Commutative Algebra

Seminar Note

Liyve

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Abstract

Note about [AM18, *Introduction to Commutative Algebra*].

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1 Rings and Ideals

1.1 Rings and Ring Homomorphisms

Definition 1.1.1 (Ring)**.** A ring A is a set with two binary operations, usually called addition and multiplication, such that:

1. $(A, +)$ is an abelian group,
2. $(A, \cdot)$ is a semigroup,
3. Multiplication is distributive over addition: for all $a, b, c \in A$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. Multiplication is commutative: for all $a, b \in A$, $a \cdot b = b \cdot a$.
5. There exists a multiplicative identity $1 \in A$ such that for all $a \in A$, $a \cdot 1 = 1 \cdot a = a$.

Definition 1.1.2 (Ring Homomorphism)**.** A ring homomorphism is a mapping $f : A \to B$ between rings A and B such that for all $a, a' \in A$:

1. $f(a + a') = f(a) + f(a')$,

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2. $f(a \cdot a') = f(a) \cdot f(a')$,
3. $f(1_A) = 1_B$.

1.2 Ideals and Quotient Rings

Definition 1.2.1 (Ideal)**.** An ideal $\mathfrak{a}$ of a ring A is a subset $\mathfrak{a} \subseteq A$ such that:

1. $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$,
2. For all $a \in \mathfrak{a}$ and $r \in A$, both ra and ar are in $\mathfrak{a}$ (i.e., $\mathfrak{a}$ is closed under multiplication by elements of A).

Definition 1.2.2 (Quotient Ring)**.** The quotient ring $A/\mathfrak{a}$ is defined as follows: Let A be a ring and $\mathfrak{a}$ an ideal of A. The set of cosets

$$A/\mathfrak{a} = \{a + \mathfrak{a} \mid a \in A\}$$

forms a ring with operations defined by

$$(a + \mathfrak{a}) + (b + \mathfrak{a}) = (a + b) + \mathfrak{a}, \quad (a + \mathfrak{a}) \cdot (b + \mathfrak{a}) = (ab) + \mathfrak{a}.$$

The natural projection $\pi : A \to A/\mathfrak{a}$ given by $\pi(a) = a + \mathfrak{a}$ is a surjective ring homomorphism with kernel $\mathfrak{a}$.

Proposition 1.2.3 (Correspondence of Ideals)**.** Let A be a ring and $\mathfrak{a} \lhd A$ an ideal. There is a bijective correspondence between the set of ideals of A containing $\mathfrak{a}$ and the set of ideals of the quotient ring $A/\mathfrak{a}$.

Explicitly, for each ideal $\mathfrak{b}$ of A with $\mathfrak{a} \subseteq \mathfrak{b}$, the image $\bar{\mathfrak{b}} = \mathfrak{b}/\mathfrak{a}$ is an ideal of $A/\mathfrak{a}$. Conversely, for each ideal $\bar{\mathfrak{b}}$ of $A/\mathfrak{a}$, its preimage under the natural projection $\pi : A \to A/\mathfrak{a}$ is an ideal of A containing $\mathfrak{a}$.

This correspondence preserves inclusion, sums, intersections, and properties such as being prime or maximal (with suitable conditions).

$$\{\mathfrak{b} \lhd A \mid \mathfrak{a} \subseteq \mathfrak{b}\} \leftrightarrow \{\bar{\mathfrak{b}} \lhd A/\mathfrak{a}\}$$

Definition 1.2.4 (Kernel)**.** Let $f : A \to B$ be a ring homomorphism. The kernel of f, denoted $\ker f$, is the set

$$\ker f = \{a \in A \mid f(a) = 0_B\}$$

where 0_B is the additive identity in B. The kernel $\ker f$ is an ideal of A.

Definition 1.2.5 (Image)**.** Let $f : A \to B$ be a ring homomorphism. The image of f, denoted $\operatorname{Im} f$, is the set

$$\operatorname{Im} f = \{f(a) \mid a \in A\}$$

which is a subring of B.

1.3 Zero-Divisors, Nilpotent Elements and Units

Definition 1.3.1 (Zero Divisor)**.** Let A be a ring. An element $a \in A$, $a \neq 0$, is called a **zero-divisor** if there exists a nonzero $b \in A$ such that $ab = 0$ or $ba = 0$.

Definition 1.3.2 (Integral Domain)**.** A ring A is called an **integral domain** if $A \neq \{0\}$ and A has no zero-divisors; that is, for all $a, b \in A$, if $ab = 0$, then either $a = 0$ or $b = 0$.

Definition 1.3.3 (Nilpotent)**.** Let A be a ring. An element $a \in A$ is called **nilpotent** if there exists a positive integer n such that $a^n = 0$.

Definition 1.3.4 (Unit)**.** An element $u \in A$ of a ring A is called a **unit** if there exists $v \in A$ such that $uv = vu = 1$, where 1 is the multiplicative identity in A. The set of all units in A is denoted by $A^\times$.

Definition 1.3.5 (Principal Ideal)**.** An ideal $\mathfrak{a}$ of a ring A is called a **principal ideal** if there exists an element $a \in A$ such that

$$\mathfrak{a} = (a) = \{ra \mid r \in A\}.$$

That is, $\mathfrak{a}$ is generated by a single element a.

Proposition 1.3.6. Let $A \neq 0$, then TFAE:

1. A is a field
2. the only ideals in A are (0) and $A(= (1))$.
3. $\forall f : A \to B \neq 0$ is injective.

Proof.

(1) $\implies$ (2) **:** Let $\mathfrak{a} \lhd A$. If $\mathfrak{a} \neq 0$, then $\exists x$ is a unit ,$x \in \mathfrak{a}$

(2) $\implies$ (3) **:** The kernel $\ker f$ is either $\{0\}$ or A. If $\ker f = A$, then f is the zero map, so $\operatorname{Im} f = \{0\}$, contradicting $B \neq 0$. Thus, $\ker f = \{0\}$, so f is injective.

(3) $\implies$ (1) **:** Let x be not a unit. $(x) \neq (1)$. Let $B = A/(x)$, $f(x) = 0 \implies x = 0$.

□

1.4 Prime Ideals and Maximal Ideals

Definition 1.4.1 (Prime Ideal)**.** An ideal $\mathfrak{p}$ in A is prime if $\mathfrak{p} \neq (1)$ and if $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Definition 1.4.2 (Maximal Ideal)**.** An ideal $\mathfrak{m}$ in A is maximal if $\mathfrak{m} \neq (1)$ and there is no ideal $\mathfrak{a}$ s.t. $\mathfrak{m} \subsetneq \mathfrak{a} \subsetneq (1)$.

