

# Commutative Algebra

Seminar Note

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## Abstract

Note about [AM18, *Introduction to Commutative Algebra*].

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## 1 Rings and Ideals

### 1.1 Rings and Ring Homomorphisms

**Definition 1.1.1** (Ring). A ring  $A$  is a set with two binary operations, usually called addition and multiplication, such that:

1.  $(A, +)$  is an abelian group,
2.  $(A, \cdot)$  is a semigroup,
3. Multiplication is distributive over addition: for all  $a, b, c \in A$ ,  $a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(a + b) \cdot c = a \cdot c + b \cdot c$ .
4. Multiplication is commutative: for all  $a, b \in A$ ,  $a \cdot b = b \cdot a$ .
5. There exists a multiplicative identity  $1 \in A$  such that for all  $a \in A$ ,  $a \cdot 1 = 1 \cdot a = a$ .

**Definition 1.1.2** (Ring Homomorphism). A ring homomorphism is a mapping  $f : A \rightarrow B$  between rings  $A$  and  $B$  such that for all  $a, a' \in A$ :

1.  $f(a + a') = f(a) + f(a')$ ,

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\*Last modified on 2025-06-25.

2.  $f(a \cdot a') = f(a) \cdot f(a')$ ,
3.  $f(1_A) = 1_B$ .

## 1.2 Ideals and Quotient Rings

**Definition 1.2.1** (Ideal). An ideal  $\mathfrak{a}$  of a ring  $A$  is a subset  $\mathfrak{a} \subseteq A$  such that:

1.  $(\mathfrak{a}, +)$  is a subgroup of  $(A, +)$ ,
2. For all  $a \in \mathfrak{a}$  and  $r \in A$ , both  $ra$  and  $ar$  are in  $\mathfrak{a}$  (i.e.,  $\mathfrak{a}$  is closed under multiplication by elements of  $A$ ).

**Definition 1.2.2** (Quotient Ring). The quotient ring  $A/\mathfrak{a}$  is defined as follows: Let  $A$  be a ring and  $\mathfrak{a}$  an ideal of  $A$ . The set of cosets

$$A/\mathfrak{a} = \{a + \mathfrak{a} \mid a \in A\}$$

forms a ring with operations defined by

$$(a + \mathfrak{a}) + (b + \mathfrak{a}) = (a + b) + \mathfrak{a}, \quad (a + \mathfrak{a}) \cdot (b + \mathfrak{a}) = (ab) + \mathfrak{a}.$$

The natural projection  $\pi : A \rightarrow A/\mathfrak{a}$  given by  $\pi(a) = a + \mathfrak{a}$  is a surjective ring homomorphism with kernel  $\mathfrak{a}$ .

**Proposition 1.2.3** (Correspondence of Ideals). Let  $A$  be a ring and  $\mathfrak{a} \triangleleft A$  an ideal. There is a bijective correspondence between the set of ideals of  $A$  containing  $\mathfrak{a}$  and the set of ideals of the quotient ring  $A/\mathfrak{a}$ .

Explicitly, for each ideal  $\mathfrak{b}$  of  $A$  with  $\mathfrak{a} \subseteq \mathfrak{b}$ , the image  $\bar{\mathfrak{b}} = \mathfrak{b}/\mathfrak{a}$  is an ideal of  $A/\mathfrak{a}$ . Conversely, for each ideal  $\bar{\mathfrak{b}}$  of  $A/\mathfrak{a}$ , its preimage under the natural projection  $\pi : A \rightarrow A/\mathfrak{a}$  is an ideal of  $A$  containing  $\mathfrak{a}$ .

This correspondence preserves inclusion, sums, intersections, and properties such as being prime or maximal (with suitable conditions).

$$\{\mathfrak{b} \triangleleft A \mid \mathfrak{a} \subseteq \mathfrak{b}\} \leftrightarrow \{\bar{\mathfrak{b}} \triangleleft A/\mathfrak{a}\}$$

**Definition 1.2.4** (Kernel). Let  $f : A \rightarrow B$  be a ring homomorphism. The kernel of  $f$ , denoted  $\ker f$ , is the set

$$\ker f = \{a \in A \mid f(a) = 0_B\}$$

where  $0_B$  is the additive identity in  $B$ . The kernel  $\ker f$  is an ideal of  $A$ .

**Definition 1.2.5** (Image). Let  $f : A \rightarrow B$  be a ring homomorphism. The image of  $f$ , denoted  $\text{Im } f$ , is the set

$$\text{Im } f = \{f(a) \mid a \in A\}$$

which is a subring of  $B$ .

## 1.3 Zero-Divisors, Nilpotent Elements and Units

**Definition 1.3.1** (Zero Divisor). Let  $A$  be a ring. An element  $a \in A$ ,  $a \neq 0$ , is called a **zero-divisor** if there exists a nonzero  $b \in A$  such that  $ab = 0$  or  $ba = 0$ .

**Definition 1.3.2** (Integral Domain). A ring  $A$  is called an **integral domain** if  $A \neq \{0\}$  and  $A$  has no zero-divisors; that is, for all  $a, b \in A$ , if  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .

**Definition 1.3.3** (Nilpotent). Let  $A$  be a ring. An element  $a \in A$  is called **nilpotent** if there exists a positive integer  $n$  such that  $a^n = 0$ .

**Definition 1.3.4** (Unit). An element  $u \in A$  of a ring  $A$  is called a **unit** if there exists  $v \in A$  such that  $uv = vu = 1$ , where  $1$  is the multiplicative identity in  $A$ . The set of all units in  $A$  is denoted by  $A^\times$ .

**Definition 1.3.5** (Principal Ideal). An ideal  $\mathfrak{a}$  of a ring  $A$  is called a **principal ideal** if there exists an element  $a \in A$  such that

$$\mathfrak{a} = (a) = \{ra \mid r \in A\}.$$

That is,  $\mathfrak{a}$  is generated by a single element  $a$ .

**Proposition 1.3.6.** Let  $A \neq 0$ , then TFAE:

1.  $A$  is a field
2. the only ideals in  $A$  are  $(0)$  and  $A (= (1))$ .
3.  $\forall f : A \rightarrow B \neq 0$  is injective.

*Proof.*

- (1)  $\implies$  (2) : Let  $\mathfrak{a} \triangleleft A$ . If  $\mathfrak{a} \neq 0$ , then  $\exists x$  is a unit,  $x \in \mathfrak{a}$   
(2)  $\implies$  (3) : The kernel  $\ker f$  is either  $\{0\}$  or  $A$ . If  $\ker f = A$ , then  $f$  is the zero map, so  $\text{Im } f = \{0\}$ , contradicting  $B \neq 0$ . Thus,  $\ker f = \{0\}$ , so  $f$  is injective.  
(3)  $\implies$  (1) : Let  $x$  be not a unit.  $(x) \neq (1)$ . Let  $B = A/(x)$ ,  $f(x) = 0 \implies x = 0$ .

□

## 1.4 Prime Ideals and Maximal Ideals

**Definition 1.4.1** (Prime Ideal). An ideal  $\mathfrak{p}$  in  $A$  is prime if  $\mathfrak{p} \neq (1)$  and if  $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$ .

**Definition 1.4.2** (Maximal Ideal). An ideal  $\mathfrak{m}$  in  $A$  is maximal if  $\mathfrak{m} \neq (1)$  and there is no ideal  $\mathfrak{a}$  s.t.  $\mathfrak{m} \subsetneq \mathfrak{a} \subsetneq (1)$ .

