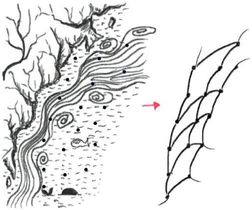
Network methods for data analysis and dynamical systems theory

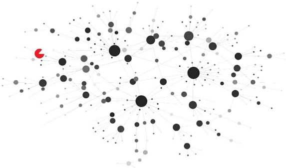
Liubov Tupikina

Ecole Polytechnique, France,
Potsdam Institute for Climate Impact Research, Germany
Bell labs seminar

August 30, 2018

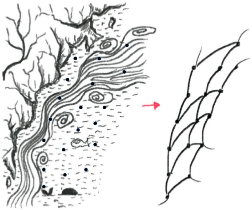
Outline



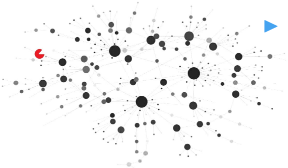


- **From dynamics to graph topology**: Method of flow-networks and correlation networks

Outline



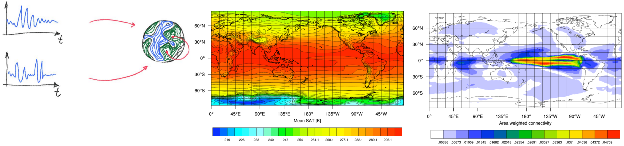
- **From dynamics to graph topology**: Method of flow-networks and correlation networks



- **From graph topology to dynamics**: Case study of the heterogeneous network dynamics

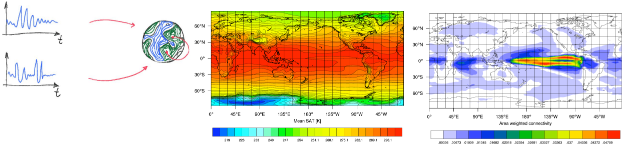
Research questions

1. How to study temperature variability in the dynamical system?



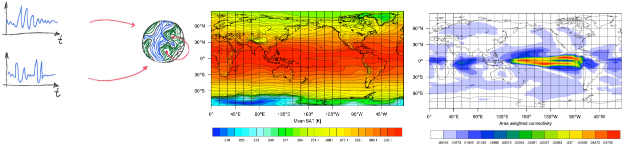
Research questions

1. How to study temperature variability in the dynamical system?
2. How to analyse time-dependent flow systems?



Research questions

1. How to study temperature variability in the dynamical system?
2. How to analyse time-dependent flow systems?
3. What is the meaning of the correlation network's patterns?

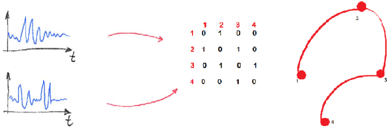


von Storch et al. (1999), Froyland et al.PRL (2007), Donges et al. EPL (2010), Tupikina et al. NPG (2014), Ser-Giacomi et al. PRE (2015)

Definition of a correlation network

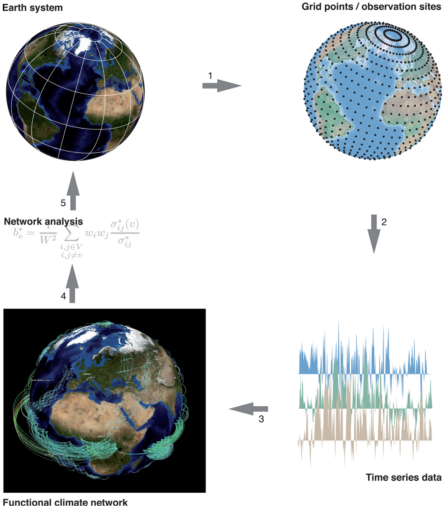
Definition

Correlation network of the set of time-series is a network, constructed from the adjacency matrix, which is the correlation matrix estimated from this set of time-series.



What do we learn about the system dynamics from topology of *correlation networks*?

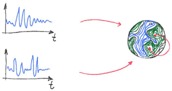
Correlation networks analysis



Idea of the flow-networks method

Definition

Given the dynamics, we define a **flow-network** as a correlation network, constructed from the set of time-series, generated from this dynamics.

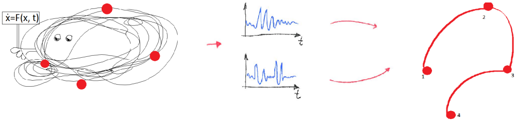


Idea: To characterize the underlying dynamics using, i.e. Pearson correlation coefficients $r_{xy} = \sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})/\sqrt{(\sum_{i=1}^{N}(x_i - \bar{x})^2)}\sqrt{(\sum_{i=1}^{N}(y_i - \bar{y})^2)}$

Idea of the flow-networks method

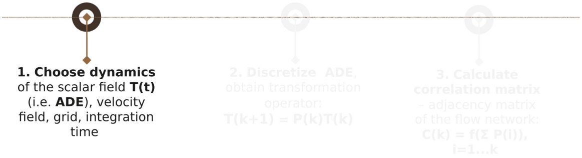
Definition

Given the dynamics, we define a **flow-network** as a correlation network, constructed from the set of time-series, generated from this dynamics.



Idea: To characterize the underlying dynamics using, i.e. Pearson correlation coefficients $r_{xy} = \sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})/\sqrt{(\sum_{i=1}^{N}(x_i - \bar{x})^2)}\sqrt{(\sum_{i=1}^{N}(y_i - \bar{y})^2)}$

Step 1 of flow-networks method: choose system

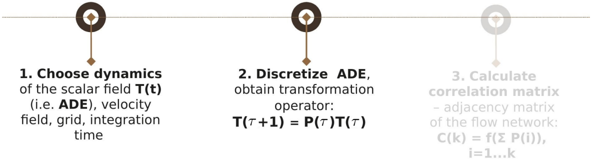


Consider **advection-diffusion equation** (ADE):

$$\frac{\partial T(\vec{x},t)}{\partial t} = \kappa \Delta T(\vec{x},t) - \vec{v}(\vec{x},t)\nabla T(\vec{x},t) + s\xi(\vec{x},t) - F(\vec{x}) - bT(\vec{x},t), \qquad (1)$$

where $\vec{v}(\vec{x},t)$ is time-dependent velocity field, where $\xi(\vec{x},t)$ is Gaussian white noise, external forcing $F(\vec{x})$, temperature decay with parameter b.

