Insights from heterogeneous random walks: from theory to practice

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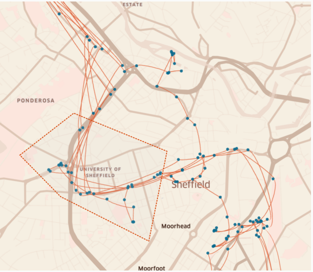
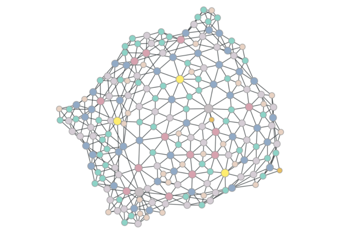
My background

What is the talk about?

Transport, random walks

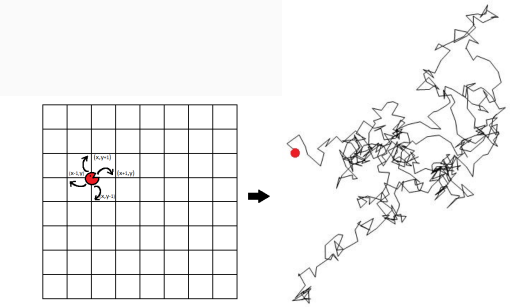


Heterogeneous networks, "real-world" examples



Random walks

Some theory: types of random walks

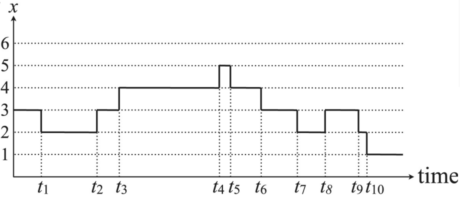


Random walk (RW) on a discrete lattice in discrete time

Continuous time random walk (CTRW) in continuous space

R.Metzler, J.Klafter, Phys.Rep. (2000), M.Jaume et al., EPJ (2017)...

a



b

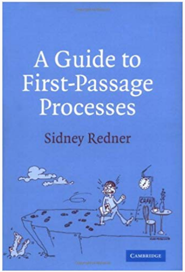
steps (n)	0	1	2	3	4	5	6	7	8	9	10
x	3	2	3	4	5	4	3	2	3	2	1

$$\begin{aligned}\hat{p}(k;s) &= \hat{p}(k;n)\hat{p}(n,s)\\ &= \frac{1-\hat{\psi}(s)}{s}\sum_{n=0}^{\infty}\hat{f}(k)^n\hat{\psi}(s)^n\\ &= \frac{1-\hat{\psi}(s)}{s}\,\frac{1}{1-\hat{f}(k)\hat{\psi}(s)}\,.\end{aligned}$$

E.W. Montroll and G.H. Weiss "Random walks on lattices" J. Math. Phys. 6 (1965) G. H. Weiss, "Aspects and Applications of the Random Walk" (1994) N. Masuda, M. Porter, R. Lambiotte "Random walks and diffusion on networks" Phys. Reports (2017)

How to estimate the first passage time for random walk?

How to estimate the first passage time for random walk?

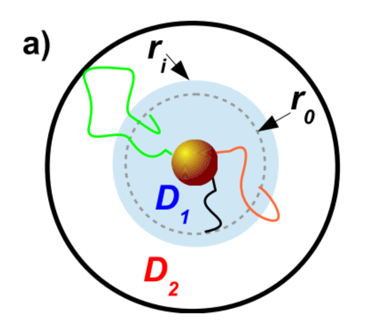


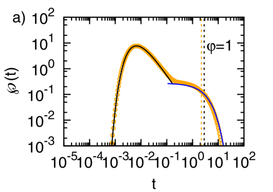
$$p_{ij}(n) = \delta_{n0}\delta_{ij} + \sum_{n'=0}^{n} F_{ij}(n')p_{jj}(n-n')$$

F_{ij} is FPT, $p_{ij}(n)$ is probability to arrive at j from i in n steps.

D. Holcman, Z. Schuss, SIAM Rev. (2014), D.Grebenkov et al., PRL (2018), R.Metzler, J.Klafter "The restaurant at the end of the random walk" (1986)

What are the main difficulties to estimate first passage time?



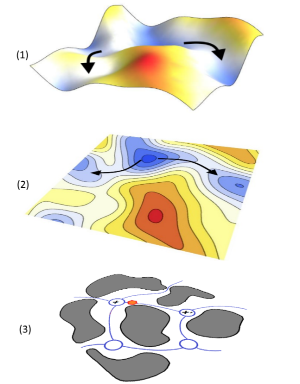


Question: What is the fist passage time (FPT) for complex domain? What is the FPT for various degrees of heterogeneity of media with ϕ and target radius x?

A. Godec et al. Nat.Pub. (2016), D.Grebenkov et al., PRL (2018)

Heterogeneous Continuous Time Random Walk Model

Model idea



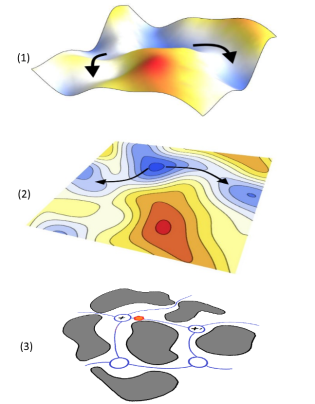
Heterogeneous Continuous Time Random Walk (**HCTRW**) model on a graph:

- graph with transition matrix Q,
- heterogeneous travel time distributions $\psi_{xx'}(t)$ between nodes x, x'.