Proposition 1.4.3.

1. $\mathfrak{p}$ is prime ideal $\Leftrightarrow$ $A/\mathfrak{p}$ is integral domain.
2. $\mathfrak{m}$ is maximal ideal $\Leftrightarrow$ $A/\mathfrak{m}$ is field. Hence maximal ideals are prime.
3. Let $f : A \to B$ is ring homomorphism. $\mathfrak{p}$ is a prime ideal in B, then $f^{-1}(\mathfrak{p})$ is prime in A.

Proof.

(1)(2) **:** Omitted. cf.[聂灵沼 21, Ch.3, Sec.4, p.110, thm.7, thm.8]

(3) **:** You can consider the preimage $f^{-1}(\mathfrak{p}) = \{a \in A \mid f(a) \in \mathfrak{p}\}$. If $xy \in f^{-1}(\mathfrak{p})$, then $f(xy) = f(x)f(y) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $f(x) \in \mathfrak{p}$ or $f(y) \in \mathfrak{p}$, so $x \in f^{-1}(\mathfrak{p})$ or $y \in f^{-1}(\mathfrak{p})$.

In particular, you can consider $A/f^{-1}(\mathfrak{p}) \cong B/\mathfrak{p}$. □

Remark. Note that if $\mathfrak{m} \lhd B$ is maximal, then $f^{-1}(\mathfrak{m})$ is a maximal ideal of A if f is surjective. In general, the preimage of a maximal ideal under a ring homomorphism need not be maximal unless the map is surjective.

Let $f : \mathbb{Z} \to \mathbb{Q}$ be the natural embedding, $\mathfrak{m} = (0)$. $\mathbb{Q}$ is a field, $\mathfrak{m}$ is maximal, but its preimage $f^{-1}(\mathfrak{m}) = (0)$ in $\mathbb{Z}$ is properly contained in (p), for any $p \in \mathbb{N}$.

Lemma 1.4.4 (Zorn's lemma)**.** Let S be a non-empty partially ordered set such that every chain (i.e., totally ordered subset) in S has an upper bound in S. Then S contains at least one maximal element; that is, there exists $m \in S$ such that if $m \leq s$ for some $s \in S$, then $m = s$.

Theorem 1.4.5 (Existence of Maximal Ideals)**.** Every nonzero ring A with 1 has at least one maximal ideal.

Proof. Let S be the set of all proper ideals of A, partially ordered by inclusion. S is nonempty since (0) is a proper ideal (as $A \neq 0$). Any chain of ideals in S has an upper bound given by the union of the chain, which is again a proper ideal. By Zorn's Lemma, S has a maximal element, which is a maximal ideal of A. □

Corollary 1.4.6 (Every Ideal is Contained in a Maximal Ideal)**.** If $\mathfrak{a}$ be a proper ideal of A, then $\exists \mathfrak{m}$ is maximal, s.t. $\mathfrak{a} \subseteq \mathfrak{m}$.

Proof. Let $\mathfrak{a}$ be a proper ideal of A (i.e., $\mathfrak{a} \neq (1)$). Consider the quotient ring $A/\mathfrak{a}$. By the existence of maximal ideals, $A/\mathfrak{a}$ has a maximal ideal $\bar{\mathfrak{m}}$. The preimage $\mathfrak{m} = \pi^{-1}(\bar{\mathfrak{m}})$ under the natural projection $\pi : A \to A/\mathfrak{a}$ is a maximal ideal of A containing $\mathfrak{a}$. □

Corollary 1.4.7 (Every Non-Unit is Contained in a Maximal Ideal)**.** Every non-unit element of A is contained in some maximal ideal of A. Let $a \in A$ be a non-unit. Then the ideal (a) generated by a is a proper ideal, i.e., $(a) \neq (1)$. By the previous corollary, there exists a maximal ideal $\mathfrak{m}$ such that $(a) \subseteq \mathfrak{m}$. Thus, $a \in \mathfrak{m}$.

Proof. Let S be the set of all proper ideals of A, partially ordered by inclusion. S is nonempty since (0) is a proper ideal (as $A \neq 0$). Any chain of ideals in S has an upper bound given by the union of the chain, which is again a proper ideal. By Zorn's Lemma, S has a maximal element, which is a maximal ideal of A. □

Definition 1.4.8 (Local Ring)**.** A ring A is called a **local ring** if it has a unique maximal ideal $\mathfrak{m}$. That is, there exists exactly one maximal ideal in A.

Definition 1.4.9 (Residue Field)**.** Let A be a local ring with unique maximal ideal $\mathfrak{m}$. The **residue field** of A is the quotient ring

$$k = A/\mathfrak{m}$$

which is a field. The natural projection $A \to k$ is called the **residue map**.

Proposition 1.4.10.

1. Let A be a ring and $\mathfrak{m} \neq (1)$, s.t. $\forall x \in A \setminus \mathfrak{m}$ is a unit. Then A is a local ring, and $\mathfrak{m}$ is maximal.
2. Let A be a ring and $\mathfrak{m}$ maximal ideal of A, s.t. $1 + \mathfrak{m}$ is a unit of A. Then A is a local ring.

Proof.

(1) **:** Every non-unit is contained in $\mathfrak{m}$. Hence $\mathfrak{m}$ is the only maximal ideal.
(2) **:** $\forall \mathfrak{n} \lhd A$. If $\mathfrak{n} \not\subseteq \mathfrak{m}$, take $x \in \mathfrak{n} \setminus \mathfrak{m}$. $(x) + \mathfrak{m} = (1)$. $\exists y \in A, m \in \mathfrak{m}, xy + m = 1 \implies xy = 1 - m$ is a unit. Then $\mathfrak{n} = (1)$. Contradiction!

□

Definition 1.4.11 (Semi-local Ring)**.** A ring A is called **semi-local** if A has only finitely many maximal ideals.

Definition 1.4.12 (PID)**.** An integral domain A is called a **principal ideal domain (PID)** if every ideal of A is principal; that is, for every ideal $\mathfrak{a} \subseteq A$, there exists $a \in A$ such that $\mathfrak{a} = (a) = \{ra \mid r \in A\}$.

Proposition 1.4.13. In PID, $\mathfrak{a}$ is prime $\Leftrightarrow$ $\mathfrak{a}$ is maximal.

Proof. If $(x) \neq (1)$ is prime. Let $(x) \subsetneq (y)$. Then $x \in (y) \implies \exists z$ s.t. $x = yz$. $y \notin (x) \implies z \in (x) \implies \exists t$, s.t. $z = xt$.$ □

1.5 Nilradical and Jacobson Radical

Proposition 1.5.1.