Step 2: estimate transformation operator for ADE



Discretize advection-diffusion equation (ADE) on a lattice with Δx for time step $\tau = t/\Delta t$, so that for each state-vector $\mathbf{T}(\tau)$:

$$\mathbf{T}(\tau + 1) = \mathbf{P}(\tau)\mathbf{T}(\tau), \tag{2}$$

then $\mathbf{P}(\tau)$ is the one-step **transformation operator**

Step 3. Construct flow-networks

1. **Choose dynamics** of the scalar field **T(t)** (i.e. **ADE**), velocity field, grid, integration time
2. **Discretize ADE**, obtain transformation operator: **T(τ+1) = P(τ)T(τ)**
3. **Calculate correlation matrix** – adjacency matrix of the flow network: **C(τ) = f(Σ P(i)), i=1...τ**

Calculate analytically **correlation matrix C**, where $\mathbf{C}_{mn} = \frac{Cov(T(\tau))_{mn}}{\sqrt{Cov(T(\tau))_{mm} Cov(T(\tau)_{nn})}}$ for covariance matrix $Cov(T(\tau))$, evaluated for all pairs of grid points m, n:

$$Cov(T(\tau)) \equiv \left\langle \left(T(\tau) - \langle T(\tau)\rangle_\xi\right)\left(T(\tau) - \langle T(k)\rangle_\xi\right)^T \right\rangle_\xi$$

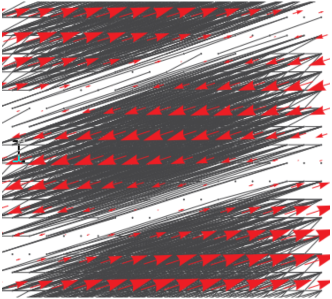
C is an adjacency matrix of a *flow-network* 

Applications of the flow-networks method

Degree centrality for node i:
$deg(i) = \sum_{j=1}^{l} e_{ij}$
for adjacent edges $e_{ij}, j = 1, ..., l$.

Clustering coefficient:
$clust(i) = \sum_{j,k=1}^{l} tr(ijk)/deg(i)$.
for triangles $tr(ijk)$.

Barthelemy Phys.Rep.(2010)



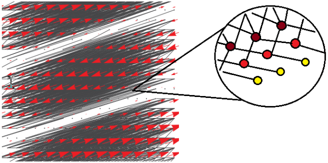
A node color is proportional to a network measure for that node, Molkenthin et al. Sci.Rep.(2014).

Applications of the flow-networks method

Degree centrality for node i:
$deg(i) = \sum_{j=1}^{l} e_{ij}$
for adjacent edges $e_{ij}, j = 1, ..., l$.

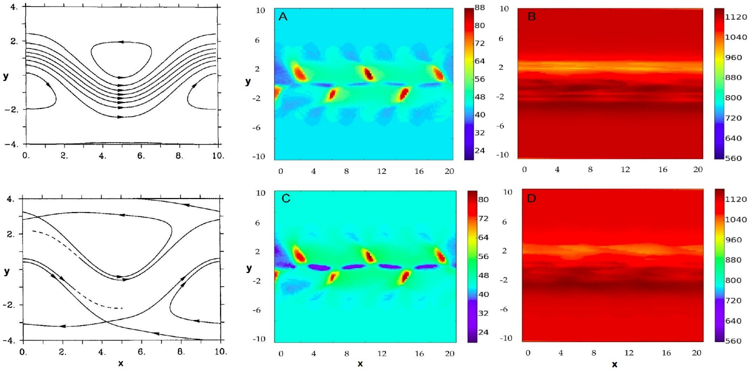
Clustering coefficient:
$clust(i) = \sum_{j,k=1}^{l} tr(ijk)/deg(i)$.
for triangles $tr(ijk)$.

Barthelemy Phys.Rep.(2010)



A node color is proportional to a network measure for that node, Molkenthin et al. Sci.Rep.(2014).

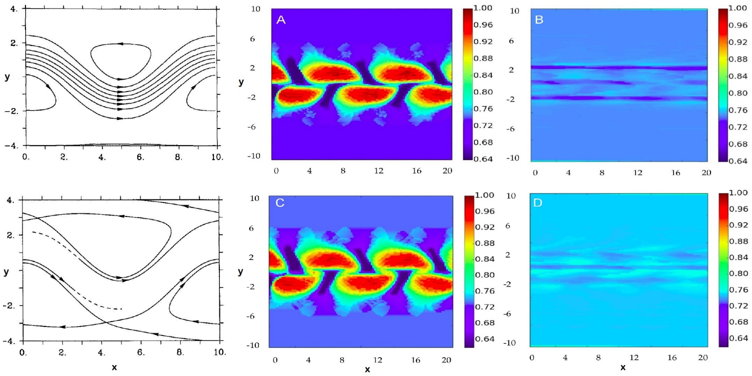
Numerical results of the flow-networks method



Degree measure of the flow-networks: for static, $\nu = 0$, (**A,B**) and fluctuating, $\nu = 0.7$, (**C,D**) flows. **A** and **C** are for the fast decay case $b = 1$, and **B** and **D** are for the slow decay, $b = 0.05$. $N \times N = 120 \times 120$ nodes, $\Delta x = 0.167$.

Tupikina, Molkenthin, Lopez, Marwan, Hernandez-Garcia, Kurths, Plos One (2016)

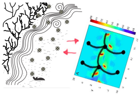
Numerical results



Clustering coefficient for the flow-networks: **A** and **B** are for the static flow, ($\nu = 0$). **C** and **D** are for the amplitude-changing case, ($\nu = 0.7$). **A** and **C** are for the fast decay case $b = 1$, and **B** and **D** are for the slow decay, $b = 0.05$.

Conclusions: correlation networks analysis

Flow-networks is a new framework to study system dynamics: stochastic forcing does not affect correlation matrix, damping parameter plays role of "memory" of the system.

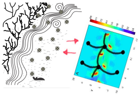


Molkenthin, Rehfeld, Stolbova, Tupikina, Marwan, Kurths, NPG (2014),

Tupikina, Molkenthin, Lopez, Marwan, Hernandez-Garcia, Kurths, Plos One (2016).

Conclusions: correlation networks analysis

Flow-networks is a new framework to study system dynamics: stochastic forcing does not affect correlation matrix, damping parameter plays role of "memory" of the system.



Molkenthin, Rehfeld, Stolbova, Tupikina, Marwan, Kurths, NPG (2014),

Tupikina, Molkenthin, Lopez, Marwan, Hernandez-Garcia, Kurths, Plos One (2016).

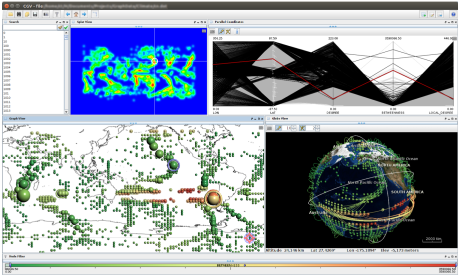
Geometrical patterns of flow-networks showed that degree and clustering fields specify flow regimes only when compared with each other. Regions with higher node-degree correspond not only to regions with higher velocity, as it was considered before, but also match with zones where the velocity-vector is static.

Tupikina, Molkenthin, Lopez, Marwan, Hernandez-Garcia, Kurths, Plos One (2016),

Molkenthin, Kutza, Tupikina, Donges, Marwan, Feudel, Kurths, Donner, Chaos (2016)

Software pyunicorn

How to analyse various types of data? How to combine both croase-graining methods and graph theoretical approach?



"Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package"
J.F. Donges, J. Heitzig, B. Beronov, M. Wiedermann, J. Runge, Q.-Y. Feng, L.Tupikina, V. Stolbova, R.V. Donner, N. Marwan, H.A. Dijkstra, and J. Kurths, Chaos 25, 113101 (2015)

Software pyunicorn

Releases | Tags

Latest release

v0.5.1

b9fea02

pyunicorn v0.5.1

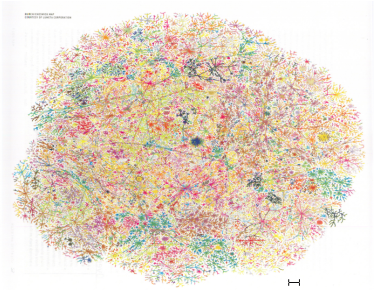
jdonges released this on 15 Nov 2015 · 227 commits to master since this release

Assets 3

- pyunicorn-0.5.1-compiled-cython.tar.gz — 3.46 MB
- Source code (zip)
- Source code (tar.gz)

Release of pyunicorn along with publication of description paper: Donges et al., Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package, Chaos 25, 113101 (2015), doi:10.1063/1.4934554, preprint: arxiv.org:1507.01571 [physics.data-an].

