Heterogeneous network models and their applications

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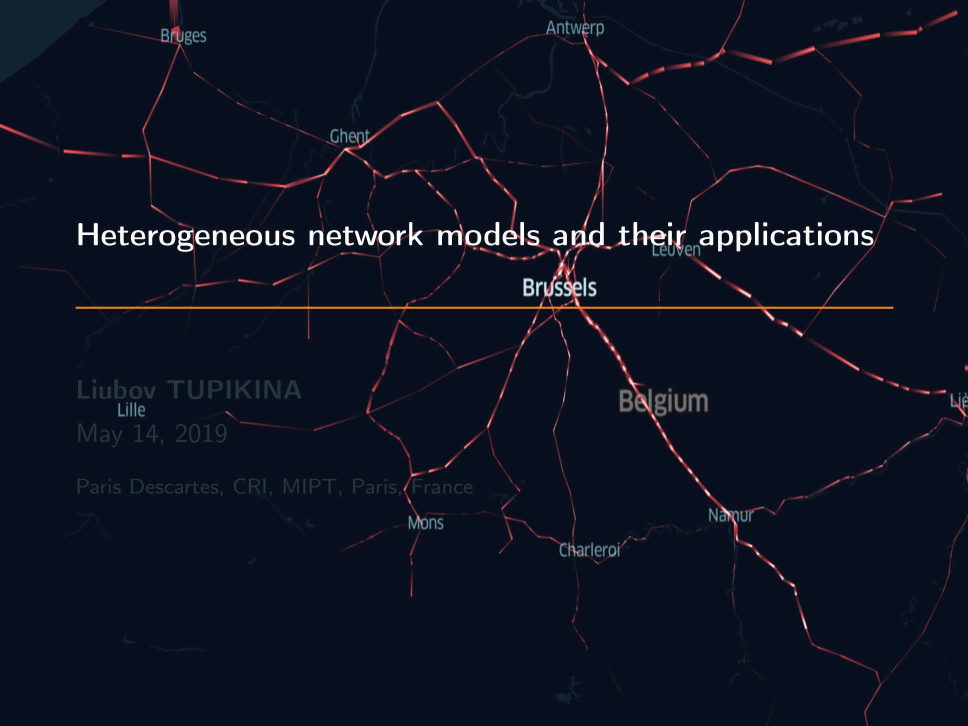
Paris Descartes, CRI, MIPT, Paris, France



My background

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Networks in Climate

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Balears, Spain, Cristina Masoller, Universitat Politècnica de Catalunya, Spain, Marcelo Ba
Universidad de la Republica, Montevideo, Uruguay

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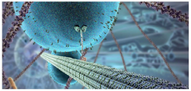
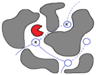


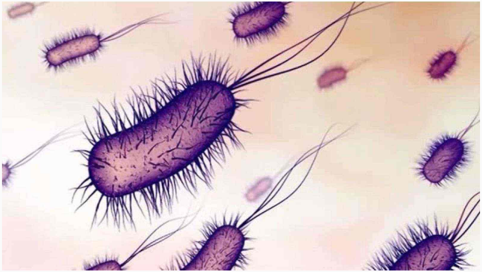
Transport of macromolecules, organelles and vesicles in living cells is a very complicated process that essentially determines and controls many biochemical reactions, growth and functioning of cells.

Stochastic processes and applications, random graph theory, diffusion on networks, spreading processes

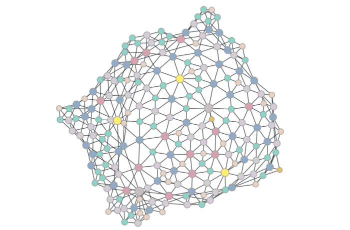


What is it about?

First passage time (FPT)



Heterogeneous networks

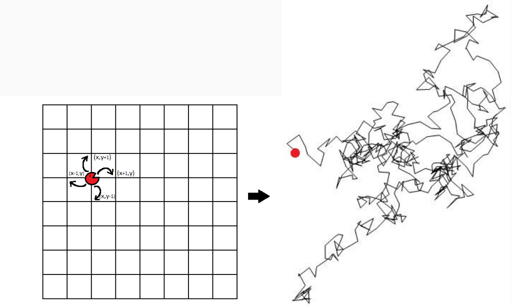


Random walks and first passage time

Random walk theory and other fields

F.Caravelli, T.Cui, LT, et al. "Spectral methods"

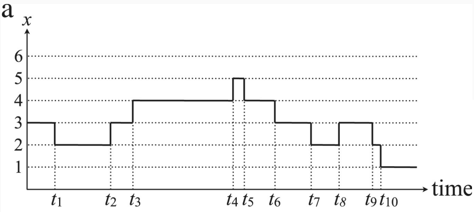
Types of Random Walks



Random walk (RW) on a discrete lattice in discrete time

Continuous time random walk (CTRW) in continuous space

R.Metzler, J.Klafter, Phys.Rep. (2000), M.Jaume et al., EPJ (2017)...



b

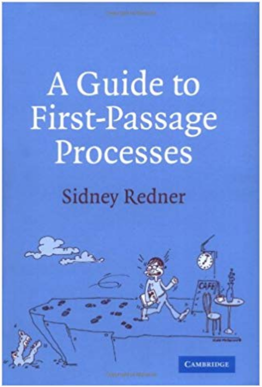
steps (n)	0	1	2	3	4	5	6	7	8	9	10
x	3	2	3	4	5	4	3	2	3	2	1

$$\begin{aligned}\hat{p}(k;s) &= \hat{p}(k;n)\hat{p}(n,s)\\ &= \frac{1-\hat{\psi}(s)}{s}\sum_{n=0}^{\infty}\hat{f}(k)^n\hat{\psi}(s)^n\\ &= \frac{1-\hat{\psi}(s)}{s}\,\frac{1}{1-\hat{f}(k)\hat{\psi}(s)}\,.\end{aligned}$$

E.W. Montroll and G.H. Weiss "Random walks on lattices" J. Math. Phys. 6 (1965) G. H. Weiss, "Aspects and Applications of the Random Walk" (1994) N. Masuda, M. Porter, R. Lambiotte "Random walks and diffusion on networks" Phys. Reports (2017)

How to estimate the first passage time for random walk?

