

# Machine Learning

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# Chapter 1

## Math

### 1.1 Convolution

- Definition

- $f * g(z) = \int_{\mathbb{R}} f(x)g(z-x)dx$ , where  $f(x), g(x)$  are functions in  $\mathbb{R}$

- Statistical Meaning

- Notation

- $X, Y$ : independent random variables, with pdf's given by  $f$  and  $g$
    - $Z = X + Y$ , with pdf given by  $h(z)$ :

- $\Rightarrow h(z) = f * g(z)$

- derivation

$$\begin{aligned} H(z) &= P(Z < z) = P(X + Y < z) \\ &= \int_x P(X = x)P(X + Y < z | X = x)dx \\ &= \int_x f(x)P(Y < z - x)dx \\ &= \int_x f(x)G_Y(z - x)dx \\ \Rightarrow h(x) &= \frac{d}{dz}H(z) = \frac{d}{dz} \int_x f(x)G_Y(z - x)dx \\ &= \int_x f(x) \frac{dG_Y(z - x)}{dz} dx \\ &= \int_x f(x)g(z - x)dx \\ &= f * g(z) \end{aligned}$$

### 1.2 Linear Algebra

#### 1.2.1 Essence

##### Vector

- Interpretation
  - Movement

- direction
    - distance
  - Numeric in High Dimensions
    - in 1- $D$ : +/- represents direction
    - in  $n$ - $D$ : +/- alone each dimension combined to represent an overall direction (direction of the  $n$ - $D$  numeric - vector)
  - Numerics Multiplication
    - Scaling
      - the number scales the distance of vector (direction remains)  
 $\Rightarrow$  such number thus also called scalar
      - $\Rightarrow$  scale alone each axis by that scalar  
 $2\mathbf{x} = 2x_1e_1 + \dots + 2x_ne_n$ ,  
 where  $e_1, \dots, e_N$  are vector defining coordinates
  - Linear Combination
    - Vector Adding: Generalization of Numerical Adding
      - in 1- $D$ : joint movement along single axis
      - in  $n$ - $D$ : joint movement along each axis  $\Rightarrow$  a joint movement in  $n$ - $D$  space
    - Definition:  $\mathbf{x} = a_1\mathbf{x}_1 + \dots + a_n\mathbf{x}_n$ 
      - the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  only altered linearly (as only being scaled)  
 $\Rightarrow$   $\mathbf{x}$  direction & size are linear combination of that in  $\mathbf{x}_1, \dots, \mathbf{x}_n$
    - Span of  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ 
      - the  $n$ - $D$  space  $S$  constructed by linear combination of  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$
    - $x_0$  linearly dependent on  $\{x_1, \dots, x_n\}$ 
      - $x_0$  can be constructed by linear combination of  $\{x_1, \dots, x_n\}$   
 (already in the span space  $S$ )
      - $\Leftrightarrow$  function  $a_0x_0 + \dots + a_nx_n = 0$  has other solution than  $a_0 = \dots = a_n = 0$
    - $x_0$  linearly INdependent with  $\{x_1, \dots, x_n\}$ 
      - $x_0$  can NOT be constructed by linear combination of  $\{x_1, \dots, x_n\}$   
 (not in the span space  $S$ , will increase the dimension of  $S$  if adopted)
      - $\Leftrightarrow$  function  $a_0x_0 + \dots + a_nx_n = 0$  has and only has solution  $a_0 = \dots = a_n = 0$
  - Special Vectors
    - $n$ - $D$  Zero Vector  $\mathbf{x}$ 
      -
    - Unit Vector
    - Basis of Vector Space  $S^n$ 
      - general basis: a set of linearly independent vectors that span the space  
 (i.e. a set of linearly independent  $n$ - $D$  vectors)
      - unit basis: a general basis where every vector is unit vector
      - orthogonal basis: a general basis where all vectors are orthogonal with each other
      - unit orthogonal basis: a basis that is also a unit basis and an orthogonal basis
- $\Rightarrow$  coordinate: the scalar to composite a vector given a specific basis

## Linear Transformation and Maps

- Linear Transformation
  - Transformation
    - a function mapping: vector  $\rightarrow$  vector
    - a vector movement: scale & rotate all possible input vectors (i.e. a vector space)
  - Transformation with Linearity  $L(\cdot)$ 
    - intuition: lines remain lines & origin remains origin
    - definition: a transformation  $L(\cdot)$  is linear if
      1. additivity:  $L(\mathbf{x}_1 + \mathbf{x}_2) = L(\mathbf{x}_1) + L(\mathbf{x}_2)$
      2. scaling:  $L(a\mathbf{x}) = aL(\mathbf{x})$ , where  $a$  is scalar
  - Features of  $L(\cdot)$  given  $\mathbf{x} = x_1e_1 + \dots + x_ne_n$ 
    - same scalar for coordinates
 
$$\begin{aligned} \Rightarrow L(\mathbf{x}) &= L(x_1e_1 + \dots + x_ne_n) \\ &= L(x_1e_1) + \dots + L(x_ne_n) \\ &= x_1L(e_1) + \dots + x_nL(e_n) \end{aligned}$$
    - $\Rightarrow$  transformed vector  $\mathbf{x}' = L(\mathbf{x})$  has the same coord under the transformed basis
- Linear Map
  - Definition
    - map  $F : V \rightarrow X$  is a linear map if it is a linear transformation, where  $V, X$  are vector spaces
- Multilinear Maps
  - Definition
    - map  $F : \underbrace{V \times \dots \times V}_{k \text{ copies}} \rightarrow X$  is multilinear/ $k$ -linear if it is linear in each slot
      - i.e.  $F(\mathbf{v}_1, \dots, a\mathbf{v}_i + b\mathbf{v}'_i, \dots, \mathbf{v}_k) = aF(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k) + bF(\mathbf{v}_1, \dots, \mathbf{v}'_i, \dots, \mathbf{v}_k)$
      - i.e. for fixed  $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_k$ ,  $F$  reduced to linear map with  $\mathbf{v}_i$  as variable (where  $V, X$  are vector spaces)
  - Alternating Maps
    - map  $F$  is alternating if, its output is  $\mathbf{0}$  whenever two vectors in inputs are identical
  - for Multilinear Map  $F$ :  $F$  Alternating  $\Leftrightarrow F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) = -F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots)$ 
    - i.e. for multilinear map  $F$ ,  $F$  alternating  $\Leftrightarrow$  swapping two inputs flips sign of output
    - proof: given multilinear and alternating map  $F$ , for any  $\mathbf{v}, \mathbf{w}$ 

$$\begin{aligned} 0 &= F(\dots, (\mathbf{v} + \mathbf{w}), \dots, (\mathbf{v} + \mathbf{w}), \dots) \\ &= F(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{w}, \dots) + F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots) \\ &= F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) + F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots) \\ &\Rightarrow F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) = -F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots) \end{aligned}$$
    - proof: given multilinear map  $F : F(\dots, \mathbf{v}, \dots, \mathbf{w}, \dots) = -F(\dots, \mathbf{w}, \dots, \mathbf{v}, \dots)$ 

$$\begin{aligned} &\Rightarrow F(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) = -F(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) \\ &\Rightarrow F(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) = 0, \text{ hence alternating} \end{aligned}$$



## Matrix

- Matrix for Linear Transformation  $L : S \rightarrow S'$

- Representing Space Transformation

- package the transformed basis under the original basis using matrix  $M$   
i.e. represent the  $e'_1, \dots, e'_n = L(e_1), \dots, L(e_n)$  under the original basis  $e_1, \dots, e_n$   
 $\Rightarrow M = [e'_1, \dots, e'_n]$ , with all transformed basis as column vectors  
 $\Rightarrow M$  represent the result of linear transformation for the basis of  $S$
    - hence, determine a linear space transformation  $L : S \rightarrow S'$  using  $e_1, \dots, e_n$   
for  $M_{m \times n}$  matrix: a linear transformation from  $n$ -D space to  $m$ -D space
      1.  $m < n$ : projecting to subspace
      2.  $m > n$ : expanding into a hyper-plane/-line/etc (constrained in hyper-space)

- Performing Space Transformation

- $\forall i \in \{1, \dots, n\}, x'_i = i^{\text{th}}$  component of  $\mathbf{x}' = L(\mathbf{x})$ , then  $x'_i = (e'_{1i} + \dots + e'_{ni})x_i$   
(as proved above)
    - $\Rightarrow$  output vector  $\mathbf{x}' = L(\mathbf{x}) = M\mathbf{x}$  under the original basis  $e_1, \dots, e_n$   
hence the rule for matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

where green and red columns the  $\{e'_1, e'_2\}$  under  $\{e_1, e_2\}$ ,  $[x, y]$  the input vector  $\mathbf{x}$

- Matrices for Composition of Linear Transformation

- Transformation  $L_1, L_2$  as Matrix  $M_1, M_2$

- as easy to prove  $M_2 \cdot (M_1 \cdot \mathbf{x}) = (M_1 \cdot M_2) \cdot \mathbf{x}$   
 $\Rightarrow L_2(L_1(\cdot)) \Leftrightarrow$  the composition of transformation defined by  $M_2 \cdot M_1$
    - $\Rightarrow \mathbf{x}$  first transformed by  $L_1$  then  $L_2 \Leftrightarrow$  transformed by  $M_2 \cdot M_1$

- Linear Transformation on Space

- given a basis matrix  $E = [e_1, \dots, e_n]$ , a transformed basis  $L_1(E) = L_1(e_1), \dots, L_1(e_n)$   
( $e_1, \dots, e_n$  as column vectors)  
 $\Rightarrow L_2(L_1(e_k))$  performs  $L_2$  transformation on the  $k^{\text{th}}$  vector of  $L_1(E)$
    - hence to perform  $L_2$  on the transformed space  $L_1(E) \Rightarrow L_2(L_1(E))$
    - given that  $L_1(E) = M_1 \Rightarrow L_2(L_1(E)) = M_2 \cdot M_1$   
(due to the derivation of linear transformation as matrix)
    - hence,  $M_2 \cdot M_1$  denotes
      1. a further  $L_2$  transformation on a transformed space  $L_1(E)$   
(all result represented under the original basis  $E$ )
      2. the final transformed basis (first  $L_1$  then  $L_2$ ) under original basis  $E$   
 $\Rightarrow$  denotes a composite transformation of  $L_2(L_1(\cdot))$

- Multiplication between Matrices

- $\Rightarrow$  composite linear transformations into single linear transformation
    - $\Rightarrow$  linear transformation on vectors/space (generalized from single vector)

- Understanding Properties

- $(AB)C = A(BC)$

- both meaning apply transformation  $C$  then  $B$  then  $A$ ...

$$\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{M_2} \overbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}^{M_1} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

where  $M_1 = [L_1(e_1), L_1(e_2)]$ ,  $M_2 = [L_2(e_1), L_2(e_2)]$ , the result  $= [L_2(L_1(e_1)), L_2(L_1(e_2))]$   
(all under basis  $E = [e_1, e_2]$ )

## Dot Product

- Projecting to 1-D
  - Unit Orthogonal Basis
    - $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n = \mathbf{x}^T \cdot \mathbf{y} = \mathbf{y}^T \cdot \mathbf{x}$   
( $\mathbf{x}, \mathbf{y}$  assumed to be column vector / matrix with single column)
  - General Basis  $E = [e_1, \dots, e_n]$ 
    - $\Rightarrow \mathbf{x} = x_1 e_1 + \dots + x_n e_n = E\mathbf{x}, \mathbf{y} = y_1 e_1 + \dots + y_n e_n = E\mathbf{y}$   
 $\Rightarrow \mathbf{x} \cdot \mathbf{y} = (E\mathbf{x}) \cdot (E\mathbf{y}) = \mathbf{x}^T E^T E \mathbf{y}$
    - understanding:
      1.  $\mathbf{x} = E\mathbf{x}$ : transfer back to a representation under unit orthogonal basis
      2. for  $E = I$ , back to the unit orthogonal case
- $\Rightarrow \mathbf{x} \cdot \mathbf{y}$ : projecting  $\mathbf{x}/\mathbf{y}$  to 1-D line using transformation  $\mathbf{y}/\mathbf{x}$   
(direction/scaling effect of projection can be alter by choice of basis  $E$  though)
- Duality
  - Dual Vector
    - the vector represent a projection (linear transformation) to 1-D line  
(hence the vector equivalent to the matrix with 1 row defining the projection)
    - $\Rightarrow$  performing transformation on a vector  $\Leftrightarrow$  taking product with the dual vector

## Exterior (Wedge) Product

- Introduction
  - Definition
    - $k$ th exterior product  $\wedge^k V$  is a vector space, with a map  $\psi : \underbrace{V \times \dots \times V}_{k \text{ times}} \rightarrow \wedge^k V$   
 note: map  $\psi$  called exterior multiplication, element  $\psi(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_k)$  called  $k$ -blade
  - Key Properties for Pair  $(\wedge^k V, \psi)$ 
    - $\psi$  is alternating multilinear map
    - for basis  $\{e_1, \dots, e_n\}$  of  $V \in \mathbb{R}^n \Rightarrow \{e_{i_1} \wedge \dots \wedge e_{i_k} | 1 \leq i_1 \leq i_k \leq n\}$  a basis for  $\wedge^k V$   
(due to its alternating multilinearity)  
 $\Rightarrow e_{i_1} \wedge \dots \wedge e_{i_k}$  can form any permutation of the same input by swapping order  
 e.g. for  $\wedge^2 V : e_1 \wedge e_1 = e_2 \wedge e_2 = \mathbf{0}, e_1 \wedge e_2 = -e_2 \wedge e_1 \Rightarrow$  not linearly independent
    - $\dim \wedge^k V = \binom{n}{k} = C_n^k$  (due to the form of its basis),  
 where  $n = \dim V$
    - sum of  $k$ -wedge is still in the vector space  $\wedge^k V$
    - any alternating multilinear map  $F : \underbrace{V \times \dots \times V}_k \rightarrow X$  factors uniquely into:  

$$\underbrace{V \times \dots \times V}_k \xrightarrow{\psi} \wedge^k V \xrightarrow{F} X, \text{ where } \psi \text{ exterior multiplication, } \underline{F} \text{ a linear map}$$

**Determinant**

- Signed Volume
- 
- Measuring Linear Transformation on Volume
  - Unit Volume
    - unit volume  $v = \|e_1 \wedge \dots \wedge e_n\|$  with basis  $e_1, \dots, e_n$   
(for orthogonal basis,  $v = \|e_1\| \times \dots \times \|e_n\|$ )
    - after transformation:  $L(v) = \|L(e_1) \wedge \dots \wedge L(e_n)\| = \|Me_1 \wedge \dots \wedge Me_n\|$   
where  $\wedge$  is the exterior product
  - Measuring Change of Unit Volume
    - $\Rightarrow \det(M) = \frac{L(v)}{v} = \|M_0 \times \dots \times M_n\|$ ,  
assuming unit orthogonal basis  $E = I$ , where  $I$  is identity matrix  
(note: interpretation of  $\det(\cdot) \leftrightarrow$  spacial interpretation of cross product  $\times$ )
    - hence, the rule of calculating  $\det(\cdot)$
- Measuring Change of Orientation
  - Orientation of Space
    - jointly defined by the direction & order of the sequence of basis vector  
i.e. the positive/negative part of each axis in sequence
  - Measuring the Change
    - $\det(M) < 0$  if axes flipped over once (for an odd times)  
 $\det(M) > 0$  if flipped for even times
    - interpretation: the flipped axis approaches 0 then expanded into the negative  
(measured by original basis  $E$ )
- Linear Dependency
  - $\det(M) = 0$ 
    - volume in current space becomes 0  
 $\Rightarrow$  dimensions decreases after transformation applied  
 $\Rightarrow$  i.e. transformed basis not able to span the current space
    - $\Rightarrow$  basis in  $M$  NOT linearly INdependent!  $\Leftrightarrow \det(M) = 0$
  - $\det(M) \neq 0$ 
    - volume still exist  
 $\Rightarrow$  dimensions remain & transformed basis still span the space
    - $\Rightarrow$  basis in  $M$  is linearly INdependent  $\Leftrightarrow \det(M) \neq 0$
- Understanding Properties
  - $\det(M_1 M_2) = \det(M_1) \det(M_2)$ :
    - left: final volume & orientation changed after transformation  $M_2$  then  $M_1$
    - right: the volume scaled by one transformation, then further by the other;  
(similar for orientation, as measured by sign)

## Cross Product

- Determinant
  - $\|\mathbf{v} \times \mathbf{w}\| = |\det([\mathbf{v}, \mathbf{w}])|$ 
    - as determinant measuring the change of unit basis  
 $\Rightarrow$  constructing matrix with target vector  $\mathbf{v}, \mathbf{w}$  as column
    - $\Rightarrow$  matrix  $[\mathbf{v}, \mathbf{w}]$  as the transformation altering the space
  - Direction of  $\mathbf{v} \times \mathbf{w}$ 
    - expanding in the perpendicular direction w.r.t the hyper-plane defined by  $\mathbf{v}, \mathbf{w}$   
 (obeying the "right hand rule")
- Linear Transformation
  -