1. The set $\mathfrak{N}$ of all nilpotent elements of A is an ideal.

$$\mathfrak{N} = \{a \in A \mid a \text{ is nilpotent}\}$$

2. And $A/\mathfrak{N}$ has no non-zero nilpotent element.

Proof.

(1) **:** If $x \in \mathfrak{N}$, then $ax \in \mathfrak{N}$, for $\forall\ a \in A$. $\forall x,\ y \in \mathfrak{N},\ \exists m,\ n,\ x^m = y^n = 0$, then

$$(x + y)^{m+n-1} = 0 \implies x + y \in \mathfrak{N}.$$

(2) **:** If $\bar{x}^n = 0$, $x^n \in \mathfrak{N} \implies \exists k,\ x^{nk} = 0 \implies x \in \mathfrak{N} \implies \bar{x} = 0$.

□

Definition 1.5.2 (Nilradical)**.** The set $\mathfrak{N}$ is called **Nilradical** of A.

Proposition 1.5.3. The nilradical $\mathfrak{N}$ of a ring A is equal to the intersection of all prime ideals of A That is, an element $a \in A$ is nilpotent if and only if a belongs to every prime ideal of A. Let

$$\mathfrak{N}' = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$$

We need to show $\mathfrak{N} = \mathfrak{N}'$

Proof.

($\mathfrak{N} \subseteq \mathfrak{N}'$) **:** If $x \in \mathfrak{N}$, then $x^n = 0 \in \mathfrak{p}$ for any $\mathfrak{p}$. It implies $x \in \mathfrak{p}$ for any $\mathfrak{p}$.
($\mathfrak{N}' \subseteq \mathfrak{N}$) **:** Suppose $\forall n > 0$, $x^n \neq 0$. Let

$$\Sigma = \{\mathfrak{a} \lhd A \mid x^n \notin \mathfrak{a}, \forall n > 0\}.$$

Let T be a totally ordered chain in Σ. Consider $\mathfrak{a}_T = \bigcup_{\mathfrak{a}\in T} \mathfrak{a}$. We claim that $\mathfrak{a}_T \in \Sigma$.

- $\mathfrak{a}_T$ is an ideal: Since T is a chain, the union of the ideals in T is again an ideal.
- For all $n > 0$, if $x^n \in \mathfrak{a}_T$, then $x^n \in \mathfrak{a}$ for some $\mathfrak{a} \in T$, contradicting the definition of Σ.

Thus, every chain in Σ has an upper bound, so by Zorn's Lemma, Σ has a maximal element, say $\mathfrak{p}$. We claim that $\mathfrak{p}$ is a prime ideal.

Suppose $a, b \notin \mathfrak{p}$. Then the ideals $\mathfrak{a}_1 = \mathfrak{p} + (a)$ and $\mathfrak{a}_2 = \mathfrak{p} + (b)$ strictly contain $\mathfrak{p}$, so by maximality, there exist $n_1, n_2 > 0$ such that $x^{n_1} \in \mathfrak{a}_1$ and $x^{n_2} \in \mathfrak{a}_2$. Thus,

$$x^{n_1} = y_1 + az_1, \quad x^{n_2} = y_2 + bz_2$$

for some $y_1, y_2 \in \mathfrak{p}$, $z_1, z_2 \in A$. Then

$$x^{n_1+n_2} = (x^{n_1})(x^{n_2}) = (y_1 + az_1)(y_2 + bz_2)$$

Expanding and using that $\mathfrak{p}$ is an ideal, all terms except abz_1z_2 are in $\mathfrak{p}$, so

$$x^{n_1+n_2} - abz_1z_2 \in \mathfrak{p} \implies x^{n_1+n_2} \in \mathfrak{p} + (ab)$$

Thus, $x^{n_1+n_2} \in \mathfrak{p} + (ab)$, so by maximality, $x^m \in \mathfrak{p} + (ab)$ for some $m > 0$, but $x^m \notin \mathfrak{p}$ by construction, so $ab \notin \mathfrak{p}$.

Therefore, $\mathfrak{p}$ is a prime ideal not containing any power of x, contradicting $x \in \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$. Thus, $\mathfrak{N} = \mathfrak{N}'$. □

Definition 1.5.4 (Jacobson Radical)**.** Let $\mathfrak{R}$ be the intersection of all maximal ideals of A:

$$\mathfrak{R} = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$$

This ideal is called the **Jacobson radical** of A.

Proposition 1.5.5. $x \in \mathfrak{R} \Leftrightarrow 1 - xy$ is a unit in A for all $y \in A$

Proof. ($\implies$): Suppose $x \in \mathfrak{R}$, but $1 - xy$ is not a unit for some $y \in A$. Then the ideal $(1 - xy)$ is proper, so it is contained in some maximal ideal $\mathfrak{m}$. Thus, $1 - xy \in \mathfrak{m}$. But $x \in \mathfrak{R} \subseteq \mathfrak{m}$, so $xy \in \mathfrak{m}$, hence $1 = (1 - xy) + xy \in \mathfrak{m}$, which is impossible since $\mathfrak{m}$ is proper. Therefore, $1 - xy$ must be a unit for all $y \in A$.

($\Longleftarrow$) **:** Suppose $x \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then the ideal generated by x and $\mathfrak{m}$ is the whole ring: $(x) + \mathfrak{m} = (1)$. This means there exist $y \in A$ and $t \in \mathfrak{m}$ such that $xy + t = 1$, or equivalently, $1 - xy = t \in \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, $1 - xy$ is not a unit only if it lies in some maximal ideal, but by assumption $x \notin \mathfrak{m}$, so $1 - xy$ cannot be non-invertible. Therefore, if $1 - xy$ is a unit for all $y \in A$, then x must be contained in every maximal ideal, i.e., $x \in \mathfrak{R}$.

□

1.6 Operations on Arbitrary Families of Ideals

Let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of ideals in a ring A.

Definition 1.6.1 (Sum of Ideals)**.** The **sum** $\sum_{i \in I} \mathfrak{a}_i$ is defined as:

$$\sum_{i \in I} \mathfrak{a}_i = \{a_1 + a_2 + \cdots + a_n \mid a_k \in \mathfrak{a}_{i_k},\ i_k \in I,\ n \geq 1\}$$

Definition 1.6.2 (Intersection of Ideals)**.** The **product** $\prod_{i \in I} \mathfrak{a}_i$ is defined as:

$$\prod_{i \in I} \mathfrak{a}_i = \left\{ \sum_{k=1}^{m} a_{1,k} \cdots a_{n,k} \mid a_{j,k} \in \mathfrak{a}_j,\ m \geq 1 \right\}$$

(For infinite families, the product is usually defined only for finite subfamilies.)