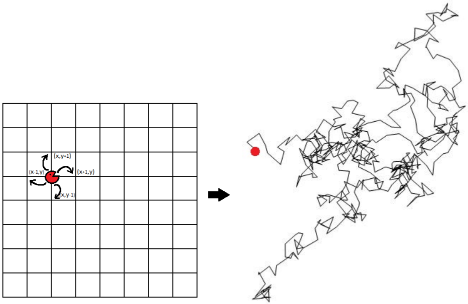
https://github.com/pik-copan/pyunicorn/releases



How a network topology is affecting dynamics on a network?

Models of random walks on heterogeneous networks "HOpS model of opinion spreading" L. Tupikina (2017) *https://arxiv.org/abs/1708.01647*
"Heterogeneous continuous time random walk model", D. Grebenkov, L. Tupikina, PhysRevE 97 012148 (2018)
https://journals.aps.org/pre/abstract/10.1103/PhysRevE.97.012148

Random Walk framework



Random walk (RW) on a discrete lattice in discrete time

Continuous time random walk (CTRW) in continuous space

R.Metzler, J.Klafter, Phys.Rep. (2000], M.Jaume et al., EPJ [2017]...

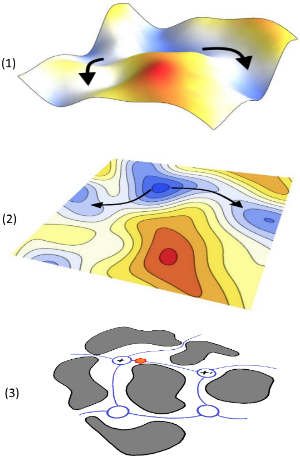
Heterogeneous Continuous Time Random Walk model

on a graph:

- graph with transition **matrix** Q,
- heterogeneous **travel time distributions** $\psi_{xx'}(t)$ between nodes x, x'.

The generalized transition matrix

$$Q_{xx'}(t) = Q_{xx'}\psi_{xx'}(t)$$



Heterogeneous Continuous Time Random Walk model

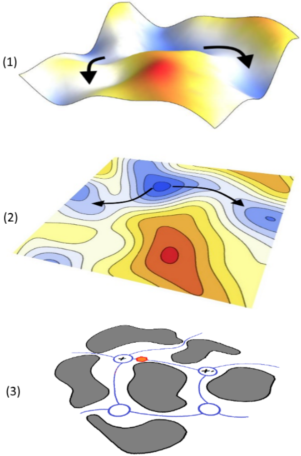
on a graph:

- graph with transition **matrix** Q,
- heterogeneous **travel time distributions** $\psi_{xx'}(t)$ between nodes x, x'.

The generalized transition matrix

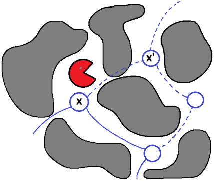
$$Q_{xx'}(t) = Q_{xx'}\psi_{xx'}(t)$$

- Grebenkov, Tupikina, "Heterogeneous continuous time random walks", PRE 97, 012148 (2018)



HCTRW is a formalism for studying diffusion in heterogeneous structures (1),(2).

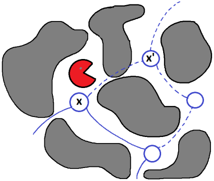
Analytical results for HCTRW model



HCTRW on a graph:

1. graph with transition matrix Q,
2. travel times $\psi_{xx'}(t)$ between nodes x, x'

Analytical results for HCTRW model



Analytic formula for HCTRW propagator $\tilde{P}_{x_0 x}(s)$:

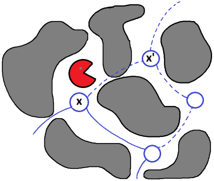
$$\boxed{\tilde{P}_{x_0 x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0 x}}, \qquad (3)$$

where $\tilde{Q}_{xx'}(s) = Q_{xx'} \tilde{\psi}_{xx'}(s)$.

HCTRW on a graph:

1. graph with transition matrix Q,
2. travel times $\psi_{xx'}(t)$ between nodes x, x'

Analytical results for HCTRW model



HCTRW on a graph:

1. graph with transition matrix Q,
2. travel times $\psi_{xx'}(t)$ between nodes x, x'

Analytic formula for HCTRW propagator $\tilde{P}_{x_0x}(s)$:

$$\boxed{\tilde{P}_{x_0x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0x}}, \qquad (3)$$

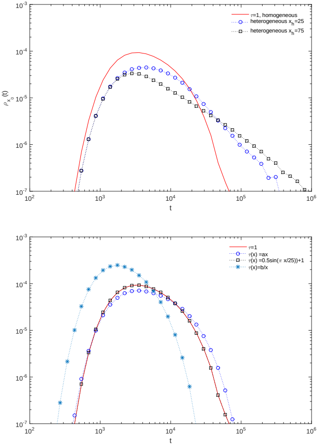
where $\tilde{Q}_{xx'}(s) = Q_{xx'} \tilde{\psi}_{xx'}(s)$.

Long-time behaviour for HCTRW propagator $P_{x_0x}(t)$:

$$\boxed{\tilde{P}_{x_0x}(s) \simeq \underbrace{p_{st}(x)}_{stat.dist.} + t(x) \sum_{k>0} \frac{u^{\dagger}_{0k}(x)\, v_{0k}(x_0)}{\lambda_{0k} + s\lambda_{1k}}} \qquad (4)$$

λ_{0k} are eigenvalues, u_{0k}, v_{0k} eigenvectors of $I - Q$.
$\lambda_{0k} + s\lambda_{1k}$ is the 1^{st} order correction for $I - Q + sT$,
$T_{xx'} = Q_{xx'} \langle \mathcal{T}_{xx'} \rangle$, $t(x) = \sum_{x'} T_{xx'}$.

Conclusions: HCTRW model



The FPT density of HCTRW on an interval, absorbing $x = 100$: (top) hetero-nodes $x_h = 25, 75$ with heavy-t.distr. $\alpha = 0.5$; (bottom) $\tau_{\pm} = 1$; $\tau_{xx\pm\mathbf{1}} = ax$; $\tau_{xx\pm\mathbf{1}} = 0.5\sin(\pi x/25) + 1$; $\tau_{xx\pm\mathbf{1}} = b/x$.

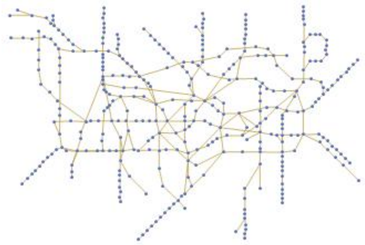
Analytic formula for HCTRW propagator $P_{x_{\mathbf{0}}x}(t)$ on a graph is derived.