The generalized transition matrix

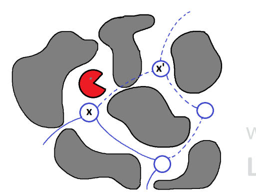
$$Q_{xx'}(t) = Q_{xx'}\psi_{xx'}(t)$$

Grebenkov, LT, PRE, 012148, 97 (2018)



Analytical results for HCTRW model

Analytic formula for HCTRW propagator $\tilde{P}_{x_0x}(s)$:



$$\boxed{\tilde{P}_{x_0x}(s) = \frac{1-\sum_{x'}\tilde{Q}(s)_{xx'}}{s}[(I-\tilde{Q}(s))^{-1}]_{x_0x}}, \qquad (1)$$

where $\tilde{Q}_{xx'}(s) = Q_{xx'}\tilde{\psi}_{xx'}(s)$.

Long-time behaviour for HCTRW propagator $P_{x_0x}(t)$:

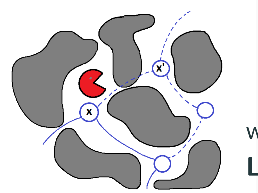
HCTRW on a graph:
transition matrix Q,
travel times $\psi_{xx'}(t)$

$$\boxed{\tilde{P}_{x_0x}(s) \simeq \underbrace{p_{st}(x)}_{stat.dist.} + t(x)\sum_{k>0}\frac{u^{\dagger}_{0k}(x)\,v_{0k}(x_0)}{\lambda_{0k}+s\lambda_{1k}}} \qquad (2)$$

λ_{0k} are eigenvalues, u_{0k}, v_{0k} eigenvectors of $I-Q$.
$\lambda_{0k}+s\lambda_{1k}$ is the 1^{st} order correction for $I-Q+sT$,
$T_{xx'} = Q_{xx'}\langle\mathcal{T}_{xx'}\rangle$, $t(x) = \sum_{x'} T_{xx'}$.

Analytical results for HCTRW model

Analytic formula for HCTRW propagator $P_{x_0 x}(s)$:



$$\boxed{\tilde{P}_{x_0 x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0 x}}, \qquad (1)$$

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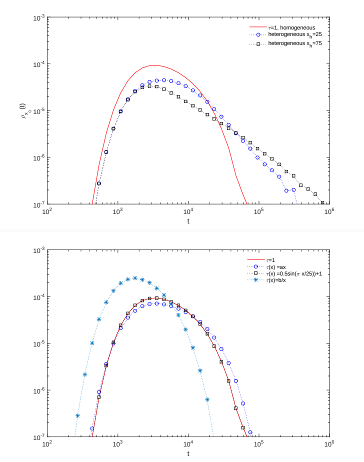
Results: I



The FPT density of HCTRW on an interval, absorbing $x = 100$: (top) hetero-nodes $x_h = 25, 75$ with heavy-t.distr. $\alpha = 0.5$; (bottom) $\tau_{\pm} = 1$; $\tau_{xx\pm\mathbf{1}} = ax$; $\tau_{xx\pm\mathbf{1}} = 0.5\sin(\pi x/25) + 1$; $\tau_{xx\pm\mathbf{1}} = b/x$.

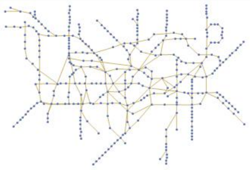
1. **Analytic formula** for HCTRW model propagator $P_{x_\mathbf{0}x}(t)$ on a graph for studying first passage times.
2. **From first passage time properties** we retrieve the effects of structural graph properties on random walk dynamics.
3. HCTRW framework allows of study asymptotic solutions.

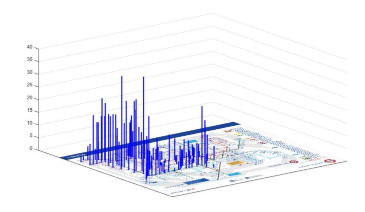
- Grebenkov, LT, PRE 012148 (2018) LT,

D.Grebenkov "Structural and temporal

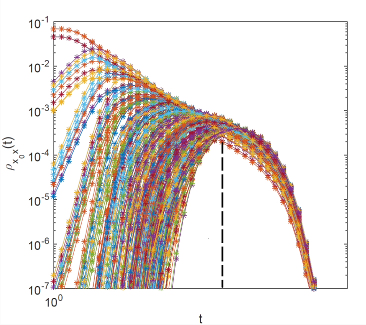
heterogeneities on networks", J.Appl.Net.(2019)

Results: II

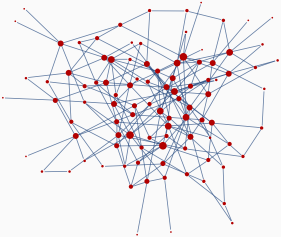


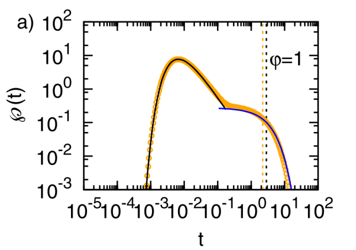


First passage time distributions $\rho_{x_0 x}(t)$ for various types of networks for different nodes x_0.