How to estimate the first passage time for random walk?



$$p_{ij}(n) = \delta_{n0}\delta_{ij} + \sum_{n'=0}^{n} F_{ij}(n')p_{jj}(n-n'),$$

where F_{ij} is FPT, $p_{ij}(n)$ is probability to arrive at j from i in n steps.

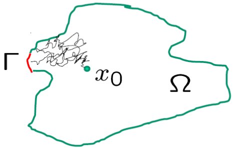
R.Metzler, J.Klafter "The restaurant at the end of the random walk" (1986)

What are the main difficulties?

$$\tau = \inf\{t > 0 \ : \ X_t \in \Gamma\}$$

$$S(x_0, t) = \mathbb{P}_{x_0}\{\tau > t\}$$

Survival probability



MFPT: $T(x_0) = \mathbb{E}_{x_0}\{\tau\}$

Mixed boundary value problem

$$D\Delta T(x_0) = -1$$

$$T(x_0) = 0 \quad \text{on target}$$

$$\frac{\partial}{\partial n} T(x_0) = 0 \quad \text{on the rest}$$

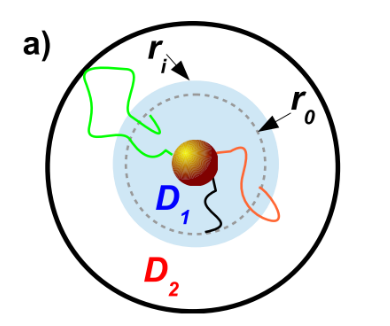
Global MFPT:

$$T = \frac{1}{|\Omega|} \int_{\Omega} dx_0 \, T(x_0)$$

D. Holcman, Z. Schuss, SIAM Rev. (2014),

D.Grebenkov et al., PRL (2018)

First passage time distribution

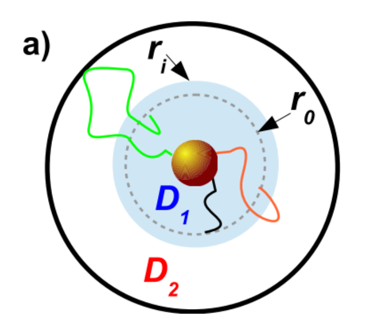


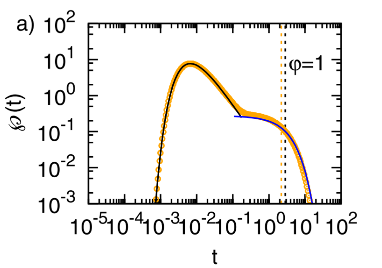
Question: What is the fist passage time (FPT) for complex domain?

What is the FPT for heterogeneous media with ϕ and target radius x?

A. Godec et al. Nat.Pub. (2016)

First passage time distribution



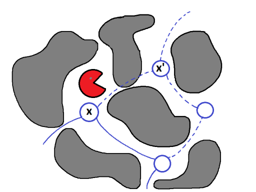


FPT densities for various degrees of heterogeneity ϕ and target radius x.

A. Godec et al. Nat.Pub. (2016)

Heterogeneous Continuous Time Random Walks

Model idea

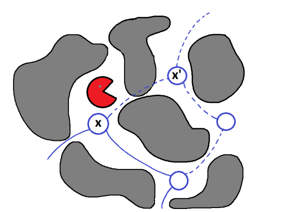


Heterogeneous Continuous Time Random Walk (HCTRW) model on a graph:

- graph with transition matrix Q,
- heterogeneous travel time distributions $\psi_{xx'}(t)$ between nodes x, x'.

The generalized transition matrix

$$Q_{xx'}(t) = Q_{xx'}\psi_{xx'}(t)$$



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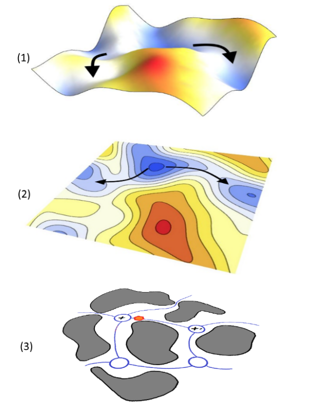
Grebenkov, LT, PRE, 012148, 97 (2018)



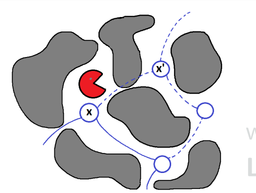
Analytical results for HCTRW model

Analytic formula for HCTRW propagator $\tilde{P}_{x_0 x}(s)$:

$$\boxed{\tilde{P}_{x_0 x}(s) = \frac{1 - \sum_{x'} \tilde{Q}(s)_{xx'}}{s} [(I - \tilde{Q}(s))^{-1}]_{x_0 x}}, \qquad (1)$$

where $\tilde{Q}_{xx'}(s) = Q_{xx'} \tilde{\psi}_{xx'}(s)$.