## Rank

- Measuring Linear Transformation on Dimension
  - Measuring the Transformed Space
    - rank of  $M = r$ : basis in  $M$  span a  $r$ -dimension (column) space  
 note: the column space of  $M$ : transformed space defined by column basis
    - full rank: the dimension of column space is as high as possible
- Null Space (Kernel) of  $M$ 
  - Vectors in Null Space
    - $\forall \mathbf{x}, M\mathbf{x} = \mathbf{0}$   
 (i.e. all the vectors in null space transformed into  $\mathbf{0}$  by  $M$ , hence the name)
    - $\mathbf{0}$  always in null space
  - Dimension of Null Space
    - $D(\text{null space}) = \text{num of column in } M - \text{rank of } M$   
 note: as focusing on column space  $\Rightarrow$  num of cols = dimension of input vectors
- Understanding Properties
  - $R(M_{m \times n}) \leq \min\{m, n\}$ 
    - $m < n$ : projecting a  $n$ -D vector to a  $m$ -D subspace, hence at most of rank  $m$
    - $m > n$ :  $M$  consists of  $n$  vectors of  $m$  dimensions, define at most  $n$ -D space
  - $R(M_{m \times n}) = \min\{m, n\} \Leftrightarrow M$  Full Rank
    - from definition, it is as high as possible
    - $m$  vectors with  $n$  dimensions, span at most a  $\min\{m, n\}$  space  
 either linearly dependent ( $m < n$ ), or not enough independent vectors ( $m > n$ )

## Inverse of Matrix

- Inverse of Linear Transformation
  - Interpretation
    - a transformation  $L^{-1}$  to inverse the effect of another transformation  $L$   
 $\Rightarrow \forall \mathbf{x}, L^{-1}L(\mathbf{x}) = \mathbf{x}$
    - $\Rightarrow M^{-1}M = I$ , where  $M$  for  $L$ ,  $M^{-1}$  for  $L^{-1}$ ,  $I$  the identity matrix

- Requirement
  - transformed basis in  $M$  still span the space!  
otherwise, transformed vectors projected onto subspace  $\Rightarrow$  information lost!
  - hence, following equivalent requirements:
    1. transformed basis in  $M$  linearly INdependent
    2.  $M$  is square and full rank
    3.  $\det(M) \neq 0$

## Linear System of Equations

- Linearity
  - Linear Combination of Variables
    - coefficients matrix  $A$ : holding the coefficient for each equation (in row)
    - variables vector  $\mathbf{x}$ : holding variables as column vector
    - constants vector  $\mathbf{v}$ : holding target constant for each equations as column vector $\Rightarrow A\mathbf{x} = \mathbf{v}$  for a set of linear equations
  - Linear Transformation Perspective
    - $\mathbf{x}/\mathbf{v}$  as original/transformed vectors
    - columns of  $A$  (coefficients for the same variable) as transformed basis $\Rightarrow$  finding a start position  $\mathbf{x}$  which, after transformation  $A$ , lands on  $\mathbf{v}$
- Solutions
  - Transformed Basis Linearly INdependent ( $\det(A) \neq 0$ )
    - hence,  $A\mathbf{x} = \mathbf{v} \Leftrightarrow \mathbf{x} = A^{-1}\mathbf{v}$   
(use the inverse transformation  $A^{-1}$  to find the input  $\mathbf{x}$  using output  $\mathbf{v}$ ) $\Rightarrow$  single unique  $\mathbf{x}$  found
  - Transformed Basis Linearly Dependent ( $\det(A) = 0$ )
    - information lost ( $\mathbf{x}$  projected to subspace)  $\Rightarrow$ 
      1. multiple solutions: basis in  $A$  linearly dependent with  $\mathbf{v}$   
 $\Rightarrow$  i.e.  $\mathbf{v}$  in the column space of  $A$   
 (special case where  $\mathbf{v} = \mathbf{0}$ : solution space = null space of  $A$ )
      2. no solution: basis in  $A$  linearly INdependent with  $\mathbf{v}$   
 $\Rightarrow$  i.e. can NOT possibly be described by basis in  $A$

## 1.2.2 Interchanging Coordinates

### N-dimensional Spherical Coordinates

- Notation
  - $N$ -dim Euclidean Space  $E_N$ 
    - $e_1, e_2, \dots, e_N$ : a group of orthonormal basis of  $E_N$
    - $\mathbf{x} = (x_1, x_2, \dots, x_N)$ : vector in  $E_n$
    - $\mathbf{x}_{i-N}$ : projection of  $\mathbf{x}$  onto subspace spanned by  $e_i, \dots, e_N$   

$$\Rightarrow \mathbf{x}_{i-N} = \sum_{n=i}^N x_n e_n$$
  - Spherical Coordinates

- $r = \|\mathbf{x}\|$ : the norm of  $\mathbf{x}$
- $\phi_i \in [0, \pi]$ : angle between  $\mathbf{x}_{i-N}$  and  $e_i$
- $r_i = \|\mathbf{x}_{i-N}\|$ : norm of projection  $\mathbf{x}_{i-N}$ , with  $r_1 = r$

- Observation

- Space  $e_1, \dots, e_N$ :

- $\cos \phi_1 = \frac{\mathbf{x}e_1}{\|\mathbf{x}\|\|e_1\|} = \frac{x_1}{r}$   
 $\Rightarrow x_1 = r \cos \phi_1$   
 $\Rightarrow \mathbf{x} = r \cos \phi_1 e_1 + \sum_{n=2}^N x_n e_n$

- Space  $e_2, \dots, e_N$ :

- from above:  $\mathbf{x}^2 = r^2 \cos^2 \phi_1 + \sum_{n=2}^N x_n^2 = r^2$   
 $\Rightarrow \sum_{n=2}^N x_n^2 = r^2 \sin^2 \phi_1$   
 $\Rightarrow \mathbf{x}_{2-N} = \sum_{n=2}^N x_n e_n$   
 $\Rightarrow \begin{cases} \|\mathbf{x}_{2-N}\|^2 = \sum_{n=2}^N x_n^2 = r^2 \sin^2 \phi_1 = r_2^2 \\ \cos \phi_2 = \frac{\mathbf{x}_{2-N} \cdot e_2}{\|\mathbf{x}_{2-N}\|\|e_2\|} = \frac{x_2}{r_2} \end{cases}$   
 $\Rightarrow \begin{cases} r_2 = r \sin \phi_1 \\ x_2 = r_2 \cos \phi_2 = r \sin \phi_1 \cos \phi_2 \end{cases} \quad (\text{as } \phi_1 \in [0, \pi])$   
 $\Rightarrow \mathbf{x}_{2-N} = r \sin \phi_1 \cos \phi_2 e_2 + \sum_{n=3}^N x_n e_n$

- Space  $e_3, \dots, e_N$ :

- from above:  $\mathbf{x}_{2-N}^2 = r^2 \sin^2 \phi_1 \cos^2 \phi_2 + \sum_{n=3}^N x_n^2 = r^2 \sin^2 \phi_1$   
 $\Rightarrow \sum_{n=3}^N x_n^2 = r^2 \sin^2 \phi_1 \sin^2 \phi_2$   
 $\Rightarrow \mathbf{x}_{3-N} = \sum_{n=3}^N x_n e_n$   
 $\Rightarrow \begin{cases} \|\mathbf{x}_{3-N}\|^2 = \sum_{n=3}^N x_n^2 = r^2 \sin^2 \phi_1 \sin^2 \phi_2 = r_3^2 \\ \cos \phi_3 = \frac{\mathbf{x}_{3-N} \cdot e_3}{\|\mathbf{x}_{3-N}\|\|e_3\|} = \frac{x_3}{r_3} \end{cases}$   
 $\Rightarrow \begin{cases} r_3 = r \sin \phi_1 \sin \phi_2 \\ x_3 = r_3 \cos \phi_3 \end{cases} \quad (\text{as } \phi_1, \phi_2 \in [0, \pi])$   
 $\Rightarrow \mathbf{x}_{3-N} = r \sin \phi_1 \sin \phi_2 e_3 + \sum_{n=4}^N x_n e_n$

- Proof for  $x_i$

- Procedure

$$\begin{aligned}
\blacksquare \mathbf{x}_{i-N} &= \sum_{n=i}^N x_n e_n \\
\Rightarrow \cos \phi_i &= \frac{\mathbf{x}_{i-N} \cdot e_i}{\|\mathbf{x}_{i-N}\| \|e_i\|} = \frac{x_i}{r_i} \\
\Rightarrow x_i &= r_i \cos \phi_i
\end{aligned}$$

- Induction

- Goal

$$\blacksquare \forall i \geq 2, r_i = r \prod_{j=1}^{i-1} \sin \phi_j$$

- Base Case ( $i = 2$ )

$$\blacksquare \text{ as in observation, } r_2 = r \sin \phi_1 = r \prod_{j=1}^{2-1} \sin \phi_j$$

- Step Case

$$\blacksquare \text{ assumption } r_i = r \prod_{j=1}^{i-1} \sin \phi_j$$

$$\blacksquare \text{ procedure: } \mathbf{x}_{i-N} = \sum_{n=i}^N x_n e_n = r_i \cos \phi_i + \sum_{n=i+1}^N x_n e_n$$

$$\Rightarrow \|\mathbf{x}_{i-N}\|^2 = r_i^2 \cos^2 \phi_i + \sum_{n=i+1}^N x_n^2 = r_i^2$$

$$\Rightarrow \|\mathbf{x}_{i+1-N}\|^2 = \sum_{n=i+1}^N x_n^2 = r_i^2 \sin^2 \phi_i = r_{i+1}^2$$

$$\Rightarrow r_{i+1} = r_i \sin \phi_i = r \prod_{j=1}^i \sin \phi_j$$

- Derivation

- $x_i$  from Combined Proofs

$$\blacksquare x_i = \begin{cases} r \cos \phi_1 & i = 1 \\ r \cos \phi_i \prod_{j=1}^{i-1} \sin \phi_j & 2 \leq i \leq N \end{cases}$$

- Last 2 Dimensions

$$\blacksquare \mathbf{x}_{(N-1)-N} = x_{N-1} \cdot e_{N-1} + x_N \cdot e_N, \text{ with } r_{N-1} = r \prod_{j=1}^{N-2} \sin \phi_j$$

$$\Rightarrow \|\mathbf{x}_{(N-1)-N}\| = f(\phi_{N-1}, \phi_N) = r_{N-1}$$

$$\Rightarrow \phi_{N-1}, \phi_N \text{ not independent!}$$

$$(\text{actually, if } e_N = e_{N-1} + \frac{\pi}{2}, \text{ then } \phi_N = \phi_{N-1} - \frac{\pi}{2})$$

$$\Rightarrow \text{define } \theta \in [0, 2\pi) \text{ instead of } \phi_{N-1}, \phi_N \in [0, \pi]$$

$$\Rightarrow x_{N-1} = r_{N-1} \sin \theta, x_N = r_{N-1} \cos \theta \text{ (interchangeable)}$$

- Final Spherical Coordinates

$$\blacksquare x_i = \begin{cases} r \cos \phi_1 & i = 1 \\ r \cos \phi_i \prod_{j=1}^{i-1} \sin \phi_j & 2 \leq i \leq N-1 \\ r \sin \theta \prod_{j=1}^{N-2} \sin \phi_j & i = N-1 \\ r \cos \theta \prod_{j=1}^{N-2} \sin \phi_j & i = N \end{cases}$$

## 1.3 Calculus

### 1.3.1 Integral

#### Interchanging Coordinates in Integral

- General Theory

- Notation

- $(x, y)$ : coordinate under Field  $D$
- $(u, v)$ : coordinate under Field  $D'$
- $T : \begin{cases} x = x(u, v), \\ y = y(u, v) \end{cases} : \text{transformation from } D \text{ to } D'$

- Assumption

- $f(x, y)$  continuous in  $D$
- transformation  $T$ 's partial 1<sup>st</sup> order derivatives continuous on  $D'$
- transformation  $T$ 's Jacobian  $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \neq 0$
- transformation  $T : D \rightarrow D'$  is 1-1 mapping

- Derivation

- take infinitely small square in  $D'$  :

$$\begin{array}{ll} M'_4(u, v + \delta v), & M'_3(u + \delta u, v + \delta v), \\ M'_1(u, v), & M'_2(u + \delta u, v) \end{array}$$

$\Rightarrow$  after transformation to  $D$  :

$$\begin{array}{ll} M_4(x(u, v + \delta v), y(u, v + \delta v)), & M_3(x(u + \delta u, v + \delta v), y(u + \delta u, v + \delta v)), \\ M_1(x(u, v), y(u, v)), & M_2(x(u + \delta u, v), y(u + \delta u, v)) \end{array}$$

$$\Rightarrow x_2 - x_1 = x(u + \delta u, v) - x(u, v) = \frac{\partial x}{\partial u}|_{(u, v)} \delta u$$

$$x_4 - x_1 = x(u, v + \delta v) - x(u, v) = \frac{\partial x}{\partial v}|_{(u, v)} \delta v$$

$$y_2 - y_1 = y(u + \delta u, v) - y(u, v) = \frac{\partial y}{\partial u}|_{(u, v)} \delta u$$

$$y_4 - y_1 = y(u, v + \delta v) - y(u, v) = \frac{\partial y}{\partial v}|_{(u, v)} \delta v$$

as  $\delta u, \delta v \rightarrow 0$ , curvilinear boundary quadrilateral  $M_1 M_2 M_3 M_4 \rightarrow$  parallelogram

$$\Rightarrow S_{M_1 M_2 M_3 M_4} = |\overrightarrow{M_1 M_2} \times \overrightarrow{M_1 M_4}| = \left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_4 - x_1 & y_4 - y_1 \end{vmatrix} \right|$$

$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u} \delta u & \frac{\partial y}{\partial u} \delta u \\ \frac{\partial x}{\partial v} \delta v & \frac{\partial y}{\partial v} \delta v \end{vmatrix} \right| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \delta u \delta v$$

$$= |J(u, v)| \delta u \delta v$$



- $\Rightarrow$  infinitely small area  $\delta\sigma = dxdy = |J(u, v)|\delta u\delta v$ 

$$\Rightarrow \int \int_D f(x, y)dxdy = \int \int_{D'} f(x(u, v), y(u, v))|J(u, v)|dudv$$
- Integral in Cartesian  $\rightarrow$  Polar
  - Result
    - $dxdy = r dr d\theta$
  - Derivation
    - from general transformation:  $x = r \cos(\theta), y = r \sin(\theta)$ 

$$\Rightarrow dxdy = |J(r, \theta)|drd\theta = r dr d\theta$$
    - from direct calculation of infinite small area in polar coordinate
 
$$\Rightarrow d\sigma = \frac{1}{2}(r + dr)^2 d\theta - \frac{1}{2}r^2 d\theta = r dr d\theta + \frac{1}{2}(dr)^2 d\theta$$

$$\Rightarrow d\sigma = r dr d\theta, \text{ when } dr, d\theta \rightarrow 0$$

## Gaussian Integral

- Gaussian Function
  - $f(x) = e^{-a(x+b)^2}$ 
    - special form:  $f(x) = e^{-(x)^2}$
    - alternative form:  $f(x) = e^{ax^2+bx+c}$
    - no indefinite integral  $\int_a^b e^{-x^2}$
    - only definite integral  $\int_{-\infty}^{+\infty} e^{-x^2}$
- Direct Integral
  - $(\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx)^2 = \int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx \int_{-\infty}^{+\infty} e^{-a(y+b)^2} dy$ 

$$= \int \int_{-\infty}^{+\infty} e^{-a[(x+b)^2+(y+b)^2]} d(x+b)d(y+b) = \int \int_{-\infty}^{+\infty} e^{-a(x^2+y^2)} dxdy$$

$$= \int_0^{2\pi} \int_0^{+\infty} e^{-ar^2} r dr d\theta$$

$$= \frac{\pi}{a}$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}, \text{ alternatively } \int_{-\infty}^{+\infty} e^{ax^2+bx+c} dx = \sqrt{\frac{\pi}{-a}} \cdot e^{\frac{b^2}{4a}+c}$$
- Even Moment of Gaussian Function
  - $\int_{-\infty}^{+\infty} x^{2n} e^{-ax^2} dx = (-1)^n \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial a^n} e^{-ax^2} dx$ 

$$= (-1)^n \frac{\partial^n}{\partial a^n} \int_{-\infty}^{+\infty} e^{-ax^2} dx \quad \text{by parameter differentiation}$$

$$= (-1)^n \sqrt{\pi} \frac{\partial^n}{\partial a^n} a^{-\frac{1}{2}}$$

$$= \sqrt{\frac{\pi}{a}} \frac{(2n-1)!!}{(2a)^n}, \quad \text{where !! is double factorial}$$

## 1.4 Probability Theory

### 1.4.1 Introduction

#### Background

- Measuring Uncertainty
  - Source of Uncertainty
    - noise in reality & observation
    - finite size of data (limited information)
- Derivation
  - Quantifying Belief
    - by Cox (1946): if numerical values used to represent degrees of belief, a simple set of axioms encoding common sense properties of such beliefs will lead uniquely to a set of rules for manipulating degrees of belief that are equivalent to the sum and product rules of probability
  - Measuring Uncertainty
    - by Jaynes (2003): probability theory can be regarded as an extension of Boolean logic to situations involving uncertainty
  - Common Destination
    - numerical quantities to measure uncertainty, derived from different sets of properties/axioms, behave precisely according to the rules of probability

