Fully General Relativistic Structure and Spectrum of ADAFs

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Contents

Chapter 1

Basic Usage

- `/src`: contains the source file for calculation of disk structure, intrinsic spectrum and observed spectrum;
- `/data`: contains the input/output data;
- `/doc` : the use manual.

To use the code, change directory to `/src` , type command `make` and three executive files are produced,

- `disk`: for disk structure;
- `spec`: for intrinsic spectrum;
- `obs`: for observed spectrum.

Then type `./disk` in shell will calculate the disk structure and analogically for `./spec` and `./obs`.

In the directory `/data`, the files are

- `datain.txt`: the input data.
- `/spec`: the SED at different radius, in which the number in the file name (e.g. spec025.txt) correspond with the line number in file `rdisk.dat`.
- `nrows.txt`: the number of rows in `rdisk.dat`.
- `soltot.dat`: the solutions of all variables of ADAFs.
- `adaf.dat`: used for `spec`.
- `sol-for-spec.dat`: used for `obs`.
- `rdisk.dat`: used for calculation of optical depth.
- `spectrum.dat specobs.dat`: the intrinsic spectrum and the observed spectrum.

Chapter 2

The Structure of ADAFs

2.1 Equation set for ADAFs

We adopt the geometrical units $G = c = 1$. The kerr metric expanded around the equatorial plane is

$$\begin{aligned} ds^2 &= -\frac{r^2\Delta}{A}dt^2 + \frac{A}{r^2}(d\phi - \omega dt)^2 + \frac{r^2}{\Delta}dr^2 + dz^2 && (2.1) \\ &= -\left(\frac{r^2\Delta}{A} - \frac{A\omega^2}{r^2}\right)dt^2 - \frac{2A\omega}{r^2}dtd\phi + \frac{A}{r^2}d\phi^2 + \frac{r^2}{\Delta}dr^2 + dz^2 && (2.2) \end{aligned}$$

Therefore

$$g_{tt} = -\left(\frac{r^2\Delta}{A} - \frac{A\omega^2}{r^2}\right), \quad g_{t\phi} = -\frac{A\omega}{r^2}, \quad g_{\phi\phi} = \frac{A}{r^2}, \quad g_{rr} = \frac{r^2}{\Delta}, \quad g_{zz} = 1 \tag{2.3}$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad A = r^4 + r^2a^2 + 2Mra^2, \quad \omega = \frac{2Mar}{A}, \quad a = \frac{J}{M} \tag{2.4}$$

The equation set are

- Continuity equation

$$\dot{M} = -2\pi\Delta^{1/2}\Sigma_0\gamma_r V \tag{2.5}$$

- Momentum equation

$$\frac{\dot{M}}{2\pi}(L-L_{in}) = rW_\phi^r \tag{2.6}$$

$$\gamma_r^2 V\frac{dV}{dr} + \frac{1}{\mu\Sigma_0}\frac{dW}{dr} = -\frac{\gamma_\phi^2 AM}{r^4\Delta}\frac{(\Omega-\Omega_K^+)(\Omega-\Omega_K^-)}{\Omega_K^+\Omega_K^-}. \tag{2.7}$$

- Energy equation

$$\frac{\dot{M}W_i}{2\pi r\Sigma_0}\frac{1}{\Gamma_i-1}\left(\frac{d\ln W_i}{dr} - \Gamma_i\frac{d\ln\Sigma_0}{dr} + \frac{\Gamma_i-1}{r}\right) = (1-\delta)\frac{\alpha W}{r}\frac{\gamma_\phi^4 A^2}{r^6}\frac{d\Omega}{dr} + \Lambda_{ie} \tag{2.8}$$

$$\frac{\dot{M}W_e}{2\pi r\Sigma_0}\frac{1}{\Gamma_e-1}\left(\frac{d\ln W_e}{dr} - \Gamma_e\frac{d\ln\Sigma_0}{dr} + \frac{\Gamma_e-1}{r}\right) = \delta\frac{\alpha W}{r}\frac{\gamma_\phi^4 A^2}{r^6}\frac{d\Omega}{dr} - \Lambda_{ie} + F^- \tag{2.9}$$

The involved variables:

$$\gamma_r = \frac{1}{\sqrt{1-V^2}}, \qquad \gamma_\phi = \sqrt{1+\frac{r^2L^2}{\mu^2\gamma_r^2 A}} \tag{2.10}$$

$$\mu = 1 + \frac{W}{\Sigma_0}\left[\left(a_i + \frac{1}{\beta}\right)\frac{W_i}{W} + \left(a_e + \frac{1}{\beta}\right)\frac{W_e}{W}\right] \tag{2.11}$$

$$a_i = \frac{1}{\gamma_i - 1} + \frac{2(1-\beta)}{\beta}, \qquad a_e = \frac{1}{\gamma_e - 1} + \frac{2(1-\beta)}{\beta} \tag{2.12}$$

$$\gamma_i = 1 + \theta_i\left[\frac{3K_3(1/\theta_i) + K_1(1/\theta_i)}{4K_2(1/\theta_i)} - 1\right]^{-1} \tag{2.13}$$

$$\gamma_e = 1 + \theta_e\left[\frac{3K_3(1/\theta_e) + K_1(1/\theta_e)}{4K_2(1/\theta_e)} - 1\right]^{-1} \tag{2.14}$$

$$\Gamma_i = 1 + \left[a_i\left(1 + \frac{d\ln a_i}{d\ln T_i}\right)\right]^{-1}, \qquad \Gamma_e = 1 + \left[a_e\left(1 + \frac{d\ln a_e}{d\ln T_e}\right)\right]^{-1} \tag{2.15}$$

$$\Omega_K^\pm = \pm\frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}}, \qquad \Omega = \omega + \frac{r^3\Delta^{1/2}L}{\mu\gamma_r\gamma_\phi A^{3/2}} \tag{2.16}$$

The vertical scale height is

$$h^2 = \frac{W}{\mu\Sigma_0}\frac{r^4}{(L/\mu)^2 - a^2[(E/\mu)^2 - 1]} \tag{2.17}$$

where

$$L = \mu u_\phi, \quad E = -\mu u_t, \quad \frac{u^\phi}{u^t} = \Omega \tag{2.18}$$

and

$$u_\phi = g_{\phi t}u^t + g_{\phi\phi}u^\phi = \left(\frac{g_{\phi t}}{\Omega} + g_{\phi\phi}\right)u^\phi, \quad u_t = g_{tt}u^t + g_{\phi t}u^\phi = \left(\frac{g_{tt}}{\Omega} + g_{t\phi}\right)u^\phi \tag{2.19}$$

