

Classification of Quadratic Forms over \mathbb{Q} ¹

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Notations

- We denote by K an arbitrary field of characteristic $\neq 2$.
- X, Y, Z are used as variables.
- $\left(\frac{a}{p}\right)$ is the Legendre symbol.
- We will often denote by the same letter an element and its class modulo.

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What is a Quadratic Form/Quadratic Space?

Definition (Quadratic Space)

Let V be a vector space (finite-dimensional) over a field K of characteristic $\neq 2$. A function $Q : V \rightarrow K$ is called quadratic form on V satisfying:

- $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in K, v \in V$,
- The function $B_Q(u, v) = Q(u + v) - Q(u) - Q(v)$ is a symmetric bilinear form on V .

A quadratic space is such a pair (V, Q) .

- Put $x \cdot y = \frac{1}{2}B_Q(x, y)$. One has $Q(x) = x \cdot x$.
- Given a basis $\{e_1, \dots, e_n\}$ of V , the quadratic form Q can be associated with a symmetric matrix $A = (a_{ij})$ where $a_{ij} = e_i \cdot e_j$.

$$\text{If } x = \sum_{i=1}^n x_i e_i \in V, \text{ then } Q(x) = \sum_{i,j=1}^n a_{ij} x_i x_j.$$

Let us consider quadratic forms in a more familiar form:

- $f(X) = \sum_{i,j=1}^n a_{ij}X_iX_j$ is a quadratic form in n variables over K , where $a_{ij} = a_{ji}$.
- The pair (K^n, f) is a quadratic space.
- The matrix $A = (a_{ij})$ is associated with f .
- Let $f(X_1, \dots, X_n)$ and $g(X_1, \dots, X_m)$ be two quadratic forms, we denote $f \oplus g$ the quadratic form

$$f(X_1, \dots, X_n) + g(X_{n+1}, \dots, X_{n+m})$$

in $n + m$ variables.

Invariant: Discriminant

Change the basis $\{e_i\}$ to another basis $\{e'_i\}$; the associated symmetric matrix A transforms as $A' = PAP^T$.

- Two quadratic forms are equivalent if their matrices are congruent under such a transformation. (*Denoted as $f \sim g$*)
- We know that any symmetric matrix can always be diagonalized by a congruence transformation.
- Without loss of generality, assume quadratic forms are of the shape

$$f \sim \sum_{i=1}^n a_i X_i^2$$

And $A' = PAP^T$ give us: $\det(A) = \det(A') \det(P)^2$.

- This means the image of $\det(A)$ in $K^\times / K^{\times 2}$ is a invariant, it's called discriminant of Q , and denoted by $d(Q)$ or simply d .

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The case over \mathbb{R}

Theorem (Sylvester's law of inertia)

Let $f = \sum_{i,j=1}^n a_{ij}X_iX_j$ be a quadratic form of rank n over \mathbb{R} . Then

$$f \sim X_1^2 + X_2^2 + \cdots + X_r^2 - X_{r+1}^2 - \cdots - X_{r+s}^2.$$

where r and s are non-negative integers, and $r + s = n$, the pair (r, s) is called signature of f .

By diagonalizing via congruence and factoring out squares on the diagonal, we see that the only invariants for classifying real quadratic forms are:

- the rank $\text{rank } f = n$.
- the signature $(r, s) := (\# \text{positive eigenvalues}, \# \text{negative eigenvalues})$.

The *rank* and *signature* are invariants.

On an arbitrary field K :

- The rank is always an invariant. Hence we may (and we shall always) reduce to classify the non-degenerate quadratic forms of rank n .
- Two quadratic forms $f = \sum_{i \neq j} a_{ij} X_i X_j$ and $f' = \sum_{i \neq j} a'_{ij} X_i X_j$ satisfy: there exist $t_{ij} \in K^{\times 2}$ s.t. $a_{ij} = t_{ij} a'_{ij}$, then $f \sim f'$.
- The distribution of diagonal elements in $K^{\times} / (K^{\times})^2$ suffices to show the equivalence.
 - $\mathbb{C}^{\times} / (\mathbb{C}^{\times})^2 \cong \{1\}$, suffices to classify by the rank.
 - $\mathbb{R}^{\times} / (\mathbb{R}^{\times})^2 \cong \{1, -1\}$, signature is also needed.
 - $\mathbb{F}_q^{\times} / (\mathbb{F}_q^{\times})^2 \cong \{1, a\}$, where $a \in \mathbb{F}_q$ isn't a square.
 - For \mathbb{Q}_p and \mathbb{Q} ?

