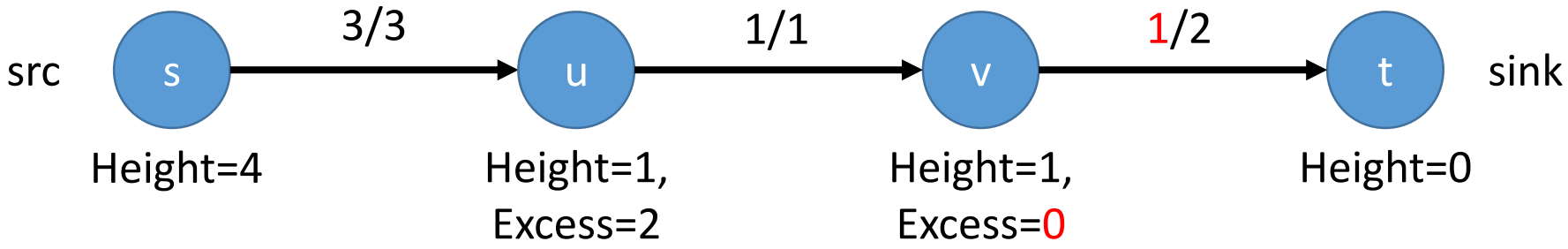




- The height of vertex ***v*** is relabeled (height becomes 1) as it has excess flow and there is no adjacent with smaller height.
- The new height of ***v*** is equal to minimum of heights of adjacent plus 1.

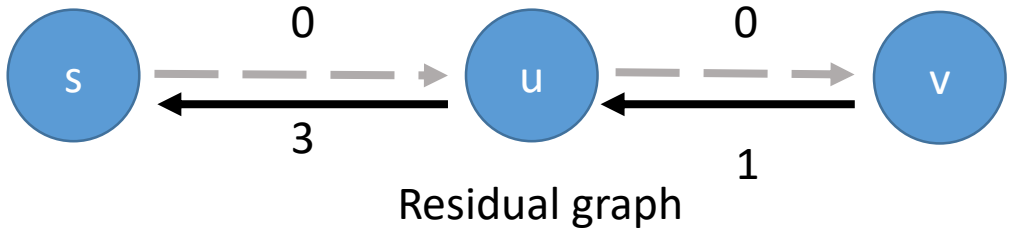
Push from ***v***

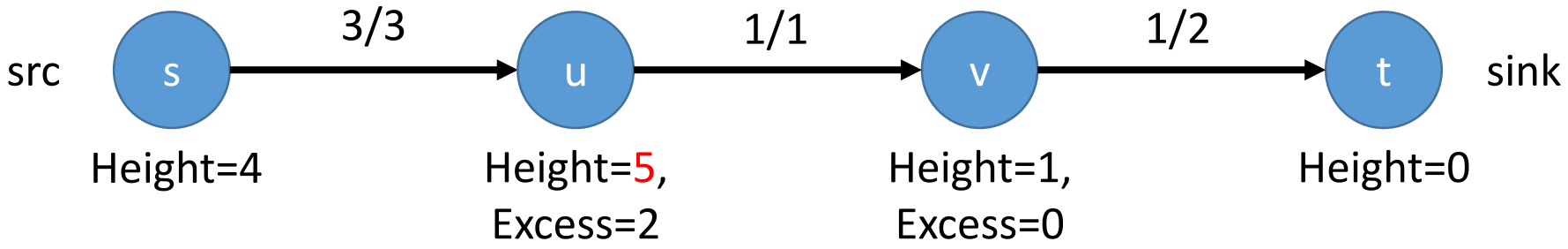




- The vertex ***v*** has excess flow and there is an adjacent with lower height, so ***push*** happens.
- Excess flow of vertex ***v*** is (1 – 0 = 1) and the capacity of edge(***v***,***t***) is 2; thus, the amount of pushed flow is 1 (minimum of excess flow and the edge capacity).