Thus we obtain

$$E = -\mu u_t = -\frac{g_{tt} + \Omega g_{t\phi}}{g_{t\phi} + \Omega g_{\phi\phi}}L = \frac{r^4\Delta/A^2 + \omega\Omega - \omega^2}{\Omega - \omega}L \tag{2.20}$$

Note that

$$\Omega - \omega = \frac{r^3\Delta^{1/2}L}{\mu\gamma_r\gamma_\phi A^{3/2}} \tag{2.21}$$

we have

$$E = \mu\gamma_r\gamma_\phi r\frac{\Delta^{1/2}}{A^{1/2}} + \omega L \tag{2.22}$$

- calculation of VdV/dr

$$\frac{V}{\sqrt{1-V^2}} = -\frac{\dot{M}}{2\pi\Delta^{1/2}\Sigma_0} \tag{2.23}$$

Since

$$\frac{V^2}{1-V^2} = \frac{\dot{M}^2}{4\pi^2}\frac{1}{\Delta\Sigma_0^2} = B \tag{2.24}$$

a simple calculation gives

$$V^2 = 1 - \frac{1}{1+B} \tag{2.25}$$

and thereby

$$2V\frac{dV}{dr} = \frac{1}{(1+B)^2}\frac{dB}{dr} \tag{2.26}$$

$$= -\frac{1}{(1+B)^2}\frac{\dot{M}^2}{4\pi^2}\frac{1}{\Delta\Sigma_0}\left[\frac{1}{\Delta}\frac{d\Delta}{dr} + \frac{2}{\Sigma_0}\frac{d\Sigma_0}{dr}\right] \tag{2.27}$$

$$= -\frac{2B}{(1+B)^2}\left(\frac{r-M}{\Delta} + \frac{1}{\Sigma_0}\frac{d\Sigma_0}{dr}\right) \tag{2.28}$$

- calculation of $d\Omega/dr$

$$\Omega = \omega + \frac{r^3\Delta^{1/2}L}{\mu\gamma_r\gamma_\phi A^{3/2}}, \quad L = \frac{2\pi r W_\phi^r}{\dot{M}} + L_{in}, \quad W_\phi^r = \alpha\frac{A^{3/2}\Delta^{1/2}\gamma_\phi^3}{r^6}W \tag{2.29}$$

$$\frac{d\Omega}{dr} = \frac{d\omega}{dr} + \frac{2\pi\alpha}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2 \Delta W}{\mu\gamma_r r^2}\right) + L_{in}\frac{d}{dr}\left(\frac{r^3\Delta^{1/2}}{\mu\gamma_r\gamma_\phi A^{3/2}}\right) \tag{2.30}$$

To sum up, we write down the equation set

$$\frac{dW_i}{dr} + \frac{dW_e}{dr} - \frac{\beta\mu\gamma_r^2 B}{(1+B)^2}\frac{d\Sigma_0}{dr} = \frac{\beta\mu\gamma_r^2 B\Sigma_0}{(1+B)^2}\frac{r-M}{\Delta} - \frac{\beta\mu\gamma_\phi^2\Sigma_0 AM}{r^4\Delta}\frac{(\Omega-\Omega_K^+)(\Omega-\Omega_K^-)}{\Omega_K^+\Omega_K^-} \tag{2.31}$$

$$\left[\frac{1}{\Gamma_i-1}\frac{\dot{M}}{\Sigma_0} - \frac{(1-\delta)4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}\frac{\Delta A^2 W}{r^8}\right]\frac{dW_i}{dr} - \left[\frac{(1-\delta)4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}\frac{\Delta A^2 W}{r^8}\right]\frac{dW_e}{dr} - \frac{\Gamma_i}{\Gamma_i-1}\frac{W_i\dot{M}}{\Sigma_0^2}\frac{d\Sigma_0}{dr}$$

$$= (1-\delta)2\pi\alpha W\frac{\gamma_\phi^4 A^2}{r^6}\left[\frac{2\pi\alpha W}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2\Delta}{\mu\gamma_r r^2}\right) + L_{in}\frac{d}{dr}\left(\frac{r^3\Delta^{1/2}}{\mu\gamma_\phi\gamma_r A^{3/2}}\right) + \frac{d\omega}{dr}\right] - \frac{W_i\dot{M}}{r\Sigma_0} + 2\pi r\Lambda_{ie} \tag{2.32}$$

$$-\left[\frac{\delta 4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}\frac{\Delta A^2 W}{r^8}\right]\frac{dW_i}{dr} + \left[\frac{1}{\Gamma_e-1}\frac{\dot{M}}{\Sigma_0} - \frac{\delta 4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}\frac{\Delta A^2 W}{r^8}\right]\frac{dW_e}{dr} - \frac{\Gamma_e}{\Gamma_e-1}\frac{W_e\dot{M}}{\Sigma_0^2}\frac{d\Sigma_0}{dr}$$

$$= \delta 2\pi\alpha W\frac{\gamma_\phi^4 A^2}{r^6}\left[\frac{2\pi\alpha W}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2\Delta}{\mu\gamma_r r^2}\right) + L_{in}\frac{d}{dr}\left(\frac{r^3\Delta^{1/2}}{\mu\gamma_\phi\gamma_r A^{3/2}}\right) + \frac{d\omega}{dr}\right] - \frac{W_e\dot{M}}{r\Sigma_0} + 2\pi r(F^- - \Lambda_{ie}) \tag{2.33}$$

Following we substitute

$$M = r_g = \frac{GM}{c^2}, \tag{2.34}$$

$$\Delta = 1 - \frac{2r_g}{r} + \frac{a^2}{r^2}, \quad A = 1 + \frac{a^2}{r^2} + \frac{2r_g a^2}{r^3}, \quad \omega = \frac{2r_g a}{r^3 A} \tag{2.35}$$