Case over \mathbb{F}_q

- Following the above discussion, we might hope that the number of squares appearing on the diagonal would serve as a sufficient criterion for equivalence. However, this is not the case.
- Consider the non-degenerate quadratic form of rank 2 in 2 variables over \mathbb{F}_q with a quadratic nonresidue a ,

$$f_1 = aX^2 + aY^2 \sim f_2 = X^2 + Y^2$$

- Do a change of basis: $X = sX' + tY'$ and $Y = tX' - sY'$. If we require $aX'^2 + aY'^2 = X^2 + Y^2$, then $s^2 + t^2 = a$. Then we must focus on the existence of solution of equation $s^2 + t^2 = a$.
- s^2 and $a - t^2$ have both $(q+1)/2$ possible values, the pigeonhole principle implies the equation has a nonzero solution.
- The discriminant $d(f_1) = d(f_2) = 1 \in \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$ is an invariant for classifying quadratic forms.

Hilbert Symbol

The existence of nonzero solutions to the equation $aX^2 + bY^2 = Z^2$ in K^3 seems to be of great importance.

Definition (Hilbert symbol)

Let $a, b \in K^\times$:

$$(a, b)_K = \begin{cases} 1 & \text{if } Z^2 - aX^2 - bY^2 = 0 \text{ has a nontrivial solution in } K^3, \\ -1 & \text{if } Z^2 - aX^2 - bY^2 = 0 \text{ has no nontrivial solution in } K^3. \end{cases}$$

The number $(a, b)_K$ is called the *Hilbert symbol* of a and b relative to K .
(When there is no ambiguity, the subscript is often omitted.)

- The symbol may also be viewed in $K^\times / (K^\times)^2$ when working with non-degenerate forms.

The Hilbert Symbol over \mathbb{Q}_p

From now on, we always assume $K = \mathbb{Q}_p$ for a prime p :

Theorem ([Ser73] p. 20, chap. 3, sec. 1.2, theorem 1))

Say $a = p^\alpha u$ and $b = p^\beta v$ are p -adic numbers where $u, v \in \mathbb{Z}_p^\times$, then

$$(a, b) = (-1)^{\alpha \cdot \beta \cdot \frac{p-1}{2}} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \quad \text{if } p \neq 2$$

$$(a, b) = (-1)^{\frac{u-1}{2} \frac{v-1}{2} + \alpha \frac{v^2-1}{8} + \beta \frac{u^2-1}{8}} \quad \text{if } p = 2$$

- This means that Hilbert symbol is a symmetric non-degenerate bilinear form.

Product Formula of Hilbert Symbol

Let $\mathbb{V} = \mathbb{P} \cup \{\infty\}$, and $\mathbb{Q}_\infty = \mathbb{R}$. If $a, b \in \mathbb{Q}^\times$, $(a, b)_v$ denotes the Hilbert symbol of their images in \mathbb{Q}_v for all $v \in \mathbb{V}$.

Proposition (Product formula)

If $a, b \in \mathbb{Q}^\times$, we have $(a, b)_v = 1$ for almost all $v \in \mathbb{V}$ and

$$\prod_{v \in \mathbb{V}} (a, b)_v = 1.$$

- We have seen that the Hilbert symbol works for rank 2, but how do we generalize to rank > 2 ?
- Let $\varepsilon(f) = \prod_{i < j} (a_i, a_j)$, which is called the Hasse invariant of f .

Two Invariants

We have reduced to work with non-degenerate diagonalized quadratic forms of rank n .

- Recall that the *discriminant*

$$d(f) = a_1 a_2 \dots a_n \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$$

is an invariant.

- Recall that the *Hasse invariant*

$$\varepsilon(f) := \prod_{1 \leq i < j \leq n} (a_i, a_j)$$

is also an invariant.

