

# Classification of Quadratic Forms over $\mathbb{Q}$

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## 1 Classification and Representation

- Some History
- From Global to Local

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## 3 Classification Results

# Notations

- $K$ : an arbitrary field with  $\text{Char } K \neq 2$ .
- $X, Y, Z$ : variables.
- $\mathbb{V} = \{v : v \in \mathbb{V}\} = \{p : \text{prime number}\} \cup \{\infty\}$ .
- $\mathbb{Q}_v$ : completion of  $\mathbb{Q}$ 
  - $\mathbb{Q}_\infty = \mathbb{R}$
  - $\mathbb{Q}_p$ : with respect to the  $p$ -adic valuation.

# Quadratic Forms

- $f(\vec{X}) = \sum_{i,j=1}^n a_{ij}X_iX_j$  is a quadratic form
  - $a_{ij} = a_{ji} \in K$ .
  - $\vec{X} = (X_1, \dots, X_n) \in K^n$
- The matrix  $A_f = (a_{ij})$  associated with  $f$  is symmetric.
- The pair  $(K^n, f)$  is a quadratic space.
  - $f \sim g$ :  $\exists P \in GL(n, K)$  s.t.  $A_f = P^T A_g P$ .
  - $f \sim g \iff (K^n, f) \cong (K^n, g)$ .

# Case over $\mathbb{R}$

- Invariants:

- Rank:  $\text{rank } f = n$ .
- Signature:  $(r, s) := (\#\text{positive eigenvalues}, \#\text{negative eigenvalues})$ .

## Theorem (Sylvester's Law of Inertia)

Let  $f = \sum_{i,j=1}^n a_{ij} X_i X_j$  be a quadratic form of rank  $n$  over  $\mathbb{R}$ . Then

$$f \sim X_1^2 + X_2^2 + \cdots + X_r^2 - X_{r+1}^2 - \cdots - X_{r+s}^2.$$

**Problem: How to Classify Quadratic Forms over  $\mathbb{Q}$ ?**

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# Background (17-18th Century)

$f$  represents  $a \in K$ :  $\exists x \in K^n \setminus \{0\}$  s.t.  $f(x) = a$ .

## Theorem (Fermat's Two-Square Theorem)

*An odd prime  $p$  can be represented as the sum of two squares if and only if  $p \equiv 1 \pmod{4}$ .*

## Theorem (Gauss's Three-Square Theorem)

*A natural number can be represented as a sum of three squares if and only if it is not of the form  $4^a(8b - 1)$  for integers  $a \geq 0$ ,  $b > 0$ .*

## Theorem (Lagrange's Four Square Theorem)

*Every natural number can be represented as the sum of at most four squares.*

# Background (19-20th Century)

## Remark

Equivalent quadratic forms represent exactly the same set of numbers.

## Theorem (Hasse-Minkowski)

*$f$  represents 0 over  $\mathbb{Q}$  iff it represents 0 over all  $\mathbb{Q}_v$ .*



# Representation of Numbers

$f(X_1, \dots, X_n)$  and  $g(X_1, \dots, X_m)$

- $f \oplus g = f(X_1, \dots, X_n) + g(X_{n+1}, \dots, X_{n+m})$

## Proposition

*Let  $a \in K^\times$ . The following are equivalent:*

- $f$  represents  $a$
- $f \sim f_1 \oplus aZ^2$  where  $f_1$  is of rank  $\text{rank } f - 1$ .
- $f_a = f \oplus -aZ^2$  represents 0.

## Corollary (Hasse-Minkowski Theorem)

$f$  represents  $a \in \mathbb{Q}^\times$  over  $\mathbb{Q}$  iff it represents  $a$  over all  $\mathbb{Q}_v$ .

- Apply the Hasse-Minkowski Theorem to  $f_a = f \oplus -aZ^2$ .

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# Decomposing Quadratic Spaces

## Theorem (Witt's Cancellation)

$f_1 \oplus g_1 \sim f_2 \oplus g_2$  and  $g_1 \sim g_2$  implies  $f_1 \sim f_2$ .