**Proposition 1.4.3.**

1.  $\mathfrak{p}$  is prime ideal  $\Leftrightarrow A/\mathfrak{p}$  is integral domain.
2.  $\mathfrak{m}$  is maximal ideal  $\Leftrightarrow A/\mathfrak{m}$  is field. Hence maximal ideals are prime.
3. Let  $f : A \rightarrow B$  is ring homomorphism.  $\mathfrak{p}$  is a prime ideal in  $B$ , then  $f^{-1}(\mathfrak{p})$  is prime in  $A$ .

*Proof.*

- (1)(2) : Omitted. cf.[聂灵沼 21, Ch.3, Sec.4, p.110, thm.7, thm.8]  
(3) : You can consider the preimage  $f^{-1}(\mathfrak{p}) = \{a \in A \mid f(a) \in \mathfrak{p}\}$ . If  $xy \in f^{-1}(\mathfrak{p})$ , then  $f(xy) = f(x)f(y) \in \mathfrak{p}$ . Since  $\mathfrak{p}$  is prime,  $f(x) \in \mathfrak{p}$  or  $f(y) \in \mathfrak{p}$ , so  $x \in f^{-1}(\mathfrak{p})$  or  $y \in f^{-1}(\mathfrak{p})$ .

In particular, you can consider  $A/f^{-1}(\mathfrak{p}) \cong B/\mathfrak{p}$ .

□

*Remark.* Note that if  $\mathfrak{m} \triangleleft B$  is maximal, then  $f^{-1}(\mathfrak{m})$  is a maximal ideal of  $A$  if  $f$  is surjective. In general, the preimage of a maximal ideal under a ring homomorphism need not be maximal unless the map is surjective.

Let  $f : \mathbb{Z} \rightarrow \mathbb{Q}$  be the natural embedding,  $\mathfrak{m} = (0)$ .  $\mathbb{Q}$  is a field,  $\mathfrak{m}$  is maximal, but its preimage  $f^{-1}(\mathfrak{m}) = (0)$  in  $\mathbb{Z}$  is properly contained in  $(p)$ , for any  $p \in \mathbb{N}$ .

**Lemma 1.4.4** (Zorn's lemma). Let  $S$  be a non-empty partially ordered set such that every chain (i.e., totally ordered subset) in  $S$  has an upper bound in  $S$ . Then  $S$  contains at least one maximal element; that is, there exists  $m \in S$  such that if  $m \leq s$  for some  $s \in S$ , then  $m = s$ .

**Theorem 1.4.5** (Existence of Maximal Ideals). Every nonzero ring  $A$  with 1 has at least one maximal ideal.

*Proof.* Let  $S$  be the set of all proper ideals of  $A$ , partially ordered by inclusion.  $S$  is nonempty since  $(0)$  is a proper ideal (as  $A \neq 0$ ). Any chain of ideals in  $S$  has an upper bound given by the union of the chain, which is again a proper ideal. By Zorn's Lemma,  $S$  has a maximal element, which is a maximal ideal of  $A$ .

□

**Corollary 1.4.6** (Every Ideal is Contained in a Maximal Ideal). If  $\mathfrak{a}$  be a proper ideal of  $A$ , then  $\exists \mathfrak{m}$  is maximal, s.t.  $\mathfrak{a} \subseteq \mathfrak{m}$ .

*Proof.* Let  $\mathfrak{a}$  be a proper ideal of  $A$  (i.e.,  $\mathfrak{a} \neq (1)$ ). Consider the quotient ring  $A/\mathfrak{a}$ . By the existence of maximal ideals,  $A/\mathfrak{a}$  has a maximal ideal  $\bar{\mathfrak{m}}$ . The preimage  $\mathfrak{m} = \pi^{-1}(\bar{\mathfrak{m}})$  under the natural projection  $\pi : A \rightarrow A/\mathfrak{a}$  is a maximal ideal of  $A$  containing  $\mathfrak{a}$ .  $\square$

**Corollary 1.4.7** (Every Non-Unit is Contained in a Maximal Ideal). Every non-unit element of  $A$  is contained in some maximal ideal of  $A$ . Let  $a \in A$  be a non-unit. Then the ideal  $(a)$  generated by  $a$  is a proper ideal, i.e.,  $(a) \neq (1)$ . By the previous corollary, there exists a maximal ideal  $\mathfrak{m}$  such that  $(a) \subseteq \mathfrak{m}$ . Thus,  $a \in \mathfrak{m}$ .

*Proof.* Let  $S$  be the set of all proper ideals of  $A$ , partially ordered by inclusion.  $S$  is nonempty since  $(0)$  is a proper ideal (as  $A \neq 0$ ). Any chain of ideals in  $S$  has an upper bound given by the union of the chain, which is again a proper ideal. By Zorn's Lemma,  $S$  has a maximal element, which is a maximal ideal of  $A$ .  $\square$

**Definition 1.4.8** (Local Ring). A ring  $A$  is called a **local ring** if it has a unique maximal ideal  $\mathfrak{m}$ . That is, there exists exactly one maximal ideal in  $A$ .

**Definition 1.4.9** (Residue Field). Let  $A$  be a local ring with unique maximal ideal  $\mathfrak{m}$ . The **residue field** of  $A$  is the quotient ring

$$k = A/\mathfrak{m}$$

which is a field. The natural projection  $A \rightarrow k$  is called the **residue map**.

**Proposition 1.4.10.**

1. Let  $A$  be a ring and  $\mathfrak{m} \neq (1)$ , s.t.  $\forall x \in A \setminus \mathfrak{m}$  is a unit. Then  $A$  is a local ring, and  $\mathfrak{m}$  is maximal.
2. Let  $A$  be a ring and  $\mathfrak{m}$  maximal ideal of  $A$ , s.t.  $1 + \mathfrak{m}$  is a unit of  $A$ . Then  $A$  is a local ring.

*Proof.*

- (1) : Every non-unit is contained in  $\mathfrak{m}$ . Hence  $\mathfrak{m}$  is the only maximal ideal.
- (2) :  $\forall \mathfrak{n} \triangleleft A$ . If  $\mathfrak{n} \not\subseteq \mathfrak{m}$ , take  $x \in \mathfrak{n} \setminus \mathfrak{m}$ .  $(x) + \mathfrak{m} = (1)$ .  $\exists y \in A, m \in \mathfrak{m}, xy + m = 1 \implies xy = 1 - m$  is a unit. Then  $\mathfrak{n} = (1)$ . Contradiction!

$\square$

**Definition 1.4.11** (Semi-local Ring). A ring  $A$  is called **semi-local** if  $A$  has only finitely many maximal ideals.

**Definition 1.4.12** (PID). An integral domain  $A$  is called a **principal ideal domain (PID)** if every ideal of  $A$  is principal; that is, for every ideal  $\mathfrak{a} \subseteq A$ , there exists  $a \in A$  such that  $\mathfrak{a} = (a) = \{ra \mid r \in A\}$ .