Applications of HCTRW to heterogeneous networks



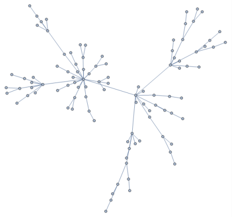


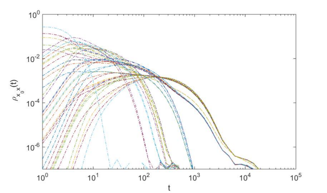
Question: What is the FPT for an arbitrary heterogeneous structure?
Solution: Given a network, heterogeneous nodes x_h, distributions of waiting times:

$$\tilde{\rho}_{x_0}(s) = \frac{\tilde{P}_{x_0 x_a}(s)}{\tilde{P}_{x_a x_a}(s)}. \tag{3}$$

LT, D.Grebenkov "Structural and temporal heterogeneities on networks", J.Appl.Net.(2019)

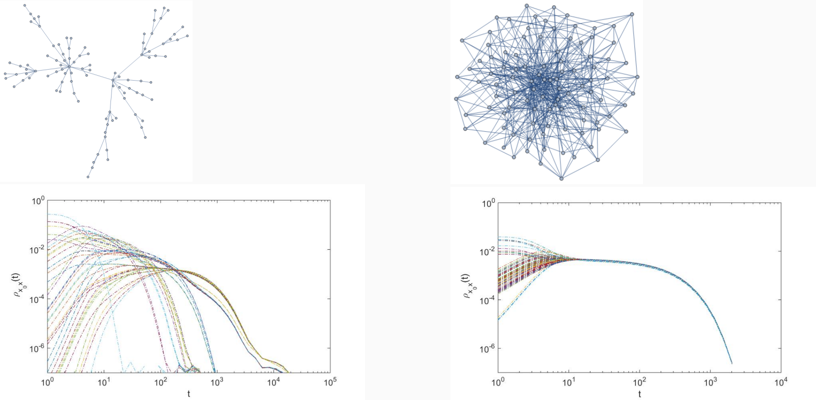
HCTRW on scale-free networks





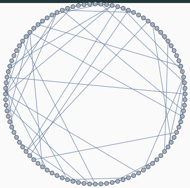
First passage time ρ_{x_0} on SF network $N = 100$, $m = 1$, $m_0 = 6$. Travel time distribution $\tilde{\psi}(s) = 1/(s^\alpha \tau^\alpha + 1), \alpha = 1, \tau = 1$, heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

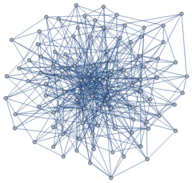
Results: HCTRW on scale-free networks

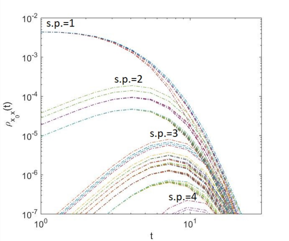


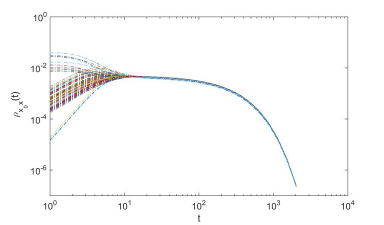
First passage time ρ_{x_0} on scale-free networks $G(N, m, m_0)$ tree (left), non-tree (right) $N = 100$. Travel time distribution $\tilde{\psi}(s) = 1/(s+1)$. Heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

Results: HCTRW and small-world effects



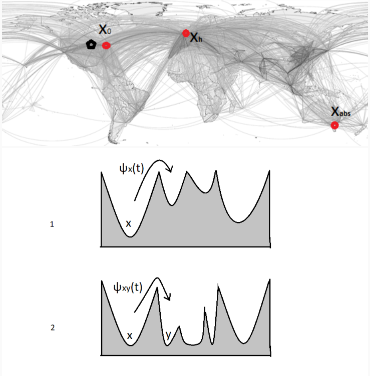






First passage time ρ_{x_0} on Watts-Strogatz (left) and SF model (right) networks $N = 100$. Travel time distribution $\tilde{\psi}(s) = 1/(s+1)$. Heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

Results: III



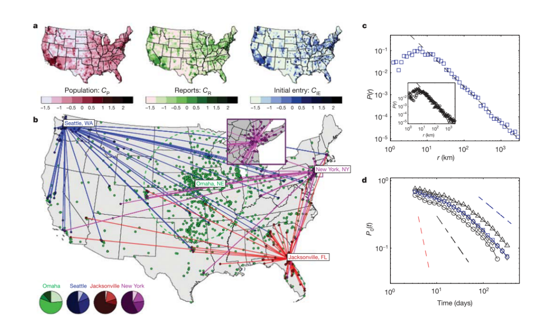
Analytic framework for HCTRW to model macroscopic behaviour.

First passage time distribution can be used as an alternative measure for dynamical properties of networks.

• LT, DG "Structural and temporal heterogeneities on networks" J.App.Net. (2019) • LT, DG "Continuous limits of heterogeneous continous time random walk" (in prep.)

Applications

Are random walks useful for mobility studies?

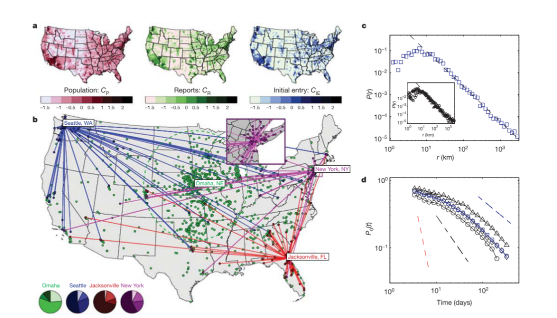


Are random walks useful for mobility studies?

1. Scaling of jump lengths for single-mode type is log-normal, for all transport types - truncated power-law
2. MSD (mean squared displacement) $MSD(t) \propto t^{\alpha}$.

Brockmann et al. "Scaling laws of human mobility"

Barbosa et al. "Human mobility: models and applications" [2017]

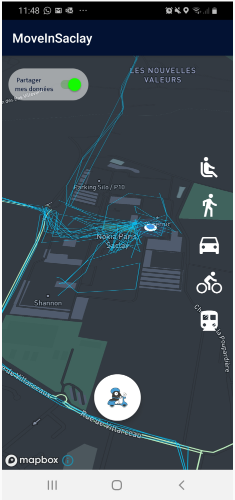


Data analysis of mobility

What about your own mobility?