#### The Basic

- Notation
  - $X, Y$ : random variable
- Discrete
  - $P(X, Y)$ : joint probability of  $X, Y$  taking their values
  - $P(X)$ : marginal probability of  $X$  taking its value
  - $P(X|Y)$ : conditional probability of  $X$  taking its value given  $Y$  observed / determined
- Continuous
  - $P(x) = P_X(x)$ : cumulative probability of value for variable  $X < x$
  - $p(x)$ : probability density,
    - where  $\lim_{\delta x \rightarrow 0} P(X \in (x, x + \delta x)) = \lim_{\delta x \rightarrow 0} p(x)\delta x \Rightarrow P(X \in (a, b)) = \int_a^b p(x)dx$
    - $\Rightarrow p(x) \geq 0$  and  $\int_{-\infty}^{+\infty} p(x) = 1$
    - $\Rightarrow P(z) = \int_{-\infty}^z p(x)dx$
- Basic Rules
  - Sum Rule
    - $P(X) = \sum_Y P(X, Y)$ , where  $X, Y$  are discrete

- $P(X) = \int_Y P(X, Y)$ , where  $X, Y$  are continuous  
(formal justification requires measure theory)
- Product Rule
  - $P(X, Y) = P(Y|X)P(X)$
  - $P(X, Y) = P(Y)P(X)$ , where  $X, Y$  are independent
- $\Rightarrow$  Bayes' Rule
  - $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \frac{P(X|Y)P(Y)}{\sum_Y P(X|Y)P(Y)}$ , where  $Y$  are discrete
  - $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \frac{P(X|Y)P(Y)}{\int_Y P(X|Y)P(Y)}$ , where  $Y$  are continuous
- Interpretation of Bayes
  - Normalization
    - the  $\sum, \int$  can be interpreted as a **normalization constant**  
 $\Rightarrow$  **posterior**  $\propto$  **likelihood**  $\times$  **prior**
  - Prior
    - $P(Y)$ : available probability of desired variable **before** anything observed  
 $\Rightarrow Y$  usually model parameters
  - Posterior
    - $P(Y|X)$ : obtained probability of desired variable **after** observation  
 $\Rightarrow$  if  $X, Y$  independent, observation has no effect  $\Rightarrow$  prior = posterior
  - Likelihood
    - $P(X|Y)$ : how probable/likely of  $X$  being observed under different setting of  $Y$
  - Prior  $\rightarrow$  Posterior
    - a process of incorporating the evidence provided by observation

### 1.4.2 Expectations and Covariances

#### Expectation

- Definition
  - Expectation of  $f(x)$  under  $p(x)$ 
    - discrete  $x$ :  $\mathbb{E}_p[f] = \sum_x p(x)f(x)$
    - continuous  $x$ :  $\mathbb{E}_p[f] = \int p(x)f(x)dx$
    - approximation with  $N$  points drawn from  $p(x)$ :  $\mathbb{E}_p[f] \simeq \frac{1}{N} \sum_{n=1}^N f(x_n)$   
(when  $N \rightarrow \infty$ ,  $\simeq$  becomes  $=$ )
  - Multivariate Expectation
    - Marginal Expectation of  $f(x, y)$  on  $x$ :  $\mathbb{E}_x[f(x, y)] = \sum_x p(x)f(x, y)$   
(hence a function of  $y$ )
    - Conditional Expectation  $f(x)$  on  $p(x|y)$ :  $\mathbb{E}[f|y] = \sum_x p(x|y)f(x)$
- Independence

- Independent  $x, y$ 
  - $\mathbb{E}_{xy}[x, y] = \sum_{x,y} p(x, y)xy = \sum_{x,y} p(x)p(y)xy = \mathbb{E}[x]\mathbb{E}[y]$

## Variance

- Definition
  - Variance of  $f(x)$ :
    - $\text{var}[f] = \mathbb{E}[(f(x) - \mathbb{E}[f(x)])^2]$ 

$$= \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$
  - Covariance

## Covariance

- Definition
  - between Variables  $x, y$ 
    - $\text{cov}[x, y] = \mathbb{E}_{x,y}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])]$ 

$$= \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y]$$
  - between Vectors  $\mathbf{x}, \mathbf{y}$  (column vectors)
    - $\text{cov}[\mathbf{x}, \mathbf{y}] = \mathbb{E}_{\mathbf{x}, \mathbf{y}}[(\mathbf{x} - \mathbb{E}[\mathbf{x}]) \cdot (\mathbf{y}^T - \mathbb{E}[\mathbf{y}^T])]$ 

$$= \mathbb{E}_{\mathbf{x}, \mathbf{y}}[\mathbf{x}\mathbf{y}^T] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{y}^T]$$

(pairwise covariance between components of  $\mathbf{x}, \mathbf{y}$ )
  - within Vector  $\mathbf{x}$ 
    - $\text{cov}[\mathbf{x}] \equiv \text{cov}[\mathbf{x}, \mathbf{x}]$ 

(pairwise covariance between its components)
- Independence Variable
  - Independent  $x, y$ 
    - $\text{cov}[x, y] = \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y] = 0$

## 1.4.3 Transformations of Random Variables

### Inverse Image

- Definition
  - Notation
    - function  $g : \mathbb{R} \rightarrow \mathbb{R}$
    - set  $A$  in  $\mathbb{R}$
  - Inverse Image on Set  $A$ 
    - $g^{-1}(A) = \{x \in \mathbb{R} | g(x) \in A\}$ 

$$\Leftrightarrow x \in g^{-1}(A) \text{ if and only if } g(x) \in A$$

interpretation: for each element in  $A$ , get its original value before  $g$  applied
- Properties
  - $g^{-1}(\mathbb{R}) = \mathbb{R}$ , as  $g$  is defined on  $\mathbb{R}$
  - $\forall \text{ set } A, g^{-1}(A^c) = g^{-1}(A)^c$ , where  $A^c$  is the complement of set  $A$

- $\forall$  collection of sets  $\{A_\lambda | \lambda \in \Lambda\}$ ,  $g^{-1}\left(\bigcup_{\lambda} A_\lambda\right) = \bigcup_{\lambda} g^{-1}(A_\lambda)$
- General Transformation  $Y = g(X)$ 
  - $P(Y \in A) = P(g(X) \in A) = P(X \in g^{-1}(A))$

### Discrete Variable

- Variable
  - $X$ : random variable with probability mass function  $P_X(x)$
  - $Y = g(X)$ , with probability mass function  $P_Y(y)$
- Probability Mass Function
  - $P_Y(y) = \sum_{x \in g^{-1}(y)} P_X(x)$ , as  $X = x$  is independent and mutually exclusive  
note:  $g^{-1}(y)$  denotes  $g^{-1}(\{y\})$
  - Example
    - uniform random variable  $X$  on  $\{-n, \dots, n\}$  with  $Y = |X|$   
 $\Rightarrow P_X(x) = \frac{1}{2n+1}$   
 $\Rightarrow P_Y(y) = \begin{cases} \frac{1}{2n+1}, & x = 0, \\ \frac{2}{2n+1}, & x \neq 0. \end{cases}$

### Continuous

- Variable
  - $X$ : random variable with cumulative distribution  $P_X(x)$ , density  $p_X(x)$
  - $Y = g(X)$ , with cumulative distribution  $P_Y(y)$ , density  $p_Y(y)$
- Cumulative Distribution
  - Strictly Monotone Increasing  $g$ 
    - $P_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = P_X(g^{-1}(y))$
  - Strictly Monotone Decreasing  $g$ 
    - $P_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y)) = 1 - P_X(g^{-1}(y))$
- Probability Density
  - Strictly Monotone  $g$  (an one-to-one function)
    - $p_Y(y) = \frac{d}{dy} P_Y(y) = \frac{dP_Y(y)}{dg^{-1}(y)} \frac{dg^{-1}(y)}{dy} = p_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ ,  
as  $g^{-1}$  has the same monotony as  $g$ , combined with the sign in  $P_Y$  to give the  $|\cdot|$

#### 1.4.4 Gaussian Distribution

##### Definition

- Univariate Gaussian
  - Variable
    - mean:  $\mu$
    - variance:  $\sigma^2 \Rightarrow$  reciprocal of the variance  $\beta = \frac{1}{\sigma^2}$  (also called precision)

- Probability Dense Function (PDF)

$$\blacksquare \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left\{ -\frac{1}{2\sigma^2}(x - \mu)^2 \right\}$$

$\Rightarrow$  satisfying probability axioms:  $\mathcal{N}(x|\mu, \sigma^2) > 0$  and  $\int_{-\infty}^{+\infty} \mathcal{N}(x|\mu, \sigma^2) dx = 1$

- Expectation

$$\blacksquare \mathbb{E}[x] = \int_{-\infty}^{+\infty} \mathcal{N}(x|\mu, \sigma^2) x dx = \mu$$

$$\Rightarrow \mathbb{E}[x^2] = \int_{-\infty}^{+\infty} \mathcal{N}(x|\mu, \sigma^2) x^2 dx$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{+\infty} x^2 \exp \left\{ -\frac{1}{2\sigma^2}(x - \mu)^2 \right\} dx$$

$$= \pi^{-\frac{1}{2}} \int_{-\infty}^{+\infty} (\sqrt{2\sigma^2}x + \mu)^2 \exp(-x^2) dx, \text{ substituting } a = \frac{x - \mu}{\sqrt{2\sigma^2}}$$

$$= \pi^{-\frac{1}{2}} (2\sigma^2 \int_{\mathbb{R}} x^2 e^{-x^2} dx + 2\mu\sqrt{2\sigma^2} \int_{\mathbb{R}} x e^{-x^2} dx + \mu^2 \int_{\mathbb{R}} e^{-x^2} dx)$$

$$= \pi^{-\frac{1}{2}} (2\sigma^2 \int_{\mathbb{R}} x^2 e^{-x^2} dx + 2\mu\sqrt{2\sigma^2} \cdot 0 + \mu^2 \sqrt{\pi})$$

$$= 2\sigma^2 \pi^{-\frac{1}{2}} \int_{\mathbb{R}} x^2 e^{-x^2} dx + \mu^2$$

$$= \sigma^2 + \mu^2, \text{ by 2nd moment of Guassian or } (xe^{-x^2})' = e^{-x^2} - 2x^2 e^{-x^2}$$

- Variance

$$\blacksquare \text{var}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

- Multivariate ( $d$ -dimensional) Gaussian

- Variable

$$\blacksquare \text{mean: } \mu \in \mathbb{R}^d$$

$$\blacksquare \text{covariance matrix: } \Sigma_{d \times d}$$

- Probability Dense Function (PDF)

$$\blacksquare \mathcal{N}_d(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\},$$

noted as  $X \sim \mathcal{N}_d(x|\mu, \Sigma)$

## Multivariate Gaussian

- Dimensionality

- Volume of High Dimensional Sphere

$$\blacksquare \text{for } n = 2k, k \in \mathbb{N}^+, V_{2k}(R) = \frac{\pi^k}{k!} R^{2k}$$

$$\blacksquare \text{for } n = 2k + 1, k \in \mathbb{N}, V_{2k+1}(R) = \frac{2^k \pi^k}{(2k+1)!!} R^{2k+1}$$

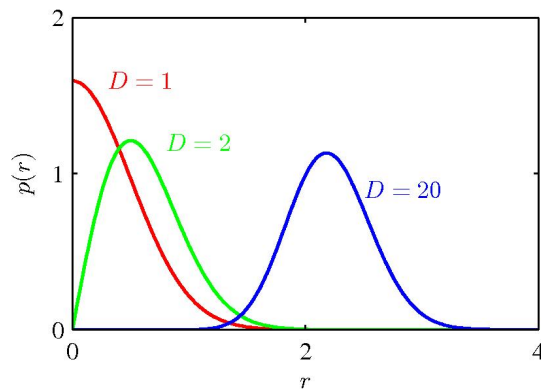
$$\blacksquare \Rightarrow \lim_{D \rightarrow +\infty} \frac{V_D(1) - V_D(1 - \epsilon)}{V_D(1)} = \lim_{D \rightarrow +\infty} 1 - 1(1 - \epsilon)^D = 1 \Rightarrow \text{the volume of a } D\text{-sphere concentrate in a thin shell near the surface!}$$

(actually, in the corner of a high dimensional cube as shown below)

- Volume of High Dimensional Cube

■

- $\Rightarrow$  volume ratio of hyper sphere and hyper cube:  $\Rightarrow$  the volume of a  $D$ -cube concentrates in its corner!  $\Rightarrow$  distance function in high dimensional space CAN be useless
- High Dimensional Distribution
- High Dimensional Gaussian
  - probability density with respect to radius  $r$  for various dimension  $D$   
 $\Rightarrow$  most density are in a thin shell at a specific  $r$



- Facing High Dimensionality
- Convolution of Gaussian
  - Integral of Gaussians  $\int G_1 G_2 dx$ 
    - $G_1 \sim \mathcal{N}_d(x|a, A), G_2 \sim \mathcal{N}_d(x|b, B)$

$$\begin{aligned}
& \Rightarrow \int \mathcal{N}_d(x|a, A) \mathcal{N}_d(x|b, B) dx \\
& = \int \frac{1}{(2\pi)^{d/2} |A|^{1/2}} e^{-\frac{1}{2}(x-a)^T A^{-1}(x-a)} \frac{1}{(2\pi)^{d/2} |B|^{1/2}} e^{-\frac{1}{2}(x-b)^T B^{-1}(x-b)} dx \\
& = \int \frac{1}{(2\pi)^{d/2} |A|^{1/2}} \frac{1}{(2\pi)^{d/2} |B|^{1/2}} e^{-\frac{1}{2}[(x-a)^T A^{-1}(x-a) + (x-b)^T B^{-1}(x-b)]} \\
& = \int \frac{1}{(2\pi)^{d/2} |A|^{1/2}} \frac{1}{(2\pi)^{d/2} |B|^{1/2}} e^{-\frac{1}{2}[(x-c)^T (A^{-1}+B^{-1})(x-c) + (a-b)^T C(a-b)]}, \\
& \quad \text{where } c = (A^{-1} + B^{-1})^{-1}(A^{-1}a + B^{-1}b), C = A^{-1}(A^{-1} + B^{-1})^{-1}B^{-1} = (A + B)^{-1} \\
& = \frac{|(A^{-1} + B^{-1})^{-1}|^{1/2}}{(2\pi)^{d/2} |A|^{1/2} |B|^{1/2}} e^{-\frac{1}{2}(a-b)^T C(a-b)} \int \frac{1}{(2\pi)^{d/2} |(A^{-1} + B^{-1})^{-1}|^{1/2}} e^{-\frac{1}{2}(x-c)^T (A^{-1}+B^{-1})(x-c)} dx \\
& = \frac{|(A^{-1} + B^{-1})^{-1}|^{1/2}}{(2\pi)^{d/2} |A|^{1/2} |B|^{1/2}} e^{-\frac{1}{2}(a-b)^T C(a-b)} \\
& = \frac{1}{(2\pi)^{d/2} (|A||B||A^{-1} + B^{-1}|)^{1/2}} e^{-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)} \\
& = \frac{1}{(2\pi)^{d/2} |ABA^{-1} + ABB^{-1}|^{1/2}} e^{-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)} \\
& = \frac{1}{(2\pi)^{d/2} |ABA^{-1} + A|} e^{-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)} \\
& = \frac{1}{(2\pi)^{d/2} |A(B+A)A^{-1}|} e^{-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)} \\
& = \frac{1}{(2\pi)^{d/2} |A+B|^{1/2}} e^{-\frac{1}{2}(a-b)^T (A+B)^{-1}(a-b)} \\
& \circ \Rightarrow \text{Convolution of Gaussians } G_1 * G_2 \\
& \quad \blacksquare G_1 \sim \mathcal{N}_d(a, A), G_2 \sim \mathcal{N}_d(b, B) \\
& G_1 * G_2(z) = \int G_1(x) G_2(z-x) dx \\
& = \int \mathcal{N}_d(x|a, A) \mathcal{N}_d(z-x|b, B) dx \\
& = \int \mathcal{N}_d(x|a, A) \cdot \frac{1}{(2\pi)^{d/2} |B|^{1/2}} e^{-\frac{1}{2}(z-x-b)^T B^{-1}(z-x-b)} dx \\
& = \int \mathcal{N}_d(x|a, A) \mathcal{N}_d(x|z-b, B) dx \\
& = \frac{1}{(2\pi)^{d/2} |A+B|^{1/2}} e^{-\frac{1}{2}(z-(a+b))^T (A+B)^{-1}(z-(a+b))} \\
& = \mathcal{N}_d(z|a+b, A+B)
\end{aligned}$$

### 1.4.5 Bayesian Interpretation of Probability

#### Contrasting Frequentist Estimator

- Posterior  $p(\mathbf{w}|\mathcal{D}) = \frac{p(\mathcal{D}|\mathbf{w})p(\mathbf{w})}{p(\mathcal{D})}$ 
  - Notation
    - $\mathcal{D}$  the observed dataset
    - $\mathbf{w}$  the vector for model parameters
  - Bayesian



- only one single dataset  $\mathcal{D}$  (the observed one)
- uncertainty expressed as distribution over  $\mathbf{w}$
- model's error: use likelihood / posterior directly (or after taking log)
- pros
  1. naturally incorporating prior knowledge as prior distribution (of  $\mathbf{w}$ )
- cons
  1. prior usually selected for mathematic convenience
- Frequentist Estimator
  - parameters  $\mathbf{w}$  already determined / fixed by 'estimator' (model)
  - error bars of the model obtained by considering the distribution over  $\mathcal{D}$
  - model's error: bootstrap procedure
    1. generate dataset(s) by drawing from the observed  $\mathcal{D}$  with replacement
    2. sampling  $L$  datasets with the same size as  $\mathcal{D}$
    3. error = variability of predictions between the sampled datasets
  - pros
    1. protect the conclusion from false prior knowledge
  - cons
    1. sensitive to observation (extreme cases), especially under small dataset