Long-time behaviour for HCTRW propagator $P_{x_0 x}(t)$:



HCTRW on a graph:
transition matrix Q,
travel times $\psi_{xx'}(t)$

$$\boxed{\tilde{P}_{x_0 x}(s) \simeq \underbrace{p_{st}(x)}_{stat.dist.} + t(x) \sum_{k>0} \frac{u^{\dagger}_{0k}(x)\, v_{0k}(x_0)}{\lambda_{0k} + s\lambda_{1k}}} \qquad (2)$$

λ_{0k} are eigenvalues, u_{0k}, v_{0k} eigenvectors of $I - Q$.
$\lambda_{0k} + s\lambda_{1k}$ is the 1^{st} order correction for $I - Q + sT$,
$T_{xx'} = Q_{xx'} \langle \mathcal{T}_{xx'} \rangle$, $t(x) = \sum_{x'} T_{xx'}$.

Analytical results for HCTRW model

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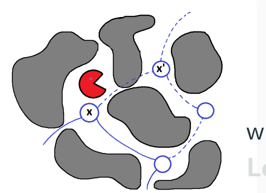

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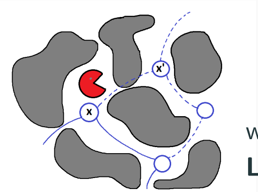

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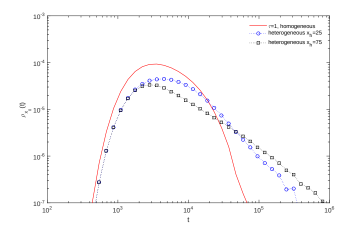
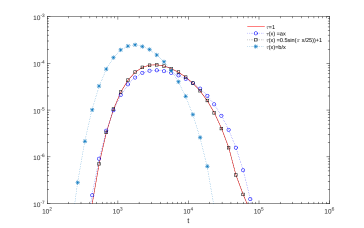
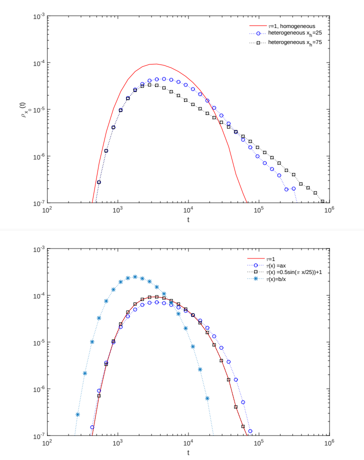
Results: I





The FPT density of HCTRW on an interval, absorbing $x = 100$: (top) hetero-nodes $x_h = 25, 75$ with heavy-t.distr. $\alpha = 0.5$; (bottom) $\tau_{\pm} = 1$; $\tau_{xx\pm\mathbf{1}} = ax$; $\tau_{xx\pm\mathbf{1}} = 0.5\sin(\pi x/25) + 1$; $\tau_{xx\pm\mathbf{1}} = b/x$.

Results: I



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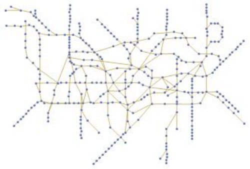
Analytic formula for HCTRW model propagator $P_{x_{\mathbf{0}}x}(t)$ on a graph allows to study first passage time properties.

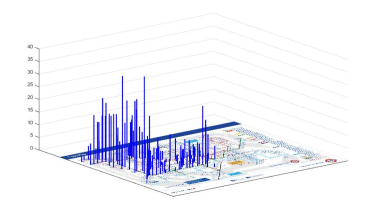
From first passage time properties we can retrieve the effects of structural graph properties on the dynamical random walk properties

HCTRW framework allows of study asymptotic solutions, also for first passage times on graphs.

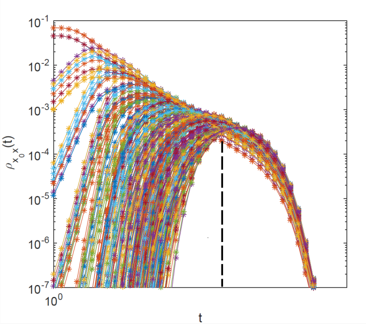
- Grebenkov, LT, PRE 012148 (2018)

Results: II

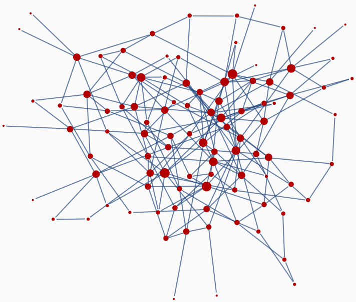


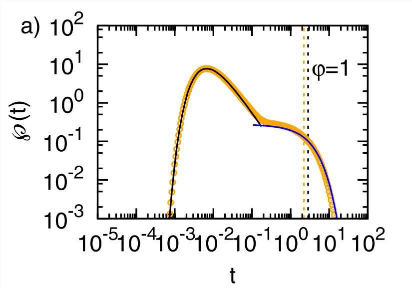


First passage time distributions $\rho_{x_0 x}(t)$ for various types of networks for different nodes x_0.



Applications of HCTRW to network models

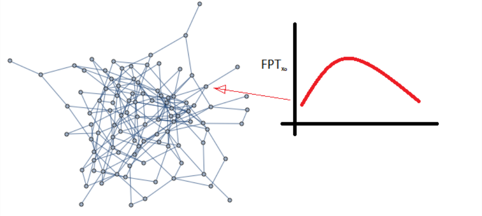




Question: What is the FPT for an arbitrary heterogeneous structure?