The equations set will be

$$\frac{dW_i}{dr} + \frac{dW_e}{dr} - \frac{\beta\mu\gamma_r^2 B}{(1+B)^2}\frac{d\Sigma_0}{dr} = \frac{\beta\mu\gamma_r^2 B\Sigma_0}{(1+B)^2}\frac{r-r_g}{r^2\Delta} - \frac{\beta\mu\gamma_\phi^2\Sigma_0 A r_g}{r^2\Delta}\frac{(\Omega-\Omega_K^+)(\Omega-\Omega_K^-)}{\Omega_K^+\Omega_K^-} \tag{2.36}$$

$$\left[\frac{1}{\Gamma_i-1}\frac{\dot{M}}{\Sigma_0} - \frac{(1-\delta)4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}r^2\Delta A^2 W\right]\frac{dW_i}{dr} - \left[\frac{(1-\delta)4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}r^2\Delta A^2 W\right]\frac{dW_e}{dr} - \frac{\Gamma_i}{\Gamma_i-1}\frac{W_i\dot{M}}{\Sigma_0^2}\frac{d\Sigma_0}{dr}$$

$$= (1-\delta)2\pi\alpha W\gamma_\phi^4 A^2 r^2\left[\frac{2\pi\alpha W}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2\Delta}{\mu\gamma_r}\right) + L_{in}\frac{d}{dr}\left(\frac{\Delta^{1/2}}{\mu\gamma_\phi\gamma_r r^2 A^{3/2}}\right) + \frac{d\omega}{dr}\right] - \frac{W_i\dot{M}}{r\Sigma_0} + 2\pi r\Lambda_{ie} \tag{2.37}$$

$$-\left[\frac{\delta 4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}r^2\Delta A^2 W\right]\frac{dW_i}{dr}+\left[\frac{1}{\Gamma_e-1}\frac{\dot{M}}{\Sigma_0}-\frac{\delta 4\pi^2\alpha^2\gamma_\phi^6}{\mu\gamma_r\beta\dot{M}}r^2\Delta A^2 W\right]\frac{dW_e}{dr}-\frac{\Gamma_e}{\Gamma_e-1}\frac{W_e\dot{M}}{\Sigma_0^2}\frac{d\Sigma_0}{dr}$$
$$=\delta 2\pi\alpha W\gamma_\phi^4 A^2 r^2\left[\frac{2\pi\alpha W}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2\Delta}{\mu\gamma_r}\right)+L_{in}\frac{d}{dr}\left(\frac{\Delta^{1/2}}{\mu\gamma_\phi\gamma_r r^2 A^{3/2}}\right)+\frac{d\omega}{dr}\right]-\frac{W_e\dot{M}}{r\Sigma_0}+2\pi r(F^- -\Lambda_{ie}) \tag{2.38}$$

where

$$\frac{d\Delta}{dr}=\frac{2r_g}{r^2}-\frac{2a^2}{r^3},\quad \frac{dA}{dr}=-\frac{2a^2}{r^3}-\frac{6r_g a^2}{r^4} \tag{2.39}$$

$$\frac{d}{dr}\left(\frac{\Delta^{1/2}}{r^2A^{3/2}}\right)=\frac{\Delta^{1/2}}{r^2A^{3/2}}\left(\frac{1}{2\Delta}\frac{d\Delta}{dr}-\frac{2}{r}-\frac{3}{2A}\frac{dA}{dr}\right) \tag{2.40}$$

$$\frac{d\omega}{dr}=-\frac{2r_g a}{r^3 A}\left(\frac{3}{r}+\frac{1}{A}\frac{dA}{dr}\right) \tag{2.41}$$

The solution are

$$\frac{d\Sigma_0}{dr}=\frac{(c_3-a_{31}c_1)(a_{22}-a_{21})-(c_2-a_{21}c_1)(a_{32}-a_{31})}{(a_{33}-a_{31}a_{13})(a_{22}-a_{21})-(a_{23}-a_{21}a_{13})(a_{32}-a_{31})}=S \tag{2.42}$$

$$\frac{dW_i}{dr}=\frac{a_{22}c_1-c_2}{a_{22}-a_{21}}-\frac{a_{22}a_{13}-a_{23}}{a_{22}-a_{21}}S \tag{2.43}$$

$$\frac{dW_e}{dr}=\frac{c_2-a_{21}c_1}{a_{22}-a_{21}}-\frac{a_{23}-a_{21}a_{13}}{a_{22}-a_{21}}S \tag{2.44}$$

Once the solution for Σ_0, W_e and W_i are obtained,

$$B=\frac{\dot{M}^2}{4\pi^2}\frac{1}{r^2\Delta\Sigma_0^2},\qquad V=\sqrt{\frac{B}{1+B}},\qquad \gamma_r=\frac{1}{\sqrt{1-V^2}} \tag{2.45}$$

$$L=L_{in}+\frac{2\pi r^2}{\dot{M}}\alpha A^{3/2}\Delta^{1/2}W\gamma_\phi^3,\qquad \gamma_\phi=\sqrt{1+\frac{L^2}{\mu^2\gamma_r^2 r^2 A}} \tag{2.46}$$

$$T_i=\frac{\mu_i W_i m_H}{k\Sigma_0},\quad T_e=\frac{\mu_e W_e m_H}{k\Sigma_0} \tag{2.47}$$

2.2 The boundary conditions

$$\Omega = 0.8\Omega_K \tag{2.48}$$

$$T_i = T_e = 0.1T_{vir}, \quad T_{vir} = (\gamma - 1)\frac{GMm_H}{kr} \tag{2.49}$$

2.3 Dimension units

$$r : R_s, \quad L : cR_s, \quad \Omega : \frac{c}{R_s}, \quad \dot{M} : \dot{M}_{Edd}, \quad \Sigma : \frac{\dot{M}_{Edd}}{cR_s} \tag{2.50}$$

$$W : \frac{\dot{M}_{Edd}c}{R_s}, \quad Q_{rad} : \frac{\dot{M}_{Edd}c^2}{R_s^2} \tag{2.51}$$

2.4 The transonic point

$$a_{22} - a_{21} = -\frac{1}{\Gamma_i - 1}\frac{\dot{M}}{\Sigma_0}, \qquad a_{32} - a_{31} = \frac{1}{\Gamma_e - 1}\frac{\dot{M}}{\Sigma_0} \tag{2.52}$$

$$a_{33} - a_{31}a_{13} = -\frac{\Gamma_e}{\Gamma_e - 1}\frac{\dot{M}W_e}{\Sigma_0^2} + \delta Y\frac{\beta\mu\gamma_r^2 B}{(1+B)^2} \tag{2.53}$$

$$a_{23} - a_{21}a_{13} = -\frac{\Gamma_i}{\Gamma_i - 1}\frac{\dot{M}W_i}{\Sigma_0^2} + \left[\frac{1}{\Gamma_i - 1}\frac{\dot{M}}{\Sigma_0} + (1-\delta)Y\right]\frac{\beta\mu\gamma_r^2 B}{(1+B)^2} \tag{2.54}$$