**Proposition 1.4.13.** In PID,  $\mathfrak{a}$  is prime  $\Leftrightarrow \mathfrak{a}$  is maximal.

*Proof.* If  $(x) \neq (1)$  is prime. Let  $(x) \subsetneq (y)$ . Then  $x \in (y) \implies \exists z$  s.t.  $x = yz$ .  $y \notin (x) \implies z \in (x) \implies \exists t$ , s.t.  $z = xt$ .  $\square$

## 1.5 Nilradical and Jacobson Radical

**Proposition 1.5.1.**

1. The set  $\mathfrak{N}$  of all nilpotent elements of  $A$  is an ideal.

$$\mathfrak{N} = \{a \in A \mid a \text{ is nilpotent}\}$$

2. And  $A/\mathfrak{N}$  has no non-zero nilpotent element.

*Proof.*

- (1) : If  $x \in \mathfrak{N}$ , then  $ax \in \mathfrak{N}$ , for  $\forall a \in A$ .  $\forall x, y \in \mathfrak{N}$ ,  $\exists m, n$ ,  $x^m = y^n = 0$ , then

$$(x + y)^{m+n-1} = 0 \implies x + y \in \mathfrak{N}.$$

(2) : If  $\bar{x}^n = 0$ ,  $x^n \in \mathfrak{N} \implies \exists k, x^{nk} = 0 \implies x \in \mathfrak{N} \implies \bar{x} = 0$ .

□

**Definition 1.5.2** (Nilradical). The set  $\mathfrak{N}$  is called **Nilradical** of  $A$ .

**Proposition 1.5.3.** The nilradical  $\mathfrak{N}$  of a ring  $A$  is equal to the intersection of all prime ideals of  $A$ . That is, an element  $a \in A$  is nilpotent if and only if  $a$  belongs to every prime ideal of  $A$ .

Let

$$\mathfrak{N}' = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$$

We need to show  $\mathfrak{N} = \mathfrak{N}'$

*Proof.*

( $\mathfrak{N} \subseteq \mathfrak{N}'$ ) : If  $x \in \mathfrak{N}$ , then  $x^n = 0 \in \mathfrak{p}$  for any  $\mathfrak{p}$ . It implies  $x \in \mathfrak{p}$  for any  $\mathfrak{p}$ .

( $\mathfrak{N}' \subseteq \mathfrak{N}$ ) : Suppose  $\forall n > 0, x^n \neq 0$ . Let

$$\Sigma = \{\mathfrak{a} \triangleleft A \mid x^n \notin \mathfrak{a}, \forall n > 0\}.$$

Let  $T$  be a totally ordered chain in  $\Sigma$ . Consider  $\mathfrak{a}_T = \bigcup_{\mathfrak{a} \in T} \mathfrak{a}$ . We claim that  $\mathfrak{a}_T \in \Sigma$ .

- $\mathfrak{a}_T$  is an ideal: Since  $T$  is a chain, the union of the ideals in  $T$  is again an ideal.
- For all  $n > 0$ , if  $x^n \in \mathfrak{a}_T$ , then  $x^n \in \mathfrak{a}$  for some  $\mathfrak{a} \in T$ , contradicting the definition of  $\Sigma$ .

Thus, every chain in  $\Sigma$  has an upper bound, so by Zorn's Lemma,  $\Sigma$  has a maximal element, say  $\mathfrak{p}$ . We claim that  $\mathfrak{p}$  is a prime ideal.

Suppose  $a, b \notin \mathfrak{p}$ . Then the ideals  $\mathfrak{a}_1 = \mathfrak{p} + (a)$  and  $\mathfrak{a}_2 = \mathfrak{p} + (b)$  strictly contain  $\mathfrak{p}$ , so by maximality, there exist  $n_1, n_2 > 0$  such that  $x^{n_1} \in \mathfrak{a}_1$  and  $x^{n_2} \in \mathfrak{a}_2$ . Thus,

$$x^{n_1} = y_1 + az_1, \quad x^{n_2} = y_2 + bz_2$$

for some  $y_1, y_2 \in \mathfrak{p}, z_1, z_2 \in A$ . Then

$$x^{n_1+n_2} = (x^{n_1})(x^{n_2}) = (y_1 + az_1)(y_2 + bz_2)$$

Expanding and using that  $\mathfrak{p}$  is an ideal, all terms except  $abz_1z_2$  are in  $\mathfrak{p}$ , so

$$x^{n_1+n_2} - abz_1z_2 \in \mathfrak{p} \implies x^{n_1+n_2} \in \mathfrak{p} + (ab)$$

Thus,  $x^{n_1+n_2} \in \mathfrak{p} + (ab)$ , so by maximality,  $x^m \in \mathfrak{p} + (ab)$  for some  $m > 0$ , but  $x^m \notin \mathfrak{p}$  by construction, so  $ab \notin \mathfrak{p}$ .

Therefore,  $\mathfrak{p}$  is a prime ideal not containing any power of  $x$ , contradicting  $x \in \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$ . Thus,  $\mathfrak{N} = \mathfrak{N}'$ . □

**Definition 1.5.4** (Jacobson Radical). Let  $\mathfrak{R}$  be the intersection of all maximal ideals of  $A$ :

$$\mathfrak{R} = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$$

This ideal is called the **Jacobson radical** of  $A$ .

**Proposition 1.5.5.**  $x \in \mathfrak{R} \iff 1 - xy$  is a unit in  $A$  for all  $y \in A$

*Proof.* ( $\implies$ ) : Suppose  $x \in \mathfrak{R}$ , but  $1 - xy$  is not a unit for some  $y \in A$ . Then the ideal  $(1 - xy)$  is proper, so it is contained in some maximal ideal  $\mathfrak{m}$ . Thus,  $1 - xy \in \mathfrak{m}$ . But  $x \in \mathfrak{R} \subseteq \mathfrak{m}$ , so  $xy \in \mathfrak{m}$ , hence  $1 = (1 - xy) + xy \in \mathfrak{m}$ , which is impossible since  $\mathfrak{m}$  is proper. Therefore,  $1 - xy$  must be a unit for all  $y \in A$ .

( $\impliedby$ ) : Suppose  $x \notin \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ . Then the ideal generated by  $x$  and  $\mathfrak{m}$  is the whole ring:  $(x) + \mathfrak{m} = (1)$ . This means there exist  $y \in A$  and  $t \in \mathfrak{m}$  such that  $xy + t = 1$ , or equivalently,  $1 - xy = t \in \mathfrak{m}$ . Since  $\mathfrak{m}$  is maximal,  $1 - xy$  is not a unit only if it lies in some maximal ideal, but by assumption  $x \notin \mathfrak{m}$ , so  $1 - xy$  cannot be non-invertible. Therefore, if  $1 - xy$  is a unit for all  $y \in A$ , then  $x$  must be contained in every maximal ideal, i.e.,  $x \in \mathfrak{R}$ . □

## 1.6 Operations on Arbitrary Families of Ideals

Let  $\{\mathfrak{a}_i\}_{i \in I}$  be a family of ideals in a ring  $A$ .

**Definition 1.6.1** (Sum of Ideals). The **sum**  $\sum_{i \in I} \mathfrak{a}_i$  is defined as:

$$\sum_{i \in I} \mathfrak{a}_i = \{a_1 + a_2 + \cdots + a_n \mid a_k \in \mathfrak{a}_{i_k}, i_k \in I, n \geq 1\}$$

**Definition 1.6.2** (Intersection of Ideals). The **product**  $\prod_{i \in I} \mathfrak{a}_i$  is defined as:

$$\prod_{i \in I} \mathfrak{a}_i = \left\{ \sum_{k=1}^m a_{1,k} \cdots a_{n,k} \mid a_{j,k} \in \mathfrak{a}_j, m \geq 1 \right\}$$

(For infinite families, the product is usually defined only for finite subfamilies.)