A. Godec, R.Metzler, "First passage time distribution beyond the mean first passage time " Nat.Pub. (2016) LT, D.Grebenkov "Structural and temporal heterogeneities on networks", J.Appl.Net.(2019)

Applications of HCTRW to network models

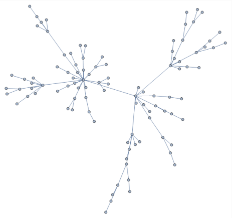


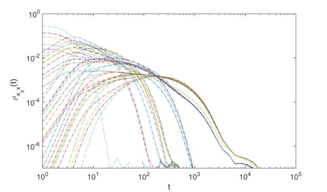
Idea: Given a network, heterogeneous nodes x_h, distributions of waiting times to characterize change of the first passage time:

$$\tilde{\rho}_{x_0}(s) = \frac{\tilde{P}_{x_0 x_a}(s)}{\tilde{P}_{x_a x_a}(s)}. \tag{3}$$

LT, D.Grebenkov "Structural and temporal heterogeneities on networks"

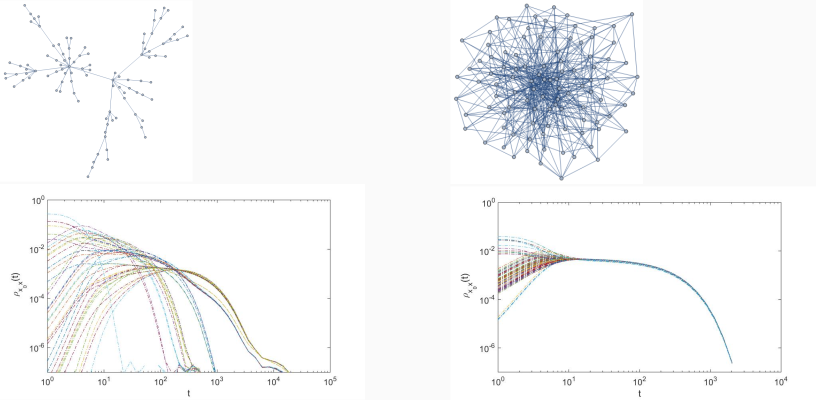
HCTRW on scale-free networks





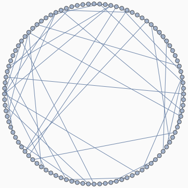
First passage time ρ_{x_0} on SF network $N = 100$, $m = 1$, $m_0 = 6$. Travel time distribution $\tilde{\psi}(s) = 1/(s^\alpha \tau^\alpha + 1), \alpha = 1, \tau = 1$, heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

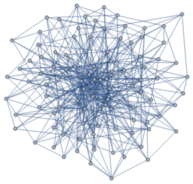
HCTRW on scale-free networks

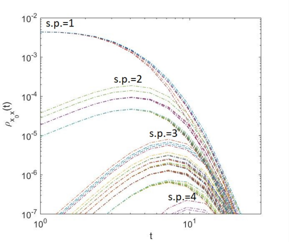


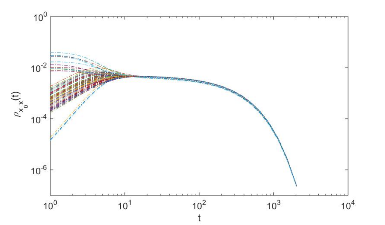
First passage time ρ_{x_0} on scale-free networks $G(N, m, m_0)$ tree (left), non-tree (right) $N = 100$. Travel time distribution $\tilde{\psi}(s) = 1/(s+1)$. Heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

First passage times and small-world effects



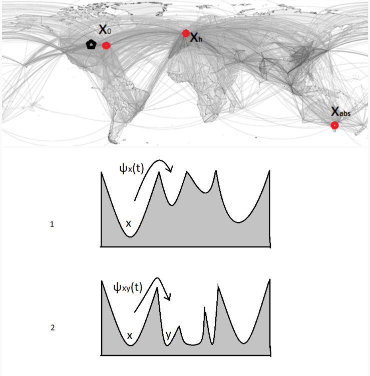






First passage time ρ_{x_0} on Watts-Strogatz (left) and SF model (right) networks $N = 100$. Travel time distribution $\tilde{\psi}(s) = 1/(s+1)$. Heterogeneous node has $\tilde{\psi}(s) = 1/(s^\alpha + 1)$ with $\alpha = 0.5$.

Results: III



Analytic framework for HCTRW to model macroscopic behaviour.

FPT distribution can be used as an alternative measure for dynamical properties of networks.

- LT, DG "Structural and temporal heterogeneities on networks" J.App.Net. (2019) • LT, DG "Continuous limits of heterogeneous continous time random walk" (in prep.)

How to describe the macroscopic behaviour based on the micropscopic dynamics?

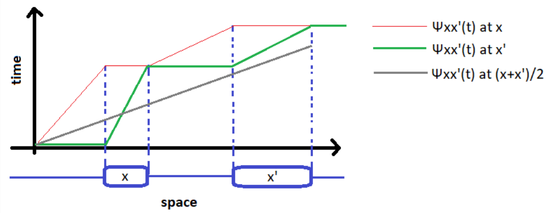
First, the proper physical meaning should be given for

$$\dot{x}(t) = f(x) + g(x,t)l(t), \tag{4}$$

where f and g are given functions, $l(t)$ is the rapidly fluctuating function [van Kampen, 1981].

Limiting processes are sometimes easier to consider and have some straightforward applications [Redner, Cambridge Uni Press (2001),] A.Chechkin, G.Gorenflo et al. "Fractional diffusion in inhomogeneous media" (2010), S.Nigris et al. "Onset of anomalous diffusion from local motion rules" PRE (2017) LT, DG "Continuous limits of heterogeneous continous time random walk" (in prep.)