Definition 1.6.3 (Product of Ideals)**.** The **intersection** $\bigcap_{i \in I} \mathfrak{a}_i$ is defined as:

$$\bigcap_{i \in I} \mathfrak{a}_i = \{a \in A \mid a \in \mathfrak{a}_i \text{ for all } i \in I\}$$

1. Distributive law:
$$\mathfrak{a}(\mathfrak{b} + i\mathfrak{c}) = \mathfrak{ab} + \mathfrak{ac}$$

2. Modular law:
$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}, \text{if } \mathfrak{a} \supseteq \mathfrak{b}\text{or } \mathfrak{a} \supseteq \mathfrak{c}$$

   In general, we have $\mathfrak{a} + \mathfrak{b}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathfrak{ab}$. Clearly,$\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, hence $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$ provided $\mathfrak{a} + \mathfrak{b} = (1)$.

Definition 1.6.4 (Coprime)**.** Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A. We call $\mathfrak{a}, \mathfrak{b}$ are coprime, when $\mathfrak{a} + \mathfrak{b} = A$.

Definition 1.6.5 (Direct Product of Rings)**.** Let $\{A_i\}_{i \in I}$ be a family of rings. The **direct product** $\prod_{i \in I} A_i$ is defined as

$$\prod_{i \in I} A_i := \{(x_i)_{i \in I} \mid x_i \in A_i \text{ for all } i \in I\}$$

with addition and multiplication defined componentwise:

$$(x_i) + (y_i) = (x_i + y_i), \quad (x_i) \cdot (y_i) = (x_i y_i)$$

for all $(x_i), (y_i) \in \prod_{i \in I} A_i$.

Let A_i be rings, and let $p_i : \prod_{j \in I} A_j \to A_i$ be the projection onto the i-th component, defined by $p_i((x_j)_{j \in I}) = x_i$.

Definition 1.6.6 (Chinese Remainder Map)**.** Let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of ideals of A. Define the canonical ring homomorphism

$$\Phi : A \to \prod_{i \in I} A/\mathfrak{a}_i, \quad a \mapsto (a + \mathfrak{a}_i)_{i \in I}$$

where each component is the natural projection $\phi_i : A \to A/\mathfrak{a}_i$, $a \mapsto a + \mathfrak{a}_i$.

This map Φ is a ring homomorphism, called the **Chinese Remainder map** associated to the family $\{\mathfrak{a}_i\}$.

Proposition 1.6.7. Let $\{\mathfrak{a}_i\}_{i=1}^n$ be a family of ideals of A.

1. $\forall i \neq j$, $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime, then $\prod_{i=1}^n \mathfrak{a}_i = \bigcap_{i=1}^n \mathfrak{a}_i$.

2. ϕ is surjective $\Leftrightarrow$ $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime.

3. ϕ is injective $\Leftrightarrow$ $\bigcap_{i=1}^n \mathfrak{a}_i = 0$.

Proof. Omitted. cf.[AM18, ch.1, sec.6, p.7, prop.1.10]. ☐

Theorem 1.6.8 (Chinese Remainder Theorem)**.** Let A be a ring and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals of A such that $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ for all $i \neq j$ (i.e., the ideals are pairwise coprime). Then the canonical map

$$\Phi : A \to \prod_{i=1}^{n} A/\mathfrak{a}_i, \quad a \mapsto (a + \mathfrak{a}_1, \ldots, a + \mathfrak{a}_n)$$

is surjective, and its kernel is $\bigcap_{i=1}^{n} \mathfrak{a}_i$. Thus,

$$A/\left(\bigcap_{i=1}^{n} \mathfrak{a}_i\right) \cong \prod_{i=1}^{n} A/\mathfrak{a}_i$$

as rings.

In particular, if $A = \mathbb{Z}$ and the $\mathfrak{a}_i = (n_i)$ with $\gcd(n_i, n_j) = 1$ for $i \neq j$, then

$$\mathbb{Z}/(n_1 n_2 \cdots n_k) \cong \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k.$$

Proof. Let $\Phi : A \to \prod_{i=1}^{n} A/\mathfrak{a}_i$ be the canonical map, $a \mapsto (a + \mathfrak{a}_1, \ldots, a + \mathfrak{a}_n)$.

- **Kernel:** $\ker \Phi = \bigcap_{i=1}^{n} \mathfrak{a}_i$, since $a \in \ker \Phi$ iff $a \in \mathfrak{a}_i$ for all i.
- **Surjectivity:** For any $(b_1 + \mathfrak{a}_1, \ldots, b_n + \mathfrak{a}_n) \in \prod_{i=1}^{n} A/\mathfrak{a}_i$, we want $a \in A$ such that $a \equiv b_i \pmod{\mathfrak{a}_i}$ for all i.

Since the ideals are pairwise coprime, for each i there exists $e_i \in A$ such that $e_i \equiv 1 \pmod{\mathfrak{a}_i}$ and $e_i \equiv 0 \pmod{\mathfrak{a}_j}$ for $j \neq i$. (This follows from the Chinese Remainder construction: for each i, let $J_i = \bigcap_{j \neq i} \mathfrak{a}_j$, then $J_i + \mathfrak{a}_i = (1)$, so $1 = x_i + y_i$ with $x_i \in J_i$, $y_i \in \mathfrak{a}_i$; set $e_i = x_i$.)

Then set $a = \sum_{i=1}^{n} b_i e_i$. For each k, $a \equiv b_k e_k \equiv b_k \pmod{\mathfrak{a}_k}$, since $e_k \equiv 1 \pmod{\mathfrak{a}_k}$ and $e_i \equiv 0 \pmod{\mathfrak{a}_k}$ for $i \neq k$.

Thus, Φ is surjective.

- **Isomorphism:** By the First Isomorphism Theorem, $A/\ker \Phi \cong \operatorname{Im} \Phi = \prod_{i=1}^{n} A/\mathfrak{a}_i$.

Therefore,

$$A/\left(\bigcap_{i=1}^{n} \mathfrak{a}_i\right) \cong \prod_{i=1}^{n} A/\mathfrak{a}_i.$$

□

Remark. The union of ideals is not necessarily an ideal unless one contain the others.

In general, the union $\mathfrak{a} \cup \mathfrak{b}$ fails to be closed under addition. For example, in $\mathbb{Z}$, the ideals (2) and (3) have union $\{\ldots, -6, -4, -3, -2, 0, 2, 3, 4, 6, \ldots\}$, but $2 \in (2)$ and $3 \in (3)$, yet $2 + 3 = 5 \notin (2) \cup (3)$.