**Definition 1.6.3** (Product of Ideals). The **intersection**  $\bigcap_{i \in I} \mathfrak{a}_i$  is defined as:

$$\bigcap_{i \in I} \mathfrak{a}_i = \{a \in A \mid a \in \mathfrak{a}_i \text{ for all } i \in I\}$$

1. Distributive law:

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$$

2. Modular law:

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}, \text{ if } \mathfrak{a} \supseteq \mathfrak{b} \text{ or } \mathfrak{a} \supseteq \mathfrak{c}$$

In general, we have  $\mathfrak{a} + \mathfrak{b}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathfrak{a}\mathfrak{b}$ . Clearly,  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ , hence  $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$  provided  $\mathfrak{a} + \mathfrak{b} = (1)$ .

**Definition 1.6.4** (Coprime). Let  $\mathfrak{a}, \mathfrak{b}$  be ideals of  $A$ . We call  $\mathfrak{a}, \mathfrak{b}$  are coprime, when  $\mathfrak{a} + \mathfrak{b} = A$ .

**Definition 1.6.5** (Direct Product of Rings). Let  $\{A_i\}_{i \in I}$  be a family of rings. The **direct product**  $\prod_{i \in I} A_i$  is defined as

$$\prod_{i \in I} A_i := \{(x_i)_{i \in I} \mid x_i \in A_i \text{ for all } i \in I\}$$

with addition and multiplication defined componentwise:

$$(x_i) + (y_i) = (x_i + y_i), \quad (x_i) \cdot (y_i) = (x_i y_i)$$

for all  $(x_i), (y_i) \in \prod_{i \in I} A_i$ .

Let  $A_i$  be rings, and let  $p_i : \prod_{j \in I} A_j \rightarrow A_i$  be the projection onto the  $i$ -th component, defined by  $p_i((x_j)_{j \in I}) = x_i$ .

**Definition 1.6.6** (Chinese Remainder Map). Let  $\{\mathfrak{a}_i\}_{i \in I}$  be a family of ideals of  $A$ . Define the canonical ring homomorphism

$$\Phi : A \rightarrow \prod_{i \in I} A/\mathfrak{a}_i, \quad a \mapsto (a + \mathfrak{a}_i)_{i \in I}$$

where each component is the natural projection  $\phi_i : A \rightarrow A/\mathfrak{a}_i, a \mapsto a + \mathfrak{a}_i$ .

This map  $\Phi$  is a ring homomorphism, called the **Chinese Remainder map** associated to the family  $\{\mathfrak{a}_i\}$ .

**Proposition 1.6.7.** Let  $\{\mathfrak{a}_i\}_{i=1}^n$  be a family of ideals of  $A$ .

1.  $\forall i \neq j, \mathfrak{a}_i, \mathfrak{a}_j$  are coprime, then  $\prod_{i=1}^n \mathfrak{a}_i = \bigcap_{i=1}^n \mathfrak{a}_i$ .
2.  $\phi$  is surjective  $\Leftrightarrow \mathfrak{a}_i, \mathfrak{a}_j$  are coprime.
3.  $\phi$  is injective  $\Leftrightarrow \bigcap_{i=1}^n \mathfrak{a}_i = 0$ .

*Proof.* Omitted. cf.[AM18, ch.1, sec.6, p.7, prop.1.10]. □

**Theorem 1.6.8** (Chinese Remainder Theorem). Let  $A$  be a ring and  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$  be ideals of  $A$  such that  $\mathfrak{a}_i + \mathfrak{a}_j = (1)$  for all  $i \neq j$  (i.e., the ideals are pairwise coprime). Then the canonical map

$$\Phi : A \rightarrow \prod_{i=1}^n A/\mathfrak{a}_i, \quad a \mapsto (a + \mathfrak{a}_1, \dots, a + \mathfrak{a}_n)$$

is surjective, and its kernel is  $\bigcap_{i=1}^n \mathfrak{a}_i$ . Thus,

$$A / \left( \bigcap_{i=1}^n \mathfrak{a}_i \right) \cong \prod_{i=1}^n A/\mathfrak{a}_i$$

as rings.

In particular, if  $A = \mathbb{Z}$  and the  $\mathfrak{a}_i = (n_i)$  with  $\gcd(n_i, n_j) = 1$  for  $i \neq j$ , then

$$\mathbb{Z}/(n_1 n_2 \cdots n_k) \cong \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k.$$

*Proof.* Let  $\Phi : A \rightarrow \prod_{i=1}^n A/\mathfrak{a}_i$  be the canonical map,  $a \mapsto (a + \mathfrak{a}_1, \dots, a + \mathfrak{a}_n)$ .

- **Kernel:**  $\ker \Phi = \bigcap_{i=1}^n \mathfrak{a}_i$ , since  $a \in \ker \Phi$  iff  $a \in \mathfrak{a}_i$  for all  $i$ .
- **Surjectivity:** For any  $(b_1 + \mathfrak{a}_1, \dots, b_n + \mathfrak{a}_n) \in \prod_{i=1}^n A/\mathfrak{a}_i$ , we want  $a \in A$  such that  $a \equiv b_i \pmod{\mathfrak{a}_i}$  for all  $i$ .

Since the ideals are pairwise coprime, for each  $i$  there exists  $e_i \in A$  such that  $e_i \equiv 1 \pmod{\mathfrak{a}_i}$  and  $e_i \equiv 0 \pmod{\mathfrak{a}_j}$  for  $j \neq i$ . (This follows from the Chinese Remainder construction: for each  $i$ , let  $J_i = \bigcap_{j \neq i} \mathfrak{a}_j$ , then  $J_i + \mathfrak{a}_i = (1)$ , so  $1 = x_i + y_i$  with  $x_i \in J_i$ ,  $y_i \in \mathfrak{a}_i$ ; set  $e_i = x_i$ .)

Then set  $a = \sum_{i=1}^n b_i e_i$ . For each  $k$ ,  $a \equiv b_k e_k \equiv b_k \pmod{\mathfrak{a}_k}$ , since  $e_k \equiv 1 \pmod{\mathfrak{a}_k}$  and  $e_i \equiv 0 \pmod{\mathfrak{a}_k}$  for  $i \neq k$ .

Thus,  $\Phi$  is surjective.

- **Isomorphism:** By the First Isomorphism Theorem,  $A/\ker \Phi \cong \text{Im } \Phi = \prod_{i=1}^n A/\mathfrak{a}_i$ .

Therefore,

$$A / \left( \bigcap_{i=1}^n \mathfrak{a}_i \right) \cong \prod_{i=1}^n A/\mathfrak{a}_i.$$

□

*Remark.* The union of ideals is not necessarily an ideal unless one contains the others.

In general, the union  $\mathfrak{a} \cup \mathfrak{b}$  fails to be closed under addition. For example, in  $\mathbb{Z}$ , the ideals  $(2)$  and  $(3)$  have union  $\{\dots, -6, -4, -3, -2, 0, 2, 3, 4, 6, \dots\}$ , but  $2 \in (2)$  and  $3 \in (3)$ , yet  $2 + 3 = 5 \notin (2) \cup (3)$ .

