Classification of Quadratic Forms over $\mathbb{Q}$

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First Attempts and General Approaches

Notations

- We denote by K an arbitrary field. All fields are assumed to be of characteristic $\neq 2$.
- $\nu_p : \mathbb{Q}_p \to \mathbb{Z}$ being the p-adic valuation.
- $\left(\frac{a}{p}\right)$ being the Legendre symbol. a is understood as $p^{-\nu_p(a)} a \bmod p$ if $a \in \mathbb{Q}_p$. Define this similarly in $\mathbb{F}_q$.
- Let $f \oplus g$ denote the direct sum of two quadratic forms f and g.
- Assume familiarity with quadratic residues and basic knowledge of p-adic numbers
- Skip most of the proofs
- Apology in advance for potential mistakes

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Review: Quadratic Forms over $\mathbb{R}$

- A quadratic form $f\colon V \to K$ may be identified by a symmetric matrix $A \in M_n(K)$ by $f(v) = v^{\mathrm{T}} A v$.
  Their equivalence is defined by *congruence*:
  $A \sim B \iff A = Q^{\mathrm{T}} B Q$.
- Real symmetric matrices may be diagonalized orthogonally.
- Scale each eigenvalue by multiplying a square. Only their sign matters.
  - the *rank* n, an invariant
  - the *signature* $(r, s) := (\#\text{positive eigenvalues}, \#\text{negative eigenvalues})$.
- Same rank and signature implies the equivalence.
- Sylvester's law of inertia: signature is also an invariant.

Some Refinement

On an arbitrary field K:

- All symmetric matrix is equivalent to a diagonal one.
  - Pick a non-isotropic vector v (exists when the form is nonzero), its orthogonal complement is a hyperplane and does not include v. Change basis and do the induction.
- The rank is always an invariant. We may (and we shall always) reduce to classify the non-degenerate quadratic forms of rank n.
- The squares $(K^\times)^2$ give us the ability to scale. Knowledge of the distribution of diagonal elements in $K^\times/(K^\times)^2$ suffices to show the equivalence[2].
  - $\mathbb{C}^\times/(\mathbb{C}^\times)^2 \cong \{1\}$, suffices to classify by the rank.
  - $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \{1, -1\}$, signature is also needed.
  - $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \cong \{1, a\}$, where $a \in \mathbb{F}_q$ is a quadratic nonresidue.
  - For $\mathbb{Q}_p$ and $\mathbb{Q}$?

[2] Though working in the refined structure $\{0\} \cup K^\times/(K^\times)^2$ is probably a better idea if one wishes to deal with the degenerate case in a uniform manner.

Another Example: Quadratic Forms over $\mathbb{F}_q$

We classify the non-degenerate quadratic forms of rank n.

- Refined signature: counting nonzero quadratic residues and nonresidues. It may serve as a sufficient criterion for equivalence.
- But it's not an invariant. $aX^2 + aY^2 \sim X^2 + Y^2$ over $\mathbb{F}_q$.
  - Do a change of basis $X = sU + tV$ and $Y = tU - sV$. If we require $aU^2 + aV^2 = X^2 + Y^2$, then $s^2 + t^2 = a$.
  - It always has a nonzero solution in $\mathbb{F}_q$: s^2 and $a - t^2$ have both $(q+1)/2$ possible values, thus must reach a common value.
- The *discriminant* $d := \left(\frac{\det(A)}{q}\right) \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ is an invariant and reveals the parity of the signature. It classifies the non-degenerate quadratic forms over $\mathbb{F}_q$.

Insight: Existence of nonzero solutions to the equation $aX^2 + bY^2 = Z^2$ in K seems to be of great importance.

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Quadratic Spaces

The structure of a quadratic space, i.e. vector space equipped with a symmetric bilinear form, is much more subtle than its positive-definite counterpart over $\mathbb{R}$ or $\mathbb{C}$. For example, for a non-degenerate quadratic space V and a subspace U of V ([Ser73] p. 28, chap. 4, sec. 1.2):

- $U \cap U^{\perp} = \mathrm{rad}(U)$, $\dim U + \dim U^{\perp} = \dim V$, $(U^{\perp})^{\perp} = U$
- $U \oplus U^{\perp} = V$ iff $U + U^{\perp} = V$ iff $\mathrm{rad}(U) = 0$
- It's much harder to show that an orthogonal basis of U expands to an orthogonal basis of V.