- If $f = a_1 X_1^2 \oplus f_1$ where $f_1 = a_2 X_2^2 + \dots + a_n X_n^2$, then we have:

$$d(f) = \prod_{i=1}^n a_i = a_1 \prod_{i=2}^n a_i = a_1 d(f_1).$$

$$\varepsilon(f) = \prod_{1 \leq i < j \leq n} (a_i, a_j) = \varepsilon(f_1) \cdot (a_1, a_2 \dots a_n) = \varepsilon(f_1) \cdot (a_1, a_1 d(f))$$

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Decomposition of Quadratic Forms

On an arbitrary field K , we say that a quadratic form f *represents* $a \in K$ if there exists a nonzero $v \in V$ such that $f(v) = a$.

- It may be viewed in $\{0\} \cup K^\times / (K^\times)^2$.

Proposition ([Ser73] p. 33, chap. 4, sec. 1.6, corollary 1))

Let $a \in K^\times$. TFAE:

- f represents a
 - $f \sim g \oplus aZ^2$ where g is of rank $\text{rank } f - 1$.
 - $f \oplus -aZ^2$ represents 0.
-
- To check if a can be represented by f , it suffices to examine when a quadratic form represents 0.
 - Suppose f_1 and f_2 can both represent some $a \in K^\times$, then we hope to reduce their rank and use induction in subsequent proofs.

When does a quadratic form represent 0, a ($a \in K^\times$)?

Theorem ([Ser73] p. 36, chap. 4, sec. 2.2, theorem 6))

f represents 0 iff:

- For $n = 2$: $d = -1$;
- For $n = 3$: $(-1, -d) = \varepsilon$;
- For $n = 4$: $d \neq 1$ or $d = 1$ and $\varepsilon = (-1, -1)$;
- For $n \geq 5$: no conditions.

By applying Theorem to $f_a = f \oplus -aZ^2$, we obtain:

Corollary ([Ser73] p. 37, chap. 4, sec. 2.2, corollary to theorem 6))

f represents $a \in K^\times / K^\times$ iff:

- For $n = 1$: $a = d$;
- For $n = 2$: $(a, -d) = \varepsilon$;
- For $n = 3$: $a \neq d$ or $a = d$ and $\varepsilon = (-1, -d)$;
- For $n \geq 4$: no conditions.

Conditions for decomposing quadratic forms

Theorem (Witt ([Ser73] p. 31, chap. 4, sec. 1.5, theorem 3))

Every injective metric-preserving map from a subspace U of a quadratic space V to another quadratic space W may be extended to a metric-preserving map from V to W .

- If a quadratic space (V, Q) has two isometric subspaces U and W , then by Witt's theorem, the isometry can be extended to an automorphism of V . By restricting this automorphism to U^\perp , we see that U^\perp and W^\perp are also isometric. The results about quadratic spaces can be translated into results about quadratic forms:

Theorem (Witt's cancellation ([Ser73] p. 34, chap. 4, sec. 1.6, theorem 4))

$f_1 \oplus g_1 \sim f_2 \oplus g_2$ and $g_1 \sim g_2$ implies $f_1 \sim f_2$.

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Quadratic Forms $f \sim g$ over \mathbb{Q}_p

Theorem ([Ser73] p. 39, chap. 4, sec. 2.3, theorem 7))

Two non-degenerate quadratic forms of rank n over \mathbb{Q}_p are equivalent iff they have the same discriminant d and Hasse invariant ε .

- f, g have same d and ε , thus there exists $a \in \mathbb{Q}_p^\times$ which both represented by f and g .
- Then $f \sim f_1 \oplus aZ^2$, where f_1 is of rank $n - 1$. Similarly for g .
- d and ε of f_1 can be determined:
 - $d(f_1) = ad(f) = ad(g) = d(g_1)$
 - $\varepsilon(f_1) = \varepsilon(f) \cdot (a, ad(f)) = \varepsilon(g) \cdot (a, ad(g)) = \varepsilon(g_1)$
- Thus f_1, g_1 share the same d and ε . QED by induction.

Classification of Quadratic Forms over \mathbb{Q}_p

The invariants d and ε are not independent; they satisfy the following relations:

- For $n = 1$: $\varepsilon = 1$;
- For $n = 2$: $d \neq -1$ or $\varepsilon = 1$;
- For $n \geq 3$: no conditions.