How to analyze it, how to understand it better, how to use it better?

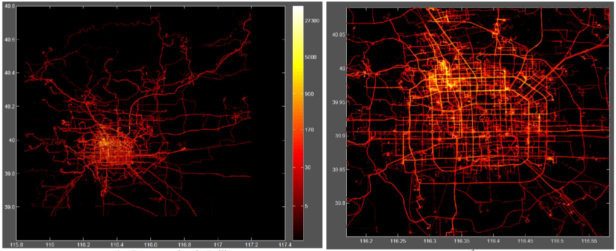
`www.moveinsaclay.app`



Data analysis of mobility

Some open data studies: Open data analysis (Geolife, 182 trajectories)

SS

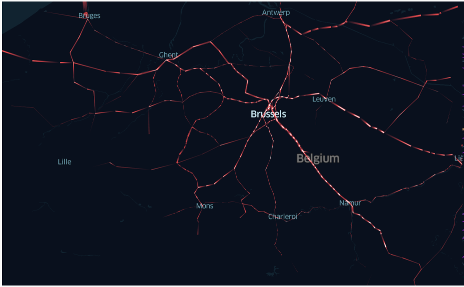


A) Data overview in Beijing
B) Within the 5th Ring Road of Beijing

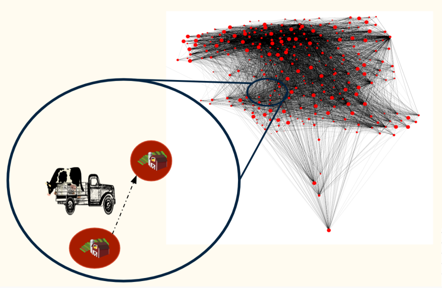
Ongoing projects on mobility

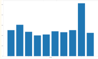
Analysis of train delays

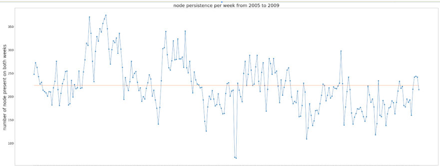
as perturbations spreading in networks, analysis of open data on bikes



Epidemiological processes on networks



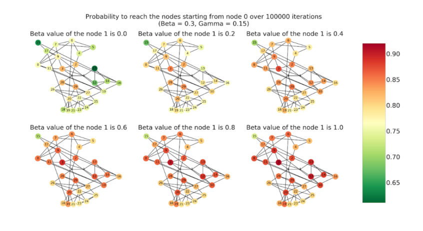


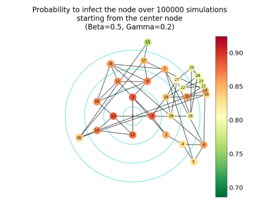


First passage time $\rho(t)$ can be used to estimate the time of infection to arrive at some farm.

Temporal network dataset of bovine moves (work in progress, INRA, Tokyo Tech)

Epidemiological model measures: applications of HCTRW



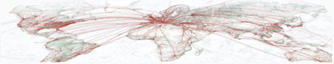


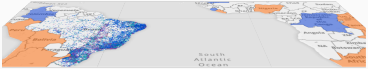
$$D_{ij}^{\mathrm{RW}}(\lambda) = -\ln\left(\sum_{k\neq j}\left(\mathbf{I}^{(j)} - e^{-\lambda}\mathbf{P}^{(j)}\right)^{-1}_{ik} e^{-\lambda} p_k^{(j)}\right). \tag{4}$$

Estimation of probability to reach the node (work in progress)

Open questions

- How to define heterogeneous network measures?
- What is suitable space-time-diffusivity for HCTRW?
- Applications to data and estimation of waiting times





Open question: Ito-Stratonovich dilemma

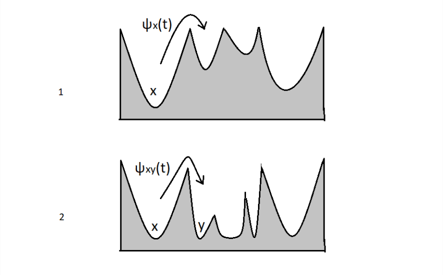
First, the proper physical meaning should be given for

$$\dot{x}(t) = f(x) + g(x, t)l(t), \tag{5}$$

where f and g are given functions, $l(t)$ is the rapidly fluctuating function [van Kampen, 1981].

Limiting processes are sometimes easier to consider and have some straightforward applications [Redner, Cambridge Uni Press (2001),] A.Chechkin, G.Gorenflo et al. "Fractional diffusion in inhomogeneous media" (2010), S.Nigris et al. "Onset of anomalous diffusion from local motion rules" PRE (2017) LT, DG "Continuous limits of heterogeneous continous time random walk" (in prep.)

Continuous limits of HCTRW dynamics



Three possible continuous dynamics of HCTRW.