Structure of Quadratic Spaces

We mention some results here without details.

Theorem (Witt ([Ser73] p. 31, chap. 4, sec. 1.5, theorem 3))

Every injective metric-preserving map from a subspace U of a quadratic space V to another quadratic space W may be extended to a metric-preserving map from V to W.

Theorem (Witt's cancellation ([Ser73] p. 34, chap. 4, sec. 1.6, theorem 4))

$f_1 \oplus g_1 \sim f_2 \oplus g_2$ *and* $g_1 \sim g_2$ *implies* $f_1 \sim f_2$.

Theorem (Witt's decomposition)

Every quadratic space V is a direct sum of: $\mathrm{rad}(V)$, an anisotropic quadratic space (i.e. its nonzero vectors has nonzero norms) and a split quadratic space (i.e. $U = U^{\perp}$, full of hyperbolas)

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Another Invariant: $\mathrm{R}(f)$

On an arbitrary field K, we say that a quadratic form f *represent*s $a \in K$ if there exists a nonzero $v \in V$ such that $f(v) = a$.

- The numbers represented by a quadratic form f, $\mathrm{R}(f)$, is an invariant.
- It may be viewed in $\{0\} \cup K^\times/(K^\times)^2$.
- Is it complete?

Insights from the $\mathrm{R}(f)$

Proposition (([Ser73] p. 33, chap. 4, sec. 1.6, corollary 1))

Let $a \in K^\times$. TFAE:

- f *represents* a
- $f \sim g \oplus (Z \mapsto aZ^2)$ *where* g *is of rank* $\operatorname{rk} f - 1$.
- $f \oplus (Z \mapsto -aZ^2)$ *represents* 0.

- Insight from line 3: To understand the $\mathrm{R}(f)$, it suffices to examine when a quadratic form represents 0.
- Insight from line 2 (*the common represented element method*): Say f_1, f_2 are nonzero and represent a common $a \in K^\times$. Reducing $Z \mapsto aZ^2$, if only g_1 and g_2 also share a common represented element...

Insights from the $\mathrm{R}(f)$

- Sadly, the $\mathrm{R}(f)$ is not always a complete invariant.
  - Otherwise all indefinite quadratic forms over $\mathbb{R}$ are equivalent, absurd.
- But we shall show that when $K = \mathbb{Q}_p$ and moreover $K = \mathbb{Q}$, it plays a subtle role in the classification of quadratic forms. This requires a more precise characterization of the $\mathrm{R}(f)$.
- In fact, if only there are some simple invariants that can fully determine the $\mathrm{R}(f)$...

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Global and Local Equivalence

- Fact: Field extensions preserve the equivalence of quadratic forms.
  - Example: Equivalence classes are finer over $\mathbb{R}$ than those over $\mathbb{C}$.
- $\mathbb{Q} \hookrightarrow \mathbb{R}$, thus the rank and the signature are invariants. But we need more information to classify.
- Other field extension of $\mathbb{Q}$? $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$

Theorem (Hasse-Minkowski ([Ser73] p. 41, chap. 4, sec. 3.1, theorem 8))

f represents 0 over $\mathbb{Q}$ iff it represents 0 over $\mathbb{R}$ and all $\mathbb{Q}_p$.

- To gain more invariants for $\mathbb{Q}$ (especially those related to the $\mathrm{R}(f)$), let's classify quadratic forms over $\mathbb{Q}_p$ first.

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Structure of $\mathbb{Q}_p^\times$

- $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$ by collecting common powers of p
- $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$ by $a \mapsto a \bmod p$
  - It splits by the explicit construction of a primitive root of order p, via Hensel's lemma / Teichmüller lift $\lim_{n\to\infty} g^{p^n}$, where g is a primitive root of $\mathbb{F}_p^\times$.