## Parameter Estimation

- Bias vs. Variance
- Taking Logarithm
  - Reason
    - monotonically increasing function  $\Rightarrow \arg \max_{\theta} f(x; \theta) = \arg \max_{\theta} \log f(x; \theta)$
    - simplify mathematical analysis  $\Rightarrow \prod \rightarrow \sum$
    - numerical stability  $\Rightarrow$  avoid  $\prod$ (small probabilities)  
(may otherwise underflow the numerical precision)
- Maximum Likelihood Estimation for Gaussian
  - Notation
    - $X = \{x_1, \dots, x_N\}$ : observed  $N$  data points
  - Assumption
    - data points are i.i.d. (identically and independently distributed)
    - underlying distribution is Gaussian  $\mathcal{N}(\mu, \sigma)$
  - Likelihood
    - $p(X|\mu, \sigma^2) = \prod_{n=1}^N p(x_n|\mu, \sigma^2)$
    - $\Rightarrow \ln p(X|\mu, \sigma) = -\frac{1}{2\sigma^2} \sum_{n=1}^N (x_n - \mu)^2 - \frac{N}{2} \ln \sigma^2 - \frac{N}{2} \ln(2\pi)$
  - Solution
    - let  $\frac{\partial}{\partial \mu} \ln p(X|\mu, \sigma^2) = 0 \Rightarrow \mu_{\text{ML}} = \frac{1}{N} \sum_{n=1}^N x_n$
    - let  $\frac{\partial}{\partial \sigma^2} \ln p(X|\mu, \sigma^2) = 0 \Rightarrow \sigma_{\text{ML}}^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \mu_{\text{ML}})^2$

- Analysis

- $\mathbb{E}[\mu_{\text{ML}}] = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N x_n\right] = \frac{1}{N} \sum_{n=1}^N \mathbb{E}[x_n] = \mu$   
 (as  $x_1, \dots, x_N$  i.i.d, drawn from  $\mathcal{N}(\mu, \sigma^2)$ , thus  $\sim \mathcal{N}(\mu, \sigma^2)$ )  
 $\Rightarrow$  unbiased estimation of mean
- $\mathbb{E}[\sigma_{\text{ML}}^2] = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N (x_n - \mu_{\text{ML}})^2\right]$   
 $= \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N (x_n^2 - 2\mu_{\text{ML}}x_n + \mu_{\text{ML}}^2)\right]$   
 $= \frac{1}{N} \sum_{n=1}^N \mathbb{E}[x_n^2] - \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^N 2x_n\mu_{\text{ML}}\right] + \mathbb{E}[\mu_{\text{ML}}^2]$   
 $= \frac{1}{N} \sum_{n=1}^N \mathbb{E}[x_n^2] - 2\mathbb{E}[\mu_{\text{ML}}^2] + \mathbb{E}[\mu_{\text{ML}}^2]$   
 $= \frac{1}{N} \sum_{n=1}^N \mathbb{E}[x_n^2] - \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}[x_i x_j]$   
 $= \frac{1}{N} \sum_{n=1}^N (\sigma^2 + \mu^2) - \frac{1}{N^2} [N(N-1)\mu^2 + N(\sigma^2 + \mu^2)]$   
 (by 2nd moment of Gaussian  $\mathbb{E}[x^2]$  and i.i.d assumption)  
 $= \left(\frac{N-1}{N}\right) \sigma^2$   
 $\Rightarrow$  biased variance !  
 $\Rightarrow$  unbiased variance  $\hat{\sigma}^2 = \frac{N}{N-1} \sigma_{\text{ML}}^2 = \frac{1}{N-1} \sum_{n=1}^N (x_n - \mu_{\text{ML}})^2$   
 interpretation:  $N-1$  degree of freedom,  
 (as calculating  $\sigma^2$  needs  $\mu$ , which help pin down  $x_N$  given  $x_1, \dots, x_{N-1}$ )

## Predictive Distribution

- Probabilistic Prediction

- Notation

- $\mathbf{x}, \mathbf{t}$ : vector of data examples and corresponding ground truth
- $\mathbf{w}$ : model parameters
- $x, t$ : new data example for prediction and its ground truth

- Prediction by Model

- $p(t|x, \mathbf{w}')$ , where  $\mathbf{w}'$  is the best fit parameters founded

- Prediction by Data

- $p(t|x, \mathbf{x}, \mathbf{t}) = \int p(t|x, \mathbf{w})p(\mathbf{w}|\mathbf{x}, \mathbf{t})d\mathbf{w}$ , where  $\mathbf{w}$  marginalized over its posterior

# Chapter 2

## Introduction

### 2.1 General Concern

#### 2.1.1 Types of Learning

##### Supervised Learning

- Overview
  - training data comprises examples of input vectors with corresponding target vectors
- Regression
  - output one or more continuous variable
- Classification
  - assign input to one of a finite number of discrete categories

##### Unsupervised Learning

- Overview
  - training data consists of a set of input vectors without target vectors
- Clustering
  - Goal: discover groups of similar examples
- Density Estimation
  - Goal: determine the distribution of data within the input space
- Dimension Reduction
  - Goal: project data into low dimension, for the purpose of such as visualization

##### Reinforcement Learning

- Overview
  - input with time series & discover optimal output by a process of trial and error
- Goal
  - find actions to take under given circumstance to maximize a reward

## 2.2 Model Selection

### 2.2.1 Project Structuring

#### Data Arrangement

- Train-Val-Test
  - Training Set
    - data used to find the model parameter estimation  
⇒ over-fit by complex model
  - Validation Set
    - data to indicate generalization ability of a range of models  
⇒ for model comparison & selection
    - may still **over-fit** if model design iterated for too many times!
  - Test Set
    - to indicate the generalization ability of finally selected model
- Constraints on Val
  - Limited Data
    - better model ⇒ more training data
    - ⇒ small validation ⇒ noisy estimation of generalization ability

#### Measuring Generalization Ability

- K-fold Cross Validation
  - Procedure
    - split all data into  $K$  folds,  $K - 1$  folds for train, 1 for validation
    - ⇒ average over all  $C_K^1$  combination to indicate the generalization ability
    - extreme case: leave-out-one ⇒  $K = N$ , where  $N$  is number of all data
  - Cons:
    - $\mathcal{O}(K)$  ⇒ slow, especially if training process already slow
    - ⇒ trade off between time vs. constraint on validation
- Information Criteria
  - Pros
    - rely only on training set
    - need only (each) single training process to compare multiple models

### 2.2.2 Over-/Under-fitting

#### Over-fitting

- Symptom
  - good performance on training set & poor generalization
  - ⇒ good at fitting training set; bad at representing modeling underlying data source
- Cause
  - too much representation ability (to fit even the noise)

- directly model the likelihood instead of posterior
- Remedy
  - larger dataset
  - regularization (model the posterior by accounting prior)

### Under-fitting

- bad at fitting training examples & modeling underlying data source
- lack of representation ability
- $\Rightarrow$  poor performance on training set & good generalization (though meaningless)

## 2.3 Decision Theory

### 2.3.1

## 2.4 Information Theory

## 2.5 Supervised Learning

- Feature normalization:  $\forall x_{ij} \in X, x_{ij} = \frac{x_{ij} - \mu_j}{\sigma_j}$ ,  $X : [instance][feature]$ , without  $[1...1]^T$  in 1st column  $X = [x_1, x_2, \dots, x_m]$ ,  $m$  instances in total
- Regularization: add penalty for  $\theta$  being large into cost function
- $J(\theta) = \dots + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$ , **bias  $\theta_0$  shouldn't be penalized**

## 2.6 Linear Regression

- Notation
  - $t$ : observed data
  - $y(\mathbf{x}, \mathbf{w}) = \sum_{i=0}^M \phi_i(\mathbf{x}) w_i = \mathbf{w}^T \phi(\mathbf{x})$  : model generating ground truth, with
    - $\mathbf{w}$ : weight vector
    - $\phi(\mathbf{x})$ : basis function for feature vector  $\mathbf{x}$ , with usually  $\phi_0(\mathbf{x}) = 1$  as bias
- Assumption
  - Deterministic Model with Observation Noise
    - $t = y(x, w) + \epsilon$ , where  $\epsilon \sim \mathcal{N}(0, \beta^{-1})$  is Gaussian noise where precision (inverse variance)  $\beta$
    - $\Rightarrow$  consequence
      1. likelihood  $p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$
      2.  $\mathbb{E}[t|\mathbf{x}] = \int t \cdot p(t|\mathbf{x}) dt = y(\mathbf{x}, \mathbf{w})$
      3. unimodal distribution  $p(t|\mathbf{x}) \Rightarrow$  extended by conditional mixture model
- Joint Likelihood

- $P(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^N \mathcal{N}(t_n | \mathbf{w}^T \phi(\mathbf{x}_n), \beta^{-1})$ , where
  - $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ ,  $\mathbf{t} = \{t_1, \dots, t_N\}$
- Log Likelihood
  - $\ln P(\mathbf{y}|\mathbf{X}, \theta, \beta) = \frac{N}{2} \ln \beta - \frac{N}{2} \ln(2\pi) - \beta \frac{1}{2} \sum_{i=1}^m (h_\theta(x^i) - y^i)^2$
- Log Posterior leads to regularization
  - Maximizing the likelihood function  $\Rightarrow$  (often) excessively complex models & over-fitting
  - Regularization term comes from the Prior:
    - assume Prior  $p(\theta) = \mathcal{N}(\theta|0, \alpha^{-1}I)$ , so that Posterior & Prior are of the same distribution to maximize log Posterior :
 
$$\Rightarrow \ln p(\theta|X) \propto -\frac{\beta}{2} \sum_{i=1}^n (y^i - h_\theta(x))^2 - \frac{\alpha}{2} \theta^T \theta + \text{const}$$
    - If  $\alpha \rightarrow 0$  (Prior is most useless), maximise Posterior is equivalent to maximizing likelihood
    - Maximize Posterior  $\Leftrightarrow$  Minimize cost function with regularization, where  $\lambda = \alpha/\beta$
- Predictive Distribution:  $p(y|x, X, Y)$ 
  - $p(y|x, X, Y) = \int p(y, \theta|x, X, Y) d\theta = \int p(y|\theta, x, X, Y) p(\theta|x, X, Y) d\theta$
  - $p(y|\theta, x, X, Y) = p(y|\theta, x) = \mathcal{N}(y|h(x, \theta), \beta^{-1})$   
 based on assumption:  $y = y(x, \theta) + \epsilon$ , where  $\epsilon$  is Gaussian noise  
 $p(\theta|x, X, Y) = p(\theta|X, Y)$  = posterior
  - $\Rightarrow p(y|x, X, Y) = \int p(y|\theta, x) p(\theta|X, Y) d\theta$
  - Expected Loss =  $(\text{bias})^2 + \text{variance} + \text{noise}$
- Notation:
  - $t = y(x, w) + \epsilon$ , where  $\epsilon$  is Gaussian noise
  - $\hat{y}$  is prediction function to approximate  $y = y(x, w)$
- Procedure:
  - $\mathbb{E}[(t - \hat{y})^2] = \mathbb{E}[t^2 - 2t\hat{y} + \hat{y}^2]$   
 $= \mathbb{E}[t^2] + \mathbb{E}[\hat{y}^2] - \mathbb{E}[2t\hat{y}]$   
 $= \text{Var}[t] + \mathbb{E}[t]^2 + \text{Var}[\hat{y}] + \mathbb{E}[\hat{y}]^2 - 2y\mathbb{E}[\hat{y}]$   
 $= \text{Var}[t] + \text{Var}[\hat{y}] + (y^2 - 2y\mathbb{E}[\hat{y}] + \mathbb{E}[\hat{y}]^2)$   
 $= \text{Var}[t] + \text{Var}[\hat{y}] + (y - \mathbb{E}[\hat{y}])^2$   
 $= \text{Var}[t] + \text{Var}[\hat{y}] + \mathbb{E}[t - \hat{y}]^2$   
 $= \sigma^2 + \text{Var}[\hat{y}] + \text{Bias}[\hat{y}]^2$   
 where  $\sigma^2 = \text{Var}[\epsilon]$  is the noise  
 (formula used:  $\text{Var}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \Leftrightarrow \mathbb{E}[x^2] = \text{Var}[x] + \mathbb{E}[x]^2$ )
  - Matrix inverse can be evil & avoid inverse operation:  
 $A = U\Lambda U^T$ , where  $\Lambda$  is diagonal matrix  
 $\Rightarrow A^{-1} = U\Lambda^{-1}U^T$   
 but number on the diagonal line of  $\Lambda$  can be small  $\Rightarrow$  maybe 0 depending on accuracy of computer

## 2.7 Bayesian Regression

- Assumption:
  - $t = y(x, w) + \epsilon$ , where  $\epsilon$  is Gaussian noise;  $y(x, w)$  approximated by  $\phi(x)w$
- Bayesian view:
- Gaussian Prior :  $p(w) = \mathcal{N}(w|m_0, S_0)$   
Reason: to be conjugate
- Likelihood :  $p(\mathbf{t}|w) = \prod_{n=1}^N \mathcal{N}(t_n|w^T \phi(x_n), \beta^{-1}) = \mathcal{N}(\mathbf{t}|\Phi w, \beta^{-1}I)$
- $\Rightarrow$  Posterior :  $p(w|\mathbf{t}) = \mathcal{N}(w|m_N, S_N)$   
where  $m_N = S_N(S_0^{-1}m_0 + \beta\Phi^T\mathbf{t})$ ,  $S_N^{-1} = S_0^{-1} + \beta\Phi^T\Phi$
- Maximum Likelihood:
  - Likelihood :  $p(\mathbf{t}|w) = \prod_{n=1}^N \mathcal{N}(t_n|\phi(x_n)w, \beta^{-1})$ 
    - meaning: how probable the observed dataset is w.r.t the model setting (under parameter  $w$ )
  - $\ln$  Likelihood =  $\sum_{n=1}^N [-\ln \frac{\beta}{\sqrt{2\pi}} - \frac{\beta}{2}(t_n - \phi(x)w)^2]$
  - $\frac{\partial}{\partial w} \ln$  Likelihood =  $\beta\Phi^T(\mathbf{t} - \Phi w)$   
let  $\frac{\partial}{\partial w} \ln$  Likelihood = 0  
 $\Rightarrow w_{ML} = (\Phi^T\Phi)^{-1}\Phi^T\mathbf{t}$
  - $\frac{\partial}{\partial \beta} \ln$  Likelihood =  $-N\beta^{\frac{1}{2}} + \beta^{\frac{3}{2}}(\mathbf{t} - \Phi w)^T(\mathbf{t} - \Phi w)$   
let  $\frac{\partial}{\partial \beta} \ln$  Likelihood = 0  
 $\Rightarrow \beta^{-1} = \frac{1}{N}(\mathbf{t} - \Phi w)^T(\mathbf{t} - \Phi w)$   
Note: solve  $w = w_{ML}$  first
- Maximum Posterior:
  - Posterior =  $p(w|\mathbf{t})$ , Prior =  $p(w)$ , Likelihood =  $p(\mathbf{t}|w)$ , Normalization =  $p(\mathbf{t})$   
 $\Rightarrow$  Posterior =  $\frac{\text{Likelihood} \cdot \text{Prior}}{\text{Normalization}}$   
 $\Rightarrow$  Posterior  $\propto$  Likelihood \* Prior
  - assume Prior  $p(w) = \mathcal{N}(w|m_0, S_0)$ ,  
so that Prior & Likelihood are conjugate  $\Rightarrow$  Gaussian Posterior
  - Likelihood  $p(\mathbf{t}|w) = \prod_{n=1}^N \mathcal{N}(t_n|\phi(x_n)w, \beta^{-1}) = \mathcal{N}(\mathbf{t}|\Phi w, \beta^{-1}I)$
  - $\Rightarrow$  Posterior  $p(w|\mathbf{t}) = \mathcal{N}(w|m_N, S_N)$ ,  
where  $m_N = S_N(S_0^{-1}m_0 + \beta\Phi^T\mathbf{t})$ ,  $S_N^{-1} = S_0^{-1} + \beta\Phi^T\Phi$   
 $\Rightarrow w_{MAP}$  = mean of the Gaussian =  $m_N$   
Note: can also get  $w_{MAP}$  from taking gradient

- Simple Prior:

Prior  $p(w) = \mathcal{N}(w|0, \alpha^{-1}I)$

$\Rightarrow$  Posterior  $p(w|\mathbf{t}) = \mathcal{N}(w|m_N, S_N)$ ,

where  $m_N = \beta(\alpha I + \beta \Phi^T \Phi)^{-1} \Phi^T \mathbf{t}$ ,  $S_N^{-1} = \alpha I + \beta \Phi^T \Phi$

$w_{MAP} \rightarrow w_{ML}$ , when  $\alpha \rightarrow 0$  (most useless Prior)

- Maximize Posterior  $\Leftrightarrow$  Minimize cost function with regularization:

Simple Prior  $\Rightarrow \ln p(w|\mathbf{t}) = -\frac{\beta}{2}(\mathbf{t} - \Phi w)^T(\mathbf{t} - \Phi w) - \frac{\alpha}{2}w^T w + \text{const}$