Proposition 1.6.9.

1. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals and let a be an ideal contained in $\bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $a \subseteq \mathfrak{p}_i$ for some i.
2. Let $a_1, \ldots, a_n$ be ideals and let $\mathfrak{p}$ be a prime ideal containing $\bigcap_{i=1}^{n} a_i$. Then $\mathfrak{p} \supseteq a_i$ for some i. If $\mathfrak{p} = \bigcap a_i$, then $\mathfrak{p} = a_i$ for some i.

Proof. Omitted. cf.[AM18, ch.1, sec.6, p.8, prop.1.11]. □

Definition 1.6.10 (Quotion of Ideals)**.** The set $(\mathfrak{a} : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} \subseteq \mathfrak{a}\}$ is **quotien** of $\mathfrak{a}$ and $\mathfrak{b}$. This set is an ideal of A.

If $\mathfrak{b} = (x)$ is a principal ideal of A, then $(\mathfrak{a} : \mathfrak{b})$ is denoted by $(\mathfrak{a} : x)$.

Definition 1.6.11 (Annihilator)**.** The set $(0 : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} = 0\}$ is called the **annihilator** of $\mathfrak{b}$. It is denoted by $\operatorname{Ann}(\mathfrak{b})$.

Proposition 1.6.12 (Zero-Divisors)**.** The set of zero-divisors of a ring A is the set

$$D = \{a \in A \mid \exists\, b \in A,\ b \neq 0,\ ab = 0 \text{ or } ba = 0\}.$$

This set is not necessarily an ideal, but it is a union of ideals of A.

$$D = \bigcup_{x \neq 0} \operatorname{Ann}(x),$$

Moreover, it is a union of prime ideals of A.

$$D = \bigcup_{\mathfrak{p} \text{ prime}} \mathfrak{p},$$

where the union is taken over all prime ideals of A.

In particular, every zero-divisor lies in some prime ideal.

Definition 1.6.13 (Radical of an Ideal)**.** Let $\mathfrak{a}$ be an ideal of a ring A. The **radical** of $\mathfrak{a}$, denoted $\sqrt{\mathfrak{a}}$ or $\mathrm{r}(\mathfrak{a})$, is defined as

$$\sqrt{\mathfrak{a}} = \{x \in A \mid \exists n > 0,\ x^n \in \mathfrak{a}\}$$

That is, x is in the radical of $\mathfrak{a}$ if some power of x lies in $\mathfrak{a}$. The radical $\sqrt{\mathfrak{a}}$ is itself an ideal of A.

If $\mathfrak{a} = (0)$, then $\sqrt{(0)}$ is the set of all nilpotent elements, i.e., the nilradical of A.

Proposition 1.6.14.

1. $\mathrm{r}(\mathfrak{a}) \supseteq \mathfrak{a}$.
2. $\mathrm{r}(\mathrm{r}(\mathfrak{a})) = \mathrm{r}(\mathfrak{a})$.
3. $\mathrm{r}(\mathfrak{a}\mathfrak{b}) = \mathrm{r}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{r}(\mathfrak{a}) \cap \mathrm{r}(\mathfrak{b})$.
4. $\mathrm{r}(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$.
5. $\mathrm{r}(\mathfrak{a} + \mathfrak{b}) = \mathrm{r}(\mathrm{r}(\mathfrak{a}) + \mathrm{r}(\mathfrak{b}))$.
6. If $\mathfrak{p}$ is prime, $\mathrm{r}(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.

Proof. Left to the reader. (Easy to check) □

Proposition 1.6.15.

$$\mathrm{r}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$$

.

Hint: Consider nilradical of the quotient ring $A/\mathfrak{a}$, and the corresponding of ideals.

Proof.1. Let $\pi : A \to A/\mathfrak{a}$ be the canonical projection. By the Correspondence Theorem, there is a bijection between the set of prime ideals of A containing $\mathfrak{a}$ and the set of prime ideals of $A/\mathfrak{a}$.

The nilradical of $A/\mathfrak{a}$, denoted $\mathfrak{N}(A/\mathfrak{a})$, is the intersection of all prime ideals of $A/\mathfrak{a}$:

$$\mathfrak{N}(A/\mathfrak{a}) = \bigcap_{\bar{\mathfrak{p}} \text{ prime in } A/\mathfrak{a}} \bar{\mathfrak{p}}$$

The preimage of this intersection under π is the intersection of all prime ideals of A containing $\mathfrak{a}$:

$$\pi^{-1}(\mathfrak{N}(A/\mathfrak{a})) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a},\ \mathfrak{p} \text{ prime}} \mathfrak{p}$$

On the other hand, $\mathfrak{N}(A/\mathfrak{a})$ consists of all elements $\bar{x} = x + \mathfrak{a}$ such that $(x + \mathfrak{a})^n = \mathfrak{a}$ for some $n \geq 1$, i.e., $x^n \in \mathfrak{a}$. Thus,

$$\pi^{-1}(\mathfrak{N}(A/\mathfrak{a})) = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \geq 1\} = \mathrm{r}(\mathfrak{a})$$

Therefore,

$$\mathrm{r}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a},\ \mathfrak{p} \text{ prime}} \mathfrak{p}$$

□

Proof.2. Let $x \in \mathrm{r}(\mathfrak{a})$. Then $x^n \in \mathfrak{a}$ for some $n > 0$. For any prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$, since $\mathfrak{p}$ is prime and $x^n \in \mathfrak{p}$, it follows that $x \in \mathfrak{p}$. Thus, x is in every prime ideal containing $\mathfrak{a}$, so $x \in \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$.

Conversely, suppose $x \notin \mathrm{r}(\mathfrak{a})$. Then $x^n \notin \mathfrak{a}$ for all $n > 0$. Consider the quotient ring $A/\mathfrak{a}$ and the image $\bar{x}$ of x. Then $\bar{x}^n \neq 0$ for all $n > 0$. By the proof of the nilradical as intersection of primes, there exists a prime ideal $\bar{\mathfrak{p}}$ of $A/\mathfrak{a}$ not containing any power of $\bar{x}$. The preimage $\mathfrak{p}$ of $\bar{\mathfrak{p}}$ under the projection $A \to A/\mathfrak{a}$ is a prime ideal of A containing $\mathfrak{a}$ but not x. Thus, $x \notin \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$.

Therefore, $\mathrm{r}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$. □

Definition 1.6.16 (Radical of a Subset)**.** More general, let $S \subseteq A$ be any subset of a ring A. The **radical** of S, denoted $\sqrt{S}$ or $\mathrm{r}(S)$, is defined as the intersection of all prime ideals of A containing S:

$$\sqrt{S} = \bigcap_{\mathfrak{p} \supseteq S,\ \mathfrak{p} \text{ prime}} \mathfrak{p}$$

Proposition 1.6.17.