Structure of $1+p\mathbb{Z}_p$ and the log / exp map

For $p \neq 2$, $\alpha \geq 1$ or $p = 2$, $\alpha \geq 2$:

$$1 + p^\alpha \mathbb{Z}_p \cong (p^\alpha \mathbb{Z}_p, +) \cong (\mathbb{Z}_p, +)$$
$$1 + p^\alpha a \mapsto \log(1 + p^\alpha a)$$

For $p = 2$, $\alpha = 1$,

- $$1 + 2\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \times (1 + 4\mathbb{Z}_2)$$
  - by $1 + 2a \mapsto a \bmod 2$
  - It splits by the explicit construction of a primitive root of order 2: $(-1, -1, \dots) = \sum_{n=0}^{+\infty} 2^n$.
- $(1 + 4\mathbb{Z}_2) \cong (4\mathbb{Z}_2, +) \cong (\mathbb{Z}_2, +)$ by the log map
- Thus
  $$1 + 2\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}_2, +)$$

Quadratic residues of $\mathbb{Q}_p$

For $p \neq 2$:

- $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{F}_p^\times \times (\mathbb{Z}_p, +)$
- 2 is a unit in $\mathbb{Z}_p$. Thus $a \in (\mathbb{Q}_p^\times)^2$ iff $\nu_p(a) \bmod 2 = 0$ and $a \bmod p \in \mathbb{F}_p^\times$ is a quadratic residue.
- $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by p and a, where $a \bmod p$ is a quadratic nonresidue.

For $p = 2$:

- $\mathbb{Q}_2^\times \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}_2, +)$
- Quadratic residues of $(\mathbb{Z}_2, +)$ are $(2\mathbb{Z}_2, +)$, which pull back to $1 + 8\mathbb{Z}_2$.
- $a \in (\mathbb{Q}_2^\times)^2$ iff $\nu_2(a) \bmod 2 = 0$ and $a \bmod 8 \equiv 1$.
- $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by 2, 3 and 5.

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The Hilbert Symbol

The Hilbert symbol over $\mathbb{Q}_p$ is defined as:

$$\langle a, b\rangle := \begin{cases} 1 & \text{if } aX^2 + bY^2 = Z^2 \text{ has a nonzero solution in } \mathbb{Q}_p \\ -1 & \text{otherwise} \end{cases}$$

The symbol may also be viewed in $\{0\} \cup \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ or even more simply in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ when working with non-degenerate forms.[3]

[3] Lots of the resources, even [Ser73], switch between these three views without enough warning. Sadly we shall also commit this usual mild sin (and have already done to other innocent invariants such as the discriminant…)

Properties of the Hilbert Symbol

- $\langle a, -a \rangle = 1$
- $\langle a, b \rangle = \langle b, a \rangle$ (symmetric)
- If $\langle a_2, b \rangle = 1$, then $\langle a_1 a_2, b \rangle = \langle a_1, b \rangle$
  - In fact, $\langle a_1 a_2, b \rangle = \langle a_1, b \rangle \langle a_2, b \rangle$ (multiplicatively bilinear)
- $\langle a, b \rangle = 1$ for all b iff $a \in \mathbb{Q}_p^2$ (nondegenerate)
- the Hilbert symbol is a non-degenerate symmetric bilinear form of the $\mathbb{F}_2$-vector space $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$
  - This is a non-trivial result and is said to be, to some extent, a generalization of the law of quadratic reciprocity in local class field theory.
  - To show above over $\mathbb{Q}_p$, we develop an explicit formula for the Hilbert symbol.

The Explicit Formula of the Hilbert Symbol

Theorem (([Ser73] p. 20, chap. 3, sec. 1.2, theorem 1))

Say $a = p^\alpha u$ *and* $b = p^\beta v$ *are* p*-adic numbers where* $u, v \in \mathbb{Z}_p^\times$*, then*

$$\langle a, b \rangle = (-1)^{\alpha \cdot \beta \cdot \frac{p-1}{2}} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \text{ if } p \neq 2$$

We omit the case $p = 2$*. It's a tedious modification of the above formula.*

	0	1	a	p
0	1	1	1	1
1		1	1	1
a			1	-1
p				$(-1)^{\frac{p-1}{2}}$

Table: Table of Hilbert symbol, $p \neq 2$

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The Hasse Invariant

Recall that we have reduced to work with non-degenerate diagonalized quadratic forms of rank n. Recall that the discriminant

$$d(f) = a_1 a_2 \ldots a_n \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$$

is an invariant.