Then we have

$$\begin{aligned}
\mathcal{D} = \quad & (a_{33} - a_{31}a_{13})(a_{22} - a_{21}) - (a_{23} - a_{21}a_{13})(a_{32} - a_{31}) \\
= \quad & -\frac{1}{\Gamma_i - 1}\frac{\dot{M}}{\Sigma_0}\left[-\frac{\Gamma_e}{\Gamma_e - 1}\frac{\dot{M}W_e}{\Sigma_0^2} + \delta Y\frac{\beta\mu\gamma_r^2 B}{(1+B)^2}\right] \\
& -\frac{1}{\Gamma_e - 1}\frac{\dot{M}}{\Sigma_0}\left\{-\frac{\Gamma_i}{\Gamma_i - 1}\frac{\dot{M}W_i}{\Sigma_0^2} + \left[\frac{1}{\Gamma_i - 1}\frac{\dot{M}}{\Sigma_0} + (1-\delta)Y\right]\frac{\beta\mu\gamma_r^2 B}{(1+B)^2}\right\}
\end{aligned} \tag{2.55}$$

where

$$\beta\mu\gamma_r^2 BY = -\frac{\alpha^2\gamma_r\gamma_\phi^6 A^2 W\dot{M}}{\Sigma_0^2} \tag{2.56}$$

$$
\begin{aligned}
-\frac{\Sigma_0}{\dot{M}}\mathcal{D} = \quad & -\frac{1}{(\Gamma_i-1)(\Gamma_e-1)}\left(\Gamma_i\frac{W_i}{W}+\Gamma_e\frac{W_e}{W}\right)\frac{\dot{M}W}{\Sigma_0^2} \\
& -\left(\frac{\delta}{\Gamma_i-1}+\frac{1-\delta}{\Gamma_e-1}\right)\frac{\alpha^2\gamma_r\gamma_\phi^6A^2}{(1+B)^2}\frac{\dot{M}W}{\Sigma_0^2} \\
& +\frac{1}{(\Gamma_i-1)(\Gamma_e-1)}\frac{\beta\mu\gamma_r^4}{(1+B)^2}\frac{B}{\gamma_r^2}\frac{\dot{M}}{\Sigma_0} \\
= \quad & \frac{\dot{M}}{\Sigma_0}[b_1V^2-(b_2+b_3)c_s^2]
\end{aligned}
\tag{2.57}
$$

Therefore, the transonic point satisfies,

$$\frac{V^2}{c_s^2}=\frac{b_2+b_3}{b_1} \tag{2.58}$$

where

$$b_1=\frac{1}{(\Gamma_i-1)(\Gamma_e-1)}\frac{\beta\mu\gamma_r^4}{(1+B)^2} \tag{2.59}$$

$$b_2=\frac{1}{(\Gamma_i-1)(\Gamma_e-1)}\left(\Gamma_i\frac{W_i}{W}+\Gamma_e\frac{W_e}{W}\right) \tag{2.60}$$

$$b_3=\left(\frac{\delta}{\Gamma_i-1}+\frac{1-\delta}{\Gamma_e-1}\right)\frac{\alpha^2\gamma_r\gamma_\phi^6A^2}{(1+B)^2} \tag{2.61}$$

2.5 Comparison

The differences of equation set under the Newtonian theory and General relativity theory.

1. Continuity equation.

   Newronian:

$$\dot{M}=-2\pi r\Sigma v_r \tag{2.62}$$

   GR:

$$\dot{M}=-2\pi\Delta^{1/2}\Sigma\gamma_rV \tag{2.63}$$

In the Newtonian approxiamtion, $\Delta = r^2 - 2Mr + a^2 \sim r^2$, $\gamma_r \sim 1$, the above equation will be

$$\dot{M} = -2\pi r \Sigma V. \tag{2.64}$$

2. Radial momentum equation.

   Newtonian:

$$v_r \frac{dv_r}{dr} + \frac{1}{\Sigma}\frac{dW}{dr} = r(\Omega^2 - \Omega_K^2) - \frac{W}{\Sigma}\frac{d\ln\Omega_k}{dr} \tag{2.65}$$

   GR:

$$\gamma_r^2 V \frac{dV}{dr} + \frac{1}{\mu\Sigma}\frac{dW}{dr} = -\frac{\gamma_\phi^2 AM}{r^4\Delta}\frac{(\Omega-\Omega_K^+)(\Omega-\Omega_K^-)}{\Omega_K^+\Omega_K^-}. \tag{2.66}$$

   In the Newtonian approxiamtion, $\gamma_r \sim 1$, $\mu \sim 1$, $A = r^4 + r^2a^2 + 2Mra^2 \sim r^4$, and

$$\Omega_K^+ = -\Omega_K^- = \frac{M^{1/2}}{r^{3/2}} \tag{2.67}$$

   the above equation will be

$$V\frac{dV}{dr} + \frac{1}{\Sigma}\frac{dW}{dr} = r(\Omega^2 - \Omega_K^2) \tag{2.68}$$

3. Azimuthal momentum equation.

   Newtonian:

$$\frac{\dot{M}}{2\pi}(L - L_{in}) = r^2\alpha W. \tag{2.69}$$

   GR:

$$\frac{\dot{M}}{2\pi}(L - L_{in}) = \frac{A^{3/2}\Delta^{1/2}\gamma_\phi^3}{r^5}\alpha W \tag{2.70}$$

   In the Newtonian approxiamtion,

$$\frac{\dot{M}}{2\pi}(L - L_{in}) = r^2\alpha W. \tag{2.71}$$

4. Energy equation.

   Newtonian:

$$\frac{\dot{M}W_i}{\Sigma}\left[\frac{1}{\gamma-1}\frac{d\ln W_i}{dr} - \frac{\gamma}{\gamma-1}\frac{d\ln\Sigma}{dr} + \frac{d\ln H}{dr}\right] = 2\pi r^2\alpha W\frac{d\Omega}{dr} + 2\pi r\Lambda_{ie} \tag{2.72}$$

$$\frac{d\Omega}{dr} = \frac{2\pi\alpha}{\dot{M}}\frac{dW}{dr} - \frac{2L_{in}}{r^3} \tag{2.73}$$