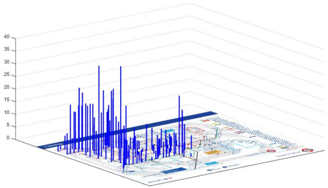
Analytic framework of HCTRW links structural graph properties and dynamical RW properties

HCTRW framework allows of study asymptotic solutions, FPT for processes on graphs.

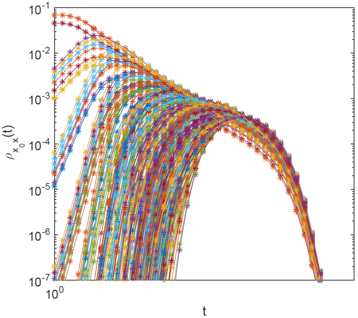
- Grebenkov, Tupikina, "Heterogeneous continuous time random walks", PRE 012148 (2018)

Applications: HCTRW model

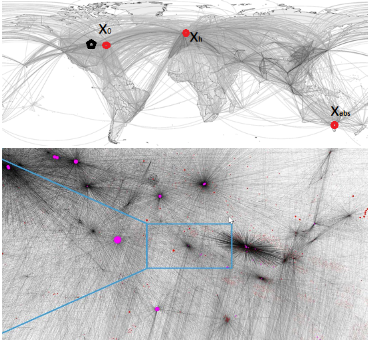




First passage time distributions $\rho_{x_0 x}(t)$ for a graph of London metro. Dependence on the initial node x_0.



Conclusions: HCTRW model



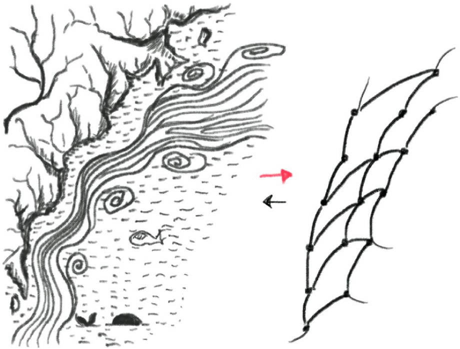
Analytic formula for HCTRW propagator $P_{x_0 x}(t)$ on a graph is derived.

Analytic framework for HCTRW links structural graph properties and dynamical RW properties

HCTRW framework allows of study asymptotic solutions, FPT for processes on graphs.

- Grebenkov, Tupikina, "Heterogeneous continuous time random walk model", PRE 97, 012148 (2018)
- Tupikina, Grebenkov "Heterogeneous continuous time random walks on networks", under rev. (2018)

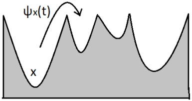
Take-home messages



- Techniques for analysis of correlation networks built from dynamical systems
- Behaviour of systems with time-varying underlying flows can be characterized using flow-networks measures
- Spatial and temporal heterogeneities of a network affect dynamics on it and can be characterized using HCTRW framework

Open questions

Connection between barrier models and HCTRW

1


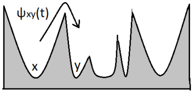
2


Illustration of HCTRW and connection to the trap (1) or barrier (2) models.

- Grebenkov, Tupikina, "Heterogeneous continuous time random walks", PRE 97, 012148 (2018)

Open questions

Connection between barrier models and HCTRW

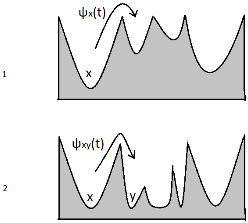


Illustration of HCTRW and connection to the trap (1) or barrier (2) models.
• Grebenkov, Tupikina, "Heterogeneous continuous time random walks", PRE 97, 012148 (2018)

How to derive continuous limits of HCTRW?

GME for the propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

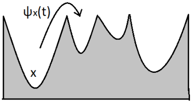
$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} Q_{\bar{x}x'}\left(sM_{x'}(s)\tilde{P}_{x_0x'}(s)\right) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s). \tag{5}$$

where $M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{1-\tilde{\mathcal{Q}}_{\bar{x}}(s)}$.

• Tupikina, Grebenkov "Continuous limits for Heterogeneous continuous time random walks", work in progr.

Open questions

Connection between barrier models and HCTRW

1 

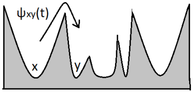
2 

Illustration of HCTRW and connection to the trap (1) or barrier (2) models.
• Grebenkov, Tupikina, "Heterogeneous continuous time random walks", PRE 97, 012148 (2018)

How to derive continuous limits of HCTRW?

GME for the propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_{\mathbf{0}}\bar{x}}(s) - P_{x_{\mathbf{0}}\bar{x}}(0) = \sum_{x'} Q_{\bar{x}x'}\left(sM_{x'}(s)\tilde{P}_{x_{\mathbf{0}}x'}(s)\right) - sM_{\bar{x}}(s)\tilde{P}_{x_{\mathbf{0}}\bar{x}}(s). \quad (5)$$

where $M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{\mathbf{1}-\tilde{\mathcal{Q}}_{\bar{x}}(s)}$.

• Tupikina, Grebenkov "Continuous limits for Heterogeneous continuous time random walks", work in progr.

HCTRW FPTD on different graphs

Main Collaborators

Nora Molkenthin,
Jonathan Donges
Cristóbal López,
Emilio Hernández-García,
Henk Dijkstra,
Norbert Marwan, Jobst Heitzig,
Jürgen Kurths, Petter Holme
Denis Grebenkov