- If  $\alpha \rightarrow 0$  (Prior is most useless), maximize Posterior is equivalent to maximizing likelihood
- Maximize Posterior equal to minimize sum-of-squares error function with the addition of a quadratic regularization term with  $\lambda = \alpha/\beta$
- Regularization term comes from the Prior

- Predictive Distribution:

- Assume: Prior :  $p(x|\alpha) = \mathcal{N}(x|0, \alpha^{-1}I)$ , ( $m_0 = 0, S_0 = \alpha^{-1}I$ )

- $p(t|x, X, \mathbf{t}) = \int p(t|w, x)p(w|X, \mathbf{t})dw$

- $\Rightarrow p(t|x, X, \mathbf{t}) = \mathcal{N}(t|m_N^T \phi(x), \sigma_N^2(x))$   
where  $\sigma_N^2(x) = \frac{1}{\beta} + \phi(x)^T S_N \phi(x)$ ;  $m_N, S_N$  from Posterior( $m_N = w_{MAP}$ )

- Sequential data:

- Posterior from previous data  $\Leftrightarrow$  the Prior for the arriving data
- Sequential data and data in one go is equivalent when finding the Posterior

- Gradient descent

- Hypothesis function:

$$\blacksquare h_\theta(x) = x\theta, \theta = [\theta_0, \theta_1, \dots, \theta_n]^T, x = [x_0, x_1, \dots, x_n], x_0 = 1$$

- $x$  is one instance

- Cost function:  $J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^i) - y^i)^2 + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$

- Update rule:  $\forall \theta_j \in \theta, \theta_j := \theta_j - \alpha \frac{\partial J(\theta)}{\partial \theta_j}, \frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m} \sum_{i=1}^m [(h_\theta(x^i) - y^i)x_j^i] + \frac{\lambda}{m} \theta_j -$

$$\frac{d}{d\theta} J(\theta) = \frac{1}{m} ((X\theta - y)^T X)^T + \frac{\lambda}{m} [0, \theta_1, \dots, \theta_m]^T \quad (\theta_0 \text{ shouldn't be penalized})$$

- simultaneously for all  $\theta_j \in \theta$
- Normal equation (mathematical solution)

$$\blacksquare \theta = (X^T X)^{-1} X^T y$$



## 2.8 Logistic Regression (Classification)

- Decision Theory:

- classes  $C_1, \dots, C_K$ , decision regions  $\mathcal{R}_1, \dots, \mathcal{R}_K$  - Minimize  $p(\text{mistake}) = \sum_{k=1}^K \left( \int_{\mathcal{R}_k} \sum_{i \neq k} p(x, C_i) dx \right)$

(can have weight on each misclassification though) - Maximize  $p(\text{correct}) = \sum_{k=1}^K \int_{\mathcal{R}_k} p(x, C_k) dx$

- Models for Decision Problems:

- Find a discriminant function - Discriminative Models: less powerful, yet less parameter = easier to learn

- Infer \*\*posterior\*\*  $p(C_k|x)$ ,  $C_k$ :  $x \in C_k$ ,  $x$  is examples in training set - Use decision theory to assign a new  $x$  - Generative Models: more powerful, but computationally expensive - Infer conditional probabilities  $p(x|C_k)$  - Infer prior  $p(C_k)$  - Find \*\*either\*\* \*\*posterior\*\*  $p(C_k|x)$ , \*\*or\*\* \*\*joint distribution\*\*  $p(x, C_k)$  (using Bayes' theorem) - Use decision theory to assign a new  $x$

- \*\*= Able to create synthetic data using  $p(x)$

- Naive Bayes on Discrete Features:

- Assumption:

- Discrete Features: data point  $x \in \{0, 1\}^D$

- Naive Bayes: all features conditioned on class  $C_k$  are independent with each other

$$\Rightarrow p(x|C_k) = \prod_{i=1}^D \mu_{ki}^{x_i} (1 - \mu_{ki})^{1-x_i}$$

1. Linear Discriminant (Least Squares Approach)

- Prediction:

-  $y(x) = xw + w_0$ , with bias  $= w_0$ , where  $w = [w_1, \dots, w_n]^T$ ,  $x = [x_1, \dots, x_n]$  -  $y(x) > 0$ :

positive confidence to assign  $x$  to current class -  $-w_0$  called threshold sometimes

- Decision Boundary  $y(x) = w^T x + w_0 = 0$ :

-  $w$  is orthogonal to vectors on the boundary:

assume  $x_1, x_2$  on the boundary

$$\Rightarrow 0 = y(x_1) - y(x_2) = (x_1 - x_2)w$$

- Distance from origin to boundary is  $-\frac{w_0}{\|w\|}$ :

assume distance is  $k$

$$\Rightarrow k \frac{w}{\|w\|} \text{ on boundary, thus } k \frac{w}{\|w\|} w + w_0 = 0$$

$$\Rightarrow k = -\frac{w_0}{\|w\|}$$

-  $y(x)$  is a signed measure of distance from point  $x$  to boundary:

assume distance is  $r$

$$\Rightarrow y(x) = \overbrace{(x_{\perp} + r \frac{w}{\|w\|})}^x w + w_0 = \overbrace{x_{\perp} w + w_0}^0 + r \|w\| = r \|w\|$$

$$\Rightarrow r = \frac{y(x)}{\|w\|}$$

- Multi-class (k-classes):

- prediction:  $x$  is of class  $C_k$  if  $\forall j \neq k, y_k(x) > y_j(x)$

$\Rightarrow y(x) = xW$ , where  $W = [w_1, \dots, w_k]$ ,  $\forall x_i \in X, x_{i0} = 1$  (bias),  $y(x)$  is 1-of-k coding

- sum-of-squares error:  $E_D(W) = \frac{1}{2} \text{tr}\{(XW - T)(XW - T)^T\}$

$$\Rightarrow \text{optimal } W = (X^T X)^{-1} X^T T$$

note that  $\text{tr}\{AB\} = A^T B^T$

2. Fisher's Linear Discriminant

- Basic idea:

- Take linear classification  $y = w^T x$  as dimensionality reduction (projection onto 1-D) - = find a projection (denoted by vector  $w$ ) which maximally preserves the class separation - = if  $y > -w_0$  then class  $C_1$ , otherwise  $C_2$

- Goal:

- Distance within one class is small - Distance between classes is large

- Mean & Variance of Projected Data:

- Mean:  $\tilde{m}_k = w^T m_k$ , where  $m_k = \frac{1}{N_k} \sum_{x \in C_k} x$  - Variance:  $\tilde{s}_k = \sum_{x \in C_k} (w^T x - w^T m_k)^2 =$

$$w^T \left[ \sum_{x \in C_k} (x - m_k)(x - m_k)^T \right] w$$

- 2-Classes to 1-D line:

- Maximize Fisher criterion:  $J(w) = \frac{|\tilde{m}_1 - \tilde{m}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$

- Between-class covariance:  $S_B = (m_1 - m_2)(m_1 - m_2)^T$

- Within-class covariance:  $S_k = \sum_{n \in C_k} (x_n - m_k)(x_n - m_k)^T$

$$\Rightarrow J(w) = \frac{w^T S_B w}{w^T S_W w}$$

- Lagrangian:  $L(w, \lambda) = w^T S_B w + \lambda(1 - w^T S_W w)$

fix  $w^T S_W w$  to 1 to avoid infinite solution (any multiple of a solution is a solution)

$$\Rightarrow \frac{\partial}{\partial w} L = 2S_B w - 2\lambda S_W w = 0$$

$$\Rightarrow S_B w = \lambda S_W w$$

$$\Rightarrow (S_W^{-1} S_B) w = \lambda w$$

To maximize  $J(w)$ ,  $w$  is the largest eigenvector of  $S_W^{-1} S_B$  if  $S_W$  invertible

- K-classes to a d-D subspace:  $N_k$  is num in class k,  $N$  is the total example num

- Between-class covariance:  $S_B = \sum_{k=1}^K N_k (m_k - m)(m_k - m)^T$ , where  $m = \frac{1}{N} \sum_{n=1}^N x_n$

reduce to  $(m_1 - m_2)(m_1 - m_2)^T$  when  $K=2$  (constant ignored)

- Within-class covariance:  $S_W = \sum_{k=1}^K S_k$ , where  $S_k = \sum_{n \in C_k} (x_n - m_k)(x_n - m_k)^T$ ,  $m_k =$

$$\frac{1}{N_k} \sum_{n \in C_k} x_n$$

- Maximize Fisher criterion:  $J(w) = \frac{\text{tr}\{W^T S_B W\}}{\text{tr}\{W^T S_W W\}}$

- Lagrangian:

Solve for each  $w_i \in W \Rightarrow (S_W^{-1} S_B) w_i = \lambda_i w_i$

$\Rightarrow W$  consists of the largest d eigenvectors

$S_W^{-1} S_B$  is not guaranteed to be symmetric  $\Rightarrow W$  might not be orthogonal

Need to minimize the whole covariance matrix ( $J(w)$  as a matrix)  $\Rightarrow$  not choosing same eigenvectors twice

- Maximum Possible Projection Directions =  $K - 1$ :

$$r(S_W^{-1} S_B) \leq \min(r(S_W^{-1}), r(S_B)) \leq r(S_B)$$

$$r(S_B) \leq \sum_K r((m_k - m)(m_k - m)^T) = K, \text{ as } r(m_k - m) = 1$$

$$\sum_K m_k = m \Rightarrow r(m_1 - m, \dots, m_K - m) = K - 1$$

$$\Rightarrow r(S_B) \leq K - 1$$

$$\Rightarrow r(S_W^{-1} S_B) \leq K - 1$$

3. Perceptron Algorithm

- Generalised linear model  $y = f(w^T \phi(x))$ , where  $\phi(x)$  is basis function;  $\phi_0(x) = 1$

- Nonlinear activation function:  $f(a) = \begin{cases} 1, & a \geq 0 \\ -1, & a < 0 \end{cases}$

- Target coding:  $t = \begin{cases} 1, & \text{if } C_1 \\ -1, & \text{if } C_2 \end{cases}$

- Cost function:

- All correctly classified patterns:  $w^T \phi(x_n) t_n > 0$
- Add the errors for all misclassified patterns (denoted as set  $\mathcal{M}$ ):

$$\Rightarrow E_P(w) = - \sum_{n \in \mathcal{M}} w^T \phi(x_n) t_n$$

- Algorithm: (Aim: minimize total num of misclassified patterns)
- loop

choose a training pair  $(x_n, t_n)$

update the weight vector  $w$ :  $w = w - \eta \nabla E_P(w) = w + \phi_n t_n$

where  $\eta=1$  because  $y = f(\cdot)$  does not depend on  $\|w\|$

- Perceptron Convergence Theorem:

- If the training set is linearly separable, the perceptron algorithm is guaranteed to find a solution in a finite number of steps

(Also is the algorithm to find whether the set is linearly separable = Halting Problem)

#### 4. Maximum Likelihood

- Assumption: -  $p(x|C_k) \sim \mathcal{N}(\mu_k, \Sigma)$ , and all  $p(x|C_k)$  share the same  $\Sigma$  -  $p(C_1) = \pi, p(C_2) = 1-\pi$ ,  $\pi$  unknown - Likelihood of whole data set  $\mathbf{X}, \mathbf{t}$ ,  $N$  is the num of data -  $p(\mathbf{X}, \mathbf{t} | \pi, \mu_1, \mu_2, \Sigma) =$

$$\prod_{n=1}^N [\pi \mathcal{N}(x_n | \mu_1, \Sigma)]^{t_n} [(1-\pi) \mathcal{N}(x_n | \mu_2, \Sigma)]^{1-t_n} \rightarrow \text{when info of label } t \text{ lost: mixture of Gaussian}$$

$$\ln(\text{Likelihood}) = \sum_{n=1}^N [t_n (\ln \pi + \ln \mathcal{N}(x_n | \mu_1, \Sigma)) + (1-t_n) (\ln(1-\pi) + \ln \mathcal{N}(x_n | \mu_2, \Sigma))] - \text{Parameters}$$

when maximum reached: -  $\pi = \frac{N_1}{N_1 + N_2}$ ,  $N_1$  is the num of class  $C_1$  -  $\mu_1 = \frac{1}{N_1} \sum_{n=1}^N t_n x_n$ ,  $\mu_2 =$

$$\frac{1}{N_2} \sum_{n=1}^N (1-t_n) x_n, \text{ (mean of each class)} - \Sigma = \frac{N_1}{N} S_1 + \frac{N_2}{N} S_2, \text{ where } S_k = \frac{1}{N_k} \sum_{n \in C_k} (x_n - \mu_k)(x_n - \mu_k)^T$$

#### 5. Logistic Regression

- Sigmoid function:  $\sigma(a) = \frac{1}{1 + e^{-a}}$

- $p(x|C_k) \sim \mathcal{N} \Rightarrow p(C_k|x) = \sigma(w^T x + w_0)$  (2-classes) (Generative model)

- Assumption:

$p(x|C_k) = \mathcal{N}(\mu_k, \Sigma)$  (can also be a number of other distributions)

$\forall k, p(x|C_k)$  shares the same  $\Sigma$

-

$$p(C_1|x) = \frac{p(x|C_1)p(C_1)}{p(x|C_1)p(C_1) + p(x|C_2)p(C_2)} = \sigma(a),$$

$$\text{where } a = \ln \frac{p(x, C_1)}{p(x, C_2)}$$

$$= \ln \frac{p(x|C_1)p(C_1)}{p(x|C_2)p(C_2)}$$

= ... (assumption applied)

$$= \ln \frac{\exp(\mu_1^T \Sigma^{-1} x - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1)}{\exp(\mu_2^T \Sigma^{-1} x - \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2)} + \ln \frac{p(C_1)}{p(C_2)}$$

$$\Rightarrow a = w^T x + w_0 \text{ where,}$$

$$w = \Sigma^{-1}(\mu_1 - \mu_2)$$

$$w_0 = -\frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 + \ln \frac{p(C_1)}{p(C_2)}$$

- $\Rightarrow p(C_1|x) = \sigma(w^T x + w_0)$

$\Rightarrow$  Find parameters in Gaussian model using Maximal Likelihood Solution

as:  $p(C_1|x) \propto p(x|C_1)p(C_1) = p(x, C_1)$

- Generalize to K-classes:

$$a_k(x) = \ln[p(x|C_k)p(C_k)], p(C_k|x) = \frac{\exp(a_k)}{\sum_i \exp(a_i)}$$

$$\Rightarrow a_k(x) = w_k^T x + w_{k0}, \text{ where } w_k = \Sigma^{-1} \mu_k; w_{k0} = -\frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + p(C_k)$$

- Assume directly  $p(C_k|x) = \sigma(w^T x + w_0)$  (2-classes) (Discriminative model)

- Assume directly:  $p(C_1|w, x) = \sigma(w^T x), x_0 = 1$

$\Rightarrow$  less parameters to fit (compared to Gaussian)

- Likelihood function:

$$p(\mathbf{t}|w, X) = \prod_{n=1}^N p_n^{t_n} (1 - p_n)^{1-t_n}, \text{ where, } p_n = p(C_1|x_n), t_n \text{ is the class of } x_n$$

Define error function :

$$E(w) = -\ln(\text{Likelihood}) = -\sum_{n=1}^N [t_n \ln p_n + (1 - t_n) \ln(1 - p_n)]$$

$$\Rightarrow \nabla E(w) = \sum_{n=1}^N (p_n - t_n) x_n$$

- Find Posterior  $p(w|\mathbf{t})$ :

Likelihood is product of sigmoid

Conjugate Prior for "sigmoid distribution" is unknown

$\Rightarrow$  Assume Prior  $p(w) = \mathcal{N}(w|m_0, S_0)$

$$\Rightarrow \ln p(w|\mathbf{t}) \propto -\frac{1}{2} (w - m_0)^T S_0^{-1} (w - m_0) + \sum_{n=1}^N [t_n \ln p_n + (1 - t_n) \ln(1 - p_n)]$$

$$\text{find } w_{MAP}, \text{ calculate } S_N = -\nabla \nabla \ln p(w|\mathbf{t}) = S_0^{-1} + \sum_{n=1}^N p_n (1 - p_n) \phi_n \phi_n^T$$

$\Rightarrow p(w|\mathbf{t}) \simeq \mathcal{N}(w|w_{MAP}, S_N)$ , via Laplace Approximation

- Laplace Approximation:

- Fit a gaussian to  $p(z)$  at its \*\*mode\*\* ( mode of  $p(z)$ : point where  $p'(z) = 0$  )

- Assume  $p(z) = \frac{1}{Z} f(z)$ , with normalization  $Z = \int f(z) dz$

Taylor expansion of  $\ln f(z)$  at  $z_0$ :  $\ln f(z) \simeq \ln f(z_0) - \frac{1}{2} A (z - z_0)^2$ ,

where  $f'(z_0) = 0, A = -\frac{d^2}{dz^2} \ln f(z)|_{z=z_0}$

Take its exponentiating:  $f(z) \simeq f(z_0) \exp -\frac{A}{2} (z - z_0)^2$

$\Rightarrow$  Laplace Approximation =  $(\frac{A}{2\pi})^{1/2} \exp -\frac{A}{2} (z - z_0)^2$ , where  $A = \frac{1}{\sigma^2}$

- Requirement:

$f(z)$  differentiable to find a critical point

$f''(z_0) < 0$  to have a maximum & so that  $\nabla \nabla \ln f(z_0) = A > 0$  as  $A = \frac{1}{\sigma^2}$

- In Vector Space: approximate  $p(z)$  for  $z \in \mathcal{R}^M$

Assume  $p(z) = \frac{1}{Z} f(z)$

Taylor expansion:  $\ln f(z) \simeq \ln f(z_0) - \frac{1}{2} (z - z_0)^T A (z - z_0)$ ,

Hessian  $A = -\nabla \nabla \ln f(z)|_{z=z_0}$

$$\Rightarrow \text{Laplace approximation} = \frac{|A|^{1/2}}{(2\pi^{M/2})} \exp -\frac{1}{2} (z - z_0)^T A (z - z_0) \quad (2.1)$$

$$= \mathcal{N}(z|z_0, A^{-1}) \quad (2.2)$$

- Gradient descent:

- Hypothesis function:  $h_\theta(x) = \sigma(x\theta) = \frac{1}{1+e^{-x\theta}}$

- Cost function:

$$J(\theta) = \frac{1}{m} \sum_{i=1}^m [-y^i \ln(h_\theta(x^i)) - (1 - y^i) \ln(1 - h_\theta(x^i))] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

- Update rule:  $\forall \theta_j \in \theta, \theta_j := \theta_j - \alpha \frac{\partial J(\theta)}{\partial \theta_j}, \frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m} \sum_{i=1}^m [(h_{\theta}(x^i) - y^i)x_j^i] + \frac{\lambda}{m} \theta_j$

## 2.9 Diagnose Machine Learning

- Size of data set - value of  $\lambda$  - Num of feature - Degree of polynomial - Build a quick implementation

### 2.9.1 Evaluating Hypothesis

1. Data set = training set (60)

- randomly split

2. Cross Validation:

- estimation of the generalization error, inaccurate because:

- finite training set - finite cross validation set

- S-fold Cross Validation:

- divide whole set of data into S sets

- for  $i \in (1, S)$

choose the  $i^{th}$  set as cross validation (or test set in some cases)

rest of the sets are called training set

run the procedure (mentioned later) on the training set

estimate generalization in CV set

3. Test set error

- Linear regression:  $J_{test}(\theta) = \frac{1}{2m_{test}} \sum_{i=1}^{m_{test}} (h_{\theta}(x_{test}^i) - y_{test}^i)^2$  - Logistic regression:  $J_{test}(\theta) =$

$-\frac{1}{m_{test}} \sum_{i=1}^{m_{test}} (y_{test}^i \log h_{\theta}(x_{test}^i) + (1 - y_{test}^i) \log (1 - h_{\theta}(x_{test}^i)))$  - Misclassification error:  $J_{test}(\theta) = \frac{1}{m_{test}} \sum_{i=1}^{m_{test}} err(h_{\theta}(x_{test}^i), y_{test}^i)$

$$err(h_{\theta}(x), y) = \begin{cases} 1 & \text{if } h_{\theta}(x) \geq 0.5 \text{ and } y = 0 \text{ or } h_{\theta}(x) < 0.5 \text{ and } y = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

4. Choosing procedure:

- Minimize training error  $J_{train}(\theta)$  - Select a model with lowest  $J_{cv}(\theta)$  - Estimate generalization error as  $J_{test}(\theta)$

### 2.9.2 Bias / Variance

1. Problems:

- High bias: underfitting - high  $J_{train}(\theta)$  and high  $J_{cv}(\theta)$  - High variance: overfitting - low  $J_{train}(\theta)$  but high  $J_{cv}(\theta)$

2. Interaction with regularization:

- Improper  $\lambda$ : - large  $\lambda$  = high bias - small  $\lambda$  = high variance - Choosing  $\lambda$ : - try

$\lambda = 0, 0.01, 0.02, 0.04, \dots, 10$  - select the model with lowest  $J_{cv}(\theta)$  without regularization term

3. Interaction with training set size:

- Normal Learning curve:

! [Normal learning curve] (./../Machine

- Learning curve with high bias:

- where getting more training data \*\*doesn't\*\* help

! [Learning curve with high bias] (./../Machine

- Learning curve with high variance:

- where getting more training data \*\*helps\*\*

! [Learning curve with high variance] (./../Machine

4. Ways to fix:

- High bias: - more features / more polynomial terms of features - decreasing  $\lambda$
- High variance:
- larger data set - fewer features - increasing  $\lambda$
- **\*\*In neural network:\*\***
- High bias = larger neural networks (more hidden layers / more units in one layer)
- High variance = smaller neural networks
- **\*\*Larger network with regularization ( $\lambda$ ) is more powerful\*\***

### 2.9.3 Error Analysis

1. Procedure: - Algorithm (trained) misclassifies  $n$  data in cross validation set - Classify these  $n$  data and rank them - Maybe more features are found 2. Feature selection =  $\hookrightarrow$  Numerical evaluation -  $\hookrightarrow$  test algorithm with / without this feature on **\*\*CV set\*\*** (compare error rate)

### 2.9.4 Skewed classes

- ## 1. Precision / Recall

- - - \*\*Actual 1\*\* - - \*\*Actual 0\*\* - - -

**\*\*Predicted 1\*\*** — True positive — False positive — — **\*\*Predicted 0\*\*** — False negative —  
 True negative —

$$\text{- Precision} = \frac{\text{True positive}}{\text{Predicted positive}} = \frac{\text{True positive}}{\text{True pos} + \text{False pos}}$$

$$\text{Recall} = \frac{\text{True positive}}{\text{Actual positive}} = \frac{\text{True positive}}{\text{True pos} + \text{False neg}}$$

- ## 2. Evaluation with precision/recall

- Predict 1 if  $h_{\theta}(x) \geq \epsilon$ , 0 if  $h_{\theta}(x) < \epsilon$

- larger  $\epsilon =$  higher precision, lower recall (more confident) - smaller  $\epsilon =$  lower precision, higher recall (avoid missing)

![[Possible Precision-Recall curve](../Machine Learning/Statistical Machine Learning/Possible Precision-Recall curve.png)]

3. Compare precision/recall num

-  $F_1Score = 2 \frac{PR}{P+R}$ ,  $P$  as precision,  $R$  as recall - higher better, on cross validation set

4. High precision & high recall:

- **\*\*large num of features (low bias) + large sets of data (low variance)\*\***

## 2.10 Latent Variable Analysis

### 2.10.1 Principal Component Analysis (PCA)

- ### 1. Motivation:

- Data compression (reduce highly related features) - Data visualization

2. Assumption:

- Gaussian distributions for both the latent and observed variables

- ### 3. Two Equivalent Definition of PCA:

- Linear projection of data onto lower dimensional linear space (principal subspace) such

that:

$\Rightarrow$  variance of projected data is maximized

$\Rightarrow$  distortion error from projection is minimized

- #### 4. Maximum Variance Formulation

- Goal:

- project data from D dimension to M while maximizing the variance of projected data

- Eigenvalues  $\lambda$  of covariance matrix  $S$  express the variance of data set  $X$  in direction of corresponding eigenvectors

- Projection Vectors:

-  $U = (u_1, \dots, u_M)$ , where  $\forall i \in \{1, \dots, M\}, u_i \in \mathbb{R}^D$  s.t.  $u_i^T u_i = 1$  (only consider direction)

- Projected Data:

- Mean =  $\bar{x}^T U$ , where  $\bar{x} = \frac{1}{N} \sum_{i=1}^N x^i$  - Variance =  $\text{tr}\{U^T S U\}$ , where  $S = \sum_{i=1}^N (x^i - \bar{x})(x^i - \bar{x})^T$  (outer product)

- Lagrangian to maximize Variance:

-  $L(U, \lambda) = \text{tr}\{U^T S U\} + \text{tr}\{(I - U^T U)\lambda\}$

constraint  $u_i^T u_i = 1$  to prevent  $u_i \rightarrow +\infty$

$$\text{For each } u_i \in U, \frac{\partial}{\partial u_i} L = 2S u_i - 2\lambda_i u_i = 0 \quad (2.4)$$

$$\Rightarrow S u_i = \lambda_i u_i \quad (2.5)$$

$$\Rightarrow U \text{ consists of eigenvectors corresponding to the first } M \text{ large eigenvalue of } S \quad (2.6)$$

( $S$  symmetric  $\Rightarrow U$  orthogonal)

5. Minimum Error Formulation:

- Introduce Orthogonal Basis Vector for  $D$  dimension:

-  $U = (u_1, \dots, u_D)$

- Data representation:

- Original:  $x^n = \sum_{i=1}^D \alpha_i^n u_i$  - Projected:  $\tilde{x}^n = \sum_{i=1}^M z_i^n u_i + \sum_{i=M+1}^D b_i u_i$

( $z_1^n, \dots, z_M^n$ ) is different for different  $x^n$ , ( $b_{M+1}, \dots, b_D$ ) is the same for all  $x^n$

- Cost function:  $J = \frac{1}{N} \sum_{n=1}^N \|x^n - \tilde{x}^n\|^2$ , where  $\tilde{x}^n = \sum_{i=1}^M z_i^n u_i + \sum_{i=M+1}^D b_i u_i$

$$\text{- Let } \begin{cases} \frac{\partial}{\partial z_j^n} J = 0 \\ \frac{\partial}{\partial b_j} J = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{N} 2(x^n - \tilde{x}^n)^T (-u_j) = \frac{2}{N} (z_j - (x^n)^T u_j) = 0 \\ \frac{1}{N} \sum_{n=1}^N 2(x^n - \tilde{x}^n)^T (-u_j) = \frac{2}{N} \sum_{n=1}^N (b_j - (x^n)^T u_j) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} z_j = (x^n)^T u_j & j \in \{1, \dots, M\} \\ b_j = \bar{x}^T u_j & j \in \{M+1, \dots, D\} \end{cases}$$

Noticing  $(x^n)^T u_j = \left(\sum_{i=1}^D \alpha_i^n u_i^T\right) u_j = a_j \Rightarrow a_j = (x^n)^T u_j$

$$\Rightarrow x^n - \tilde{x}^n = \sum_{i=M+1}^D [(x^n - \bar{x})^T u_i] u_i$$

$$\Rightarrow J = \frac{1}{N} \sum_{n=1}^N \left( \sum_{i=M+1}^D [(x^n - \bar{x})^T u_i] u_i \right)^T \left( \sum_{i=M+1}^D [(x^n - \bar{x})^T u_i] u_i \right) \quad (2.7)$$

$$= \frac{1}{N} \sum_{n=1}^N \left( \sum_{i=M+1}^D u_i^T ((x^n - \bar{x})^T u_i) \right) \left( \sum_{i=M+1}^D ((x^n - \bar{x})^T u_i) u_i \right) \quad (2.8)$$

$$= \frac{1}{N} \sum_{n=1}^N \sum_{i=M+1}^D u_i^T (x^n - \bar{x})^T u_i u_i^T (x^n - \bar{x}) u_i \quad u_i \text{ orthogonal to each other} \quad (2.9)$$

$$= \sum_{i=M+1}^D u_i^T \left( \frac{1}{N} \sum_{n=1}^N (x^n - \bar{x})^T (x^n - \bar{x}) \right) u_i \quad \|u_i\| = 1 \quad (2.10)$$

$$(2.11)$$

$$\Rightarrow J = \sum_{i=M+1}^D u_i^T S u_i, \text{ where } S = \frac{1}{N} \sum_{n=1}^N (x^n - \bar{x})^T (x^n - \bar{x})$$

- Lagrangian to Minimize  $J$ :

$$- L(u_{M+1}, \dots, u_D, \lambda_{M+1}, \dots, \lambda_D) = \sum_{i=M+1}^D u_i^T S u_i + \sum_{i=M+1}^D \lambda_i (1 - u_i^T u_i)$$

constraint  $\|u_i\| = 1$  to prevent  $u_i = 0$

$$\text{For each } u_i, \frac{\partial}{\partial u_i} L = 2S u_i - 2\lambda_i u_i = 0$$

$$\Rightarrow S u_i = \lambda_i u_i$$

$\Rightarrow$  To minimize  $J$ , take eigenvectors with the first  $(D - M)$  small eigenvalue orthogonal to (out of) subspace

$\Leftrightarrow$  define subspace with eigenvectors with the first  $M$  large eigenvalue

$$\text{Intuition: } \widetilde{x}_n = \sum_{i=1}^M ((x^n)^T u_i) u_i + \sum_{i=M+1}^D (\bar{x}^T u_i) u_i \quad (2.12)$$

$$= \bar{x} + \sum_{i=1}^M [(x^n - \bar{x})^T u_i] u_i \quad (2.13)$$

1. Singular Value Decomposition - SVD:

- Introduce matrix  $A_{m \times n}$

-  $(A^T A)_{n \times n}$  symmetric matrix (actually, Gram matrix  $\rightarrow$  semi-definite) -

eigenvalue decomposition:

$$(2.14)$$

$$A^T A = V D V^T, \text{ } V \text{ is normalized } (v_i^T v_i = 1) \text{ with column as eigenvector} \quad (2.15)$$

$$- AV = (Av_1, \dots, Av_n)_{m \times n}$$

$$- \text{Let } r(A) = r$$

$$\Rightarrow r(A^T A) = r(A) = r \quad (2.16)$$

$$r(AV) = \min\{r(A), r(V)\} = \min\{r, n\} = r \quad (2.17)$$

- Reduce  $AV$  to basis  $(Av_1, \dots, Av_r)$

- Let  $U = (u_1, \dots, u_r) = \left( \frac{Av_1}{\sqrt{\lambda_1}}, \dots, \frac{Av_r}{\sqrt{\lambda_r}} \right)$ ,  $\lambda_i$  is  $i$ -thh eigenvalue of  $A^T A$

- Orthogonal:  $\forall i \neq j, u_i^T u_j = \frac{1}{\sqrt{\lambda_i \lambda_j}} v_i^T A^T A v_j = \frac{\lambda_j}{\sqrt{\lambda_i \lambda_j}} v_i^T v_j = 0$

- Unit:  $\|u_i\| = \frac{\|Av_i\|}{\sqrt{\lambda_i}} = \frac{\sqrt{\langle Av_i, Av_i \rangle}}{\sqrt{\lambda_i}} = 1$

$\Rightarrow U$  is standard orthogonal (orthonormal) basis

-  $AV = U \Sigma$ , where  $\Sigma = D^{\frac{1}{2}}$



- Expand  $U$  to orthonormal in  $\mathbb{R}^m : (u_i, \dots, u_m)$
- Expand corresponding part in  $\Sigma$  with 0
- $A = U\Sigma V^T$ , with singular value in  $\Sigma$  in decreasing order

## 2. SVD with PCA:

- $X$  is data matrix in row (centered - zero mean)
- Eigenvectors of covariance matrix  $S = X^T X$  are in  $V$ , where  $X = U\Sigma V^T$
- When using  $S = U\Sigma V^T \Rightarrow U = V \wedge S = V\Sigma V^T$

reduced to eigenvalue decomposition

- $S = VDV^T$  with  $V$  orthonormal:

Eigenvalues  $\lambda$  of covariance matrix  $S$  express the variance of data set  $X$  in direction of corresponding eigenvectors

- Projection:

-  $\tilde{X} = XV_M$ , where  $V_M$  contains first  $M$ -large eigenvectors - Projection direction is **not** unique

## 3. Reconstruction (approximate):

- Data is projected onto  $k$  dimension using SVD with  $S = U\Sigma V^T$  -  $x_{approx} = U_{reduce} \cdot z$ ,  $U_{reduce}$  is  $n \times k$  matrix,  $z$  is  $k \times 1$  vector - [Reconstruction from data Compression] (./../Machine

## 4. Choosing $k$ (num of principal components):

- choose the **smallest**  $k$  making  $\frac{J}{V} \leq 0.01 = 1\%$

- $[U, S, V] = \text{svd}(\text{Sigma}) = 1 - \frac{\sum_{i=1}^k S_{ii}}{\sum_{i=1}^n S_{ii}}$ ,  $S$  is diagonal matrix

= check  $\frac{J}{V}$  before compress data

## 5. Data Preprocessing:

- PCA *vs.* Normalization: - Normalization: Individually normalized but still correlated - PCA: create decorrelated data - whitening - Whitening: projection with normalization -  $S = VDV^T$ , where  $S$  is Gram matrix over  $X^T$  -  $\forall n, y_n = D^{-\frac{1}{2}} V^T (x^n - \bar{x})$ , where  $\bar{x}$  is the mean of  $X$

$\Rightarrow y^n$  has zero mean (2.18)

$$\text{cov}(\{y^n\}) = \frac{1}{N} \sum_{n=1}^N y_n y_n^T = D^{-\frac{1}{2}} V^T S V D^{-\frac{1}{2}} = I \quad (2.19)$$

## 6. Tips for PCA:

- Do NOT use PCA to prevent overfitting, use regularization instead - Try original data before implement PCA - Train PCA only on training set

## 2.10.2 Independent Component Analysis (ICA)

1. Goal: - Recover original signals from a mixed observed data - Source signal  $S \in \mathbb{R}^{N \times K}$ ; mixing matrix  $A$ ; Observed data  $X = SA$  - Maximizes statistical independence - Find  $A^{-1}$  to maximizes independence of columns of  $S$  2. Assumption: - At most one signal is Gaussian distributed - Ignore amplitude and order of recovered signals - Have at least as many observed mixtures as signals -  $A$  invertible 3. Independence *vs.* Uncorrelatedness - Independence  $\Rightarrow$  Uncorrelatedness -  $p(x_1, x_2) = p(x_1)p(x_2) \Rightarrow \mathbb{E}(x_1 x_2) - \mathbb{E}(x_1)\mathbb{E}(x_2) = 0$  4. Central Limit Theorem 5. FastICA algorithm

## 2.10.3 t-SNE

1. Problem & Focus 2. Compared to PCA: - No whitening function to use for new data - PCA can only capture linear structure inside the data - t-SNE preserves the local distances in the original data

### 2.10.4 Anomaly Detection

1. Problem to solve:

- Given dataset  $x^1, x^2, \dots, x^m$ , build density estimation model  $p(x)$  -  $p(x^{test} < \epsilon) =_i x^{test}$  anomaly

2. Hypothesis function:

$$- p(x) = \prod_{i=1}^n p(x_i), x \in R^n, \forall i \in [1, n], x_i \sim N(\mu_i, \sigma_i^2) - \mu = \frac{1}{m} \sum_{i=1}^m x^i, \sigma^2 = \frac{1}{m} \sum_{i=1}^m (x^i - \mu)^2 -$$

assume  $x_1, \dots, x_n$  independent from each other

3. Multivariate Gaussian:

$$- p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right),$$

$x \in R^n, \mu \in R^n, \Sigma \in R^{n \times n}$ , where  $\Sigma$  is covariance matrix

$$- \mu = \frac{1}{m} \sum_{i=1}^m x^i, \Sigma = \frac{1}{m} \sum_{i=1}^m (x^i - \mu)(x^i - \mu)^T - x_1, \dots, x_n \text{ can be correlated but **not** linearly}$$

dependent - need  $m > n$  ( $m \geq 10nsuggested$ ) or else  $\Sigma$  non-invertible

4. Algorithm:

- choose features - compute  $\mu, \sigma$  - compute  $p(x)$  for new example, anomaly if  $p(x) < \epsilon$

5. Evaluation (real-number):

- Labeled data into normal/anomalous set

(okay if some anomalies slip into normal set)

- training set: unlabeled data from normal set (60- CV set: labeled data from normal (20-

test set: labeled data from normal (20

- Use evaluation metrics (skewed data)

6. When to use:

- Anomaly detection: - Very small num of positive data (0-20 commonly); Large num of negative data - Difficult to learn from positive data (not enough data, too many features...)

