Presentation Speech Draft

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Good [morning/afternoon/evening], everyone. I am Shubin Xue from Beijing Institute of Technology. Today, it's my pleasure to present the **Classification of Quadratic Forms over** $\mathbb{Q}$.

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My presentation will be divided into three main parts:

First, I will discuss the connection between classification problem and representation problem.

Second, I will classify the quadratic forms over the completion of $\mathbb{Q}$.

The final part is the classification results over $\mathbb{Q}_p$ and $\mathbb{Q}$.

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Before we begin, let me outline some notation:

- K is an arbitrary field.
- X, Y, Z capital letters are used as variables.
- $\mathbb{V}$ is the set of all prime numbers add with infinity.
- $\mathbb{Q}$ is the completion of $\mathbb{Q}$.
  - $\mathbb{Q}_\infty$ is just $\mathbb{R}$.
  - $\mathbb{Q}_p$ is completion with respect to the p-adic valuation.

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- We call the sum is quadratic form with $a_{ij} = a_{ji}$ and X is a vector in K - verctor space.
- The matrix A_f corresponding to f is a symmetric.
- The pair is called a quadratic space.
  - We say f is equivalent to g if their matrices are congruent.
  - The quadratic forms are equivalent if and only if the spaces are isomorphic.

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The case over $\mathbb{R}$ is well-konwn.

The classification over $\mathbb{R}$ is completely determined by the rank and the signature, which is the Sylvester's law of inertia.

So, the problem is: how do we classify quadratic forms over $\mathbb{Q}$?

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In order to solve the problem, I will share some history with you.

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We say f represents a element if there is a nonzero x such that f of x is equal to a.

As early as the early development of number theory, mathematicians began to pay attention to the problem of representing numbers by quadratic forms.

In the 17th century, Fermat proposed the two-square theorem, an odd prime p can be represented as the sum of two squares if and only if $p \equiv 1$ (modolo 4). Gauss prove the three-square theorem. And Lagrange show that every natural number can be represented as the sum of at most four squares.

The sum of squares problem is actually the representation problem for diagonal quadratic forms with coefficient 1.

But unfortunately, the connection between the representation problem and the classification problem was not yet noticed.

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In the 19th century, Gauss pointed out that representations of numbers are invariants for the classification of quadratic forms.

Later, in the 20th century, Hasse, building on Minkowski's work, proposed the "local-global principle" for quadratic forms over the rational numbers.

f represents 0 over $\mathbb{Q}$ if and only if represents 0 over all $\mathbb{Q}_v$.

This theorem established the link between classification and representation.

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We define the direct sum of two forms by plusing them independently.

Then an element a can be represented if and only if f can be decomposed by f_1 and aZ^2 or f_a represents 0.

The item 3 allows us to reduce the problem of representing a to the problem of representing 0.

And the item 2 allows us to decompose the quadratic form by rank.

Applying the Hasse-Minkowski theorem to f_a, we can directly obtain that f represents a over $\mathbb{Q}$ if and only if represents a over all local fields of $\mathbb{Q}$.

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Next, I want to show you why the Hasse-Minkowski theorem establishes the link between "classification" and "representation".

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If two forms are equivalent, and one summand is the same, then the other summand must also be equivalent.

You can consider two isomorphic subspaces U and W. Extend the isomorphism to an automorphism of V. Then, we restrict to the orthogonal complement. Obtain an isomorphism of the complement.

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Now, I will show you when two forms are equivalent over $\mathbb{Q}$.

Non-degenerate quadratic forms of rank n over $\mathbb{Q}$ are equivalent if and only if they are equivalent over all $\mathbb{Q}_v$.

The necessity is trivial, so we consider the converse.

- We have f is equivalent to g over all $\mathbb{Q}_v$, so the numbers represented by f and g over $\mathbb{Q}_v$ is the same.
- By the Hasse-Minkowski theorem, the numbers represented by f and g over $\mathbb{Q}_v$ is also the same.
- Take a can be represented over $\mathbb{Q}$ by both f and g. We can decompose f by f_1 and aZ^2 and similar for g.
- By Witt's cancellation theorem, we know that f_1 is equivalent to g_1 over all $\mathbb{Q}_v$.
- By induction, f_1 is equivalent to g_1 over $\mathbb{Q}$. Hence, f is equivalent to g over $\mathbb{Q}$.