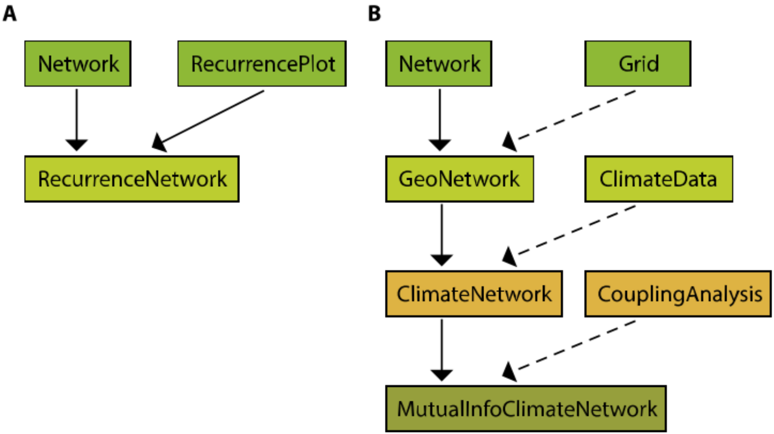
Some references

1. L. Tupikina, K. Rehfeld, N. Molkenthin, V. Stolbova, N. Marwan, and J. Kurths, "Characterizing the evolution of climate networks", Nonlin. Processes Geophys., 21, 705-711 (2014)
2. H. Bunina, L. Tupikina, "Authomorphisms of the semigroup of nonnegative invertible matrices of order 2 over rings", Journal of Mathematical Sciences, 183, 305-313 (2012)
3. N. Molkenthin, K. Rehfeld V. Stolbova L. Tupikina and J. Kurths, "On the influence of spatial sampling on climate networks", Nonlin. Processes Geophys., 21, 651-657 (2014)
4. J.F. Donges, J. Heitzig, B. Beronov, M. Wiedermann, J. Runge, Q.-Y. Feng, L. Tupikina, V. Stolbova, R.V. Donner, N. Marwan, H.A. Dijkstra, and J. Kurths, "Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package", Chaos 25, 113101-1-25 (2015)
5. L.Tupikina, N. Molkenthin, C.Lopez, E.Hernandez-Garca, N.Marwan and J.Kurths, "Correlation networks from flows. The case of forced and time-dependent advection-diffusion dynamics", Plos One, 0153703 (2016)
6. H.Kutza, N. Molkenthin, L. Tupikina, N.Marwan, J.Donges, U.Feudel, J. Kurths, R.Donner, "A geometric perspective on spatially embedded networks. Quantification of edge anisotropy and application to flow networks", Chaos 27, 035802 (2017)
7. **L.Tupikina**, "Dynamics on networks: case of Heterogeneous opinion state model (on graphs)", arxiv 1708.01647
8. P.Holme, L.Tupikina "Explicit solutions for SIS model" arxiv.org 1802.08849
9. R.Donner, M.Lindner, N.Molkenthin, **L.Tupikina** "From nonlinear dynamics to complex systems: A Mathematical modeling approach" Springer book chapter, Cham, pp. 197-226 (2019)
10. D.Grebenkov, **L.Tupikina**, "Heterogeneous continuous time random walk" Phys. Rev. E 97, 012148 (2018)
11. **L.Tupikina**, D.Grebenkov "Continuous limits of Heterogeneous continuous time random walk", (in prep.)
12. M. Kang, **L. Tupikina**, G. Berthelot, C. Nicolaides, D. Grebenkov, B. Sapoval "Universality of resistor scale-free networks" (in prep.)
13. M.Stella, M.Porter, L.Tupikina, C. Cramer, "Multilayer network between science and education", Proceedings of NetSci (2018)
14. F.Caravelli, T. Cui, **L.Tupikina** "A perspective on graph spectra", book chapter, lecture notes (online)

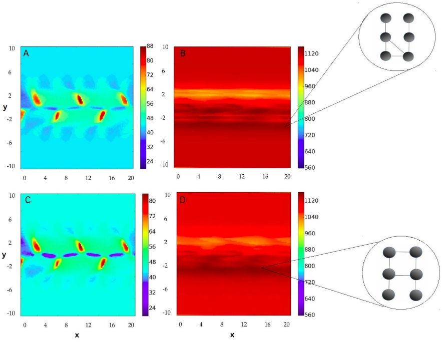
"Thank you for your attention"-slide

Contact: lyubov78@gmail.com, tupikina.liubov@polytechnique.edu
Programs and articles on random walks, flow networks:
https://liubovkmatematike.wordpress.com
https://networkscied.wordpress.com

Additional slides for I part


A
Network
RecurrencePlot
RecurrenceNetwork
B
Network
Grid
GeoNetwork
ClimateData
ClimateNetwork
CouplingAnalysis
MutualInfoClimateNetwork

Numerical results:
Geometrical patterns of flow-networks

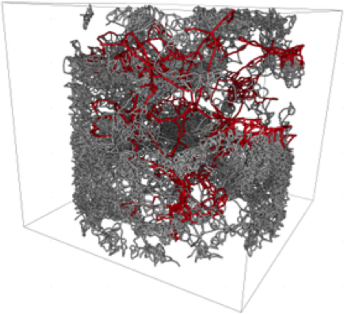


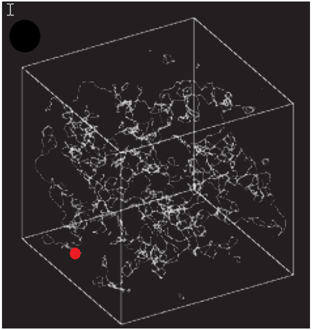
Degree of the flow-networks constructed for static (A,B) and shifting (C,D) flows.

Identical degree sequences do not necessarily lead to the same network patterns.

Tupikina et al. "Correlation networks from flows", Plos One (2016)

Real-world motivation





Approaches to **study diffusion in heterogeneous media**:
numerical simulations, coarse-graining, multiscaling reduction:
Levitz et al. 24202 EPJ [2012], Sahimi, 016316 PRE [2012]...

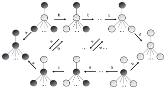
Take-home messages

Proposition (Flow-networks properties)

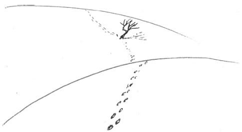
*Presented method of flow-networks provides analytic approach to study flow dynamics. Analytic formula for **correlation matrix** of flow-networks for time-dependent velocity field has been derived. From this formula follow that noise and forcing have no influence on correlation function.*

Proposition (Heterogeneous Opinion-State model on networks)

Analytical solutions for HOpS dynamics on star-networks have been obtained: For linear and star topologies HOpS dynamics is equivalent to random walk on a graph of a diagram of states.



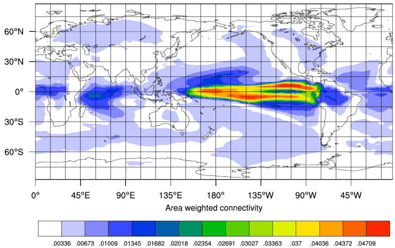
Outlook?

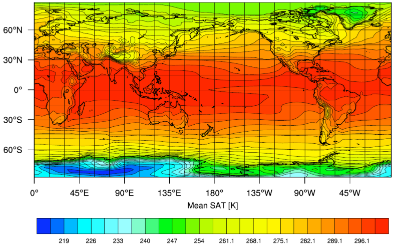


To study relation between *flow-networks* and Lagrangian networks, which should broaden connection between trajectories and densities description of the system.

To study analytic solutions for the HOpS model on arbitrary network topologies. Further applications of random walks theory for analysis of dynamics on networks.

Motivation





Plots for degree centrality constructed for correlation networks from temperature time-series.

Questions: Do spatial patterns in correlation network measures correspond to the strongest velocity flow?

How to verify correlation network measures analytically? Berezin et al. SciRep. (2012), TL et al. NPG (2014), Donges et al. Chaos (2016)

Where are the limits of flow-networks method?

Flow networks method may not be applicable when nonlinear measures of statistical dependence Deza et al.Chaos(2015), Quiroga et al.(2002), replace the correlation function, or when networks are constructed from variables from wave propagation (such as Kelvin or Rossby waves), such as sea surface height or geopotential, Arizmendi et al.NPG(2014).

Studying dynamics on networks for regular structures does not imply general analytic solution. However analytic solution for some irregular structures can be approximated using solution for regular structures.

Necessary conditions for the correlation calculation

I Eigenvalues of the transformation matrix $\mathbf{P}_k$ for $\forall k$ are smaller than 1.
Number of time-steps k:
$bk\Delta t >> 1.$

II Courant ***stability criteria***:

$$\kappa \Delta t / \Delta x^2 << 1,$$

$$max(v(\vec{x}, t))\Delta t << \Delta x.$$

Gardiner "Handbook of Stochastic Methods" (1997),

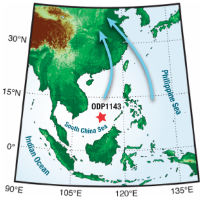
Toral, Colet "Stochastic Numerical Methods" (2014).