1. $\mathrm{r}(\bigcap_\alpha E_\alpha) = \bigcap_\alpha \mathrm{r}(E_\alpha)$.
2. $D = \bigcap_{x \neq 0} \mathrm{r}(\mathrm{Ann}(x))$.
3. $\mathrm{r}(\mathfrak{a}),\ \mathrm{r}(\mathfrak{b})$ are coprime $\implies \mathfrak{a},\ \mathfrak{b}$ are coprime.

1.7 Extension and Contraction of Ideals

Let $f : A \to B$ be a ring homomorphism.

Definition 1.7.1 (Extension)**.** Given an ideal $\mathfrak{a} \subseteq A$, the **extension** of $\mathfrak{a}$ to B is the ideal

$$\mathfrak{a}^e = f(\mathfrak{a})B = \left\{ \sum_{i=1}^{n} f(a_i)b_i \mid a_i \in \mathfrak{a},\, b_i \in B,\, n \geq 1 \right\}$$

That is, $\mathfrak{a}^e$ is the ideal of B generated by the image of $\mathfrak{a}$.

Definition 1.7.2 (Contraction)**.** Given an ideal $\mathfrak{b} \subseteq B$, the **contraction** of $\mathfrak{b}$ to A is the ideal

$$\mathfrak{b}^c = f^{-1}(\mathfrak{b}) = \{a \in A \mid f(a) \in \mathfrak{b}\}$$

Proposition 1.7.3.

1. The extension of an ideal is always an ideal; the contraction of an ideal is always an ideal.
2. If $\mathfrak{a} \subseteq A$, then $\mathfrak{a} \subseteq (\mathfrak{a}^e)^c$.
3. If $\mathfrak{b} \subseteq B$, then $(\mathfrak{b}^c)^e \subseteq \mathfrak{b}$.
4. The set $C = \{\mathfrak{a}^e \mid \mathfrak{a} \lhd A\}$, and $E = \{\mathfrak{b}^c \mid \mathfrak{b} \lhd B\}$, then $C = \{\mathfrak{a} \mid \mathfrak{a}^{ec} = \mathfrak{a}\}$, and $E = \{\mathfrak{b} \mid \mathfrak{b}^{ce} = \mathfrak{b}\}$.
5. There is a correspondence between ideals of A and ideals of B that are stable under extension and contraction, i.e., there is a bijective between E and C.
6. If f is surjective, then every ideal of B is the extension of its contraction.
7. The contraction of a prime ideal of B is a prime ideal of A.
8. The extension of a prime ideal of A need not be prime in B.

Proof. Left to the reader. (Easy to check) cf.[AM18, ch.1, sec.7, p.10, prop.1.17] □

1.8 Spectrum and Zariski Topology

This section all of proofs will be omitted, since we have discussed in seminar

Definition 1.8.1 (Spectrum of a Ring)**.** The **spectrum** of a ring A, denoted $\operatorname{Spec} A$, is the set of all prime ideals of A:

$$\operatorname{Spec} A = \{\, \mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal} \,\}$$

Proposition 1.8.2 (Toplogy Structure of Spectum)**.** Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid E \subseteq \mathfrak{p} \,\}$. Then we have: - If $\mathfrak{a}$ is the ideal generated by E, $V(E) = V(\mathfrak{a}) = V(\mathrm{r}(\mathfrak{a}))$. - $V(0) = \operatorname{Spec} A$; $V(1) = \varnothing$. - $V\left(\bigcup_\alpha \mathfrak{a}_\alpha\right) = \bigcap_\alpha V(\mathfrak{a}_\alpha)$. - $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

Definition 1.8.3 (Zariski Topology)**.** The spectrum $\operatorname{Spec} A$ is equipped with the **Zariski topology**, where the closed sets are of the form

$$V(E) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid E \subseteq \mathfrak{p} \,\}$$

for some subset $E \subseteq A$.

In particular, for an ideal $\mathfrak{a} \subseteq A$, $V(\mathfrak{a}) = \{\, \mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p} \,\}$.

Proposition 1.8.4 (Open set of Spectum)**.** For each $f \in A$, let X_f denote complement of $V(f)$ in $X = \operatorname{Spec} A$. 0. The basic open sets are complements of $V(f)$ for $f \in A$: X_f. The basic open sets is a basis of Zariski topology.

1. $X_f \cap X_g = X_{fg}$.
2. $X_f = \emptyset \Leftrightarrow f$ is nilpotent.
3. $X_f = X \Leftrightarrow f$ is a unit.
4. $X_f = X_g \Leftrightarrow \mathrm{r}((f)) = \mathrm{r}((g))$.
5. Each X_f is quasi-compact.
6. An open subset of X is quasi-compact if and only if it is a finite union of basic open sets $X_{f_1}, \ldots, X_{f_n}$ for some $f_1, \ldots, f_n \in A$.

Proposition 1.8.5 (Closures of Spectum)**.** Denote a prime ideal of A by a letter x or y when thinking of it as a point of $X = \operatorname{Spec} A$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$.

1. The set $\{x\}$ is closed in $\operatorname{Spec} A \Leftrightarrow \mathfrak{p}$ is maximal.
2. $\overline{\{x\}} = V\mathfrak{p}_x$.
3. $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$
4. X is a T_0-space.

Remark. The Zariski topology is generally not Hausdorff; its closed sets are typically large. The points corresponding to maximal ideals are called **closed points**.

Proposition 1.8.6 (Irreducible)**.** A topology space X is said **irreducible** if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-emtpy open set is dense in X.

1. $\operatorname{Spec} A$ is irreducible if and only if the nilradical of A is a prime ideal.
2. If Y is an irreducible subspace of X, then the closure $\overline{Y}$ of Y in X is irreducible.
3. Every irreducible subspace of X is contained in a maximal irreducible subspace.
4. The maximal irreducible subspaces of X are closed and cover X. They are called the **irreducible components** of X.
5. The irreducible components of $\operatorname{Spec} A$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Remark. Let $A \neq 0$ is ring. Then A has the minimal prime ideal with respect to inclusion. (You can consider Zorn's lemma to prove this remark)

Definition 1.8.7 (Induced Map on Spectra)**.** The map $f : A \to B$ induces a map on spectra:

$$f^* : \operatorname{Spec} B \to \operatorname{Spec} A, \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$$

where $\operatorname{Spec} A$ denotes the set of all prime ideals of A.

