Presentation Speech Draft

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Good [morning/afternoon/evening], everyone. I am Shubin Xue from Beijing Institute of Technology. Today, It's my pleasure to present my talk on Classification of Quadratic Forms over $\mathbb{Q}$.

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My presentation will be divided into two main parts:

First, I will discuss the primary motivations and standard approaches to this problem.

Second, I will show the classification of quadratic forms over the field $\mathbb{Q}_p$.

The third part is an appendix about classification results and the proof of the Hasse-Minkowski theorem.

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Before we begin, let me outline some notation:

- K will denote an arbitrary field.
- X, Y, Z are used as variables.
- $\mathbb{P}$ denotes the set of all prime numbers. $\mathbb{Q}_p$ is the p-adic completion of $\mathbb{Q}$.
- $\mathbb{V}$ denotes the set of all primes together with infinity. $\mathbb{Q}_\infty$ is just $\mathbb{R}$.
- We will often denote by the same letter an element and its class modulo.

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Now, let us begin with the first part.

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In the part, I hope to show you the relationship between the classification problem and the representation problem.

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First, let us clarify our object of study:

The sum of $a_{ij}X_iX_j$ is a quadratic form.

The matrix A_f corresponding to f is a symmetric matrix. $f(X)$ can be written as a product of vectors and matrices.

The pair (K^n, f) is called a quadratic space. We say f is equivalent to g if there exists an invertible matrix P such that A_f is congruent to A_g.

If f is a quadratic form in n variables and g is a quadratic form in m variables, we denote by $f \oplus g$ the quadratic form in $n + m$ variables given by their direct sum.

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As early as the early development of number theory, mathematicians began to pay attention to the representation problem of quadratic forms.

In the 17th century, Fermat proposed the two-square theorem, which was later proved by Euler. And Euler also raised the three-square problem, but the connection between the representation problem and the classification problem was not yet noticed.

In the 19th century, mathematicians gradually discovered subtle connections between these problems.

Gauss was the first to introduce the idea of classification into the study of quadratic forms. He pointed out that represention of numbers are invariants for the classification of quadratic forms.

Later, in the 20th century, Hasse, building on Minkowski's work, proposed the "local-global principle" for quadratic forms over the rational numbers.

f represents 0 over $\mathbb{Q}$ if and only if it represents 0 over all $\mathbb{Q}_v$.

This fully established the link between "classification" and "representation".

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If there exists a nonzero element $x \in K^n$ such that $f(x) = a$, we say that f represents a.

For an element a in K, f represents a if and only if - $f \sim f_1 \oplus aZ^2$ - $f_a = f \oplus -aZ^2$ represents 0.

Item 3 allows us to reduce the problem of f representing a to the problem of f_a representing 0.

Item 2 allows us to decompose the quadratic form by rank.

Applying the Hasse-Minkowski theorem to f_a, we can directly obtain that f represents a over $\mathbb{Q}$ if and only if it represents a over all p-adic fields.

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Next, I want to show you why the Hasse-Minkowski theorem establishes the link between "classification" and "representation".

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If two quadratic forms are equivalent, and one summand is the same, then the other summand must also be equivalent.

We can consider this from the perspective of quadratic spaces. In fact, if there are isomorphic subspaces, then the isomorphism can be extended to an automorphism of the whole space. Finally, by restricting to the orthogonal complement of the subspace, we obtain an isomorphism of the complement.

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Now, let us have a look at the conditions for equivalence over $\mathbb{Q}$.

Non-degenerate quadratic forms of rank n over $\mathbb{Q}$ are equivalent if and only if they are equivalent over all $\mathbb{Q}_v$.

Necessity of this result is clear, so we only need to consider sufficiency.

We can take a rational number a represented by f.

Then f represents a over all $\mathbb{Q}_v$. Since We have f and g are equivalent over all $\mathbb{Q}_v$, then g represents a over all $\mathbb{Q}_v$.

By the Hasse-Minkowski theorem, a can be represented over $\mathbb{Q}$ by both f and g. Thus, we can decompose $f \sim f_1 \oplus aZ^2$ and $g \sim g_1 \oplus aZ^2$. (We have $f \sim g$ over all $\mathbb{Q}_v$, and clearly $aZ^2 \sim aZ^2$.) By Witt's cancellation theorem, we know that $f_1 \sim g_1$ over all $\mathbb{Q}_v$. By induction, $f_1 \sim g_1$ over $\mathbb{Q}$. Hence, $f \sim g$ over $\mathbb{Q}$.

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Now we have successfully reduced the global classification problem to a local one, so next we need to work over local fields.

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Let's first look at some general ideas.

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It suffices to classify non-degenerate diagonal quadratic forms of rank n with square-free coefficients.

- The rank is always an invariant.
- Symmetric matrix can always be diagonalized.
- Squared factors in the coefficients do not affect the classification of quadratic forms.

On the other hand, by considering the determinant of the matrix, we notice that the image of $\det(A)$ in $K^\times/(K^\times)^2$ is an invariant. We call it the discriminant.

Our work has now been reduced to squared-free, so we need to know its group structure.

For $\mathbb{R}$, it's $\{1, -1\}$.

For $\mathbb{Q}_p$, it's the Klein four-group K_4.

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The classification over $\mathbb{R}$ is completely determined by the rank and the signature, which is the famous Sylvester's law of inertia.

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Now, let us have a look at the invariants that determine the representation over $\mathbb{Q}_p$.

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We start with a simple example: binary quadratic forms over $\mathbb{Q}_p$ for $p \neq 2$.

Take a represented by both f and g.

f can be decomposed as $f_1 \oplus aZ^2$, where f_1 has rank 1, so its coefficient must be ad. In this way, f is completely determined.

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The table shows the classification results of binary quadratic forms. The same color indicates an equivalence class, while black indicates distinct equivalence classes. The boxed quadratic form cannot represent 1.

From the table, we can see that when the discriminant d is fixed, the classification of binary quadratic forms is completely determined by whether they can represent 1.

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Recall the equivalent characterization of f representing a: $Z^2 \oplus -f$ represents 0.

This leads us to another invariant that determines the classification of binary quadratic forms: the Hilbert symbol.

In the table above, the boxed quadratic form corresponds to a Hilbert symbol of -1, so we can directly obtain the value of the Hilbert symbol.

It is easy to observe from the table that the Hilbert symbol is a non-degenerate bilinear form.

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Now let us generalize the Hilbert symbol to higher dimensions, defining the Hasse invariant as the product of Hilbert symbols.

We can directly compute the discriminant of f_1, and use the bilinearity of the Hilbert symbol to obtain its Hasse invariant.

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Here are the conditions for a quadratic form to represent numbers over $\mathbb{Q}_p$.

The set of numbers represented by a quadratic form f is completely determined by its discriminant and Hasse invariant.

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Finally, let us state the conditions for equivalence of quadratic forms over $\mathbb{Q}_p$.

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Non-degenerate quadratic forms of rank n over $\mathbb{Q}_p$ are completely classified by their Hasse invariant and discriminant.

If f and g have the same discriminant d and ε, then they can both represent some a. Thus, f can be decomposed as $f_1 \oplus aZ^2$, and similarly for g.

A direct calculation shows that f_1 and g_1 have the same invariants. By induction on the rank, $f_1 \sim g_1$, so $f \sim g$.

(My presentation ends here. Thank you for your attention.)

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