GR:

$$\frac{\dot{M}W_i}{\Sigma}\left[\frac{1}{\Gamma_i - 1}\frac{d\ln W_i}{dr} - \frac{\Gamma_i}{\Gamma_i - 1}\frac{d\ln\Sigma}{dr} + \frac{1}{r}\right] = 2\pi\alpha W\frac{\gamma_\phi^4 A^2}{r^6}\frac{d\Omega}{dr} + 2\pi r\Lambda_{ie} \tag{2.74}$$

$$\frac{d\Omega}{dr} = \frac{2\pi\alpha\gamma_\phi^2\Delta}{\mu\gamma_r\dot{M}r^2}\frac{dW}{dr} + \frac{2\pi\alpha W}{\dot{M}}\frac{d}{dr}\left(\frac{\gamma_\phi^2\Delta}{\mu\gamma_r r^2}\right) + L_{in}\frac{d}{dr}\left(\frac{r^3\Delta^{1/2}}{\mu\gamma_r\gamma_\phi A^{3/2}}\right) + \frac{d\omega}{dr} \tag{2.75}$$

Chapter 3

The Spectrum of ADAFs

3.1 Comptonization

3.1.1 Klein-Nishina formular

In the electron rest frame,

$$\sigma_{KN} = \frac{3}{8}\sigma_T \left(\frac{\nu}{\nu_i}\right)^2 \left(\frac{\nu}{\nu_i} + \frac{\nu_i}{\nu} - 1 + \alpha^2\right), \tag{3.1}$$

where α is cosine of the scattering angle. Now for a motive electron, using coordinate conversion formular in the special relativitiy can easily obtains,

$$\sigma_{KN} = \frac{3}{8}\sigma_T \left(\frac{x'}{x}\right)^2 \left(\frac{x'}{x} + \frac{x}{x'} - 1 + \alpha^2\right), \tag{3.2}$$

where $x = \gamma\omega'(1-\beta\mu), x' = x/[1+(1-\alpha)x]$. Keep in mind that α in equ (2) is still defined in the electron rest frame while ω' is energy of the incoming photon in the lab frame. The energy of outgoing photon reads,

$$\omega = \frac{\gamma[\gamma(1-\beta\mu)+\gamma\beta\cos\theta(\mu-\beta)+\beta\sin\theta(1-\mu^2)^{1/2}\cos\phi}{1+\gamma(1-\beta\mu)(1-\cos\theta)\omega'}\omega' \tag{3.3}$$

3.1.2 Scattering rate

The angle-averaged scattering rate is given by

$$R(\omega,\gamma) = c\int_{-1}^{1}\frac{d\mu}{2}(1-\beta\mu)\int_{-1}^{1}d\alpha\sigma_{KN} \tag{3.4}$$

Standard asympotic forms for the rate R are,

- $\gamma\omega << 1$

$$R(\omega,\gamma) = c\sigma_T\left[1-\frac{2\gamma\omega}{3}(3+\beta^2)\right] \tag{3.5}$$

- $\gamma\omega >> 1$

$$R(\omega,\gamma) = \frac{3c\sigma_T}{8\gamma\omega}\left\{\left[1-\frac{2}{\gamma\omega}-\frac{2}{(\gamma\omega)^2}\right]\ln(1+\gamma\omega)+\frac{1}{2}-\frac{4}{\gamma\omega}-\frac{1}{2(1+2\gamma\omega)^2}\right\} \tag{3.6}$$

3.1.3 Mean scattered photon energy

$$\langle\omega\rangle(\omega',\gamma) = \frac{c}{R(\omega',\gamma)}\int_{-1}^{1}\frac{d\mu}{2}(1-\beta\mu)\int_{-1}^{1}d\alpha\omega\sigma_{KN} \tag{3.7}$$

3.1.4 Dispersion about $\langle\omega\rangle$

$$\langle\omega^2\rangle(\omega',\gamma) = \frac{c}{R(\omega',\gamma)}\int_{-1}^{1}\frac{d\mu}{2}(1-\beta\mu)\int_{-1}^{1}d\alpha\omega^2\sigma_{KN} \tag{3.8}$$

where

$$\omega^2 = \frac{\gamma^2\omega^2\{[\gamma(1-\beta\mu)+\gamma\beta\alpha(\mu-\beta)]^2+\frac{1}{2}\beta^2(1-\alpha^2)(1-\beta^2)\}}{[1+\gamma(1-\beta\mu)(1-\alpha)\omega]^2} \tag{3.9}$$

The dispersion is then given by

$$\langle\Delta\omega^2\rangle = \langle\omega^2\rangle - \langle\omega\rangle^2 \tag{3.10}$$

3.1.5 Scattered-photon distribution

$$P(\omega;\omega',\gamma) = \frac{1}{2D(\omega',\gamma)} H[D(\omega',\gamma) - |\omega - \langle\omega\rangle|] \tag{3.11}$$

where $H(x)$ is the Heaviside function, and $D(\omega',\gamma)$ is defined as

$$D(\omega',\gamma) = \min\{[\sqrt{3\langle\Delta\omega^2\rangle}, (\langle\omega\rangle - \omega_{min})\} \tag{3.12}$$

3.1.6 Kinetic equation

$$\frac{dn(\omega)}{dt} = -n(\omega)\int d\gamma N(\gamma)R(\omega,\gamma) + \int\int d\omega' d\gamma P(\omega;\omega',\gamma)R(\omega',\gamma)N(\gamma)n(\omega') \tag{3.13}$$

Acctually, in the calculation we only handle the second integeration,

$$\begin{aligned} \frac{dn(\omega)}{dt} &= \int\int d\omega' d\gamma P(\omega;\omega',\gamma)R(\omega',\gamma)N(\gamma)n(\omega') \\ &= (\ln 10)^2 \int\int d\log\omega' d\log\gamma P(\omega;\omega',\gamma)R(\omega',\gamma)N(\gamma)n(\omega')\omega'\gamma \end{aligned} \tag{3.14}$$