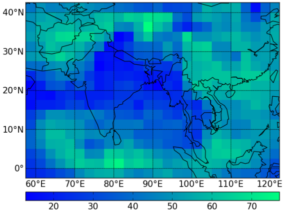
Method of flow-networks: threshold correlation matrix

Threshold estimated correlation matrix $\mathbf{C}(k)$ with threshold γ to get adjacency matrix $\mathbf{A}(k)$ of flow-network:

$$\begin{aligned} \mathbf{A}(k)_{ij} = 1 &: |\mathbf{C}(k)_{ij}| \geqslant \gamma \\ \mathbf{A}(k)_{ij} = 0 &: |\mathbf{C}(k)_{ij}| < \gamma \end{aligned} \tag{6}$$

Other types of functional networks





Correlation networks for Indian Peninsular temperature variability

LT, K. Rehfeld, N. Molkenthin, V. Stolbova, N. Marwan, and J. Kurths "Characterizing the evolution of climate networks", NPG, (2014) Additional slides for II part

CTRW framework

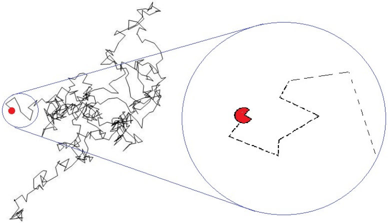
The Montroll-Weiss formula for the propagator $P_x(t)$ of continuous time random walk (CTRW)

$$\hat{\tilde{P}}_k(s) = \frac{1-\tilde{\psi}(s)}{s}\frac{1}{1-\tilde{\psi}(s)\hat{f}(k)} \qquad (7)$$

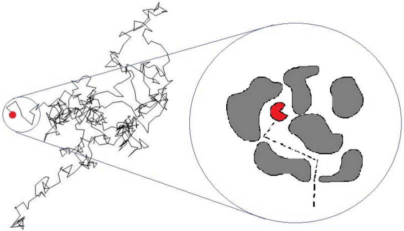
$\tilde{\psi}(s)$ is Laplace transform of PDF of the waiting time between steps,
$\hat{f}(k)$ is the characteristic function of jump distribution in Fourier space.

Montroll et al. [1965, 1969, 1973]

Can CTRW be heterogeneous?

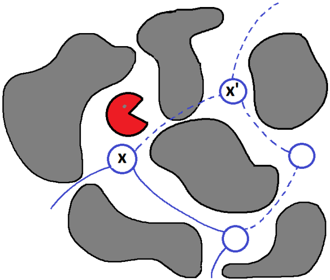


Can CTRW be heterogeneous?



We introduce **Heterogeneous Continuous Time Random Walk** (HCTRW) model on a graph

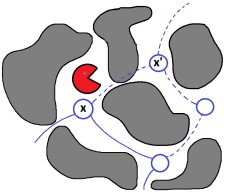
HCTRW model on a graph



Heterogeneous Continuous Time Random Walk (HCTRW) on a graph we define by:

1. graph with transition matrix Q,
2. travel times $\psi_{xx'}(t)$ between nodes x, x'

Result for HCTRW model on a graph



HCTRW model on a graph:
transition matrix Q,
travel times $\psi_{xx'}(t)$
between nodes x, x'

T., G. [in prep. 2017]

Analytic formula for the propagator for HCTRW $P_{x_0 x}(t)$:

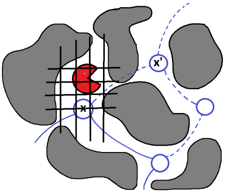
$$\boxed{\tilde{P}_{x_0 x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0 x}}, \qquad (8)$$

where the generalized transition matrix $\tilde{Q}_{xx'}(s) = Q_{xx'}\tilde{\psi}_{xx'}(s)$.

Compare (2) with Montroll-Weiss formula

$$\hat{\tilde{P}}_k(s) = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \tilde{\psi}(s)\hat{f}(k)} \qquad (9)$$

Result for HCTRW model on a graph



HCTRW model on a graph:
transition matrix Q,
travel times $\psi_{xx'}(t)$
between nodes x, x'

T., G. [in prep. 2017]

Analytic formula for the propagator for HCTRW $P_{x_0 x}(t)$:

$$\boxed{\tilde{P}_{x_0 x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0 x}}, \quad (8)$$

where the generalized transition matrix $\tilde{Q}_{xx'}(s) = Q_{xx'} \tilde{\psi}_{xx'}(s)$.

Compare (2) with Montroll-Weiss formula

$$\hat{\tilde{P}}_k(s) = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \tilde{\psi}(s) \hat{f}(k)} \quad (9)$$

Long-time behaviour for HCTRW

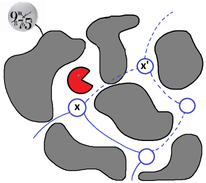
Long-time behaviour for **HCTRW**

Idea: For $s \to 0$ ($t \to \infty$) we use perturbation theory. Then for travel times $\tilde{\psi}_{xx'}(s)$ with finite moments $\langle \tau_{xx'} \rangle$:

$$\tilde{\psi}_{xx'}(s) = 1 - s\langle \tau_{xx'} \rangle + o(s). \tag{10}$$

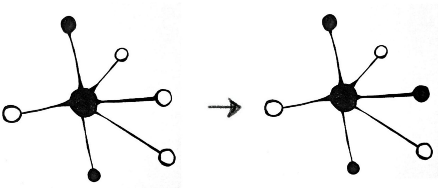
Then $I - \tilde{Q}(s) \to I - Q + sT$ ($s \to 0$), where matrix T:

$$T_{xx'} = Q_{xx'} \langle \tau_{xx'} \rangle. \tag{11}$$



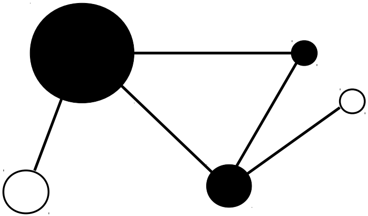
HCTRW has very heterogeneous clocks [T.,G. 2017] even more than for CTRW, J.Klafter, I.Sokolov, Oxford Press [2011])

II part:



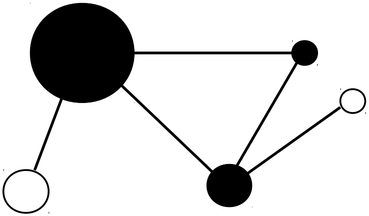
1. How simple *local* dynamical rule may create complex *global* dynamics?
2. Analytic solutions for dynamical network for graphs with tree-like topology

Heterogeneous Opinion Status Model Setup



We introduce new **H**eterogeneous **Op**inion **S**tatus Model on network (HOpS), representing dynamics with a simple stochastic rule on a heterogeneous network.

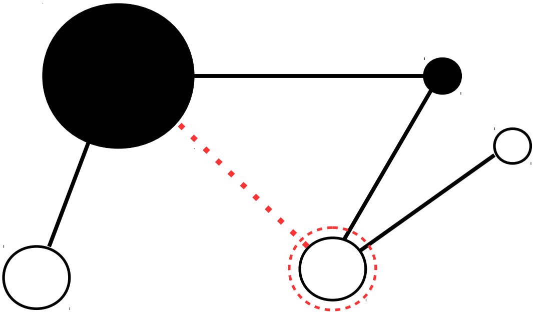
Heterogeneous Opinion Status Model Setup



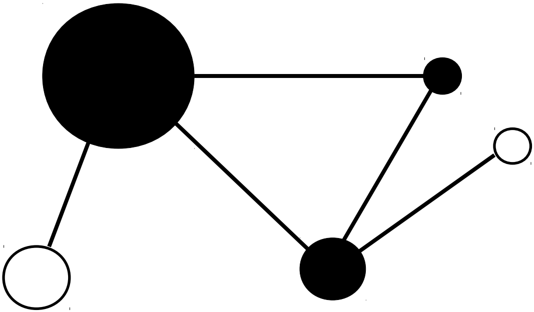
All nodes have two variables: opinion (black, white), status (node size on Figure). Status is fixed $st_i = const \in \mathbb{R}$.