**Proposition 1.6.9.**

1. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be prime ideals and let  $\mathfrak{a}$  be an ideal contained in  $\bigcup_{i=1}^n \mathfrak{p}_i$ . Then  $\mathfrak{a} \subseteq \mathfrak{p}_i$  for some  $i$ .
2. Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$  be ideals and let  $\mathfrak{p}$  be a prime ideal containing  $\bigcap_{i=1}^n \mathfrak{a}_i$ . Then  $\mathfrak{p} \supseteq \mathfrak{a}_i$  for some  $i$ .  
If  $\mathfrak{p} = \bigcap \mathfrak{a}_i$ , then  $\mathfrak{p} = \mathfrak{a}_i$  for some  $i$ .

*Proof.* Omitted. cf.[AM18, ch.1, sec.6, p.8, prop.1.11].

□

**Definition 1.6.10** (Quotient of Ideals). The set  $(\mathfrak{a} : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} \subseteq \mathfrak{a}\}$  is **quotient** of  $\mathfrak{a}$  and  $\mathfrak{b}$ . This set is an ideal of  $A$ .

If  $\mathfrak{b} = (x)$  is a principal ideal of  $A$ , then  $(\mathfrak{a} : \mathfrak{b})$  is denoted by  $(\mathfrak{a} : x)$ .

**Definition 1.6.11** (Annihilator). The set  $(0 : \mathfrak{b}) = \{x \in A \mid x\mathfrak{b} = 0\}$  is called the **annihilator** of  $\mathfrak{b}$ . It is denoted by  $\text{Ann}(\mathfrak{b})$ .

**Proposition 1.6.12** (Zero-Divisors). The set of zero-divisors of a ring  $A$  is the set

$$D = \{a \in A \mid \exists b \in A, b \neq 0, ab = 0 \text{ or } ba = 0\}.$$

This set is not necessarily an ideal, but it is a union of ideals of  $A$ .

$$D = \bigcup_{x \neq 0} \text{Ann}(x),$$

Moreover, it is a union of prime ideals of  $A$ .

$$D = \bigcup_{\mathfrak{p} \text{ prime}} \mathfrak{p},$$

where the union is taken over all prime ideals of  $A$ .

In particular, every zero-divisor lies in some prime ideal.

**Definition 1.6.13** (Radical of an Ideal). Let  $\mathfrak{a}$  be an ideal of a ring  $A$ . The **radical** of  $\mathfrak{a}$ , denoted  $\sqrt{\mathfrak{a}}$  or  $r(\mathfrak{a})$ , is defined as

$$\sqrt{\mathfrak{a}} = \{x \in A \mid \exists n > 0, x^n \in \mathfrak{a}\}$$

That is,  $x$  is in the radical of  $\mathfrak{a}$  if some power of  $x$  lies in  $\mathfrak{a}$ . The radical  $\sqrt{\mathfrak{a}}$  is itself an ideal of  $A$ .

If  $\mathfrak{a} = (0)$ , then  $\sqrt{(0)}$  is the set of all nilpotent elements, i.e., the nilradical of  $A$ .

**Proposition 1.6.14.**

1.  $r(\mathfrak{a}) \supseteq \mathfrak{a}$ .
2.  $r(r(\mathfrak{a})) = r(\mathfrak{a})$ .
3.  $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$ .
4.  $r(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$ .
5.  $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$ .
6. If  $\mathfrak{p}$  is prime,  $r(\mathfrak{p}^n) = \mathfrak{p}$  for all  $n > 0$ .

*Proof.* Left to the reader. (Easy to check) □

**Proposition 1.6.15.**

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}, \mathfrak{p} \text{ prime}} \mathfrak{p}$$

**Hint:** Consider nilradical of the quotient ring  $A/\mathfrak{a}$ , and the corresponding of ideals.

*Proof.1.* Let  $\pi : A \rightarrow A/\mathfrak{a}$  be the canonical projection. By the Correspondence Theorem, there is a bijection between the set of prime ideals of  $A$  containing  $\mathfrak{a}$  and the set of prime ideals of  $A/\mathfrak{a}$ .

The nilradical of  $A/\mathfrak{a}$ , denoted  $\mathfrak{N}(A/\mathfrak{a})$ , is the intersection of all prime ideals of  $A/\mathfrak{a}$ :

$$\mathfrak{N}(A/\mathfrak{a}) = \bigcap_{\bar{\mathfrak{p}} \text{ prime in } A/\mathfrak{a}} \bar{\mathfrak{p}}$$

The preimage of this intersection under  $\pi$  is the intersection of all prime ideals of  $A$  containing  $\mathfrak{a}$ :

$$\pi^{-1}(\mathfrak{N}(A/\mathfrak{a})) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}, \mathfrak{p} \text{ prime}} \mathfrak{p}$$

On the other hand,  $\mathfrak{N}(A/\mathfrak{a})$  consists of all elements  $\bar{x} = x + \mathfrak{a}$  such that  $(x + \mathfrak{a})^n = \mathfrak{a}$  for some  $n \geq 1$ , i.e.,  $x^n \in \mathfrak{a}$ . Thus,

$$\pi^{-1}(\mathfrak{N}(A/\mathfrak{a})) = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \geq 1\} = r(\mathfrak{a})$$

Therefore,

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}, \mathfrak{p} \text{ prime}} \mathfrak{p}$$

□



*Proof.2.* Let  $x \in r(\mathfrak{a})$ . Then  $x^n \in \mathfrak{a}$  for some  $n > 0$ . For any prime ideal  $\mathfrak{p} \supseteq \mathfrak{a}$ , since  $\mathfrak{p}$  is prime and  $x^n \in \mathfrak{p}$ , it follows that  $x \in \mathfrak{p}$ . Thus,  $x$  is in every prime ideal containing  $\mathfrak{a}$ , so  $x \in \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$ .

Conversely, suppose  $x \notin r(\mathfrak{a})$ . Then  $x^n \notin \mathfrak{a}$  for all  $n > 0$ . Consider the quotient ring  $A/\mathfrak{a}$  and the image  $\bar{x}$  of  $x$ . Then  $\bar{x}^n \neq 0$  for all  $n > 0$ . By the proof of the nilradical as intersection of primes, there exists a prime ideal  $\bar{\mathfrak{p}}$  of  $A/\mathfrak{a}$  not containing any power of  $\bar{x}$ . The preimage  $\mathfrak{p}$  of  $\bar{\mathfrak{p}}$  under the projection  $A \rightarrow A/\mathfrak{a}$  is a prime ideal of  $A$  containing  $\mathfrak{a}$  but not  $x$ . Thus,  $x \notin \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$ .

Therefore,  $r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a} \text{ prime}} \mathfrak{p}$ .  $\square$

**Definition 1.6.16** (Radical of a Subset). More general, let  $S \subseteq A$  be any subset of a ring  $A$ . The **radical** of  $S$ , denoted  $\sqrt{S}$  or  $r(S)$ , is defined as the intersection of all prime ideals of  $A$  containing  $S$ :

$$\sqrt{S} = \bigcap_{\mathfrak{p} \supseteq S, \mathfrak{p} \text{ prime}} \mathfrak{p}$$

**Proposition 1.6.17.**

1.  $r(\bigcap_{\alpha} E_{\alpha}) = \bigcap_{\alpha} r(E_{\alpha})$ .
2.  $D = \bigcap_{x \neq 0} r(\text{Ann}(x))$ .
3.  $r(\mathfrak{a}), r(\mathfrak{b})$  are coprime  $\implies \mathfrak{a}, \mathfrak{b}$  are coprime.

## 1.7 Extension and Contraction of Ideals

Let  $f : A \rightarrow B$  be a ring homomorphism.