For simplicity, we take $dt = H/c$, consequently,

$$n(\omega) = (\ln 10)^2 \frac{H}{c} \int\int d\log\omega' d\log\gamma P(\omega;\omega',\gamma)R(\omega',\gamma)N_e(\gamma)n(\omega')\omega'\gamma \tag{3.15}$$

$$= (\ln 10)^2 N_e H\sigma_T \int\int d\log\omega' d\log\gamma P(\omega;\omega',\gamma)\frac{R(\omega',\gamma)}{c\sigma_T}\frac{N(\gamma)}{N_e} n(\omega')\omega'\gamma \tag{3.16}$$

$$= (\ln 10)^2 \tau_e \int\int d\log\omega' d\log\gamma P(\omega;\omega',\gamma)\frac{R(\omega',\gamma)}{c\sigma_T}\frac{N(\gamma)}{N_e} n(\omega')\omega'\gamma \tag{3.17}$$

3.1.7 Electron distribution

The relativistic Maxwell-Boltzmann distribution is gien by (Özel et al 2000)

$$N(\gamma) = N_{th}\gamma^2\beta\frac{\exp\left(-\gamma/\theta_e\right)}{\theta_e K_2(1/\theta_e)} \tag{3.18}$$

3.2 Radiation mechanisms

3.2.1 Synchrotron emission

$$\chi_{syn} = 4.43 \times 10^{-30} \frac{4\pi n_e \nu}{K_2(1/\theta_e)} I \left(\frac{4\pi m_e c \nu}{3eB\theta_e^2} \right) \tag{3.19}$$

where

$$I(x) = \frac{4.0505}{x^{1/6}} \left(1 + \frac{0.4}{x^{1/4}} + \frac{0.5316}{x^{1/2}} \right) \exp(-1.8899 x^{1/3}) \tag{3.20}$$

3.2.2 Bremstralung emission

$$\chi_{br} = q_{br}^{-} G \exp\left(\frac{h\nu}{kT_e} \right) \tag{3.21}$$

where G is the Gaunt factor, q_{br}^{-} is the bremsstrahlung emission per unit volume , which reads

$$q_{br}^{-} = 1.48 \times 10^{-22} n_e^2 F(\theta) \tag{3.22}$$

$$\begin{aligned} F(\theta) \quad &= 4 \left(\frac{2\theta_e}{\pi^3} \right)^{1/2} (1 + 1.781\theta_e^{1.34}) + 1.73\theta_e^{3/2} (1 + 1.1\theta_e + \theta_e^2 - 1.25\theta_e^{5/2}) \qquad \theta_e < 1 \\ &= \left(\frac{9\theta_e}{2\pi} \right) [\ln(0.48 + 1.123\theta_e) + 1.5] + 2.30\theta_e (\ln 1.123\theta_e + 1.28) \qquad \theta_e > 1 \end{aligned} \tag{3.23}$$

Gaunt factor is

$$G = \frac{h}{kT_e} \left(\frac{3}{\pi} \frac{kT_e}{h\nu} \right)^{1/2} \qquad \frac{kT_e}{h\nu} < 1 \tag{3.24}$$

$$G = \frac{h}{kT_e} \frac{\sqrt{3}}{\pi} \ln \left(\frac{4}{\xi} \frac{kT_e}{h\nu} \right)^{1/2} \qquad \frac{kT_e}{h\nu} > 1 \tag{3.25}$$

3.2.3 Compton scattering

The energy enhancement factor η is

$$\eta = \exp[s(A-1)][1-P(j_m+1, As)] + \eta_{max}P(j_m+1, s) \tag{3.26}$$

where P is the incomplete gamma function and

$$A = 1+4\theta_e+16\theta_e^2, \qquad s = \tau_{es}+\tau_{es}^2 \tag{3.27}$$

$$\eta_{max} = \frac{3kT_e}{h\nu}, \qquad j_m = \frac{\ln\eta_{max}}{\ln A}, \qquad \tau_{es} = 2n_e\sigma_T H \tag{3.28}$$

With the energy enhancement factor η, the local radiative cooling rate Q^-_{rad} is given by

$$Q^-_{rad} = \int d\nu 2\eta F_\nu \tag{3.29}$$

3.3 ADAF model

3.3.1 Self-simular solution

$$\begin{aligned}
v &= -2.12\times 10^10\alpha c_1 r^{-1/2}\ \text{cm s}^{-1} \\
\Omega &= 7.19\times 10^4 c_2 m^{-1} r^{-3/2}\ \text{s}^{-1} \\
c_s^2 &= 4.50\times 10^{20} c_3 r^{-1}\ \text{cm s}^{-1} \\
\rho &= 3.79\times 10^{-5}\alpha^{-1}c_1^{-1}c_3^{-1/2}m^{-1}\dot{m}r^{-3/2}\ \text{g cm}^{-3} \\
p &= 1.71\times 10^{16}\alpha^{-1}c_1^{-1}c_3^{1/2}m^{-1}\dot{m}r^{-5/2}\ \text{g cm}^{-1}\ \text{s}^{-2} \\
q^+ &= 1.84\times 10^{21}\epsilon' c_3^{1/2}m^{-2}\dot{m}r^{-4}\ \text{ergs cm}^{-3}\ \text{s}^{-1} \\
\tau_{es} &= 2n_e\sigma_T H = 12.4\alpha^{-1}c_1^{-1}\dot{m}r^{-1/2}
\end{aligned} \tag{3.30}$$

3.3.2 Global structure

3.3.3 Radiation transfer

$$F_\nu = \frac{2\pi}{\sqrt{3}} B_\nu [1 - \exp(-2\sqrt{3}\tau_\nu^*)] \tag{3.31}$$

$$\tau_\nu^* = \frac{\sqrt{\pi}}{2} \kappa_\nu(0) H \tag{3.32}$$

$$\kappa_\nu = \frac{\chi_\nu}{4\pi B_\nu} = \frac{\chi_{\nu,syn} + \chi_{\nu,br}}{4\pi B_\nu} \tag{3.33}$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \tag{3.34}$$

Chapter 4

The ray tracing method

4.1 Jacobian Elliptic Function

We denote $F(\phi, k)$ as the Incomplete elliptic integral of first kind.