Opinion $op_i \in \{0, 1\}$, black or white.

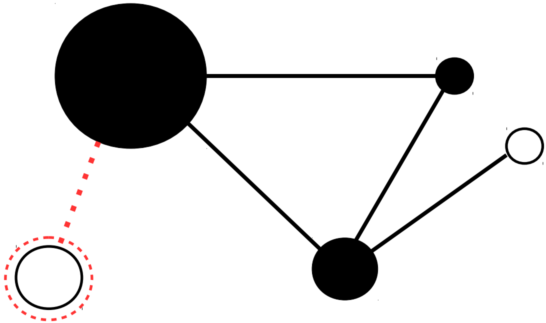
Heterogeneous Opinion Status Model on network



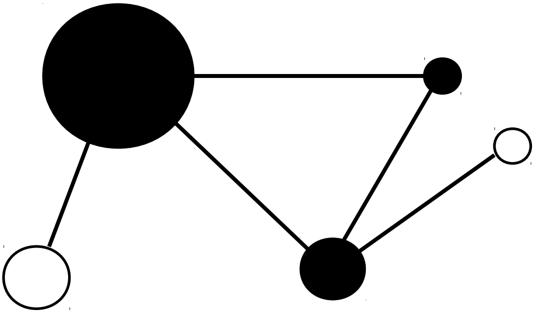
Heterogeneous Opinion Status Model time step



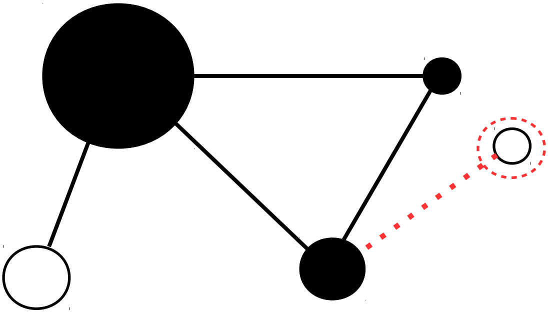
Heterogeneous Opinion Status Model time step



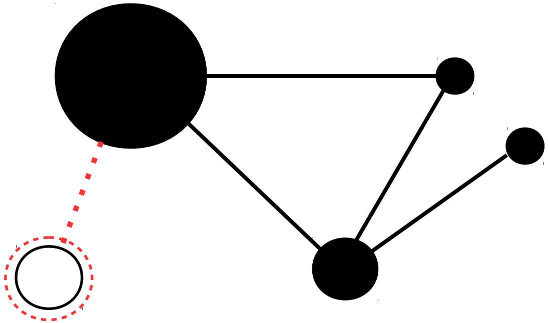
Heterogeneous Opinion Status Model time step



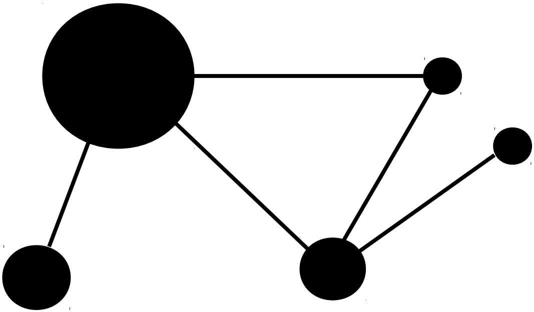
Heterogeneous Opinion Status Model time step



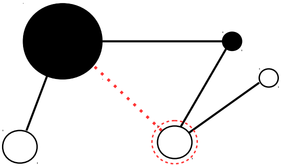
Heterogeneous Opinion Status Model time step



Heterogeneous Opinion Status Model time step



Heterogeneous Opinion Status Model time step

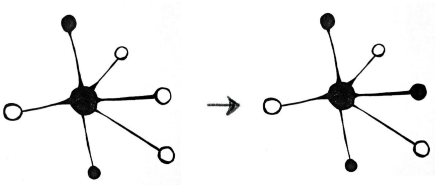


The time-step of the HOpS model:

1. **Choose active node i and its neighbor j are randomly;**

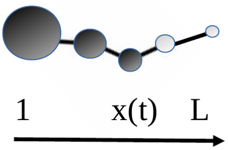
2. **Change opinion of an active node with probability $p = \tanh \sigma |st_j - st_i|$, i.e. depending on the status difference of nodes i and j.**

How to find *analytical solutions* for a Heterogeneous Opinion State model on a network with arbitrary topology?
Which network topologies to consider first?



Results: HOpS Model for symmetric tree-like networks

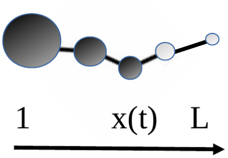
Results: HOpS Model on linear networks



Dynamics of the HOpS model on the line with decreasing node statuses $\{st_i\}, i \in [1, L]$ is a *bounded asymmetric random walk* $0 \leq x(t) \leq L$ with probabilities of right and left steps a, b.

$0 \leq a, b \leq 1$. $a = \tanh \sigma |st_i - st_{i+1}|$, $b = 1 - \tanh \sigma |st_i - st_{i+1}|$. Special initial condition: $x(t)$ left nodes are black, $L - x(t)$ nodes are white.

Results: HOpS Model



Dynamics of the HOpS model on linear network is equavalent to *bounded asymmetric random walk* $0 \leq x(t) \leq L$ with probability of RW to be at i-site at $t+1$ $P(x(t+1)=i)$:

$$\left\{ \begin{array}{c} 0 : |x(t+1)-x(t)| > 1 \\ a : x(t+1)-x(t) = 1 \\ b : x(t+1)-x(t) = -1 \\ 1-a-b : x(t+1)-x(t) = 0 \end{array} \right\} \quad (12)$$

HOpS Model on a linear network

Proposition

The HOpS model dynamics on a linear network *is described by solution of Gambler's ruin problem, which is a problem to find probability of a bounded random walker on* $[1, L]$ *to reach one of two absorbing borders, starting from position* x_0 *with probabilities* a, b *to walk right (left). An asymptotic solution for an asymmetric bounded random walk on a linear network is given by a probability:*

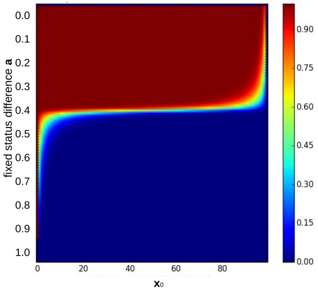
$p(x_0, a) = \frac{(a^{x_0}(1-a)^{L-x_0} - a^L)}{((1-a)^L - a^L)}$.