Skeleton of Proof:

- $n = 1$: $f = aX^2$ has $\varepsilon = 1$ and $d = a$ is arbitrary.
- $n = 2$: $f = aX^2 + bY^2$ has $\varepsilon = (a, b) = (a, -ab)$. If $d = ab = -1$, then $\varepsilon = 1$. Conversely:
 - if $d = -1$, $\varepsilon = 1$: take $f = X^2 - Y^2$
 - if $d \neq -1$, since the Hilbert symbol is non-degenerate, there exists $a \in \mathbb{Q}_p^\times$ such that $(a, -d) = \varepsilon$. Take $f = aX^2 + adY^2$.
 - (when $d = -1$, $f = X^2 - Y^2 = aX^2 + adY^2$)
- $n = 3$: Choose $a \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$ and $a \neq d$. There exists form g of rank 2 s.t. $d(g) = ad, \varepsilon(g) = \varepsilon(a, -d)$. The form $f = g \oplus aZ^2$ works.
- $n > 3$: Take $f = g(X_1, X_2, X_3) \oplus X^2 \oplus \dots \oplus X_n^2$.

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Quadratic Forms $f \sim g$ over \mathbb{Q}

Theorem (Hasse-Minkowski)

f represents 0 over \mathbb{Q} iff it represents 0 over \mathbb{R} and all \mathbb{Q}_p .

Theorem ([Ser73] p. 44, chap. 4, sec. 3.3, theorem 9))

Two non-degenerate quadratic forms of rank n over \mathbb{Q} are equivalent iff they are equivalent over each \mathbb{Q}_v .

- Suppose $f \sim g$ over \mathbb{Q}_v for all v , then there exists $a \in \mathbb{Q}$ represented by both f and g .
- Thus $f \sim aZ^2 \oplus f_1$, $g \sim aZ^2 \oplus g_1$, where $\text{rank } f_1 = \text{rank } g_1 = n - 1$.
- By Witt's cancellation theorem, we have $f_1 \sim g_1$ over \mathbb{Q}_v for all $v \in \mathbb{V}$.
- By induction on rank n , $f_1 \sim g_1$ over \mathbb{Q} , thus $f \sim g$ over \mathbb{Q} .

Classification of Quadratic Forms over \mathbb{Q}

Proposition (Conclusion over \mathbb{Q})

The invariants d_v and ε_v are not independent; they satisfy the following relations:

- $\varepsilon_v = 1$ for almost $v \in \mathbb{V}$, and $\prod_{v \in \mathbb{V}} \varepsilon_v = 1$.
- $\varepsilon_v = 1$ if $n = 1$ and if $n = 2$ and if the image d_v of d in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$ is equal to -1 .
- $r, s \geq 0$ and $r + s = \text{rank}$.
- $d_\infty = (-1)^s$
- $\varepsilon_\infty = (-1)^{s(s-1)/2}$

Let d , $(\varepsilon_v)_{v \in \mathbb{V}}$, and (r, s) verify the relations above, then there exists a quadratic form of rank n over \mathbb{Q} having for invariants d , $(\varepsilon_v)_{v \in \mathbb{V}}$, and (r, s) .

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Lemma

Let $f_i \in \mathbb{Z}_p[X_1, \dots, X_m]$ be homogeneous polynomials with p -adic integer coefficients. TFAE:

- The f_i have a non trivial common zero in $(\mathbb{Q}_p)^m$
- The f_i have a common primitive zero (i.e. solution $(z, x, y) \not\equiv (0, 0, 0) \pmod{p}$) in $(\mathbb{Z}_p)^m$
- For all $n > 1$, the f_i have a common primitive zero in $(\mathbb{Z}/p^n\mathbb{Z})^m$.

Lemma

Let $a, b \in K^\times$ and let $K_b = K(\sqrt{b})$. For $(a, b) = 1 \iff a \in N(K_b^\times)$ of norms of elements of K_b^\times .

Lemma

Let $f = g \oplus -h$, TFAE:

- f represents 0
- There exists $a \in K^\times$ which is represented by g and h .