$$sn^{-1}(\sin\phi|k^2) = F(\phi, k) \tag{4.1}$$

$$cn^{-1}(\cos\phi|k^2) = F(\phi, k) \tag{4.2}$$

$$tn^{-1}(\tan\phi|k^2) = F(\phi, k) \tag{4.3}$$

4.2 Integrals

$$R(r) = r^4 + (a^2 - \lambda^2 - Q)r^2 + 2[Q + (\lambda - a)^2]r - a^2 Q \tag{4.4}$$

$$\Theta(\mu) = Q + (a^2 - \lambda^2 - Q)\mu^2 - a^2\mu^4 \tag{4.5}$$

4.2.1 r-component

The four roots for $R(r) = 0$ are

$$r_a = \frac{1}{2}N - \frac{1}{2}\sqrt{-M+2D-4C/N} \tag{4.6}$$

$$r_b = \frac{1}{2}N + \frac{1}{2}\sqrt{-M+2D-4C/N} \tag{4.7}$$

$$r_c = -\frac{1}{2}N - \frac{1}{2}\sqrt{-M+2D+4C/N} \tag{4.8}$$

$$r_d = -\frac{1}{2}N + \frac{1}{2}\sqrt{-M+2D+4C/N} \tag{4.9}$$

where

$$C = (a-\lambda)^2 + Q, \qquad D = \frac{2}{3}(Q+\lambda^2-a^2) \tag{4.10}$$

$$E = \frac{9}{4}D^2 - 12a^2Q, \qquad F = -\frac{27}{4}D^3 - 108a^2QD + 108C^2 \tag{4.11}$$

$$M = \frac{1}{3}\left(\frac{F-\sqrt{F^2-4E^3}}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{F+\sqrt{F^2-4E^3}}{2}\right)^{1/3} \tag{4.12}$$

$$N = \sqrt{M+D} \tag{4.13}$$

1. Case I: four real roots (Byrd & Friedman 1954, eq [258.00]).

   We denote the four real roots by r_1, r_2, r_3 and r_4, with $r_1 \geq r_2 \geq r_3 \geq r_4$.

$$\int_{r_1}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{r_1}^{r} \frac{dr}{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}} \tag{4.14}$$

$$= \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}} sn^{-1}\left(\sqrt{\frac{(r_2-r_4)(r-r_1)}{(r_1-r_4)(r-r_2)}}\,\middle|\, m_4\right) \tag{4.15}$$

   where

$$m_4 = \frac{(r_1-r_4)(r_2-r_3)}{(r_1-r_3)(r_2-r_4)} \tag{4.16}$$

2. Case II: two real roots and two complex roots (Byrd & Friedman 1954, eq [260.00]).

   We asume two complex roots are r_1 and r_2 ($r_1 = \overline{r_2}$), two real roots are r_3 and r_4 with $r_3 > r_4$.

$$\int_{r_1}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{r_1}^{r} \frac{dr}{\sqrt{(r-r_1)(r-\overline{r_1})(r-r_3)(r-r_4)}} \tag{4.17}$$

$$= \frac{2}{\sqrt{AB}} cn^{-1}\left[\sqrt{\frac{(A-B)r+r_3A-r_4A}{(A+B)r-r_3A-r_4A}}\,\middle|\, m_2\right] \tag{4.18}$$

   where

$$u = \frac{r_1+\overline{r_1}}{2} \qquad v^2 = -\frac{(r_1-\overline{r_1})^2}{4} \tag{4.19}$$

$$A^2 = (r_3-u)^2+v^2, \qquad B^2 = (r_4-u)^2+v^2 \tag{4.20}$$

$$m_2 = \frac{(A+B)^2-(r_3-r_4)^2}{4AB} \tag{4.21}$$

3. Case III: four complex roots (Byrd & Friedman 1954, eq [267.00]).

   The four complex roots are r_1, r_2, r_3 and r_4 ($r_1 = \overline{r_2}$, $r_3 = \overline{r_4}$). We introduce denotations:

$$u_1 = \frac{r_1+\overline{r_1}}{2} \qquad v_1^2 = -\frac{(r_1-\overline{r_1})^2}{4} \qquad u_2 = \frac{r_3+\overline{r_3}}{2} \qquad v_2^2 = -\frac{(r_3-\overline{r_3})^2}{4} \tag{4.22}$$

$$A^2 = (u_1-u_2)^2+(v_1+v_2)^2 \qquad B^2 = (u_1-u_2)^2+(v_1-v_2)^2 \tag{4.23}$$

$$g_1^2 = \frac{4v_1^2-(A-B)^2}{(A+B)^2-4v_1^2} \qquad r_0 = u_1 - v_1 * g_1 \tag{4.24}$$

$$\int_{r_0}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{r_0}^{r} \frac{dr}{\sqrt{(r-r_1)(r-\overline{r_1})(r-r_3)(r-\overline{r_3})}} \tag{4.25}$$

$$= \frac{2}{A+B} tn^{-1}\left(\frac{r-u_1+v_1g_1}{v_1+g_1u_1-g_1r}\,\middle|\, m_c\right) \tag{4.26}$$

   where

$$m_c = \frac{4AB}{(A+B)^2} \tag{4.27}$$

4.2.2 θ-component

1. $Q \geq 0$ (Byrd & Friedman 1954, eq [213.00])

$$\Theta(\mu) = a^2(\mu_-^2 + \mu^2)(\mu_+^2 - \mu^2) \quad (0 \leq \mu^2 \leq \mu_+^2) \tag{4.28}$$

$$\mu_\pm^2 = \frac{1}{2a^2}\left\{\left[\left(\lambda^2 + Q - a^2\right)^2 + 4a^2 Q\right]^{1/2} \mp \left(\lambda^2 + Q - a^2\right)\right\} \tag{4.29}$$

$$\int_\mu^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} = \frac{1}{\sqrt{a^2(\mu_+^2 + \mu_-^2)}} cn^{-1}\left(\frac{\mu}{\mu_+}\middle| m_\mu\right) \tag{4.30}$$

where

$$m_\mu = \frac{\mu_+^2}{\mu_+^2 + \mu_-^2} \tag{4.31}$$

2. $Q < 0$ (Byrd & Friedman 1954, eq [218.00])

$$\Theta(\mu) = a^2(\mu^2 - \mu_-^2)(\mu_+^2 - \mu^2) \quad (\mu_-^2 \leq \mu^2 \leq \mu_+^2) \tag{4.32}$$