**Definition 1.7.1** (Extension). Given an ideal  $\mathfrak{a} \subseteq A$ , the **extension** of  $\mathfrak{a}$  to  $B$  is the ideal

$$\mathfrak{a}^e = f(\mathfrak{a})B = \left\{ \sum_{i=1}^n f(a_i)b_i \mid a_i \in \mathfrak{a}, b_i \in B, n \geq 1 \right\}$$

That is,  $\mathfrak{a}^e$  is the ideal of  $B$  generated by the image of  $\mathfrak{a}$ .

**Definition 1.7.2** (Contraction). Given an ideal  $\mathfrak{b} \subseteq B$ , the **contraction** of  $\mathfrak{b}$  to  $A$  is the ideal

$$\mathfrak{b}^c = f^{-1}(\mathfrak{b}) = \{a \in A \mid f(a) \in \mathfrak{b}\}$$

**Proposition 1.7.3.**

1. The extension of an ideal is always an ideal; the contraction of an ideal is always an ideal.
2. If  $\mathfrak{a} \subseteq A$ , then  $\mathfrak{a} \subseteq (\mathfrak{a}^e)^c$ .
3. If  $\mathfrak{b} \subseteq B$ , then  $(\mathfrak{b}^c)^e \subseteq \mathfrak{b}$ .
4. The set  $C = \{\mathfrak{a}^e \mid \mathfrak{a} \triangleleft A\}$ , and  $E = \{\mathfrak{b}^c \mid \mathfrak{b} \triangleleft B\}$ , then  $C = \{\mathfrak{a} \mid \mathfrak{a}^{ec} = \mathfrak{a}\}$ , and  $E = \{\mathfrak{b} \mid \mathfrak{b}^{ce} = \mathfrak{b}\}$ .
5. There is a correspondence between ideals of  $A$  and ideals of  $B$  that are stable under extension and contraction, i.e., there is a bijective between  $E$  and  $C$ .
6. If  $f$  is surjective, then every ideal of  $B$  is the extension of its contraction.
7. The contraction of a prime ideal of  $B$  is a prime ideal of  $A$ .
8. The extension of a prime ideal of  $A$  need not be prime in  $B$ .

*Proof.* Left to the reader. (Easy to check) cf.[AM18, ch.1, sec.7, p.10, prop.1.17]  $\square$

## 1.8 Spectrum and Zariski Topology

This section all of proofs will be omitted, since we have discussed in seminar

**Definition 1.8.1** (Spectrum of a Ring). The **spectrum** of a ring  $A$ , denoted  $\text{Spec } A$ , is the set of all prime ideals of  $A$ :

$$\text{Spec } A = \{ \mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal} \}$$

**Proposition 1.8.2** (Topology Structure of Spectrum). Let  $A$  be a ring and let  $X$  be the set of all prime ideals of  $A$ . For each subset  $E$  of  $A$ , let  $V(E) = \{ \mathfrak{p} \in \text{Spec } A \mid E \subseteq \mathfrak{p} \}$ . Then we have: - If  $\mathfrak{a}$  is the ideal generated by  $E$ ,  $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$ . -  $V(0) = \text{Spec } A$ ;  $V(1) = \emptyset$ . -  $V(\bigcup_{\alpha} \mathfrak{a}_{\alpha}) = \bigcap_{\alpha} V(\mathfrak{a}_{\alpha})$ . -  $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ .

**Definition 1.8.3** (Zariski Topology). The spectrum  $\text{Spec } A$  is equipped with the **Zariski topology**, where the closed sets are of the form

$$V(E) = \{ \mathfrak{p} \in \text{Spec } A \mid E \subseteq \mathfrak{p} \}$$

for some subset  $E \subseteq A$ .

In particular, for an ideal  $\mathfrak{a} \subseteq A$ ,  $V(\mathfrak{a}) = \{ \mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p} \}$ .

**Proposition 1.8.4** (Open set of Spectrum). For each  $f \in A$ , let  $X_f$  denote complement of  $V(f)$  in  $X = \text{Spec } A$ . The basic open sets are complements of  $V(f)$  for  $f \in A$ :  $X_f$ . The basic open sets is a basis of Zariski topology.

1.  $X_f \cap X_g = X_{fg}$ .
2.  $X_f = \emptyset \Leftrightarrow f$  is nilpotent.
3.  $X_f = X \Leftrightarrow f$  is a unit.
4.  $X_f = X_g \Leftrightarrow r((f)) = r((g))$ .
5. Each  $X_f$  is quasi-compact.
6. An open subset of  $X$  is quasi-compact if and only if it is a finite union of basic open sets  $X_{f_1}, \dots, X_{f_n}$  for some  $f_1, \dots, f_n \in A$ .

**Proposition 1.8.5** (Closures of Spectrum). Denote a prime ideal of  $A$  by a letter  $x$  or  $y$  when thinking of it as a point of  $X = \text{Spec } A$ . When thinking of  $x$  as a prime ideal of  $A$ , we denote it by  $\mathfrak{p}_x$ .

1. The set  $\{x\}$  is closed in  $\text{Spec } A \Leftrightarrow \mathfrak{p}$  is maximal.
2.  $\overline{\{x\}} = V\mathfrak{p}_x$ .
3.  $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$
4.  $X$  is a  $T_0$ -space.

*Remark.* The Zariski topology is generally not Hausdorff; its closed sets are typically large. The points corresponding to maximal ideals are called **closed points**.

**Proposition 1.8.6** (Irreducible). A topology space  $X$  is said **irreducible** if  $X \neq \emptyset$  and if every pair of non-empty open sets in  $X$  intersect, or equivalently if every non-empty open set is dense in  $X$ .

1.  $\text{Spec } A$  is irreducible if and only if the nilradical of  $A$  is a prime ideal.
2. If  $Y$  is an irreducible subspace of  $X$ , then the closure  $\overline{Y}$  of  $Y$  in  $X$  is irreducible.
3. Every irreducible subspace of  $X$  is contained in a maximal irreducible subspace.
4. The maximal irreducible subspaces of  $X$  are closed and cover  $X$ . They are called the **irreducible components** of  $X$ .
5. The irreducible components of  $\text{Spec } A$  are the closed sets  $V(\mathfrak{p})$ , where  $\mathfrak{p}$  is a minimal prime ideal of  $A$ .

*Remark.* Let  $A \neq 0$  is ring. Then  $A$  has the minimal prime ideal with respect to inclusion. (You can consider Zorn's lemma to prove this remark)

**Definition 1.8.7** (Induced Map on Spectra). The map  $f : A \rightarrow B$  induces a map on spectra:

$$f^* : \text{Spec } B \rightarrow \text{Spec } A, \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$$

where  $\text{Spec } A$  denotes the set of all prime ideals of  $A$ .

## 1.9 Affine Algebraic Varieties

Let  $k$  be a field. An **affine algebraic variety** over  $k$  is a subset  $V \subseteq k^n$  defined as the common zeros of a set of polynomials:

$$V = V(S) = \{x \in k^n \mid f(x) = 0 \ \forall f \in S\}$$

for some subset  $S \subseteq k[x_1, \dots, x_n]$ .

The set of all polynomials vanishing on  $V$  is an ideal:

$$I(V) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in V\}$$

There is a correspondence between affine varieties and radical ideals of  $k[x_1, \dots, x_n]$  (Hilbert's Nullstellensatz).

The coordinate ring of  $V$  is defined as

$$k[V] = k[x_1, \dots, x_n]/I(V)$$

which encodes the algebraic structure of  $V$ .