We derived **the Generalized Master Equation** for the HCTRW propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

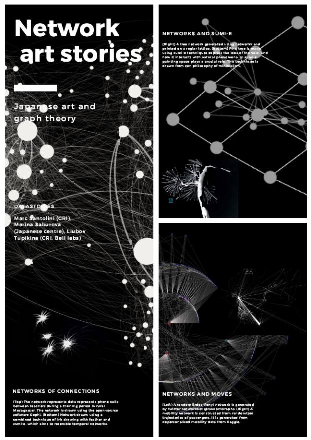
$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \Big(Q_{\bar{x}x'} sM_{x'}(s)\tilde{P}_{x_0x'}(s) \Big) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s). \quad (6)$$

Work in progress: Barrier models correspond to different conventions of diffusion equation • LT, DG "Continuous limits for HCTRW" (on arxiv soon)

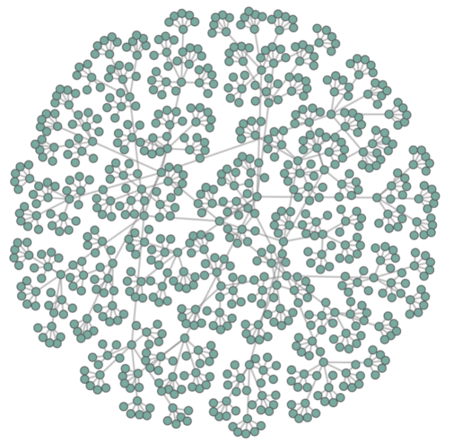
Real-world networks are much more complex...

lab.moovel.com

Network data visualisation



CEUdatastories



NetworkseminaratCRI

"Thank you for your attention"-slide

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Programs and articles on random walks, network models:
liubovkmatematike google sites and `https://github.com/Liyubov/heterogeneous-dynamics-on-networks`
`cri-paris.org` for project Welearn
`Moveinsaclay` for mobility application

Thanks to colleagues

P.Holme, F.Ianelli, N. Molkenthin,
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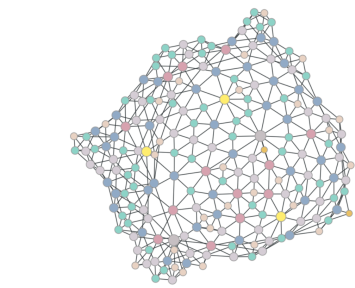
H. Dijkstra, J. Kurths,
J. Heitzig, N.Marwan,
A.Raygorodsky,
C. Masoller, G.Simon,
V.Bansaye, E.Vergu...

Take home message

- Heterogeneous random walks models allow to discover additional dynamical regimes.
- Discrete processes on networks (epidemics spreading etc.) can be described using these characteristics.
- First passage time characteristics capture more information about the process than the averaged quantities.



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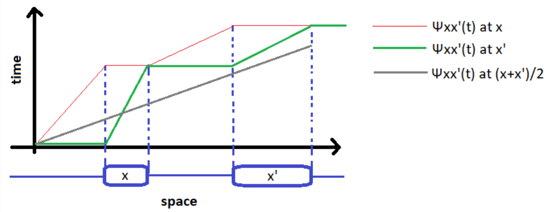
D.Grebenkov, **L.Tupikina**, "Heterogeneous continuous time random walk" Phys. Rev. E 97, 012148 (2018)
L.Tupikina, D.Grebenkov "Continuous limits of Heterogeneous continuous time random walk", (on arxiv soon.)
F.Caravelli, T. Cui, **LT** "A perspective on graph spectra", book chapter, lecture notes (online)

For additional information: `https://liubovkmatematike.wordpress.com`

Continuous limits of HCTRW dynamics



Conjecture:

There are three possible continuous dynamics of HCTRW. The general form of Fokker-Planck equation (FP) can be written as:

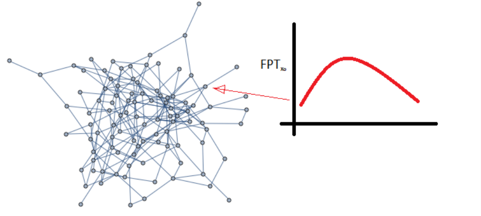
$$\frac{dP_{x_0x}(s)}{dt} = \frac{d}{dx}\left[-f(x)P_{x_0x}(s) + \alpha\left(\frac{d}{dx}D(x)\right)P_{x_0x}(s) + D\frac{dP_{x_0x}(s)}{dx}\right] \quad (7)$$

where different values of α correspond to different diffusion formalisms.

How we can apply this further?

Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.

Applications of HCTRW to networks

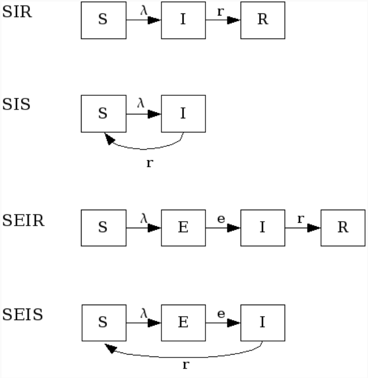


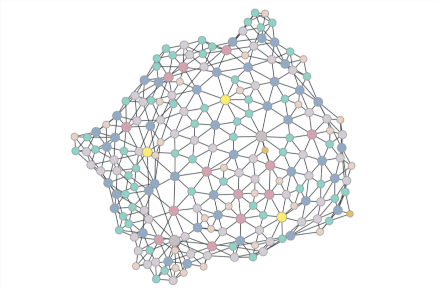
Solution: Given a network, heterogeneous nodes x_h, distributions of waiting times to characterize change of the first passage time:

$$\tilde{\rho}_{x_0}(s) = \frac{\tilde{P}_{x_0 x_a}(s)}{\tilde{P}_{x_a x_a}(s)}. \tag{8}$$

LT, D.Grebenkov "Structural and temporal heterogeneities on networks"

Epidemiological models





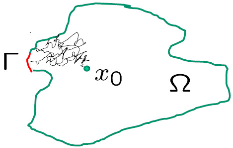
Pastor-Satorras, Castellano, Mieghem, Vespignani "Epidemic processes in complex networks", (2014).