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Now we have successfully reduced the global classification problem to a local one, so next we need to work over local fields.

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Let's first look at some general ideas.

It suffices to classify non-degenerate diagonal quadratic forms of rank n with square-free coefficients. Because:

- The rank is always an invariant.
- Symmetric matrices can always be diagonalized.
- Squared factors in the coefficients don't change the classification.

On the other hand, by considering the determinant of the matrix, we notice that the image of determinant of A in $K^\times/(K^\times)^2$ is an invariant. We call it the discriminant.

Our work has now been reduced to square-free, so we need to know its group structure.

For $\mathbb{R}$, it's 1 and negative 1.

For $\mathbb{Q}_p$ with p not equal 2, in fact, it's the Klein four-group K_4.

And for $\mathbb{Q}_2$, it's the $\mathbb{Z}$ mod two cubed.

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In order to classify the forms over $\mathbb{Q}_p$, we need to introduce an invariant.

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We start with a simple example: binary quadratic forms over $\mathbb{Q}_p$ for $p \neq 2$.

In this case, forms are equivalent if and only if they have the same discriminant d and a common represented number a.

Take a represented by both f.

f can be decomposed as f_1 and aZ^2, where f_1 has rank 1, so its coefficient must be ad. So f is completely determined.

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This is a table of the discriminant of binary forms.

The same color means an equivalence class, while black means distinct equivalence classes.

We use box to label the quadratic forms don't represent 1.

From the table, we can see that when the discriminant d is fixed, the classification is completely determined by whether can represent 1.

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We know, f representing 1 is equivalent to $Z^2 - aX^2 - bY^2$ representing 0.

This leads us to define another invariant: the Hilbert symbol.

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In the previous table, the boxed entry have Hilbert symbol -1, so we can directly read off the value of the Hilbert symbol.

From this table, you can notice the Hilbert symbol is a symmetric non-degenerate bilinear form.

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Now let us generalize the Hilbert symbol to higher dimensions, defining the Hasse invariant as the product of Hilbert symbols.

We can directly compute the discriminant of f_1, and use the bilinearity of the Hilbert symbol to obtain its Hasse invariant.

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Here are the conditions for an element can be represented.

For fixed rank, the set of numbers represented is determined by the discriminant and Hasse invariant.

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Finally, let us when two forms are equivalent over $\mathbb{Q}_p$.

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Non-degenerate quadratic forms of rank n over $\mathbb{Q}_p$ are completely classified by their Hasse invariant and discriminant.

If f and g have the same discriminant d and ε, then their represented numbers are the same. Take a represented by f and g. f can be decomposed as f_1 direct sum aZ^2.

A direct compute shows that f_1 and g_1 have the same invariants.

By induction on the rank, f_1 is equivalent to g_1, so f is equivalent to g.

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Now we already know when two quadratic forms over $\mathbb{Q}$ are completely equivalent.

But what possible equivalence classes exist over $\mathbb{Q}$ and $\mathbb{Q}_p$?

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For fixed (d, ε), here are all possible quadratic forms over $\mathbb{Q}_p$.

When n is 1: f is of the form dX^2.

When n is 2: f is of the form $aX^2 + adY^2$ for any a.

When n is greater or equal to 3: f is of the form $aX_1^2 + bX_2^2 + abdX_3^2$ plus the sum of X_i^2 for a and b satisfy the following conditions.

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In the end, the classification over $\mathbb{Q}$ need to consider the global property of Hilbert symbol.

The puoduct formula tell us the Hilbert symbol take 1 for almost v, and the product must be 1. Thus the invariants must satisfy the first condition.

The second condition from results over $\mathbb{Q}_p$.

And the last three conditions from the results over $\mathbb{R}$.

I think I'll stop here. This completes the classification of quadratic forms over $\mathbb{Q}$. Thank you.