Continuous limits of HCTRW dynamics



There are three possible continuous dynamics of HCTRW.
We derived **the Generalized Master Equation** for the HCTRW propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \left(Q_{\bar{x}x'} sM_{x'}(s)\tilde{P}_{x_0x'}(s) \right) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s). \quad (5)$$

- LT, DG "Continuous limits for Heterogeneous continuous time random walks" (on arxiv soon)

Relation between HCTRW and "barrier" models

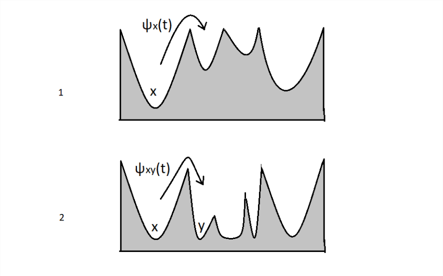
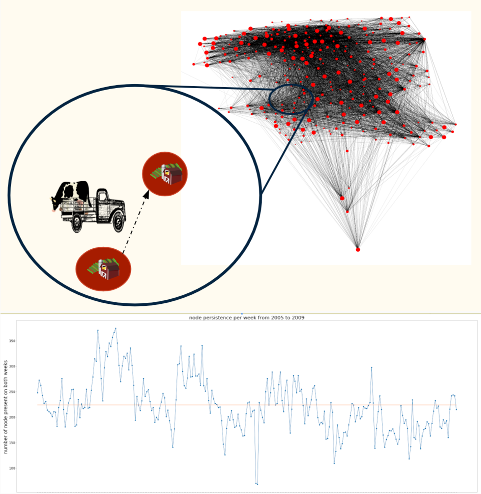


Illustration of HCTRW and connection to the trap (1) or barrier (2) models.

Idea: Different barrier models correspond to different conventions of diffusion equation, HCTRW travel times $\psi_{xx'}(t)$ are related to different types of barrier models, for which one can calculate continuous limit.

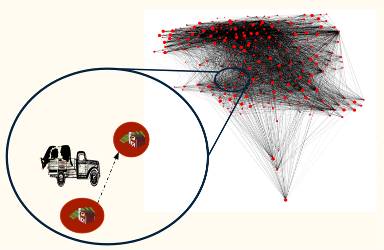
Applications

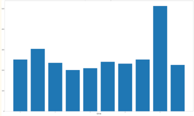
Epidemiological models and data analysis

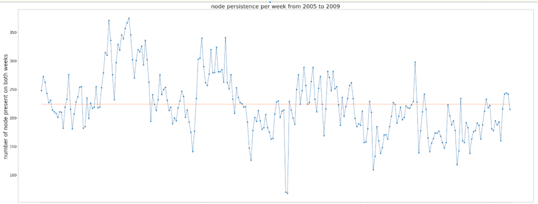


Temporal network dataset of bovine moves (work in progress, INRA, Tokyo Tech)

Epidemiological models and data analysis

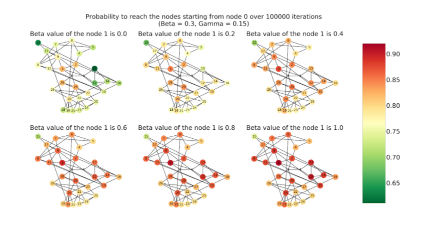


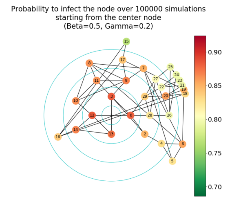




Availability of edges = number of Mondays during the year, when the link is available during Mondays (work in progress, INRA)

Epidemiological model measures: applications of HCTRW





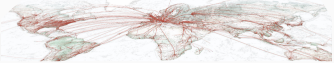
$$D_{ij}^{\mathrm{RW}}(\lambda) = -\ln\left(\sum_{k\neq j}\left(\mathbf{I}^{(j)} - e^{-\lambda}\mathbf{P}^{(j)}\right)^{-1}_{ik} e^{-\lambda} p_k^{(j)}\right). \qquad (6)$$

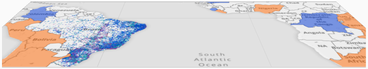
Estimation of probability to reach the node (work in progress)

Open questions

- How to define heterogeneous network measures?
- What is suitable space-time-diffusivity for HCTRW?
- Active search of data

Lecturers without borders network data analysis





Thanks to colleagues

P.Holme, F.Ianelli, N. Molkenthin,
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H. Dijkstra, J. Kurths,
J. Heitzig, N.Marwan,
A.Raygorodsky,
C. Masoller, G.Simon,
V.Bansaye, E.Vergu

"Thank you for your attention"-slide

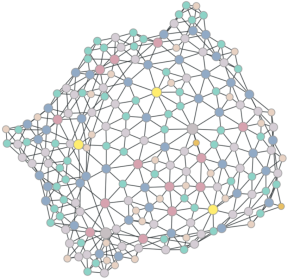
Contact: liubov.tupikina@cri-paris.org
Programs and articles on random walks, network models:
cri-pari.org/ - tupikina, https://scied.network

Real-world motivation

lab.moovel.com

Take home message

- Heterogeneous random walks models allow to discover additional dynamical regimes.
- Discrete processes on networks (epidemics spreading etc.) can be described using these characteristics.
- First passage time characteristics capture more information about the process than the averaged quantities.