## 2 Modules

### 2.1 Modules and Module Hom

**Definition 2.1.1** (Module). Let  $A$  be a ring. An  **$A$ -module**  $M$  is an abelian group  $(M, +)$  together with an action  $A \times M \rightarrow M$ ,  $(a, m) \mapsto am$ , such that for all  $a, b \in A$  and  $m, n \in M$ :

1.  $a(m + n) = am + an$
2.  $(a + b)m = am + bm$
3.  $(ab)m = a(bm)$
4.  $1m = m$  (if  $A$  has 1)

**Definition 2.1.2** (Submodule). A **submodule**  $N$  of an  $A$ -module  $M$  is a subgroup  $N \leq M$  such that  $an \in N$  for all  $a \in A$ ,  $n \in N$ .

**Definition 2.1.3** (Module Homomorphism). Let  $M, N$  be  $A$ -modules. A map  $f : M \rightarrow N$  is an  **$A$ -module homomorphism** if for all  $m, m' \in M$  and  $a \in A$ :

- $f(m + m') = f(m) + f(m')$
- $f(am) = af(m)$

The set of all  $A$ -module homomorphisms from  $M$  to  $N$  is denoted  $\text{Hom}_A(M, N)$ .

Moreover, the set  $\text{Hom}_A(M, N)$  forms an abelian group under pointwise addition:

$$(f + g)(m) = f(m) + g(m)$$

for all  $f, g \in \text{Hom}_A(M, N)$  and  $m \in M$ .

If  $A$  is commutative, then  $\text{Hom}_A(M, N)$  is itself an  $A$ -module, with scalar multiplication defined by

$$(af)(m) = a \cdot f(m)$$

for  $a \in A$ ,  $f \in \text{Hom}_A(M, N)$ , and  $m \in M$ .

## 2.2 Submodules and Quotient Modules

**Definition 2.2.1** (Quotient Module). If  $N \leq M$  is a submodule, the **quotient module**  $M/N$  is the abelian group of cosets  $m + N$  with  $A$ -action  $a(m + N) = am + N$ .

**Theorem 2.2.2** (Correspondence Theorem for Submodules). Let  $M$  be an  $A$ -module and  $N \leq M$  a submodule. There is a bijective correspondence between the set of submodules of  $M$  containing  $N$  and the set of submodules of the quotient module  $M/N$ .

**Definition 2.2.3** (Kernel, Image and Cokernel). Let  $f : M \rightarrow N$  be an  $A$ -module homomorphism.

- The **kernel** is  $\ker f = \{m \in M \mid f(m) = 0\}$ , a submodule of  $M$ .
- The **image** is  $\operatorname{Im} f = \{f(m) \mid m \in M\}$ , a submodule of  $N$ .
- The **cokernel** is  $\operatorname{Coker} f = N / \operatorname{Im} f$ .

**Proposition 2.2.4** (First Isomorphism Theorem). Let  $f : M \rightarrow N$  be an  $A$ -module homomorphism. Then

$$M / \ker f \cong \operatorname{Im} f$$

as  $A$ -modules.

*Proof.* Define  $\varphi : M / \ker f \rightarrow \operatorname{Im} f$  by  $\varphi(m + \ker f) = f(m)$ . This map is well-defined,  $A$ -linear, and bijective.  $\square$

## 2.3 Operation of Submodule

Let  $M$  be an  $A$ -module, and let  $\{N_i\}_{i \in I}$  be a family of submodules of  $M$ .

**Definition 2.3.1** (Sum of Submodules). The **sum** of submodules  $\{N_i\}$  is defined as:

$$\sum_{i \in I} N_i = \{n_1 + \cdots + n_k \mid n_j \in N_{i_j}, i_j \in I, k \geq 1\}$$

This is the smallest submodule of  $M$  containing all the  $N_i$ .

**Definition 2.3.2** (Intersection of Submodules). The **intersection** of submodules  $\{N_i\}$  is:

$$\bigcap_{i \in I} N_i = \{m \in M \mid m \in N_i \text{ for all } i \in I\}$$

This is the largest submodule contained in all the  $N_i$ .

**Proposition 2.3.3** (Lattice Structure). The set of submodules of  $M$  forms a lattice under sum and intersection:

- $N_1 + N_2$  is the least upper bound (join) of  $N_1$  and  $N_2$ .
- $N_1 \cap N_2$  is the greatest lower bound (meet).

**Proposition 2.3.4** (Second Isomorphism Theorem). Let  $M$  be an  $A$ -module, and let  $N, P$  be submodules of  $M$ . Then

$$(N + P) / P \cong N / (N \cap P)$$

as  $A$ -modules.

*Proof.* Define the map  $\varphi : N \rightarrow (N + P) / P$  by  $\varphi(n) = n + P$ . This is an  $A$ -module homomorphism with kernel  $N \cap P$ , and it is surjective. By the First Isomorphism Theorem,  $N / (N \cap P) \cong (N + P) / P$ .  $\square$

**Proposition 2.3.5** (Third Isomorphism Theorem). Let  $M$  be an  $A$ -module, and let  $N \subseteq P \subseteq M$  be submodules. Then

$$(M / N) / (P / N) \cong M / P$$

as  $A$ -modules.

*Proof.* Consider the natural map  $\varphi : M/N \rightarrow M/P$  given by  $m + N \mapsto m + P$ . This is a well-defined  $A$ -module homomorphism with kernel  $P/N$ . By the First Isomorphism Theorem,  $(M/N)/(P/N) \cong M/P$ .  $\square$

**Definition 2.3.6** (Submodule Generated by a Subset). Given a subset  $S \subseteq M$ , the submodule generated by  $S$  is:

$$\langle S \rangle = \left\{ \sum_{j=1}^n a_j s_j \mid a_j \in A, s_j \in S, n \geq 1 \right\}$$

This is the smallest submodule of  $M$  containing  $S$ .

**Definition 2.3.7** (Product of Ideal and Submodule). Let  $A$  be a ring,  $M$  an  $A$ -module,  $\mathfrak{a} \subseteq A$  an ideal, and  $N \leq M$  a submodule. The **product**  $\mathfrak{a}N$  is defined as the submodule of  $M$  generated by all products  $an$  with  $a \in \mathfrak{a}$ ,  $n \in N$ :

$$\mathfrak{a}N = \left\{ \sum_{i=1}^k a_i n_i \mid a_i \in \mathfrak{a}, n_i \in N, k \geq 1 \right\}$$

This is the smallest submodule of  $M$  containing all elements  $an$  with  $a \in \mathfrak{a}$ ,  $n \in N$ .

**Definition 2.3.8** (Quotient of Submodules).  $N, P \leq M$ , then  $(N : P) := \{ a \in A \mid aP \subseteq N \}$  is an ideal of  $A$ .

**Definition 2.3.9** (Annihilator of a Module). Let  $M$  be an  $A$ -module. The **annihilator** of  $M$  is

$$\text{Ann}_A(M) := (0 : M) = \{ a \in A \mid am = 0 \text{ for all } m \in M \}$$

which is an ideal of  $A$ .

**Proposition 2.3.10.** If  $\mathfrak{a} \subseteq \text{Ann}(M)$ , then  $M$  is also  $A/\mathfrak{a}$ -module. The multiplication defined by  $\bar{a}m = am$ , It's easy to check well-defined.