What are the main difficulties?

$$\tau = \inf\{t > 0 \ : \ X_t \in \Gamma\}$$

$$S(x_0, t) = \mathbb{P}_{x_0}\{\tau > t\}$$

Survival probability



MFPT: $T(x_0) = \mathbb{E}_{x_0}\{\tau\}$

Mixed boundary value problem

$$D\Delta T(x_0) = -1$$

$$T(x_0) = 0 \quad \text{on target}$$

$$\frac{\partial}{\partial n} T(x_0) = 0 \quad \text{on the rest}$$

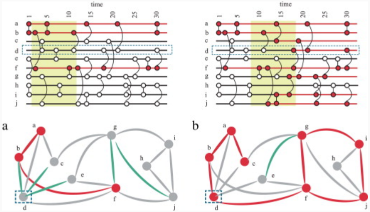
Global MFPT:

$$T = \frac{1}{|\Omega|} \int_{\Omega} dx_0 \, T(x_0)$$

D. Holcman, Z. Schuss, SIAM Rev. (2014),

D.Grebenkov et al., PRL (2018)

HCTRW on various types of networks



Question: How to map HCTRW to a random walk on a temporal [2,3] (stochastic) graph?

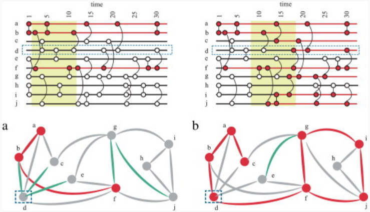
[2] P.Holme et al. "Temporal networks" (2012), [3] R.Lambiotte et al. arxiv 1305.0543 (2013)

Idea: Model on a temporal (stochastic) graph is the simplest representation of a road navigation system. Stationary distributions when $\psi_{xx'}(t)$ are non-finite mean distributions.

$\psi_{xx'}(t) = \tau_{xx'}(t) \prod_{y \neq x'} (1 - \int_0^t \tau_{yx}(t')dt')$,

for $\tau_{xx'}(t)$ - activation distribution of link xx'.

HCTRW on various types of networks



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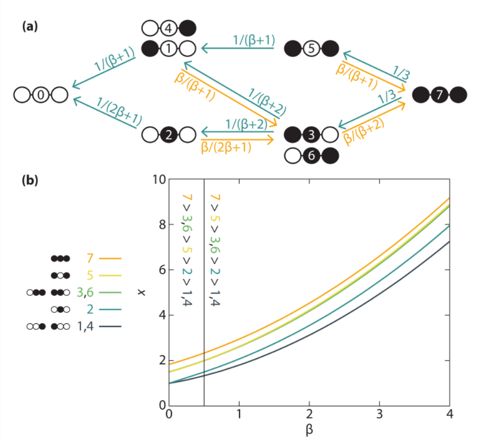
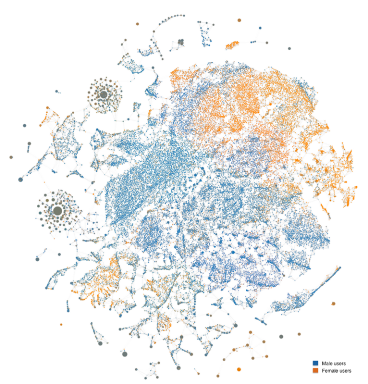
Epidemiological models: SIS on small graphs



Panel (a): the four equivalence classes of states of the SIS model at a triangle. The same idea is applied to larger networks using croase-graining. Panel (b): the expected extinction times x derived from (a) as a function of the infection rate β.

P.Holme, L.Tupikina "Explicit solutions for SIS model" arxiv.org 1802.08849

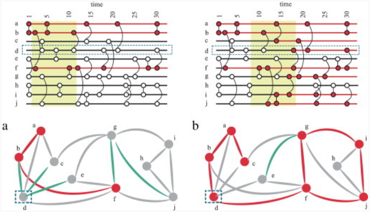
Applications of random walk theory



D. Brockmann, L. Hufnagel, T. Geisel "The scaling laws of human travel", 04292 Nature (2006)
F. Iannelli, M. Sebastian Mariani, I, M. Sokolov "Network centrality based on reaction-diffusion dynamics(...)", PRE (2018)

Real-world motivation

HCTRW on various types of networks



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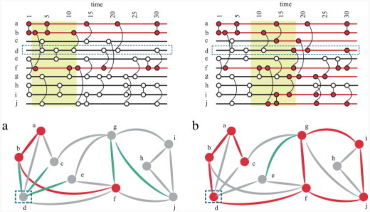
HCTRW on various types of networks



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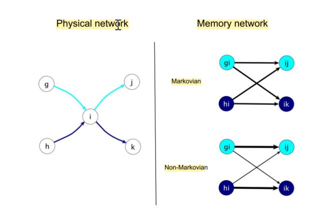


Memory network for nodes $((ij),(ik)\dots)$, $i\in(1,\dots N)$, with $p(ij\to jk)=\frac{W(ij\to jk)}{\sum_l W(ij\to jl)}$ M. Rosval, R.Lambiotte et al. Nat.Com.(2014).

Questions: How to model processes with memory on graphs?

Ideas: HCTRW propagator $P_{x_0x}(t)$ for estimation of first passage time properties on graph, MFPT as a network measure, using memory non-Markovian network as the underlying HCTRW transition matrix Q.

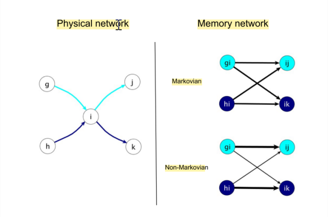
Dynamics evolution for memory network:

$P(i|t+1)=\sum_k P(ki|t+1)=\sum_{jk}P(jk|t)p(jk\to ki)$

Dynamics evolution for simple network:

$P(i|t+1)=\sum_j P(j|t)p(j\to i)$.