Quasi-stationary distribution is calculated using transition matrix of Markov chain with absorbing states:

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a & c & b & \dots & 0 \\ 0 & a & c & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (13)$$

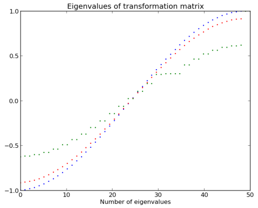
where $c = 1 - a - b$.

Numerical results for HOpS model.



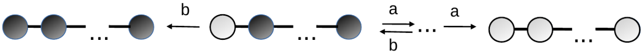
Simulations of the HOpS model dynamics on a linear network.

Horizontal axis is initial number of black nodes: $x_0 \in [0, 100]$; **vertical axis is transition probability** $a = \tanh \Delta_{st} + 0.5$. Colorbar corresponds to the probability $p(x_0, a)$ for the HOpS model to come to one certain equilibrium, when all nodes are black.



Spectrum of the transformation P **of the HOpS model dynamics on linear network for different parameter** a**, status distribution or probability of "domain wall" transition.** $a = 0.5$ (blue dots), $a = 0.7$ (red dots), $a = 0.9$ (green dots). $L = 50$.

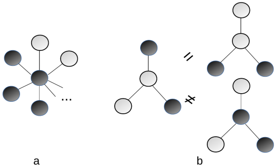
HOpS model dynamics on tree graphs



Discrete phase space of the HOpS model on linear network.

Starting from a special initial state: $L - x(0)$ nodes from the left border are black, and $x(0)$ nodes from the right border are white at time step $t = 0$, a phase space of the HOpS model can be structured in a linear way. Phase space of the HOpS model on a simple star with $k = 3$ leaves. The arrows between two model configurations represent the transition probabilities $a = \tanh \sigma |st_j - st_i|$ and $b = 1 - a$.

Results: HOpS model on star networks



We define **equivalence classes** of states (b), simplifying the phase space of the HOpS model on complex star (a).

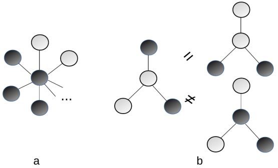
Results: HOpS model on star networks

Lemma (1)

A phase space of the HOpS model is associated with a digraph of model states, where each link is associated with transition between model states in discrete phase space.

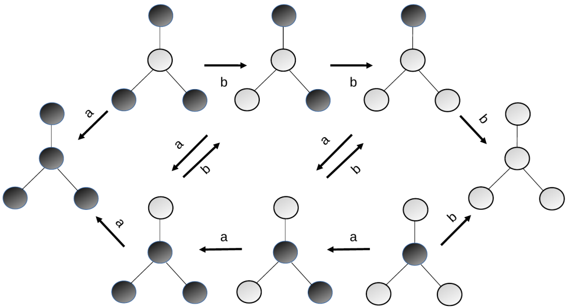
Lemma (2)

The set of (non-equal) states of the HOpS model on a finite underlying network is finite, any directed path must eventually enter a directed cycle, called a limit cycle (see graph of states as for SDS in [Mortveit, DS 2010]).



We define **equivalence classes** of states (b), simplifying the phase space of the HOpS model on complex star (a).

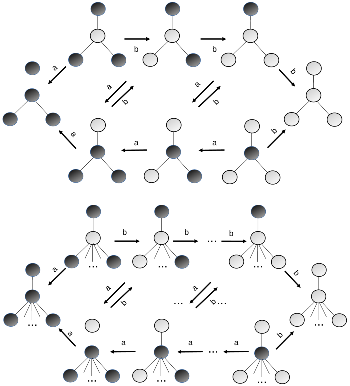
Results: graph of states of the HOpS model



Discrete phase space of the HOpS model on star network.

Phase space of the HOpS model on a simple star with $k = 3$ leaves. The arrows between model configurations represent transition probabilities equal a and b. This new representation of dynamical network model describes all possible model states.

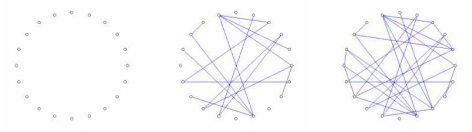
Results: HOpS model on star networks



Discrete phase spaces of the HOpS model on star networks.

Phase space of the HOpS model on a simple star with $k = 3$ (top) and arbitrary number of leaves (bottom). The arrows between two states of the HOpS model represent transition probabilities equal a and b.

Numerical results for random network models



$p = 0$
(a)

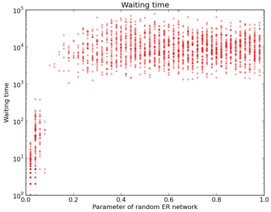
$p = 0.1$
(b)

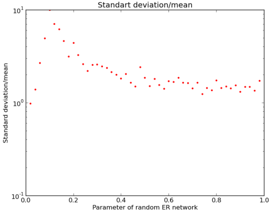
$p = 0.2$
(c)

Erdos-Renyi (ER) network $G(N, p)$:

p is probability to form an edge between two nodes, independent from every other edge. If $Np < 1$, then a graph in $G(N, p)$ will almost surely have no connected components of size larger than $O(log(N))$. $\frac{\ln N}{N}$ is a sharp threshold for the connectedness of $G(N, p)$.

Numerical results for random network models





Waiting time of the HOpS model to reach equilibrium on ER networks on $N = 90$. Each point is the HOpS model realization for ER network with parameter p (top); The variance/mean values of waiting time for each ER network parameter, *log-log* plot (bottom).

Conclusions: II part

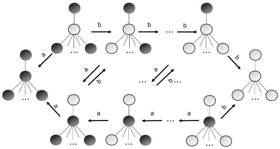
Proposition (HOpS model properties)

Dynamics of the Heterogeneous Status-Opinion Model on underlying network G can be classified depending on network topology and is characterized by a discrete phase space graph. We described two cases:

*1. Analytical solution for the **HOPS model on linear networks** is obtained from solution of Gambler's ruin problem.*

*2. Analytical solution for the **HOPS model on star-networks** is obtained from discrete phase space*

diagram and can be modeled coupled recurrence equations.



Results: Phase space of dynamical network models

Idea of graphical visualisation of phase space of **dynamical network models** is used also in **sequential dynamical systems** (SDS).

Def. SDS:

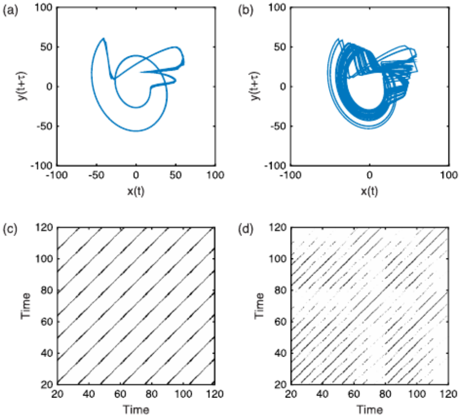
(a) a finite loop-free graph G with vertex set $\{1; \ldots N\}$ where each vertex has a binary state,

(b) a vertex labeled multi-set of functions $(F_{i;G} : F_n^2 \to F_n^2)_i$,

(c) a permutation $\pi \in S_n$. The function F_i updates the state of vertex i.

Mortveit Disc.Math.(2001), M.Macauley Cell.Aut.(2007), Kromer et al. PRE(2015)

Other types of networks



Recurrence networks N. Marwan, and J. Kurths, Chaos, (2015)