**Definition 2.3.11.** If  $\text{Ann}(M) = 0$ , then  $A$ -module  $M$  is faithful.

If  $\text{Ann}(M) = \mathfrak{a}$ , then  $M$  is faithful as a  $A/\mathfrak{a}$ -module.

## 2.4 Direct Sum and Direct Product

**Definition 2.4.1** (Direct Sum and Direct Product of Modules). Let  $\{M_i\}_{i \in I}$  be a family of  $A$ -modules.

- The **direct product**  $\prod_{i \in I} M_i$  is the set of all tuples  $(m_i)_{i \in I}$  with  $m_i \in M_i$ , with addition and scalar multiplication defined componentwise.
- The **direct sum**  $\bigoplus_{i \in I} M_i$  is the subset of the direct product consisting of tuples  $(m_i)_{i \in I}$  such that  $m_i = 0$  for all but finitely many  $i$ .

Both  $\prod_{i \in I} M_i$  and  $\bigoplus_{i \in I} M_i$  are  $A$ -modules.

## 2.5 Finitely Generated Module

**Definition 2.5.1** (Finitely Generated Module). An  $A$ -module  $M$  is **finitely generated** if there exist elements  $m_1, \dots, m_n \in M$  such that every  $m \in M$  can be written as

$$m = a_1 m_1 + \dots + a_n m_n$$

for some  $a_1, \dots, a_n \in A$ . In other words,  $M = \langle m_1, \dots, m_n \rangle$ .

**Definition 2.5.2** (Free Module). Let  $A$  be a ring and  $S$  a set. The **free  $A$ -module** on  $S$ , denoted  $F = \bigoplus_{s \in S} A$ , is the set of all functions  $f : S \rightarrow A$  such that  $f(s) = 0$  for all but finitely many  $s \in S$ . Equivalently, elements of  $F$  are finite formal sums

$$\sum_{i=1}^n a_i e_{s_i}$$

where  $a_i \in A$ ,  $s_i \in S$ , and  $e_s$  is the function with  $e_s(t) = \delta_{s,t}$ .

$F$  is an  $A$ -module with addition and scalar multiplication defined componentwise.

If  $S$  is finite with  $n$  elements, then  $F \cong A^n$  as  $A$ -modules.

A module  $M$  is **free** if it is isomorphic to a free module on some set  $S$ ; that is,  $M \cong \bigoplus_{s \in S} A$  for some  $S$ .

**Proposition 2.5.3.** An  $A$ -module  $M$  is finitely generated if and only if there exists an integer  $n \geq 0$  and a submodule  $N \leq A^n$  such that  $M \cong A^n/N$ .

*Proof Sketch:* If  $M$  is finitely generated by  $m_1, \dots, m_n$ , define a surjective  $A$ -module homomorphism  $\varphi : A^n \rightarrow M$  by  $\varphi(a_1, \dots, a_n) = a_1 m_1 + \dots + a_n m_n$ . Then  $M \cong A^n / \ker \varphi$ . Conversely, any quotient of  $A^n$  is finitely generated.  $\square$

**Proposition 2.5.4.** A quotient of a finitely generated module is finitely generated.

*Proof.* Hint: Let  $M$  be generated by  $m_1, \dots, m_n$  and  $N \leq M$ . Then  $M/N$  is generated by the images of  $m_1, \dots, m_n$  in  $M/N$ .  $\square$

**Theorem 2.5.5** (Hamilton-Cayley Theorem). Let  $M$  be a finitely generated  $A$ -module. Let  $\mathfrak{a} \triangleleft A$ , and let  $\phi : M \rightarrow M$  be an  $A$ -module endomorphism such that  $\phi(M) \subseteq \mathfrak{a}M$ . Then there exist  $a_1, \dots, a_n \in \mathfrak{a}$  (for some  $n$ ) such that

$$\phi^n + a_1 \phi^{n-1} + \dots + a_n = 0$$

as endomorphisms of  $M$ .

*Proof.* Let  $M$  be generated by  $m_1, \dots, m_n$ . Since  $\phi(M) \subseteq \mathfrak{a}M$ , for each  $i$ ,

$$\phi(m_i) = \sum_{j=1}^n a_{ij} m_j$$

with  $a_{ij} \in \mathfrak{a}$ . Let  $A = (a_{ij})$  be the  $n \times n$  matrix over  $\mathfrak{a}$  representing  $\phi$  in this basis.

Consider the  $A$ -module homomorphism  $\Phi : M^n \rightarrow M^n$  given by  $\Phi = \phi \cdot I - A$ , where  $I$  is the identity. By the Cayley-Hamilton theorem for modules, the characteristic polynomial  $f(x) = x^n + a_1 x^{n-1} + \dots + a_n$  with  $a_i \in \mathfrak{a}$  annihilates  $\phi$ :

$$f(\phi) = \phi^n + a_1 \phi^{n-1} + \dots + a_n = 0$$

as endomorphisms of  $M$ .  $\square$

**Corollary 2.5.6.** Let  $M$  be a finitely generated  $A$ -module and  $\mathfrak{a} \triangleleft A$  such that  $\mathfrak{a}M = M$ . Then there exists  $x \in A$  with  $x \equiv 1 \pmod{\mathfrak{a}}$  such that  $xM = 0$ .

*Proof.* Take  $\phi = \text{id}$ . There exists  $1 + a_1 + a_2 + \dots + a_n = 0$  since Theorem 2.5.5, let  $x = 1 + a_1 + a_2 + \dots + a_n$ .  $\square$

**Theorem 2.5.7** (Nakayama's lemma). Let  $M$  be a finitely generated  $A$ -module and  $\mathfrak{a} \triangleleft A$ , if  $\mathfrak{a} \subseteq \mathfrak{R}$ , then  $\mathfrak{a}M = M$  implies  $M = 0$ .

*Proof.* By Corollary 2.5.6, if  $\mathfrak{a}M = M$  and  $\mathfrak{a} \subseteq \mathfrak{R}$ , then there exists  $x \in A$  with  $x \equiv 1 \pmod{\mathfrak{a}}$  such that  $xM = 0$ . That is,  $x = 1 + a$  for some  $a \in \mathfrak{a}$ , and  $xM = 0$ .

But  $1 + a$  is a unit in  $A$  (since  $a \in \mathfrak{R}$  and Proposition 1.5.5). Therefore,  $x$  is invertible, so  $xM = 0$  implies  $M = 0$ .  $\square$

**Corollary 2.5.8.** Let  $M$  be a finitely generated  $A$ -module,  $N$  is a submodule of  $M$ ,  $\mathfrak{a} \triangleleft A$ , if  $\mathfrak{a} \subseteq \mathfrak{R}$ , then  $M = \mathfrak{a}M + N$  implies  $N = M$ .

*Proof.* Consider the quotient module  $M/N$ . Since  $M = \mathfrak{a}M + N$ , we have

$$M/N = (\mathfrak{a}M + N)/N \cong \mathfrak{a}M/(\mathfrak{a}M \cap N) \subseteq \mathfrak{a}(M/N)$$

so  $M/N = \mathfrak{a}(M/N)$ . By Theorem 2.5.7, since  $\mathfrak{a} \subseteq \mathfrak{R}$  and  $M/N$  is finitely generated, it follows that  $M/N = 0$ , i.e.,  $M = N$ .  $\square$

## References

- [AM18] Michael F Atiyah and Ian Grant Macdonald. *Introduction to commutative algebra*. CRC Press, 2018.
- [聂灵沼 21] 聂灵沼. 代数学引论. 高等教育出版社, 2021.