HCTRW on various types of networks



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Dynamics evolution for simple network:

$P(i|t+1) = \sum_j P(j|t) p(j \to i)$.

Continuous limits of HCTRW dynamics

Main references:

J. Jacod and A.S. Shiryaev **"Limit theorems for stochastic processes"**, 2nd Edition. Springer (2002)

V. Bansaye, M.-E. Caballero, S. Méléard, **"Scaling limits of general population processes"** hal-01702458 (2018)

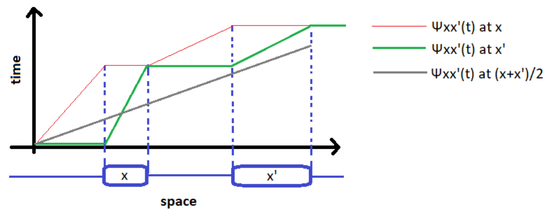
I.M.Sokolov **"Ito, Stratonovich(...)"** Chem.Phys.359-363 (2010)

B. H. Hughes, **"Random Walks and Random Environments"**, Clarendon Press, Oxford (1996)

Stratonovich, R.L. "Application of the Markov processes theory to optimal filtering", Radio Engineering and Electronic Physics (1960)

Filtration theorems

Continuous limits of HCTRW dynamics: Conjecture



There are three possible continuous dynamics of HCTRW:
when time of jump between nodes is defined in x from the distribution $\psi_{xx'}(t)$.
when time of jump between nodes is defined in x' after the jump
when time of jump between nodes is defined between x, x' during the jump

- Grebenkov, Tupikina, "Heterogeneous continuous time random walks",arXiv:1712.01020 (PRE to appear).
- Tupikina, Grebenkov "Continuous limits for Heterogeneous continuous time random walks", work in progr.

Continuous limits of HCTRW dynamics: Derivation

In Laplace space we express loss flux as

$$\tilde{j}^{-}_{\bar{x}}(s) = \tilde{\mathcal{Q}}_{\bar{x}}(s) P_{x_0\bar{x}}(0) + \tilde{\mathcal{Q}}_{\bar{x}}(s)(s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) + \tilde{j}^{-}_{\bar{x}}(s)) \quad (9)$$

where $P_{x_0\bar{x}}(0)$ are initial conditions, $\tilde{\mathcal{Q}}_{\bar{x}}(s) = \sum_{x'} Q_{\bar{x}x'}\tilde{\psi}_{\bar{x}x'}(s)$.
For convenience we introduce $M_{\bar{x}}(s)$ **function**

$$M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{1 - \tilde{\mathcal{Q}}_{\bar{x}}(s)}, \quad (10)$$

which is connected with the rate of steps. Note, kernel implicitly depends on adjacent sites of $\bar{x}$ (not like in CTRW limit).

Continuous limits of HCTRW dynamics: result

Then **the GME** for the propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \Big(Q_{\bar{x}x'} sM_{x'}(s)\tilde{P}_{x_0x'}(s) \Big) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s), \tag{11}$$

where $M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{1-\tilde{\mathcal{Q}}_{\bar{x}}(s)}$, $\tilde{\mathcal{Q}}_{\bar{x}}(s) = \sum_{x'} Q_{\bar{x}x'}\tilde{\psi}_{\bar{x}x'}(s)$.

Question:

What is the right continuous limit for Eq.(17) for various types of $\psi_{xx'}(t)$?

Example: For simple CTRW GME becomes Pauli master equation:

$$\dot{P}_{x_0\bar{x}}(t) = \frac{1}{\tau}[\sum_{x'} Q_{\bar{x}x'} P_{x_0x'}(t) - P_{x_0\bar{x}}(t)], \tag{12}$$

which then gives

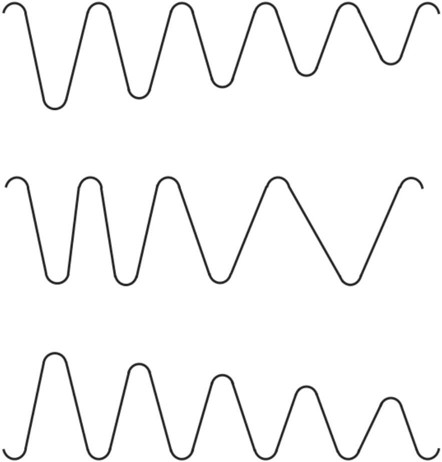


Fig. 1. Different situations corresponding to a position-dependent diffusion coefficient, which grows from left to right: The trap model, the accordion model and the barrier model, from top to bottom. These situations need for using the Ito, Stratonovich and Hänggi rules respectively when describing them within the Langevin scheme.

I.M.Sokolov **"Ito, Stratonovich(...)"** Chem.Phys.359-363 (2010)

Continuous limits of HCTRW dynamics: Analytical result

The **Generalized Master Equation** (GME) is based on two balance conditions:
(i) the local balance between the gain flux $j_{\bar{x}}^{+}(t)$ and loss flux $j_{\bar{x}}^{-}(t)$ from $\bar{x}$ site;
(ii) the balance for transitions (particle conservation or continuity)

Continuous limits of HCTRW dynamics: Analytical result

Balance equation

$$\frac{dP_{x_0\bar{x}}(t)}{dt} = j^{+}_{\bar{x}}(t) - j^{-}_{\bar{x}}(t), \tag{14}$$

is transformed to

$$j^{-}_{\bar{x}}(t) = \mathcal{Q}_{\bar{x}}(t)P_{x_0\bar{x}}(0) + \int_0^t \mathcal{Q}_{\bar{x}}(t-t')\Big(\frac{dP_{x_0\bar{x}}(t')}{dt} + j^{-}_{\bar{x}}(t')\Big)dt', \tag{15}$$

where $\mathcal{Q}_{\bar{x}}(t) = \sum_{x'} Q_{\bar{x}x'}\psi_{\bar{x}x'}(t)$.