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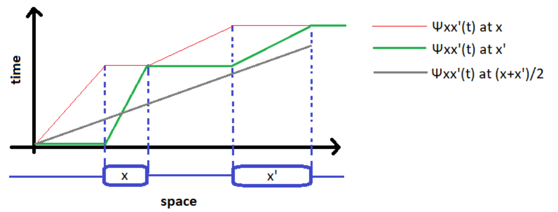
P.Holme, L.Tupikina "Explicit solutions for SIS model" arxiv.org 1802.08849
R.Donner, M.Lindner, N.Molkenthin, **L.Tupikina** "From nonlinear dynamics to complex systems: A Mathematical modeling approach" Springer book chapter, Cham, pp. 197-226 (2019)
D.Grebenkov, **L.Tupikina**, "Heterogeneous continuous time random walk" Phys. Rev. E 97, 012148 (2018)
L.Tupikina, D.Grebenkov "Continuous limits of Heterogeneous continuous time random walk", (on arxiv soon.)
F.Caravelli, T. Cui, **LT** "A perspective on graph spectra", book chapter, lecture notes (online)

For additional information: `https://liubovkmatematike.wordpress.com`

Continuous limits of HCTRW dynamics



Conjecture:

There are three possible continuous dynamics of HCTRW. The general form of Fokker-Planck equation (FP) can be written as:

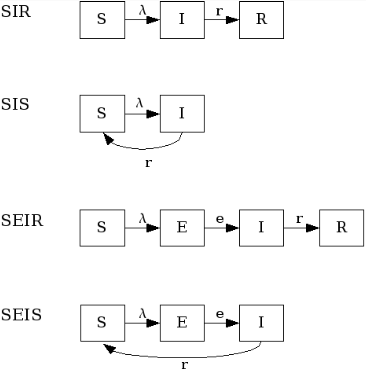
$$\frac{dP_{x_0 x}(s)}{dt} = \frac{d}{dx}\left[-f(x)P_{x_0 x}(s) + \alpha\left(\frac{d}{dx}D(x)\right)P_{x_0 x}(s) + D\frac{dP_{x_0 x}(s)}{dx}\right] \quad (7)$$

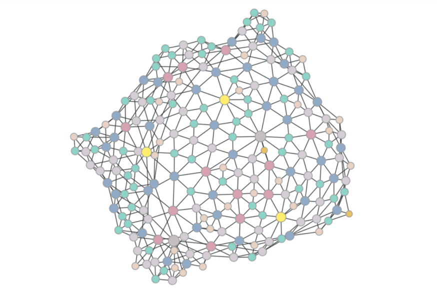
where different values of α correspond to different diffusion formalisms.

How we can apply this further?

Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.

Epidemiological models





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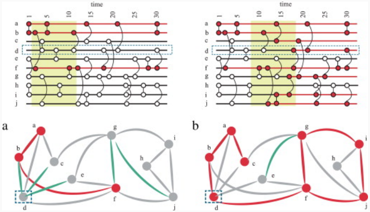
HCTRW on various types of networks



Question: How to map HCTRW to a random walk on a temporal [2,3] (stochastic) graph?

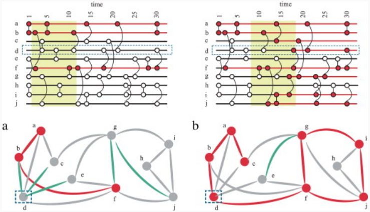
[2] P.Holme et al. "Temporal networks" (2012), [3] R.Lambiotte et al. arxiv 1305.0543 (2013)

Idea: Model on a temporal (stochastic) graph is the simplest representation of a road navigation system. Stationary distributions when $\psi_{xx'}(t)$ are non-finite mean distributions.

$\psi_{xx'}(t) = \tau_{xx'}(t) \prod_{y \neq x'} (1 - \int_0^t \tau_{yx}(t')dt')$,

for $\tau_{xx'}(t)$ - activation distribution of link xx'.

HCTRW on various types of networks



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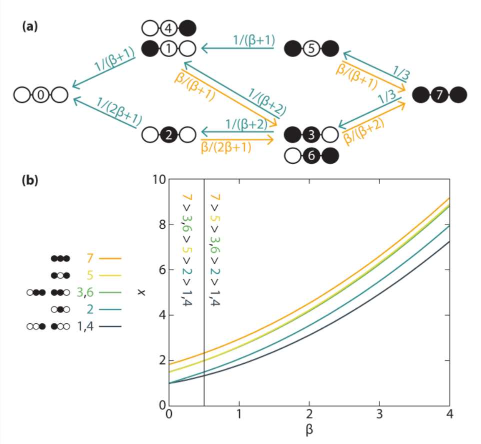
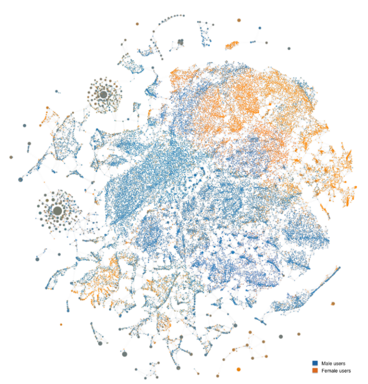
Epidemiological models: SIS on small graphs



Panel (a): the four equivalence classes of states of the SIS model at a triangle. The same idea is applied to larger networks using croase-graining. Panel (b): the expected extinction times x derived from (a) as a function of the infection rate β.