- Define the *Hasse invariant* $\varepsilon(f) := \prod_{1 \leq i < j \leq n} \langle a_i, a_j \rangle$
- It is an invariant:

$$\varepsilon(f) = \prod_{1 \leq i < j \leq n} \langle a_i, a_j \rangle = \varepsilon(f_1) \prod_{2 \leq j \leq n} \langle a_1, a_j \rangle = \varepsilon(f_1) \cdot \langle a_1, a_1 d(f) \rangle$$

  Thus ε is preserved under *contiguous* change of orthogonal bases (fixes one of the vector of the basis)

  - For $n \geq 3$, orthogonal bases are transitive under contiguous change ([Ser73] p. 30, sec. 4.1.4, theorem 2)

d and ε Determine the $\mathrm{R}(f)$

Theorem (([Ser73] p. 36, chap. 4, sec. 2.2, theorem 6))

For a non-degenerate quadratic form f of rank n over $\mathbb{Q}_p$, the $\mathrm{R}(f)$ is determined by the discriminant $d := d(f)$ and the Hasse invariant $\varepsilon := \varepsilon(f)$. Or, in detail, f represents 0 iff:

- *For $n = 2$: $d = -1$*
- *For $n = 3$: $\langle -1, -d \rangle = \varepsilon$*
- *For $n = 4$: $d \neq 1$ or $d = 1$ and $\varepsilon = \langle -1, -1 \rangle$*
- *For $n = 5$: no conditions*

Recall that f represents $a \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ iff $f \oplus (Z \mapsto -aZ^2)$ represents 0, thus above fully characterizes the $\mathrm{R}(f)$.

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Classification of Quadratic Forms over $\mathbb{Q}_p$

Theorem (([Ser73] p. 39, chap. 4, sec. 2.3, theorem 7))

Two non-degenerate quadratic forms of rank n over $\mathbb{Q}_p$ are equivalent iff they have the same discriminant d and Hasse invariant ε.

- f, g have same d and ε, thus have the same $\mathrm{R}(f)$. Say they both represent $a \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$.
- Then $f \sim f_1 \oplus (Z \mapsto aZ^2)$, where f_1 is of rank $n-1$.
- d and ε of f_1 can be determined:
  - $d(f_1) = ad(f)$
  - $\varepsilon(f_1) = \varepsilon(f) \cdot (a, ad(f))$ (shown when discussing the invariance of ε)
- The same for g. Thus f_1, g_1 share the same d and ε (thus also their $\mathrm{R}(f)$). QED by induction.

Classification of Quadratic Forms over $\mathbb{Q}$

Theorem (([Ser73] p. 39, chap. 4, sec. 2.3, theorem 7))

Two non-degenerate quadratic forms of rank n over $\mathbb{Q}$ are equivalent iff they are equivalent over $\mathbb{R}$ and over each $\mathbb{Q}_p$.

- Say f, g are equivalent over each local field ($\mathbb{Q}_p$ and $\mathbb{R}$), thus they share the same $\mathrm{R}(f)$ locally.
- By Hasse-Minkowski theorem, they also share the same $\mathrm{R}(f)$ globally over $\mathbb{Q}$.
- Then $f \sim f_1 \oplus (Z \mapsto aZ^2)$ globally, where f_1 is of rank $n-1$. The same for g.
- $f_1 \sim g_1$ locally by Witt's cancellation theorem. QED by induction.

Problem Remains

- Proof of the Hasse-Minkowski theorem
  - essentially needs some understanding of the global property of the Hilbert symbol, which we have not discussed (cf. [Ser73])
- Refine the theory for degenerate quadratic forms (relatively easy)
- Enumerate all the equivalence classes of quadratic forms over $\mathbb{Q}_p$ and $\mathbb{Q}$ (cf. [Ser73])
- To what extent can we use the common represented element method to classify quadratic forms over other fields?
- For which fields, the $\mathrm{R}(f)$ of a quadratic form is a complete invariant? (At least $\mathbb{R}$ fails. $\mathbb{Q}$, $\mathbb{Q}_p$?)
- What can we say about $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$?
- Classification of quadratic forms over commutative rings (e.g. $\mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$)

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic*. Vol. 7. Graduate Texts in Mathematics. New York, NY: Springer, 1973. ISBN: 978-0-387-90041-4 978-1-4684-9884-4. DOI: 10.1007/978-1-4684-9884-4.