- $(V, Q)$ : A quadratic space.
- $(U, Q)$  and  $(W, Q)$ : Isometric subspaces.

$$\begin{array}{ccc} V & \xrightarrow{\cong} & V \\ \text{Lift} \downarrow & & \downarrow \\ U & \xrightarrow{\cong} & W \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\cong} & V \\ \text{Restrict} \downarrow & & \downarrow \\ U^\perp & \xrightarrow{\cong} & W^\perp \end{array}$$

Figure: Witt's Cancellation Theorem

$$f \sim g \text{ over } \mathbb{Q}$$

## Theorem (Minkowski)

*Two non-degenerate quadratic forms of rank  $n$  over  $\mathbb{Q}$  are equivalent iff they are equivalent over each  $\mathbb{Q}_v$ .*

- By the Hasse-Minkowski Theorem, the numbers represented by both  $f$  and  $g$  over  $\mathbb{Q}$  is also the same.
- Take  $a$  represented,  $f \sim aZ^2 \oplus f_1$  over  $\mathbb{Q}$  and  $\mathbb{Q}_v$ . Similarly for  $g$ .
- By Witt's cancellation, we have  $f_1 \sim g_1$  over  $\mathbb{Q}_v$  for all  $v \in \mathbb{V}$ .
- By induction on rank  $n$ ,  $f_1 \sim g_1$  over  $\mathbb{Q}$ , thus  $f \sim g$  over  $\mathbb{Q}$ .

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# General Ideas over $K$

- Reduced Form:  $f \sim \sum_{i=1}^n a_i X_i^2$ , where  $a_i \in K^\times / (K^\times)^2$ .
  - Invariant rank: non-degenerate.
  - Symmetric matrices: diagonal.
  - $\sum a_i b_i^2 X_i^2 \sim \sum a_i X_i^2$ : square-free.
- If  $f \sim g$ ,  $\det(A_f) = \det(P^T A_g P) = \det(A_g) \det(P)^2$ 
  - Invariant discriminant:  $d = \det(A)$  in  $K^\times / (K^\times)^2$ .
- $\mathbb{R}^\times / (\mathbb{R}^\times)^2 \cong \{1, -1\}$ .
- $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 \cong K_4 = \{1, a, p, ap\}$  ( $p \neq 2$ ).
- $\mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ .

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## 3 Classification Results

# Binary Quadratic Forms over $\mathbb{Q}_p$ for $p \neq 2$

## Theorem (Classification of Binary Quadratic Forms over $\mathbb{Q}_p$ for $p \neq 2$ )

*Two binary quadratic forms over  $\mathbb{Q}_p$  for  $p \neq 2$  are equivalent if and only if they have the same discriminant  $d$  and a common represented number  $a$ .*

- Take  $a$  represented by  $f$ .
- $f \sim f_1 \oplus aZ^2$  where  $\text{rank } f_1 = 1$ .
- Then  $f_1 = adX^2$ . Thus  $f$  is determined.



# Binary Quadratic Forms over $\mathbb{Q}_p$ for $p \neq 2$

- Entry: the discriminant of  $\alpha X^2 + \beta Y^2$
- $a$ : same color, same equivalent class
  - mutually distinct if colored black
- $\boxed{a}$ : boxed quadratic forms don't represent 1

$\alpha \backslash \beta$	1	$a$	$p$	$ap$
1	1	$a$	$p$	$ap$
$a$		1	$\boxed{ap}$	$\boxed{p}$
$p$			1	$\boxed{a}$
$ap$				1

(a)  $p \equiv 1 \pmod{4}$

$\alpha \backslash \beta$	1	$a$	$p$	$ap$
1	1	$a$	$p$	$ap$
$a$		1	$\boxed{ap}$	$\boxed{p}$
$p$			1	$a$
$ap$				1

(b)  $p \equiv 3 \pmod{4}$

**Table:** Classification of binary quadratic forms over  $\mathbb{Q}_p$  for  $p \neq 2$

# Hilbert Symbol

$f = aX^2 + bY^2$  represents 1

$\Longleftrightarrow$

$Z^2 - aX^2 - bY^2$  represents 0.

- Hilbert symbol:

$$(a, b) = \begin{cases} 1 & Z^2 - aX^2 - bY^2 \text{ represents } 0, \\ -1 & \text{Otherwise.} \end{cases}$$

# Computation of Hilbert Symbol

$(\cdot, \cdot)$	1	$a$	$p$	$ap$
1	1	1	1	1
$a$		1	-1	-1
$p$			1	-1
$ap$				1

(a)  $p \equiv 1 \pmod{4}$

$(\cdot, \cdot)$	1	$a$	$p$	$ap$
1	1	1	$p$	$ap$
$a$		1	-1	-1
$p$			-1	1
$ap$				-1

(b)  $p \equiv 3 \pmod{4}$

*Table:* Hilbert Symbol of  $\mathbb{Q}_p$  for  $p \neq 2$

- Hilbert symbol is a symmetric non-degenerate bilinear form.