Theorem

Let $(a_i)_{i \in I}$ be a finite family of elements in \mathbb{Q}^\times and let $(\varepsilon_{i,v})_{i \in I, v \in \mathbb{V}}$ be a family of numbers equal to ± 1 . In order that there exists $x \in \mathbb{Q}^\times$ such that $(a_i, x)_v = \varepsilon_{i,v}$ for all $i \in I$ and $v \in \mathbb{V}$ iff the following conditions be satisfied:

- Almost all the $\varepsilon_{i,v} = 1$
- $\prod_{v \in \mathbb{V}} \varepsilon_{i,v} = 1$ for all $i \in I$
- For all $v \in \mathbb{V}$ there exists $x_v \in \mathbb{Q}_p^\times$ such that $(a_i, x_v)_v = \varepsilon_{i,v}$ for all $i \in I$.

Theorem (Approximation Theorem)

Let $S \subseteq \mathbb{V}$ be a finite set. The image of \mathbb{Q} in $\prod_{v \in S} \mathbb{Q}_v$ is dense.

Lemma

All quadratic forms in at least 3 variables over \mathbb{F}_q have a non trivial zero.

Lemma

Suppose $p \neq 2$. Let f be a quadratic form with coefficients in \mathbb{Z}_p whose discriminant $\det(a_{ij})$ is invertible. Let $a \in \mathbb{Z}_p$, every primitive solution of the equation $f(x) \equiv a \pmod{p}$ lifts to a true solution.

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Theorem (Hasse-Minkowski)

f represents 0 over \mathbb{Q} iff it represents 0 over \mathbb{R} and all \mathbb{Q}_p .

- The necessity is trivial. W.L.O.G., $f = \sum_{i=1}^n a_i X_i^2$, $a_i \in \mathbb{Q}^\times$. By replacing f by $a_1 f$, we can suppose $a_i = 1$
- $n = 2$: Suppose $f = X_1^2 - aX_2^2$
 - f_∞ represents 0 implies $a > 0$. Let $a = \prod_{p \text{ prime}} p^{\nu_p(a)}$.
 - f_v represents 0 implies that $\nu_p(a)$ is even. Then a is a square, f represents 0 over \mathbb{Q} .

- $n = 3$: Suppose $f = X_1^2 - aX_2^2 - bX_3^2$, we can assume a, b are square-free and $|a| \leq |b|$. Proceed by induction on $m = |a| + |b|$.
- If $m = 2$, then $f = X_1^2 \pm X_2^2 \pm X_3^2$.
 - f_∞ represents 0 implies $f \neq X_1^2 + X_2^2 + X_3^2$.
 - In other cases, f represents 0 by $f(1, 1, 0)$.
- If $m > 2$, then $b \geq 2$, let $b = \pm p_1 \cdots p_k$.
- We need to show a is a square modulo p_i for all i .

- It is obvious if $a \equiv 0 \pmod{p_i}$.
- Otherwise, a is a p_i -adic unit.
- By hypothesis, $f = X_1^2 - aX_2^2 - bX_3^2$ represents 0, i.e. $z^2 - ax^2 - by^2$ has a nontrivial zero in $(\mathbb{Q}_{p_i})^3$.
- By the lemma, $z^2 - ax^2 - by^2$ has a primitive zero (z, x, y) in $(\mathbb{Z}_{p_i})^3$.
- We have $z^2 - ax^2 \equiv 0 \pmod{p_i}$.
- If $x \equiv 0 \pmod{p_i}$, then $z \equiv 0 \pmod{p_i}$.
- Then $p_i^2 \mid by^2 = z^2 - ax^2$, but $\nu_{p_i}(b) = 1$ implies $y \equiv 0 \pmod{p_i}$.
- Thus $(z, x, y) \equiv (0, 0, 0) \pmod{p_i}$, which is a contradiction, hence $x \not\equiv 0 \pmod{p_i}$.
- Moreover, $a = \left(\frac{z}{x}\right)^2$ is a square modulo p_i .
- Since $\mathbb{Z}/b\mathbb{Z} \cong \prod_{i=1}^k \mathbb{Z}/p_i\mathbb{Z}$, a is a square modulo b .