Continuous limits of HCTRW dynamics: example

Idea:
Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.
I.e. for $\psi_{\bar{x}x'}(t) = \psi_{\bar{x}}(t)$

$$M_{\bar{x}}(s) = \frac{\mathcal{Q}_{\bar{x}}(s)}{1 - \mathcal{Q}_{\bar{x}}(s)} = \frac{\tilde{\psi}_{\bar{x}}(s) \sum_{x'} Q_{\bar{x}x'}}{1 - \tilde{\psi}_{\bar{x}}(s) \sum_{x'} Q_{\bar{x}x'}} = \frac{\tilde{\psi}_{\bar{x}}(s)}{1 - \tilde{\psi}_{\bar{x}}(s)}.$$

The exponential travel times with parameter $\tau_{\bar{x}}$: $\tilde{\psi}_{\bar{x}}(s) = \frac{1}{1+s\tau_{\bar{x}}}$, then

$$\frac{d}{dt} P_{x_0 x}(t) = \frac{d^2}{dx^2} D(x) P_{x_0 x}(t), \tag{16}$$

where $D(x) = \lim_{\bar{x}\to x} \frac{a^2}{2\tau_{\bar{x}}}$.

Continuous limits of HCTRW dynamics: result

We derived **the Generalized Master Equation** for the HCTRW propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \Big(Q_{\bar{x}x'} s M_{x'}(s) \tilde{P}_{x_0 x'}(s) \Big) - s M_{\bar{x}}(s) \tilde{P}_{x_0\bar{x}}(s). \quad (17)$$

How we can use it? Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.

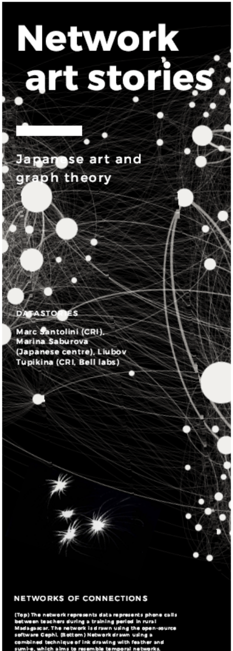
Epidemiological models: open questions

Epidemiological models on temporal networks;

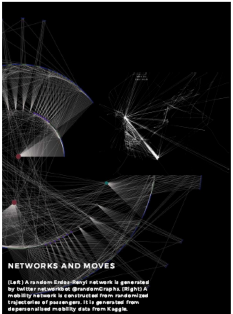
stable states of Ising models (theory of Ising models applied to node-state models);

epidemiological network measures: reproductive number etc.

P.Holme, J.Saramaki, Springer (2013)
S. Shlosman Comm. Math. Phys. Volume 102, Number 4 (1986)
P. Hoscheit, S. Geeraert, G.Beaunee, H. Monod, C. Gilligan, J. Filipe, E. Vergu, M. Moslonka-Lefebvre, J. Compl.Netw. 1-21 (2016)


Network
art stories
Japanese art and
graph theory
DATASTORIES
Marc Santolini (CRI), Marina Saburova (Japanese centre), Liubov Tupikina (CRI, Bell labs)
NETWORKS OF CONNECTIONS
(Top) The network represents data represents phone calls between teachers during a training period in rural Madagascar. The network is drawn using the open-source software Gephi. (Bottom) Network drawn using a combined technique of ink drawing with feather and sumi-e, which aims to resemble temporal networks.

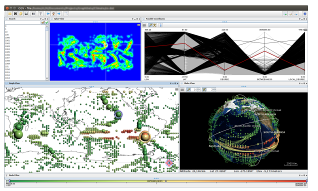

NETWORKS AND SUMI-E
(Right) A tree network generated using networkx and printed on a regiar lattice. (Bottom) Pine tree is made using sumi-e techniques express the idea of the void, and how it interacts with natural phenomena. In sumi-e painting space plays a crucial role. This technique is drawn from zen philosophy of minimalism.

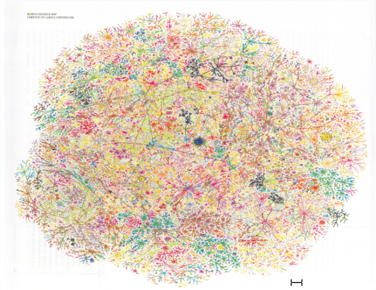

NETWORKS AND MOVES
(Left) A random Erdos-Renyi network is generated by twitter networkbot @randomGraphs. (Right) A mobility network is constructed from randomized trajectories of passengers. It is generated from depersonalised mobility data from Kaggle.

Software pyunicorn

How to analyse **various types of data using network approach?**
How to combine both croase-graining methods and graph theoretical approach?

"Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package"
J.F. Donges, J. Heitzig, B. Beronov, M. Wiedermann, J. Runge, Q.-Y. Feng, L.Tupikina, V. Stolbova, R.V. Donner, N. Marwan, H.A. Dijkstra, and J. Kurths, Chaos 25, 113101 (2015)





Models of random walks on heterogeneous networks "HOpS model of opinion spreading" L. Tupikina (2017) *https://arxiv.org/abs/1708.01647*
"Heterogeneous continuous time random walk model", D. Grebenkov, L. Tupikina, PhysRevE 97 012148 (2018)
https://journals.aps.org/pre/abstract/10.1103/PhysRevE.97.012148