- Future anomalies may look nothing like given data - Supervised Learning: - Larger num of positive & negative data - Enough positive data for algorithm to learn - Future positive example is likely to be similar to given data

7. Example:

- Anomaly detection: - Fraud detection, Manufacturing, Monitoring machines in data center... - Supervised learning: - Email spam classification (enough data), Weather prediction (sunny/rainy/etc), Cancer classification...

8. Tips:

- Non-gaussian feature: transformation / using other distribution - Choosing features: compare anomaly data with normal data

### 2.10.5 Recommender System

1. Problem Formulation:

-  $r_{i,j} = 1$  if item  $i$  is rated by user  $j$

-  $y_{i,j}$  = rating of item  $i$  given by user  $j$

-  $\theta^j$  = parameter vector for user  $j$

-  $x^i$  = feature vector for movie  $i$

=\_i for user  $j$ , movie  $i$ , ( $r_{i,j} = 0$ ), predict rating  $x^i \theta^j$

2. Content Based Recommendations:

- Treat each user as a separate linear regression problem with the feature vectors of its rated items as training set

\*\*Assume features for each items ( $x^i$ ) are available and known\*\*

=\_i given  $X$  estimate  $\Theta$

- Cost Function for  $\theta_j$ :

$$J(\theta^j) = \frac{1}{2} \sum_{i:r_{i,j}=1} (x^i \theta^j - y_{i,j})^2 + \frac{\lambda}{2} \sum_{k=1}^n (\theta_k^j)^2, \theta^j \in R^{n+1} (\theta_0 \text{ not regularized})$$

- Cost Function for  $\Theta$ :

$$J(\Theta) = \frac{1}{2} \sum_{j=1}^{n_u} \sum_{i:r_{i,j}=1} (x^i \theta^j - y_{i,j})^2 + \frac{\lambda}{2} \sum_{j=1}^{n_u} \sum_{k=1}^n (\theta_k^j)^2,$$

$\theta^j \in R^{n+1}$  ( $\theta_0$  not regularized),  $n_u$  is num of users

$$\text{- Update Rule: } \forall \theta_k^j \in \theta^j, \theta_k^j := \theta_k^j - \alpha \frac{\partial J(\Theta)}{\partial \theta_k^j}, \frac{\partial J(\Theta)}{\partial \theta_k^j} = \sum_{i:r_{i,j}=1} (x^i \theta^j - y_{i,j}) x_k^i + \lambda \theta_k^j, \text{ for } k \neq 0 \ (\theta^j \in R^{n+1})$$

3. Collaborative Filtering

- **Assume preference of each users ( $\theta^j$ ) are available and known**

=> given  $\Theta$  estimate  $X$

$$\text{- Cost Function for } x^i: J(x^i) = \frac{1}{2} \sum_{j:r_{i,j}=1} (x^i \theta^j - y_{i,j})^2 + \frac{\lambda}{2} \sum_{k=1}^n (x_k^i)^2 \text{ - Cost Function for } X:$$

$$J(X) = \frac{1}{2} \sum_{i=1}^{n_m} \sum_{j:r_{i,j}=1} (x^i \theta^j - y_{i,j})^2 + \frac{\lambda}{2} \sum_{i=1}^{n_m} \sum_{k=1}^n (x_k^i)^2$$

$x^j \in R^{n+1}$  ( $x_0$  not regularized),  $n_m$  is num of items - Update Rule:  $\forall x_k^i \in x^i, x_k^i := x_k^i - \alpha \frac{\partial J(X)}{\partial x_k^i}$ ,

$$\frac{\partial J(X)}{\partial x_k^i} = \sum_{j:r_{i,j}=1} (\theta^j x^i - y_{i,j}) \theta_k^j + \lambda x_k^i, \text{ for } k \neq 0 \ (x^i \in R^{n+1})$$

- Basic Idea:

- Randomly initialize  $\Theta$

- loop:

Estimate  $X$

Estimate  $\Theta$

- Cost Function:

$$J(X, \Theta) = \frac{1}{2} \sum_{(i,j):r_{i,j}=1} (x^i \theta^j - y_{i,j})^2 + \frac{\lambda}{2} \sum_{i=1}^{n_m} \sum_{k=1}^n (x_k^i)^2 + \frac{\lambda}{2} \sum_{j=1}^{n_u} \sum_{k=1}^n (\theta_k^j)^2, x \in R^n, \theta \in R^n$$

(the sum term in  $J(\Theta)$ ,  $J(X)$ , and  $J(X, \Theta)$  is the same)

- Update Rule:

$$\text{- } \forall x_k^i \in x^i, x_k^i := x_k^i - \alpha \frac{\partial J(X, \Theta)}{\partial x_k^i}, \frac{\partial J(X, \Theta)}{\partial x_k^i} = \frac{\partial J(X)}{\partial x_k^i} = \sum_{j:r_{i,j}=1} (\theta^j x^i - y_{i,j}) \theta_k^j + \lambda x_k^i, x^i \in R^n$$

$$\text{- } \forall \theta_k^j \in \theta^j, \theta_k^j := \theta_k^j - \alpha \frac{\partial J(X, \Theta)}{\partial \theta_k^j}, \frac{\partial J(X, \Theta)}{\partial \theta_k^j} = \frac{\partial J(\Theta)}{\partial \theta_k^j} = \sum_{i:r_{i,j}=1} (\theta^j x^i - y_{i,j}) x_k^i + \lambda \theta_k^j, \theta^j \in R^{n+1}$$

- **Algorithm**

- Initialize  $X, \Theta$  to \*\*small random values\*\*

=> for symmetry breaking (similar to random initialization in neural network)

=> so that algorithm learns features  $x^1, \dots, x^{n_m}$  that are different from each other

- Minimize  $J(X, \Theta)$

- Predict  $y_{i,j} = x^i \theta^j$  ( $Y = X\Theta$ )

- Finding Related Item to Recommend

-  $\|x^i - x^j\|$  is small => item  $i$  and  $j$  is similar

- Mean Normalization:

- Problem: if user  $j$  hasn't rated any movie,  $\theta^j = [0, \dots, 0]$

=> predicted rating of user  $j$  on all item = 0

=> useless prediction

- Algorithm (row version):

compute vector  $\mu, \forall \mu_i \in \mu, \mu_i = \text{mean of } Y_i$ , where  $Y_i$  is the  $i^{th}$  row in  $Y$

manipulate  $Y$ :  $\forall y_{i,j} \in Y \wedge r_{i,j} = 1, y_{i,j} - \mu_i = \tilde{y}_{i,j}$  the mean of each row in  $Y$  is 0

predict rating for user  $j$  on item  $i = x^i \theta^j + \mu_i$

- For item  $i$  with no rating
- => apply column version of mean normalization
- (but user with no rating is generally more important)

## 2.11 Large Scale Machine Learning

### 2.11.1 Gradient Descent with Large Dataset

1. Stochastic Gradient Descent - Problem in Big Data: - Updating  $\theta$  becomes computationally expensive in batch gradient decent

- Cost Function: - Cost function on single data:  $cost(\theta, (x^i, y^i)) = \frac{1}{2}(h_{\theta}(x^i) - y^i)^2$  - Overall

$$\text{Cost Function: } J_{train}(\theta) = \frac{1}{m} \sum_{i=1}^m cost(\theta, (x^i, y^i))$$

- Procedure:
- Randomly shuffle dataset
- Repeat
- for  $i \in [1, m]$

$$\theta_j = \theta_j - \alpha \frac{\partial}{\partial \theta_j} cost(\theta, (x^i, y^i)) \quad (2.20)$$

$$= \theta_j - \alpha (h_{\theta}(x^i) - y^i) \cdot x_j^i$$

$$(\text{for } j = 0, \dots, n) \quad (2.21)$$

$$\Rightarrow \text{make progress with each single data} \quad (2.22)$$

- Convergence:
- Wanting  $\theta$  to converge => slowly decrease  $\alpha$  over time (but more parameters)

$$(\text{E.g } \alpha = \frac{\text{const}_1}{\text{iteration num} + \text{const}_2})$$

- Compute  $cost(\theta, (x^i, y^i))$  before updating

For every  $k$  update iterations, plot average  $cost(\theta, (x^i, y^i))$  over the last  $k$  examples

- Checking curves:

Increasing  $k$  result in smoother line and less noise, but the result is more delayed

Use smaller learning rate  $\alpha$  will generally have slight benefit

Curve goes up => smaller  $\alpha$

- vs Batch Gradient Descent:

- use 1 example un each update iteration => make progress earlier => faster - Result may not be the optimal but in its neighbourhood

1. Mini-batch Gradient Descent - Use  $b$  examples in each update iteration - vs Batch Gradient Descent: - start to make progress earlier => faster - Result may not be the optimal but in its neighbourhood - vs Stochastic Gradient Descent: - can partially parallelize computation over  $b$  examples => faster under a good vectorized implementation & appropriate  $b$  - introduce extra parameter  $b$

### 2.11.2 Online Learning

1. Situation: - Has too many data (can be considered as infinite) - When data comes in as a continuous stream - Can adapt to changing user preference
2. Procedure: - Use one example only once (Similar to stochastic gradient decent in this sense)

### 2.11.3 Map-reduce

1. In Batch Gradient Descent:

- Update rule  $\theta_j = \theta_j - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^i) - y^i) x_j^i$  - Parallelize the computation of  $\sum_{i=1}^m (h_{\theta}(x^i) - y^i) x_j^i$  by dividing the data set into multiple sections
- 2. Ability to reduce:
  - Contain operation over the whole data set
 (Neural Network can be map-reduced)

## 2.12 Building Machine Learning System

- Under the example of Photo OCR (Optical Character Recognition)

### 2.12.1 Pipeline

1. Break ML system into modules
2. Example:
  - Image  $\rightarrow$  Text detection  $\rightarrow$  Character segmentation  $\rightarrow$  Character recognition
  - Text Detection:
    - Sliding window detection:
      - set different sizes of the window (mostly rectangle), for each size:
        - take a image patch
        - resize the patch into desired size
        - run ML algorithm on the small patch
        - slide the window by step\_size (eventually through the image)
    - Expansion: expand the related region to create a bigger region
  - Character Segmentation:
    - 1-D sliding window
    - Character Recognition

### 2.12.2 Getting More Data

1. **\*\*Artificial Data Synthesis\*\*** - Creating New Data: Use available resource and combine them - Example (in Character Recognition): Paste different fonts in the randomly chosen backgrounds - Amplify Data Set: Introduce distortions to the original data set - Need to identify the appropriate distortion - Usually adding purely/random/meaningless noise
  - Prerequisite: - Having a low bias/high variance hypothesis is
    1. Collect/Label Data Manually - Usually a surprise to find how little time it needs to get 10,000 data - Calculate the time it needs before decide to/not to collect the data

### 2.12.3 Ceilling Analysis

1. Aim:
    - Decide which modules might be the best use of time to improve
  2. Procedure:
    - Draw a table with 2 column (Component - Accuracy)
    - Component: the modules simulated to be perfect (100- Accuracy: the accuracy of the entire system on the test set (define by chosen evaluation matrix)
- | — **Perfect Component** — | **Accuracy** — | : — : — :  |
|---------------------------|----------------|--|
| — : — — none —            | $f$ —          | module 1 — $f + \epsilon_1$ —                                      |
|                           |                | module 1,2 — $f + \epsilon_1 + \epsilon_2$ —                       |
| .                         |                |  |
| .                         | — .            |  |
| .                         |                |  |
| .                         | — —            | module 1,2,...,n — $f + \epsilon_1 + \dots + \epsilon_n = 100\%$ — |
- $\Rightarrow$  Improving module  $x$  will gain at most  $\epsilon_x$  improvement in the overall performance

- Choose the module with most significant  $\epsilon$  to improve

## Chapter 3

# Linear Regression

## Chapter 4

# Linear Classification



## Chapter 5

# Kernel Methods

## Chapter 6

# Graphical Models

## Chapter 7

# Mixture Models and EM

## Chapter 8

# Approximate Inference

## Chapter 9

# Sampling Methods

## Chapter 10

# Continuous Latent Variable

## Chapter 11

# Sequential Data

# Chapter 12

## Deep Learning

### 12.1 Interview of Fame

#### 12.1.1 Geoffrey Hinton

##### Knowledge Embedding

- BP
  - psychology view: knowledge in vectors
  - semantic AI: knowledge graph
  - BP algorithm can interpret & convert between feature vector and graph representation (with some embedding)
- Boltzmann Machine
  - Learning Algorithm on Density Net
    - same information in forward & backward propagation to learn feature embedding
  - Restricted Boltzmann Machine (RBM)
    - ways of learning in deep dense net with fast inference
    - iterative learning (adding layer after the above trained)
    - $\text{ReLU} \Leftrightarrow$  a stack of sigmoid functions (approximately) in RBM
    - ReLU units initialized to identity for efficient learning
- EM
  - EM with Approximate E Step
- vs. Symbolic AI
  - Symbolic AI: symbolic logic-like expression to do reasoning
  - yet, maybe state vector to represent knowledge

##### Brain Science

- Brain: Nets Implemented by Evolution
  - trying to train without BP
  - doing BP (get derivatives) with re-construction error (auto-encoder)



## Memory in Nets

- Fast Weights for Short-term Memory
- Capsule Net
  - structured knowledge representation in each unit (feature with sets of property)
  - $\Rightarrow$  enable nets to vote rather than filtering - thus better generalization

## Unsupervised Learning

- Importance
  - better than human eventually (as supervised learning has limited maximum)
  - GAN as a breakthrough

## ”Slow” Feature

- Non-linear Transform to Find Linear Transform
  - find a latent representation containing linear transform to do the work
  - e.g. change viewpoints: pixels  $\rightarrow$  coordinates  $\rightarrow$  linear transform  $\rightarrow$  back to pixels

## Relations between Computers

- showing computer data to work
  - instead of programming it to work

### 12.1.2 Pieter Abbeel

## Deep Reinforcement Learning

- Overall Challenge
  - Representation
  - Exploration Problem
  - Credit Assignment
  - Worst Case Performance
- Advantage (Deep Nets in RL)
  - network capturing the representation (state vector)
- Question in DRL
  - how to learn safely
  - how to keep learning (under small negative samples) e.g. better than human
  - can we learn the reinforcement learning program (RL in the RL)
  - long time horizon
  - use experience across tasks
- Success of DRL
  - simulated robot inventing walking...  $\Rightarrow$  single general algorithms to learn

### 12.1.3 Ian

#### Generative Adversarial Networks

- Generative Models
  - Resembling
    - trained to optimized the distribution behind training data  
(then sampled from that distribution to get more imaginary training data)
    - $\Rightarrow$  produce data to resemble the training data
  - Usage
    - semi-supervised learning
    - data augmentation
    - simulating scientific experiment
  - Previous Ways
    - Boltzmann Machine
    - Sparse Coding
  - Now: Generative Adversarial Networks (GANs)
  - Future
    - increase reliability of GANs (stabilizing)