1.9 Affine Algebraic Varieties

Let k be a field. An **affine algebraic variety** over k is a subset $V \subseteq k^n$ defined as the common zeros of a set of polynomials:

$$V = V(S) = \{ x \in k^n \mid f(x) = 0 \ \forall f \in S \}$$

for some subset $S \subseteq k[x_1, \dots, x_n]$.

The set of all polynomials vanishing on V is an ideal:

$$I(V) = \{ f \in k[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in V \}$$

There is a correspondence between affine varieties and radical ideals of $k[x_1, \dots, x_n]$ (Hilbert's Nullstellensatz).

The coordinate ring of V is defined as

$$k[V] = k[x_1, \dots, x_n]/I(V)$$

which encodes the algebraic structure of V.

2 Modules

2.1 Modules and Module Hom

Definition 2.1.1 (Module)**.** Let A be a ring. An A**-module** M is an abelian group $(M, +)$ together with an action $A \times M \to M$, $(a, m) \mapsto am$, such that for all $a, b \in A$ and $m, n \in M$:

1. $a(m + n) = am + an$
2. $(a + b)m = am + bm$
3. $(ab)m = a(bm)$
4. $1m = m$ (if A has 1)

Definition 2.1.2 (Submodule)**.** A **submodule** N of an A-module M is a subgroup $N \leq M$ such that $an \in N$ for all $a \in A$, $n \in N$.

Definition 2.1.3 (Module Homomorphism)**.** Let M, N be A-modules. A map $f : M \to N$ is an A**-module homomorphism** if for all $m, m' \in M$ and $a \in A$:

- $f(m + m') = f(m) + f(m')$
- $f(am) = af(m)$

The set of all A-module homomorphisms from M to N is denoted $\mathrm{Hom}_A(M, N)$.

Moreover, the set $\mathrm{Hom}_A(M, N)$ forms an abelian group under pointwise addition:

$$(f + g)(m) = f(m) + g(m)$$

for all $f, g \in \mathrm{Hom}_A(M, N)$ and $m \in M$.

If A is commutative, then $\mathrm{Hom}_A(M, N)$ is itself an A-module, with scalar multiplication defined by

$$(af)(m) = a \cdot f(m)$$

for $a \in A$, $f \in \mathrm{Hom}_A(M, N)$, and $m \in M$.

2.2 Submodules and Quotient Modules

Definition 2.2.1 (Quotient Module)**.** If $N \leq M$ is a submodule, the **quotient module** M/N is the abelian group of cosets $m + N$ with A-action $a(m + N) = am + N$.

Theorem 2.2.2 (Correspondence Theorem for Submodules)**.** Let M be an A-module and $N \leq M$ a submodule. There is a bijective correspondence between the set of submodules of M containing N and the set of submodules of the quotient module M/N.

Definition 2.2.3 (Kernel, Image and Cokernel)**.** Let $f : M \to N$ be an A-module homomorphism.

- The **kernel** is $\ker f = \{m \in M \mid f(m) = 0\}$, a submodule of M.
- The **image** is $\operatorname{Im} f = \{f(m) \mid m \in M\}$, a submodule of N.
- The **cokernel** is $\operatorname{Coker} f = N/\operatorname{Im} f$.

Proposition 2.2.4 (First Isomorphism Theorem)**.** Let $f : M \to N$ be an A-module homomorphism. Then

$$M/\ker f \cong \operatorname{Im} f$$

as A-modules.

Proof. Define $\varphi : M/\ker f \to \operatorname{Im} f$ by $\varphi(m + \ker f) = f(m)$. This map is well-defined, A-linear, and bijective. ☐

2.3 Operation of Submodule

Let M be an A-module, and let $\{N_i\}_{i\in I}$ be a family of submodules of M.

Definition 2.3.1 (Sum of Submodules)**.** The **sum** of submodules $\{N_i\}$ is defined as:

$$\sum_{i\in I} N_i = \left\{n_1 + \cdots + n_k \mid n_j \in N_{i_j},\ i_j \in I,\ k \geq 1\right\}$$

This is the smallest submodule of M containing all the N_i.

Definition 2.3.2 (Intersection of Submodules)**.** The **intersection** of submodules $\{N_i\}$ is:

$$\bigcap_{i\in I} N_i = \{m \in M \mid m \in N_i \text{ for all } i \in I\}$$

This is the largest submodule contained in all the N_i.

Proposition 2.3.3 (Lattice Structure)**.** The set of submodules of M forms a lattice under sum and intersection:

- $N_1 + N_2$ is the least upper bound (join) of N_1 and N_2.
- $N_1 \cap N_2$ is the greatest lower bound (meet).

Proposition 2.3.4 (Second Isomorphism Theorem)**.** Let M be an A-module, and let N, P be submodules of M. Then

$$(N + P)/P \cong N/(N \cap P)$$

as A-modules.

Proof. Define the map $\varphi : N \to (N+P)/P$ by $\varphi(n) = n+P$. This is an A-module homomorphism with kernel $N \cap P$, and it is surjective. By the First Isomorphism Theorem, $N/(N \cap P) \cong (N + P)/P$. ☐

Proposition 2.3.5 (Third Isomorphism Theorem)**.** Let M be an A-module, and let $N \subseteq P \subseteq M$ be submodules. Then

$$(M/N)/(P/N) \cong M/P$$

as A-modules.

Proof. Consider the natural map $\varphi : M/N \to M/P$ given by $m + N \mapsto m + P$. This is a well-defined A-module homomorphism with kernel P/N. By the First Isomorphism Theorem, $(M/N)/(P/N) \cong M/P$. $\square$

Definition 2.3.6 (Submodule Generated by a Subset)**.** Given a subset $S \subseteq M$, the submodule generated by S is:

$$\langle S \rangle = \left\{ \sum_{j=1}^{n} a_j s_j \mid a_j \in A,\ s_j \in S,\ n \geq 1 \right\}$$

This is the smallest submodule of M containing S.

Definition 2.3.7 (Product of Ideal and Submodule)**.** Let A be a ring, M an A-module, $\mathfrak{a} \subseteq A$ an ideal, and $N \leq M$ a submodule. The **product** $\mathfrak{a}N$ is defined as the submodule of M generated by all products an with $a \in \mathfrak{a}$, $n \in N$:

$$\mathfrak{a}N = \left\{ \sum_{i=1}^{k} a_i n_i \mid a_i \in \mathfrak{a},\ n_i \in N,\ k \geq 1 \right\}$$

This is the smallest submodule of M containing all elements an with $a \in \mathfrak{a}$, $n \in N$.