- There exist t, b' integers such that $t^2 = a + bb'$.
- We can choose t such that $|t| \leq |\frac{b}{2}|$. $bb' = t^2 - a$ is a norm from $K(\sqrt{a})$ where $K = \mathbb{Q}$ or \mathbb{Q}_p .
- By above lemma $(a, bb') = 1$, hence $(a, b) = 1 \iff (a, b') = 1$.
- That means $f = X_1^2 - aX_2^2 - bX_3^2$ represents 0 iff $f' = X_1^2 - aX_2^2 - b'X_3^2$ represents 0.
- $|b'| = |\frac{t^2 - a}{b}| \leq |\frac{t^2}{b}| + |\frac{a}{b}| \leq \frac{|b|}{4} + 1 \leq |b|$.
- Write $b' = u^2 b''$, where b'' is square-free. We have $|b''| \leq |b|$.
- The inductive hypothesis applies to $f'' = X_1^2 - aX_2^2 - b''X_3^2$, so it represents 0, and the same is true for f .

- $n = 4$: Suppose $f = aX_1^2 + bX_2^2 - (cX_3^2 + dX_4^2)$. There exists $a \in K^\times$ which is represented by $aX_1^2 + bX_2^2$ and $cX_3^2 + dX_4^2$.
 - $(x_v, -ab)_v = (a, b)_v$ and $(x_v, -cd)_v = (c, d)_v$ for all $v \in \mathbb{V}$
 - By above theorem there exists $x \in \mathbb{Q}^\times$ s.t. $(x, -ab)_v = (a, b)_v$ and $(x, -cd)_v = (c, d)_v$ for all $v \in \mathbb{V}$
 - This means $aX_1^2 + bX_2^2$ and $cX_3^2 + dX_4^2$ represents x in \mathbb{Q}_p
 - i.e. $aX^2 + bY^2 - xZ^2$ represents 0 in all \mathbb{Q}_v also \mathbb{Q} , and the same argument applied to $cX_3^2 + dX_4^2$, the fact that f represents 0 in \mathbb{Q} follows from this.

- $n \geq 5$: we use induction on n . Suppose $f = h \oplus -g$ with $h = a_1X_1^2 + a_2X_2^2$, $g = -(a_3X_3^2 + \cdots + a_nX_n^2)$.
- Let $S = \{p \in \mathbb{V} \mid \nu_p(a_i) \neq 0, i \geq 3\} \cup \{2, \infty\}$, it is a finite set.
- Let $v \in S$, f_v represents 0, so there exists $a_v \in \mathbb{Q}_v^\times$ which is represented by both h and g in \mathbb{Q}_v .
- That is, there exist $x_1^{(v)}, x_2^{(v)} \in \mathbb{Q}_v$ such that $h(x_1^{(v)}, x_2^{(v)}) = a_v$, and $x_3^{(v)}, \dots, x_n^{(v)} \in \mathbb{Q}_v$ such that $g(x_3^{(v)}, \dots, x_n^{(v)}) = a_v$.
- The set $\mathbb{Q}_v^{\times 2}$ is open in \mathbb{Q}_v^\times , so $\prod a_v \mathbb{Q}_v^{\times 2}$ is also open in $\prod_{v \in S} \mathbb{Q}_v^\times$, and h is a continuous map.

- By the Approximation Theorem, there exists $a \in \mathbb{Q}^\times$ such that $a \in a_v \mathbb{Q}_v^{\times 2}$ for all $v \in S$.
- Thus, $(x_1, x_2) \in (\mathbb{Q})^2$ s.t. $h(x_1, x_2) = a$, and $a/a_v \in \mathbb{Q}^{\times 2}$ for all $v \in S$.
- Consider $f_1 = aZ^2 \oplus -g$.
 - if $v \in S$, g represents a_v , also a since $a/a_v \in \mathbb{Q}^{\times 2}$.
 - if $v \notin S$, the coefficients are v -adic units, the $d(g)$ is also a unit. And because $v \neq 2$, we have $\varepsilon(g) = 1$.
- By above lemma, there exist a solution, and it lifts a true solution.
- In all case we see f_1 represents 0 in \mathbb{Q}_v , and $\text{rank } f_1 = n - 1$.
- By inductive hypothesis: f_1 represents 0 in \mathbb{Q} . i.e. g represents a , and h represents a .
- The proof is complete.

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