### 12.1.4 Research

#### Topics

- Unsupervised Learning
- Reinforcement Learning
- AI Security
  - Anti Inducing
    - NOT to be fooled/induced to do unappropriated things  
(even if algorithm is right)
  - Built-in Security
- Fairness in AI
  - Dealing Societal Issue
  - Reflecting Preferred Bias

#### Advices

- Reading
  - read a little bit & find somewhere intuitively not right
    - good intuition: eventually work; bad intuition: not working no matter what it is doing
    - if other doubts your idea as bullshit  $\Rightarrow$  a sign for real good result
  - a supervisor with similar belief
  - PhD vs. Company
    - amount of mentoring

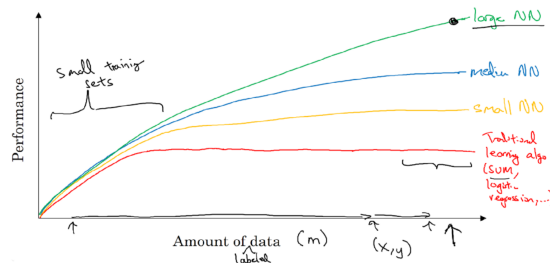
- faster if dedicated supervisor available
- Practice
  - open-source learning resource
  - implement the paper
  - work on a projected and open source it  
⇒ the stage (e.g. github) will bring people to you

## 12.2 Basic Neural Network

### 12.2.1 Advantages

#### Large/Big Data

- Larger Maximum Capability
  - Curve given Amount of Data



- Reasons
  - the scale of data (labeled)
  - the scale of neural network (computability)
  - the scale of efficiency: e.g. ReLu, faster parallel algorithm

#### Flexibility

- Different Structures for Different Tasks
  - Same Data & Task
    - changing settings/structures of deep learning model can make a difference (v.s. SVM, etc.)
- Ability to Choose Basis Functions
  - Functional View
    - $y(\mathbf{x}, \mathbf{w}) = f(\mathbf{w}^T \phi(\mathbf{x}))$ , where  $\phi$  is basis function,  $f(\cdot)$  is net as a function
  - Learning  $\phi$ : choose embedding  $\Rightarrow$  choose basis function
  - Learning  $\mathbf{w}$ : choose which feature / basis functions more useful

**Power of Depth**

- Deep Representation
  - Low-level  $\rightarrow$  High-level
    - multiple layers to choose & combine useful information (creating new feature/basis)  
 $\Rightarrow$  next layer use chosen/combined simple basis to build more complex one
    - $\Rightarrow$  an hierarchy from low-level information to high-level information
- Circuit Theory
  - Power of Combination
    - functions that can be compactly represented by a depth  $k$  architecture might require an exponential number of computational nodes using a depth  $k - 1$  architecture  
 (from the perspective of factorization)

**Yet, start from the SHALLOW (logistic regression) before trying the deep**

**12.2.2 Problem**

( $n$  units in one hidden layer)

**Weight-space Symmetries**

- Symmetries in Activation Function
  - $\mathcal{O}(2^n)$ , e.g.  $\arctan(-x) = -\arctan(x) \Rightarrow$  changing signs of all input & output has the same mapping (reduce effective data)
- Positional Combination in One Layer
  - $\mathcal{O}(n!)$  exchange unit with each other (together with their input output weights)  $\Rightarrow$  mapping stay the same

$\Rightarrow \mathcal{O}(n!2^n)$  overall weight-space symmetries

**Non-convex Error Function**

- Multiple Critical Points
  - at least  $\mathcal{O}(n!2^n)$  critical points ( $\nabla E(w) = 0$ , where  $E(w)$  is error function) due to weight-space symmetries
- Expensive in Finding Critical Point
  - expensive for even local optima with gradient decent
  - as expensive as  $\mathcal{O}(n^3)$  if using Laplace approximation

**Gradient Vanishing**

- Saturated Function
  - sigmoid/tanh function: gradient  $\rightarrow 0$  when input  $\rightarrow \pm\infty$
- Possible Solutions
  - Xavier initialization, ReLU activation, ...

### 12.2.3 Learning

#### Forward-Backward Propagation

- Representation
  - Layers
    - input layer
    - hidden layer(s): layer with NO ground truth (for the associated weights) available  
note: input & hidden layers have associated biases as well (usually)
    - output layer
  - Neuron (Unit)
    - $s_l$ : num of units in layer  $l$
    - $w^l$ : weight matrix of mapping from layer  $l$  to  $l + 1$ , with shape of  $(s_{l+1}, s_l + 1)$
    - $a_j^l$ : activation of unit  $j$  at layer  $l$
    - $h(\cdot)$ : activation function (usually shared)
    - $z_j^l$ : output of unit  $j$  at layer  $l$  (represent parameterized basis)
  - Intuition
    - all stacked vertically (vertical vector)  
⇒ horizontally for different examples; vertically for different units
- Forward Propagation (Inference)
  - Activation  $a^{j+1} = w^j \cdot [z_0^j, \dots, z_{s_j}^j]^T$ , with  $z_0 = 1$
  - Unit Output  $z^{j+1} = h(a^{j+1}) = [z_1^{j+1}, \dots, z_{s_{j+1}}^{j+1}]^T$
- Backward Propagation
  - Loss  $\mathcal{L}(W) =$
- Practice of Back Prop
  - Caching Intermediate Result
    - naturally cached: input  $a^0 = x$ , weights  $w$  and bias  $b$
    - activation input/output  $a/z$   
(since will be used in back-prop)
  - Auto Difference
    - achievement: calculate the derivatives along the forward prop !

## 12.3 Empirical Experiment Practice

### 12.3.1 Designing Networks

### 12.3.2 Hyperparameters

## 12.4 Operations & Layers Structure

### 12.4.1 Operations in Network

#### Activations

- Sigmoid  $a = \sigma(z)$

- Pros
  - mapping to  $(0, 1)$ , with  $\sigma(0) = 0.5$
- Cons
  - gradient vanishing:  $\sigma(z)' = \sigma(z)(1 - \sigma(z)) \Rightarrow \lim_{z \rightarrow \pm\infty} \sigma(z)' \rightarrow 0$   
(as the gradient passed through (via chain rule)  $= \frac{a}{z} \frac{z}{w}$ )
- Tangent  $a = \tanh(z)$ 
  - Pros
    - empirically, almost always better than sigmoid (in hidden layers)
    - maps to  $(-1, 1)$ , with  $\tanh(0) = 0 \Rightarrow$  help centering data (0-mean)  
 $\Rightarrow$  make the learning of next layer easier
  - Cons
    - still, gradient vanishing when  $z \rightarrow \pm\infty$
- Rectified Linear Unit ReLu  $\max(0, z)$ 
  - Derivation: approximated by a stack of sigmoid
    -
  - Pros
    - mitigate gradient vanishing:  $\forall z > 0, a = z \Rightarrow$  learn much faster  
 $\Rightarrow$  the default choice!
  - Cons
    - undefined behavior at  $x = 0$  (actually, gradient becomes the sub-gradient)
    - gradient totally vanished for  $x < 0$
    - $\Rightarrow$  dead units: weights learned/initialized to always output negatives  
 $\Rightarrow$  activation always output 0  
 $\Rightarrow$  the unit always output 0
- Leaky Relu  $a = \max(\alpha z, z), \alpha \rightarrow 0^+$  (e.g.  $\alpha = 0.01$ )
  - Pros
    - mitigate the gradient vanishing problem for  $(-\infty, +\infty)$
    - avoid dead units problem
 (yet not that popular as ReLU)
- Linear (Identity) Activation  $a = z$ 
  - Pros
    - used in regression to output real number  $\in (-\infty, +\infty)$
    - used in compression net
  - Cons
    - stacked units with linear activation  $\Leftrightarrow$  single linear transformation
    - logistic regression with linear activation in hidden layer is NO more expressive than logistic regression with no hidden layer !

## 12.4.2 Operations on Network

### Initialization

- Random Initialization for Weights
  - Practice
    - weights initialized to a random variable in a small range e.g.  $(-0.03, 0.03)$
  - Pros
    - avoid symmetry problem:  
if identical initialization for weights  $\Rightarrow$  units in same layer computing exactly same function  
 $\Rightarrow$  get the same learning step propagated back  
 $\Rightarrow$  then always compute exactly the same function (by induction)
    - avoid gradient vanishing: especially for gradient of sigmoid/tanh activation
  - Cons
    - NOT concern various nets: sampling in a fixed range may not work for all nets
- Xavier Initialization for Weights
  - Practice
    -
  - Pros
    - empirically much better:
- Zero Initialization for Bias
  - Reason
    - default to use 0 bias  
(can NOT used for weights as explained)

## 12.4.3 Layers

### Convolution Layer

- Normal Convolution
- Atrous Convolution
- Deconvolution

### Pooling Layer

- Normal Pooling
- Unpooling
- Spatial Pyramid Pooling (SPP)
- Region of Interest Pooling (RoI Pooling)
  - Input
    - feature maps from CNN
    - RoIs i.e. proposal region (from selective search etc.) projected on feature map
  - Operation

- divide each RoI with grid of desired size (proportional to the RoI size)
- max pooling from each cell
- ⇒ single-size SPP for each RoI
- Output
  - a fixed size feature maps for each RoI

## 12.5 Architectures

### 12.5.1 Encoder-Decoder Architecture

#### Description

- Encoder
  - Functionality
    - downsample/encode input into rich feature maps/vectors
  - Implementation
    - visual input: CNN backbone
    - natural expression input: RNN backbone
- Decoder
  - Functionality
    - upsample/decode rich feature maps back to the original size
    - actually, impose requirement onto the encoder
  - Implementation
    - visual output: CNN backbone
    - natural expression output: RNN backbone
- Connection
  - Functionality
    - combine high level information with low level information
    - image → image: outline refinement ...
    - language → language: sentence style capturing
  - Implementation
    - concatenation

#### Extension

- Multiple Encoder
  - Functionality
    - project different information into the same space
    - combine those information via some shared layers at the end
- Multiple Decoder
  - Functionality
    - impose multiple requirements to the encoder (via auxiliary loss)



## 12.6 Advanced Topic

### 12.6.1 Machine Reading Comprehension

#### Problem Formulation

- 

#### RNN with Attention

#### Convolution with Self-attention - QAnet

### 12.6.2 Image Caption

#### Problem Formulation

- Input
  - Image
    - visual input as the target of description
- Goal
  - Natural Expression
    - description of the image in natural language, e.g. English

#### Baseline Approach & Previous Work

- Neural Image Caption
  - Visual Information
    - encoded by CNN backbone into a 1-D vector
  - Word Information
    - a set of word selected beforehand
    - word embedding performed
  - Language Generation
    - generated by an LSTM decoder
    - combining info: visual encoding as initial state of LSTM
    - process: LSTM gives each word a to-be-selected probability at each time step
  - Inference
    - sampling: sample each word according to the distribution given by LSTM
    - beam search: iteratively consider extending  $k$  best sentence of length  $t$  to  $t + 1$   
 $\Rightarrow$  select  $k$  best sentence of length  $t + 1$  from all resulted sentences  
 (beam search selected in the paper)

### 12.6.3 Referring Segmentation

#### Problem Formulation

- Input
  - Image
    - visual input for segmentation
  - Natural Language Expression

- expression to denote the interested object(s)/stuff(s)
- Goal
  - Segmentation Mask of Referred Object(s)
    - currently (till early 2019), mostly binary segmentation
- Related Area
  - NLP + CV
    - referring localization
    - image caption

### Baseline Approach & Previous Work

- Segmentation from Natural Language Expressions
  - Spatial Info
    - FCN-32s to encode the image into 2-D feature maps (the last conv layer)
  - Language Info
    - LSTM to encode the sentence into 1-D vector (the last hidden state)
  - Combining Info and Output
    - per-pixel info: concat [coordinates of current pixel (coord info), language info]
    - tile the per-pixel info into a feature map, then concat to the spatial info (per-pixel info concatenated at every pixel of spatial info)
    - followed by a series of conv and finally a deconv for upsampling
  - Training
    - per-pixel cross-entropy loss
  - Pros
    - special spatial info: coord of each pixel
    - standard info combination: concatenation
  - Cons
    - no powerful spatial info encoder: FCN-32s instead of Resnet/Unet...
    - weak upsampler, compared to encoder-decoder architecture
    - language info comes late: after downsampling
    - weak language info: only integrated once
- Recurrent Multimodal Interaction for Referring Image Segmentation
  - Spatial Info
    - DeepLab-101 as encoder (Resnet as backbone, with atrous conv)
    - then tiled (concat at every pixel) by coord info (coordinate of current pixel)
  - Language Info
    - word embedding  $w_t$  for  $t = 1, \dots, T$
    - LSTM scanning the sentence, with hidden state  $h_t$  at time  $t$
    - language info  $l_t = \text{concat} [h_t, w_t]$
  - Combining Info
    - $l_t$  tiled to spatial info, at each time step  
 $\Rightarrow$  creating combined feature maps  $F_t$  (of shape [height, wide, channel])

- combined feature maps  $F_1, \dots, F_T$  fed to an convolutional LSTM, where the ConvLSTM shares weight over both space and time
  - $\Rightarrow$  feature vector of  $F_t[i, j]$  is the input of the ConvLSTM at time  $t$
  - $\Rightarrow$  conv in ConvLSTM implemented as  $1 \times 1$  conv
- a series of conv following the last hidden state of the ConvLSTM
- Output
  - bilinear interpolated to original input size
  - optionally post-processed by dense CRF, using pydensecrf (hence inference only)
- Pros
  - more powerful spatial info extractor: DeepLab-101
  - better language info: integrated at every time step, maintained by an ConvLSTM
- Cons
  - weak architecture for spatial info: still no upsampling (blur segmentation)
  - no spatial relation considered in ConvLSTM (?)
  - weak language representation (better with pos tag, word2vec, word dict, biLSTM, and maybe even attention)
  - language info still comes late: still after downsampling

### Current State-of-The-Art (early 2019)

- Key-Word-Aware Network for Referring Expression Image Segmentation
  - Spatial Info
    - DeepLab-101 as encoder for comparability
    - then tiled by coord info (coordinate of each pixel)
  - Language Info
    - LSTM scanning sentence, each hidden state as word info
  - Combining Info
    - attention mask from combined info (spatial info with language info tiled) (at each time step)
    - attention weighting over spatial info at each time step
      - $\Rightarrow$  an 1-D global encoding for each time step (via weighted mean over space)
      - $\Rightarrow$  filling feature maps: global encoding if attention here  $>$  threshold; else **0**
      - $\Rightarrow$  summing filled feature maps over time for the global spatial maps  $c$
    - attention weighting over tiled language info at each time step, correspondingly
      - $\Rightarrow$  tiled language info maps summed over time for the global language maps  $q$
    - concat [spatial info,  $c$ ,  $q$ ], followed by  $1 \times 1$  conv
  - Output
    - upsampling performed
  - Pros
    - attention introduced: from combined info
    - better combination: attention masked interact with both spatial & language info
  - Cons
    - blur segmentation: no encoder-decoder architecture
    - attention mask obtained sequentially: only last mask has complete language info
    - language info comes late: after downsampling

- Referring Image Segmentation via Recurrent Refinement Networks
  - Spatial Info
    - DeepLabv3 ResNet-101 as encoder
    - last feature maps tiled (concat at each pixel) with coord info
  - Language Info
    - LSTM scanning sentence, generating word info at each time step
    - last hidden layer as language info
  - Combining Info
    - combined info = spatial info tiled with language info
    - selecting set of feature maps from downsampling stages
    - all selected feature maps resized and fed to  $1 \times 1$  conv  
 $\Rightarrow$  to match the dimensions of combined info
    - convolutional LSTM applied to refine the combined info  
 (with matched selected feature maps as input at each time step)
  - Output
    - a conv after final hidden state of ConvLSTM for segmentation
    - upsampled to original image size
  - Pros
    - ConvLSTM integrating info at downsampling stage  $\Rightarrow$  segmentation refined
  - Cons
    - no upsampling: blur segmentation, mitigated by ConvLSTM though  
 (yet no language info introduced in refinement)
    - CNN fixed during training: relying on ConvLSTM
    - single info combination: only by tiling  
 (though, currently performing the best in all dataset)

## Research Direction

- Integrating Encoder-Decoder Architecture
  - Upsampling
    - similar to Unet, concat low-level spatial info
    - introduce language info as well  
 (e.g. early combination, explicit introducing, ...)
- Early Info Combination
  - Tiling at First Conv
    - downsampling more responsible for language info processing  
 $\Rightarrow$  hopefully get more fine-tuning alone with conv filters
    - can be used with pre-trained net:  

$$ReLU(conv_1 * X_1 + conv_2 * X_2) = ReLU([conv_1, conv_2] * [X_1, X_2])$$
  - Multiple Entries
    - combining info at different stages of downsampling / upsampling
- Attention
  - Attention from Combined Info

- as key-word-aware net
- Attention on Language Info
  - 1-D spatial pyramid pooling / attention mask on the sentence encoding
- Language Info Throughout Network
  - Encoder-Decoder for Language Info
    - network asked to recover language info after processing combined info (potentially via a separate branch only at training time)  
⇒ auxiliary loss
  - Language as Conv Filter
    - Language Info, through a subnet, becoming a set of conv filters  
⇒ then imposed in downsampling, tunnel, upsampling or bridge stage(s)
- Data Augmentation
  - Translation Module
    - using the same image
    - expression translated to a middle language and then back to English  
⇒ language info trained more finely