Definition 2.3.8 (Quotient of Submodules)**.** $N, P \leq M$, then $(N : P) := \{\ a \in A \mid aP \subseteq N\ \}$ is an ideal of A.

Definition 2.3.9 (Annihilator of a Module)**.** Let M be an A-module. The **annihilator** of M is

$$\operatorname{Ann}_A(M) := (0 : M) = \{a \in A \mid am = 0 \text{ for all } m \in M\}$$

which is an ideal of A.

Proposition 2.3.10. If $\mathfrak{a} \subseteq \operatorname{Ann}(M)$, then M is also $A/\mathfrak{a}$-module. The multiplication defined by $\bar{a}m = am$, It's easy to check well-defined.

Definition 2.3.11. If $\operatorname{Ann}(M) = 0$, then A-module M is faithful.
If $\operatorname{Ann}(M) = \mathfrak{a}$, then M is faithful as a $A/\mathfrak{a}$-module.

2.4 Direct Sum and Direct Product

Definition 2.4.1 (Direct Sum and Direct Product of Modules)**.** Let $\{M_i\}_{i \in I}$ be a family of A-modules.

- The **direct product** $\prod_{i \in I} M_i$ is the set of all tuples $(m_i)_{i \in I}$ with $m_i \in M_i$, with addition and scalar multiplication defined componentwise.
- The **direct sum** $\bigoplus_{i \in I} M_i$ is the subset of the direct product consisting of tuples $(m_i)_{i \in I}$ such that $m_i = 0$ for all but finitely many i.

Both $\prod_{i \in I} M_i$ and $\bigoplus_{i \in I} M_i$ are A-modules.

2.5 Finitely Generated Module

Definition 2.5.1 (Finitely Generated Module)**.** An A-module M is **finitely generated** if there exist elements $m_1, \dots, m_n \in M$ such that every $m \in M$ can be written as

$$m = a_1 m_1 + \cdots + a_n m_n$$

for some $a_1, \dots, a_n \in A$. In other words, $M = \langle m_1, \dots, m_n \rangle$.

Definition 2.5.2 (Free Module)**.** Let A be a ring and S a set. The **free A-module** on S, denoted $F = \bigoplus_{s \in S} A$, is the set of all functions $f : S \to A$ such that $f(s) = 0$ for all but finitely many $s \in S$. Equivalently, elements of F are finite formal sums

$$\sum_{i=1}^{n} a_i e_{s_i}$$

where $a_i \in A$, $s_i \in S$, and e_s is the function with $e_s(t) = \delta_{s,t}$.

F is an A-module with addition and scalar multiplication defined componentwise.

If S is finite with n elements, then $F \cong A^n$ as A-modules.

A module M is **free** if it is isomorphic to a free module on some set S; that is, $M \cong \bigoplus_{s\in S} A$ for some S.

Proposition 2.5.3. An A-module M is finitely generated if and only if there exists an integer $n \geq 0$ and a submodule $N \leq A^n$ such that $M \cong A^n/N$.

Proof Sketch: If M is finitely generated by $m_1, \dots, m_n$, define a surjective A-module homomorphism $\varphi : A^n \to M$ by $\varphi(a_1, \dots, a_n) = a_1 m_1 + \cdots + a_n m_n$. Then $M \cong A^n / \ker \varphi$. Conversely, any quotient of A^n is finitely generated. $\square$

Proposition 2.5.4. A quotient of a finitely generated module is finitely generated.

Proof. Hint: Let M be generated by $m_1, \dots, m_n$ and $N \leq M$. Then M/N is generated by the images of $m_1, \dots, m_n$ in M/N. $\square$

Theorem 2.5.5 (Hamilton-Cayley Theorem)**.** Let M be a finitely generated A-module. Let $\mathfrak{a} \lhd A$, and let $\phi : M \to M$ be an A-module endomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. Then there exist $a_1, \dots, a_n \in \mathfrak{a}$ (for some n) such that

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$$

as endomorphisms of M.

Proof. Let M be generated by $m_1, \dots, m_n$. Since $\phi(M) \subseteq \mathfrak{a}M$, for each i,

$$\phi(m_i) = \sum_{j=1}^{n} a_{ij} m_j$$

with $a_{ij} \in \mathfrak{a}$. Let $A = (a_{ij})$ be the $n \times n$ matrix over $\mathfrak{a}$ representing ϕ in this basis.

Consider the A-module homomorphism $\Phi : M^n \to M^n$ given by $\Phi = \phi \cdot I - A$, where I is the identity. By the Cayley-Hamilton theorem for modules, the characteristic polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with $a_i \in \mathfrak{a}$ annihilates ϕ:

$$f(\phi) = \phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$$

as endomorphisms of M. $\square$

Corollary 2.5.6. Let M be a finitely generated A-module and $\mathfrak{a} \lhd A$ such that $\mathfrak{a}M = M$. Then there exists $x \in A$ with $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.

Proof. Take $\phi = \mathrm{id}$. There exists $1 + a_1 + a_2 + \cdots + a_n = 0$ since Theorem 2.5.5, let $x = 1 + a_1 + a_2 + \cdots + a_n$. $\square$

Theorem 2.5.7 (Nakayama's lemma)**.** Let M be a finitely generated A-module and $\mathfrak{a} \lhd A$, if $\mathfrak{a} \subseteq \mathfrak{R}$, then $\mathfrak{a}M = M$ implies $M = 0$.

Proof. By Corollary 2.5.6, if $\mathfrak{a}M = M$ and $\mathfrak{a} \subseteq \mathfrak{R}$, then there exists $x \in A$ with $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$. That is, $x = 1 + a$ for some $a \in \mathfrak{a}$, and $xM = 0$.

But $1 + a$ is a unit in A (since $a \in \mathfrak{R}$ and Proposition 1.5.5). Therefore, x is invertible, so $xM = 0$ implies $M = 0$. $\square$

Corollary 2.5.8. Let M be a finitely generated A-module, N is a submodule of M, $\mathfrak{a} \lhd A$, if $\mathfrak{a} \subseteq \mathfrak{R}$, then $M = \mathfrak{a}M + N$ implies $N = M$.

Proof. Consider the quotient module M/N. Since $M = \mathfrak{a}M + N$, we have

$$M/N = (\mathfrak{a}M + N)/N \cong \mathfrak{a}M/(\mathfrak{a}M \cap N) \subseteq \mathfrak{a}(M/N)$$

so $M/N = \mathfrak{a}(M/N)$. By Theorem 2.5.7, since $\mathfrak{a} \subseteq \mathfrak{R}$ and M/N is finitely generated, it follows that $M/N = 0$, i.e., $M = N$. $\square$

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