$$\mu_\pm^2 = \frac{1}{2a^2}\left\{\left(|Q| + a^2 - \lambda^2\right) \pm \left[\left(|Q| + a^2 - \lambda^2\right)^2 - 4a^2|Q|\right]^{1/2}\right\} \tag{4.33}$$

$$\int_\mu^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} = \frac{1}{a\mu_+} sn^{-1}\left(\sqrt{\frac{\mu_+^2 - \mu^2}{\mu_+^2 - \mu_-^2}}\middle| m_\mu\right) \tag{4.34}$$

where

$$m_\mu = \frac{\mu_+^2 - \mu_-^2}{\mu_+^2} \tag{4.35}$$

4.3 TeV Trajectory

$$\tau_r = \pm \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r)}} = \pm \int_{\mu_e}^{\mu_{obs}} \frac{d\mu}{\sqrt{\Theta(\mu)}} = \tau_\mu \tag{4.36}$$

where

$$R(r) = r^4 + (a^2 - \lambda^2 - Q)r^2 + 2[Q + (\lambda - a)^2]r - a^2 Q \tag{4.37}$$

$$\Theta(\mu) = Q + (a^2 - \lambda^2 - Q)\mu^2 - a^2\mu^4 \tag{4.38}$$

Now we know the initial condition r_e, μ_e, μ_obs and asume $\alpha = 0$ (that is to say $\lambda = 0$), we need solve the β to calculate another constant of motion $Q = \beta^2 + (\alpha^2 - a^2)\cos^2\theta_\mathrm{obs}$.

4.3.1 Calculating τ_μ

$$\tau_\mu = \int_{\mu_e}^{\mu_{\text{obs}}} \frac{d\mu}{\sqrt{\Theta(\mu)}} = \int_{\mu_e}^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} - \int_{\mu_{\text{obs}}}^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} \quad \text{without turning point} \tag{4.39}$$

$$= \int_{\mu_e}^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} + \int_{\mu_{\text{obs}}}^{\mu_+} \frac{d\mu}{\sqrt{\Theta(\mu)}} \quad \text{with turning point} \tag{4.40}$$

4.3.2 Solving r_e Given τ_μ

$$\tau_r = \int_{r_e}^{\infty} \frac{dr}{\sqrt{R}} = \int_{r_+}^{\infty} \frac{dr}{\sqrt{R}} + \int_{r_+}^{r_e} \frac{dr}{\sqrt{R}} = \tau_\infty + \tau_e \quad \text{with turning point} \tag{4.41}$$

$$= \int_{r_+}^{\infty} \frac{dr}{\sqrt{R}} - \int_{r_+}^{r_e} \frac{dr}{\sqrt{R}} = \tau_\infty - \tau_e \quad \text{without turning point} \tag{4.42}$$

where $r_+ = r_1$ for Case I, $r_+ = r_3$ for Case II and $r_+ = r_0$ for Case III.

1. Case I: four real roots.

$$\tau_\infty = \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}} sn^{-1}\left(\sqrt{\frac{r_2-r_4}{r_1-r_4}} \middle| m_4\right) \tag{4.43}$$

$$\tau_e = \frac{2}{\sqrt{(r_1-r_3)(r_2-r_4)}} sn^{-1}\left[\frac{(r_2-r_4)(r_e-r_1)}{(r_1-r_4)(r_e-r_2)} \middle| m_4\right] \tag{4.44}$$

The solution is

$$r_e = \frac{r_1(r_2-r_4) - r_2(r_1-r_4)sn^2(\xi_4|m_4)}{(r_2-r_4)-(r_1-r_4)sn^2(\xi_4|m_4)} \tag{4.45}$$

where

$$\xi_4 = \frac{1}{2}(\tau_\mu - \tau_\infty)\sqrt{(r_1-r_3)(r_2-r_4)} \tag{4.46}$$

2. Case II: two real roots and two complex roots.

$$\tau_\infty = \frac{1}{\sqrt{AB}} cn^{-1}\left(\frac{A-B}{A+B} \middle| m_2\right) \tag{4.47}$$

The solution is

$$r_e = \frac{r_4 A - r_3 B - (r_4 A + r_3 B) cn(\xi_2 | m_2)}{(A-B) - (A+B) cn(\xi_2 | m_2)} \tag{4.48}$$

where

$$\xi_2 = (\tau_\mu - \tau_\infty)\sqrt{AB} \tag{4.49}$$

3. Case III: four complex roots.

$$\tau_\infty = \frac{2}{A+B} tn^{-1}\left(-\frac{1}{g_1} \middle| m_c\right) \tag{4.50}$$

$$\tau_e = \frac{2}{A+B} tn^{-1}\left(\frac{r_e - u_1 + v_1 g_1}{v_1 + g_1 u_1 - g_1 r_e} \middle| m_c\right) \tag{4.51}$$

The solution is

$$r_e = \frac{u_1 - v_1 g_1 + (v_1 + u_1 g_1) tn(\xi_c | m_c)}{1 + g_1 tn(\xi_c | m_c)} \tag{4.52}$$

where

$$\xi_c = \frac{1}{2}(\tau_\mu - \tau_\infty)(A+B) \tag{4.53}$$

4.3.3 Solving β

Specifying a β, then we get λ and Q. We can calculate directly τ_μ, and go on to solve the r_e. If r_e matches the initial condition, we find out the proper β.

Chapter 5

Mathematical functions

5.1 Bessel functions

The approximation of Bessel functions for large x is

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{4n^2 - 1}{8x} + \ldots\right) \tag{5.1}$$

Therefore for small θ,

$$K_3(1/\theta) = \sqrt{\frac{\pi}{2}} \theta^{1/2} e^{-1/\theta} \left(1 + \frac{35\theta}{8}\right), \tag{5.2}$$

$$K_2(1/\theta) = \sqrt{\frac{\pi}{2}} \theta^{1/2} e^{-1/\theta} \left(1 + \frac{15\theta}{8}\right), \tag{5.3}$$

$$K_1(1/\theta) = \sqrt{\frac{\pi}{2}} \theta^{1/2} e^{-1/\theta} \left(1 + \frac{3\theta}{8}\right), \tag{5.4}$$

and

$$\gamma = 1 + \theta \left[\frac{3K_3(1/\theta) + K_1(1/\theta)}{4K_2(1/\theta)} - 1\right]^{-1} = 1 + \frac{8 + 15\theta}{12}. \tag{5.5}$$