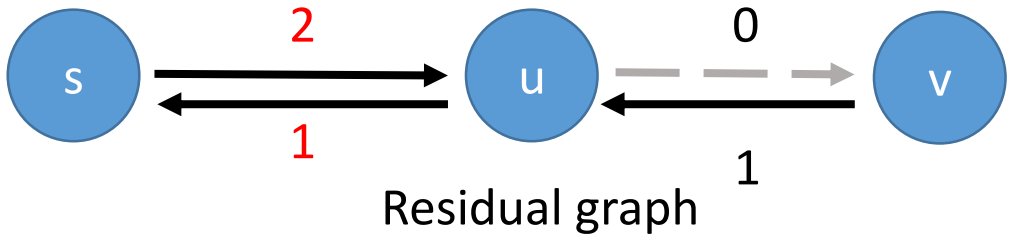
Relabel *u*

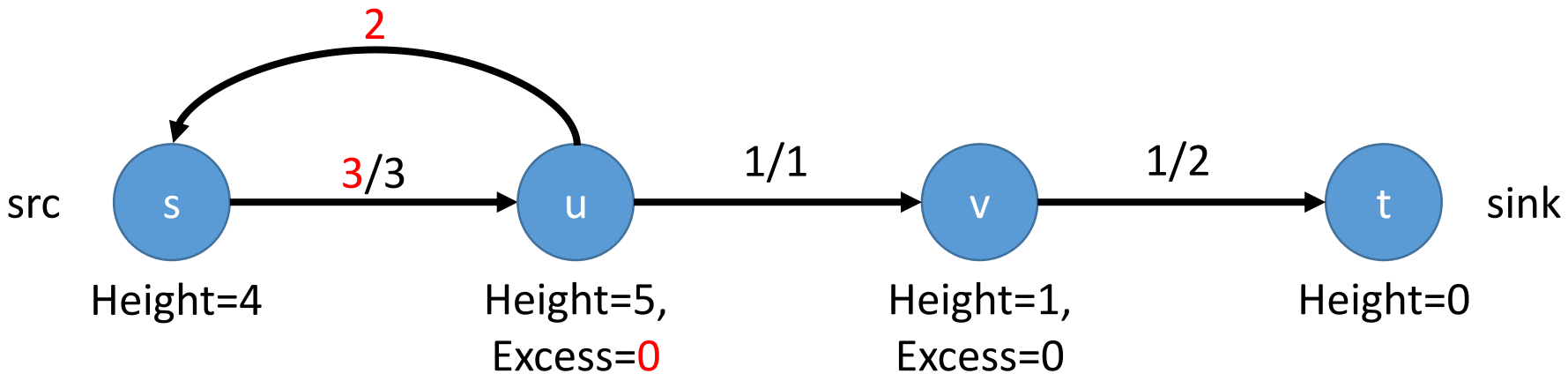




- The height of vertex ***u*** is relabeled (height becomes to 5), since it has excess flow and there is no adjacent with smaller height.
- The new height of ***u*** is equal to minimum of heights of adjacent plus 1.
- **Note vertex *v* is not an adjacent of *u* in the residual graph!**

Push from *u*

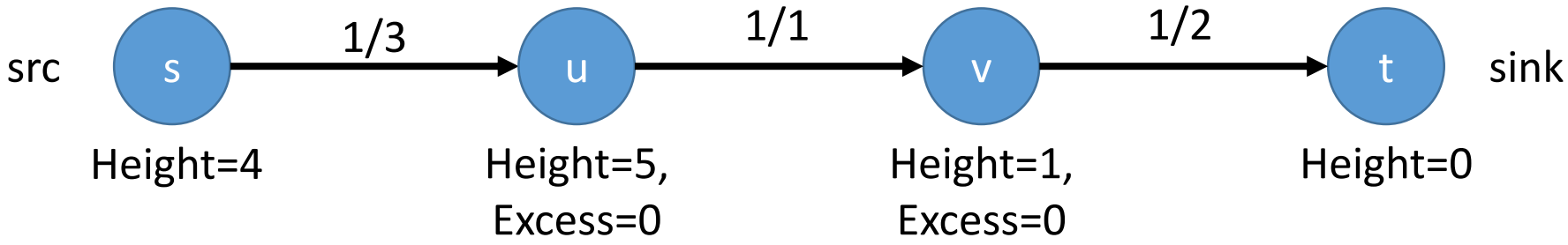




- The vertex ***u*** has excess flow and there is an adjacent with lower height, so ***push*** happens.

- Excess flow of vertex ***u*** is (3 – 1 = 2) and the capacity of edge(***u***,***s***) is 3; thus, the amount of pushed flow is 2 (minimum of excess flow and the edge capacity).

End of push-relabel



- Maximum flow achieved since there is no relabel possible and no vertex has excess flow.
- Maximum flow is 1.

References

- https://en.wikipedia.org/wiki/Flow_network
- https://en.wikipedia.org/wiki/Maximum_flow_problem
- https://www.geeksforgeeks.org/push-relabel-algorithm-set-1-introduction-and-illustration/
- https://www.geeksforgeeks.org/ford-fulkerson-algorithm-for-maximum-flow-problem/

Thanks !

Q & A