P.Holme, L.Tupikina "Explicit solutions for SIS model" arxiv.org 1802.08849

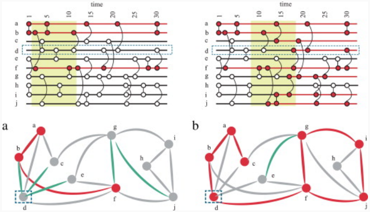
Applications of random walk theory



D. Brockmann, L. Hufnagel, T. Geisel "The scaling laws of human travel", 04292 Nature (2006)
F. Iannelli, M. Sebastian Mariani, I, M. Sokolov "Network centrality based on reaction-diffusion dynamics(...)", PRE (2018)

Real-world motivation

HCTRW on various types of networks



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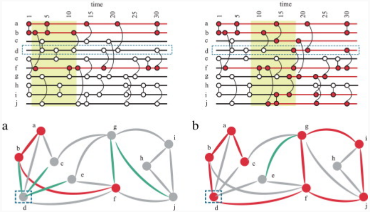
HCTRW on various types of networks



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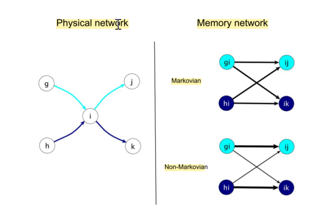


Memory network for nodes $((ij),(ik)\ldots)$, $i \in (1,\ldots N)$, with $p(ij \to jk) = \frac{W(ij \to jk)}{\sum_l W(ij \to jl)}$ M. Rosval, R.Lambiotte et al. Nat.Com.(2014).

Questions: How to model processes with memory on graphs?

Ideas: HCTRW propagator $P_{x_0 x}(t)$ for estimation of first passage time properties on graph, MFPT as a network measure, using memory non-Markovian network as the underlying HCTRW transition matrix Q.

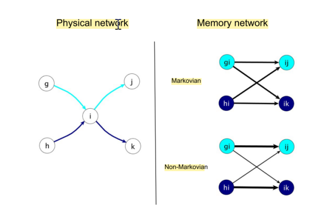
Dynamics evolution for memory network:

$P(i|t+1) = \sum_k P(ki|t+1) = \sum_{jk} P(jk|t)p(jk \to ki)$

Dynamics evolution for simple network:

$P(i|t+1) = \sum_j P(j|t)p(j \to i)$.

HCTRW on various types of networks



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Continuous limits of HCTRW dynamics

Main references:

J. Jacod and A.S. Shiryaev **"Limit theorems for stochastic processes"**, 2nd Edition. Springer (2002)

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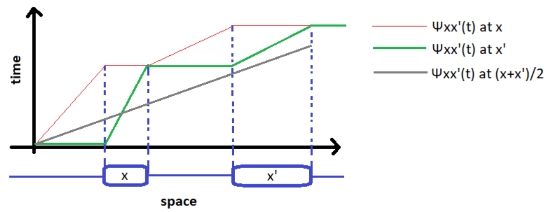
Filtration theorems

Continuous limits of HCTRW dynamics: Conjecture



There are three possible continuous dynamics of HCTRW:
when time of jump between nodes is defined in x from the distribution $\psi_{xx'}(t)$.
when time of jump between nodes is defined in x' after the jump
when time of jump between nodes is defined between x, x' during the jump

• Grebenkov, Tupikina, "Heterogeneous continuous time random walks",arXiv:1712.01020 (PRE to appear). • Tupikina, Grebenkov "Continuous limits for Heterogeneous continuous time random walks", work in progr.

Continuous limits of HCTRW dynamics: Derivation

In Laplace space we express loss flux as

$$\tilde{j}^{-}_{\bar{x}}(s) = \tilde{\mathcal{Q}}_{\bar{x}}(s) P_{x_0\bar{x}}(0) + \tilde{\mathcal{Q}}_{\bar{x}}(s)(s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) + \tilde{j}^{-}_{\bar{x}}(s)) \quad (8)$$

where $P_{x_0\bar{x}}(0)$ are initial conditions, $\tilde{\mathcal{Q}}_{\bar{x}}(s) = \sum_{x'} Q_{\bar{x}x'}\tilde{\psi}_{\bar{x}x'}(s)$.
For convenience we introduce $M_{\bar{x}}(s)$ **function**

$$M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{1 - \tilde{\mathcal{Q}}_{\bar{x}}(s)}, \quad (9)$$

which is connected with the rate of steps. Note, kernel implicitly depends on adjacent sites of $\bar{x}$ (not like in CTRW limit).

Continuous limits of HCTRW dynamics: result

Then **the GME** for the propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \Big(Q_{\bar{x}x'} sM_{x'}(s)\tilde{P}_{x_0x'}(s) \Big) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s), \tag{10}$$

where $M_{\bar{x}}(s) = \frac{\tilde{\mathcal{Q}}_{\bar{x}}(s)}{1-\tilde{\mathcal{Q}}_{\bar{x}}(s)}$, $\tilde{\mathcal{Q}}_{\bar{x}}(s) = \sum_{x'} Q_{\bar{x}x'}\tilde{\psi}_{\bar{x}x'}(s)$.

Question:

What is the right continuous limit for Eq.(16) for various types of $\psi_{xx'}(t)$?

Example: For simple CTRW GME becomes Pauli master equation:

$$\dot{P}_{x_0\bar{x}}(t) = \frac{1}{\tau}[\sum_{x'} Q_{\bar{x}x'} P_{x_0x'}(t) - P_{x_0\bar{x}}(t)], \tag{11}$$

which then gives

$$\frac{dP_x(t)}{dt} = \frac{d^2}{dx^2}(D(x)D^{1-\beta(x)}P_x(t)) \tag{12}$$

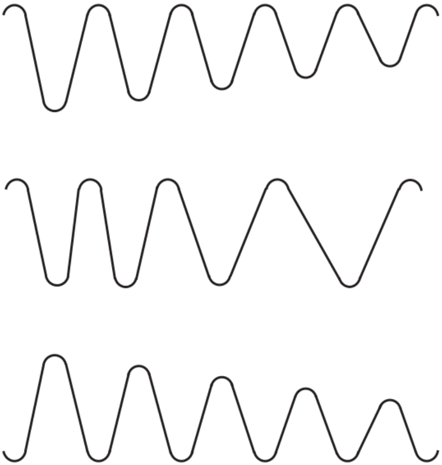


Fig. 1. Different situations corresponding to a position-dependent diffusion coefficient, which grows from left to right: The trap model, the accordion model and the barrier model, from top to bottom. These situations need for using the Ito, Stratonovich and Hänggi rules respectively when describing them within the Langevin scheme.