# Hasse Invariant

- $f = a_1X_1^2 + \cdots + a_nX_n^2$ .
- Hasse invariant:  $\varepsilon(f) = \prod_{i < j} (a_i, a_j)$
- $f_1 = a_2X_2^2 + \cdots + a_nX_n^2$ .
- $d(f) = \prod_{i=1}^n a_i = a_1 \prod_{i=2}^n a_i = a_1 d(f_1)$ .
- $\varepsilon(f) = \prod_{1 \leq i < j \leq n} (a_i, a_j) = \varepsilon(f_1) \cdot (a_1, a_2 \cdots a_n) = \varepsilon(f_1) \cdot (a_1, a_1 d(f))$ .

# Representation of Numbers over $\mathbb{Q}_p$

$f$  represents 0 over  $\mathbb{Q}_p$  iff:

- For  $n = 2$ :  $d = -1$ ;
- For  $n = 3$ :  $(-1, -d) = \varepsilon$ ;
- For  $n = 4$ :  $d \neq 1$  or  $d = 1$  and  $\varepsilon = (-1, -1)$ ;
- For  $n \geq 5$ : no conditions.

By applying the result to  $f_a = f \oplus -aZ^2$ , we obtain:

$f$  represents  $a \in \mathbb{Q}_p^\times$  iff:

- For  $n = 1$ :  $a = d$ ;
- For  $n = 2$ :  $(a, -d) = \varepsilon$ ;
- For  $n = 3$ :  $a \neq d$  or  $a = d$  and  $\varepsilon = (-1, -d)$ ;
- For  $n \geq 4$ : no conditions.

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$$f \sim g \text{ over } \mathbb{Q}_p$$

## Theorem

*Two non-degenerate quadratic forms of rank  $n$  over  $\mathbb{Q}_p$  are equivalent iff they have the same discriminant  $d$  and Hasse invariant  $\varepsilon$ .*

- $f, g$  have same  $d$  and  $\varepsilon$ , thus there exists  $a \in \mathbb{Q}_p^\times$  which both represented by  $f$  and  $g$ .
- Then  $f \sim f_1 \oplus aZ^2$ , where  $f_1$  is of rank  $n - 1$ . Similarly for  $g$ .
- $d$  and  $\varepsilon$  of  $f_1$  can be determined:
  - $d(f_1) = a \cdot d(f) = a \cdot d(g) = d(g_1)$
  - $\varepsilon(f_1) = \varepsilon(f) \cdot (a, ad(f)) = \varepsilon(g) \cdot (a, ad(g)) = \varepsilon(g_1)$
- Thus  $f_1, g_1$  share the same  $d$  and  $\varepsilon$ . QED by induction.

# Classification of Quadratic Forms over $\mathbb{Q}_p$

Fix  $(d, \varepsilon)$ , all possible quadratic forms over  $\mathbb{Q}_p$ :

- $n = 1$ :  $f = dX^2$
- $n = 2$ :  $f = aX^2 + adY^2$ , for
  - $a \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$
- $n \geq 3$ :  $f = aX_1^2 + bX_2^2 + abdX_3^2 + \sum_{i>3} X_i^2$ , for
  - $a : a \neq -d$
  - $b : (b, -ad) \cdot (a, -d) = \varepsilon$



# Classification of Quadratic Forms over $\mathbb{Q}$

## Theorem (Product Formula)

$(a, b)_v = 1$  for almost all  $v \in \mathbb{V}$  and  $\prod_{v \in \mathbb{V}} (a, b)_v = 1$

The invariants  $d_v$  and  $\varepsilon_v$  satisfy the following relations:

- $\varepsilon_v = 1$  for almost  $v \in \mathbb{V}$ , and  $\prod_{v \in \mathbb{V}} \varepsilon_v = 1$ .
- $\varepsilon_v = 1$  if  $n = 1$  and if  $n = 2$  and if the image  $d_v$  of  $d$  in  $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$  is equal to  $-1$ .
- $r, s \geq 0$  and  $r + s = \text{rank}$ .
- $d_\infty = (-1)^s$
- $\varepsilon_\infty = (-1)^{s(s-1)/2}$

# Thank You!

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