I.M.Sokolov **"Ito, Stratonovich(...)"** Chem.Phys.359-363 (2010)

Continuous limits of HCTRW dynamics: Analytical result

The **Generalized Master Equation** (GME) is based on two balance conditions:
(i) the local balance between the gain flux $j_{\bar{x}}^{+}(t)$ and loss flux $j_{\bar{x}}^{-}(t)$ from $\bar{x}$ site;
(ii) the balance for transitions (particle conservation or continuity)

Continuous limits of HCTRW dynamics: Analytical result

Balance equation

$$\frac{dP_{x_0\bar{x}}(t)}{dt} = j^{+}_{\bar{x}}(t) - j^{-}_{\bar{x}}(t), \tag{13}$$

is transformed to

$$j^{-}_{\bar{x}}(t) = \mathcal{Q}_{\bar{x}}(t)P_{x_0\bar{x}}(0) + \int_0^t \mathcal{Q}_{\bar{x}}(t-t')\Big(\frac{dP_{x_0\bar{x}}(t')}{dt} + j^{-}_{\bar{x}}(t')\Big)dt', \tag{14}$$

where $\mathcal{Q}_{\bar{x}}(t) = \sum_{x'} Q_{\bar{x}x'}\psi_{\bar{x}x'}(t)$.

Continuous limits of HCTRW dynamics: example

Idea:

Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.
I.e. for $\psi_{\bar{x}x'}(t) = \psi_{\bar{x}}(t)$

$$M_{\bar{x}}(s) = \frac{\mathcal{Q}_{\bar{x}}(s)}{1 - \mathcal{Q}_{\bar{x}}(s)} = \frac{\tilde{\psi}_{\bar{x}}(s) \sum_{x'} Q_{\bar{x}x'}}{1 - \tilde{\psi}_{\bar{x}}(s) \sum_{x'} Q_{\bar{x}x'}} = \frac{\tilde{\psi}_{\bar{x}}(s)}{1 - \tilde{\psi}_{\bar{x}}(s)}.$$

The exponential travel times with parameter $\tau_{\bar{x}}$: $\tilde{\psi}_{\bar{x}}(s) = \frac{1}{1+s\tau_{\bar{x}}}$, then

$$\frac{d}{dt} P_{x_0 x}(t) = \frac{d^2}{dx^2} D(x) P_{x_0 x}(t), \tag{15}$$

where $D(x) = \lim_{\bar{x}\to x} \frac{a^2}{2\tau_{\bar{x}}}$.

Continuous limits of HCTRW dynamics: result

We derived **the Generalized Master Equation** for the HCTRW propagator $\tilde{P}_{x_0\bar{x}}(s)$ in Laplace domain

$$s\tilde{P}_{x_0\bar{x}}(s) - P_{x_0\bar{x}}(0) = \sum_{x'} \left(Q_{\bar{x}x'} sM_{x'}(s)\tilde{P}_{x_0x'}(s) \right) - sM_{\bar{x}}(s)\tilde{P}_{x_0\bar{x}}(s). \tag{16}$$

How we can use it? Consider various types of $\psi_{\bar{x}x'}(t)$ and substitute into expression $M_{\bar{x}}(s)$.

Epidemiological models: open questions

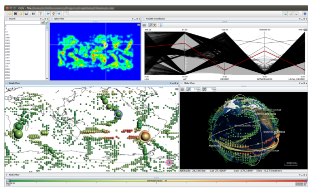
Epidemiological models on temporal networks;
stable states of Ising models (theory of Ising models applied to node-state models);
epidemiological network measures: reproductive number etc.

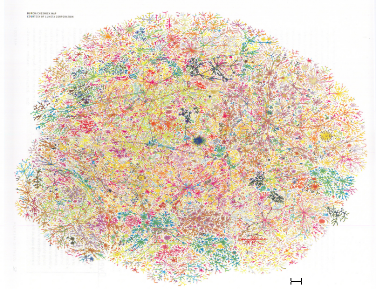
P.Holme, J.Saramaki, Springer (2013)
S. Shlosman Comm. Math. Phys. Volume 102, Number 4 (1986)
P. Hoscheit, S. Geeraert, G.Beaunee, H. Monod, C. Gilligan, J. Filipe, E. Vergu, M. Moslonka-Lefebvre, J. Compl.Netw. 1-21 (2016)

Software pyunicorn

How to analyse **various types of data using network approach?**
How to combine both croase-graining methods and graph theoretical approach?

"Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package"
J.F. Donges, J. Heitzig, B. Beronov, M. Wiedermann, J. Runge, Q.-Y. Feng, L.Tupikina, V. Stolbova, R.V. Donner, N. Marwan, H.A. Dijkstra, and J. Kurths, Chaos 25, 113101 (2015)





Models of random walks on heterogeneous networks "HOpS model of opinion spreading" L. Tupikina (2017) *https://arxiv.org/abs/1708.01647* "Heterogeneous continuous time random walk model", D. Grebenkov, L. Tupikina, PhysRevE 97 012148 (2018) *https://journals.aps.org/pre/abstract/10.1103/PhysRevE.97.012148*