Final Assignment of Selected Topics in Probability

The Existence and Uniqueness of Regular Conditional Probability

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Abstract

This assignment explores the concept of regular conditional probability, focusing on its existence and uniqueness. However, the proofs for existence and uniqueness were not provided in [1]. As a result, rigorously establishing these proofs became the central focus of the assignment.

## 1 Foreword

During last semester’s study of *Probability Theory (bilingual course)*, students at the **Qiushi College** were learning *Probability Theory* using the textbook [2] . This textbook does not introduce probability theory through the language of measure theory, nor does it cover the concept of conditional expectation. However, during a proof in *Mathematical Statistics* in the second week of this semester, conditional expectation appeared in the textbook [3] as an integral of conditional probability. This led me to hypothesize that the concept of conditional expectation mentioned in the textbook [3] must be the same as what we studied last semester [1]. After discussions with classmates, we failed to reach a satisfactory conclusion. It was not until the fourth week, during the *Selected Topics in Probability*, that the concept of regular conditional probability was briefly introduced. Unfortunately, [1] claimed that this topic had little relevance to the subsequent course content and thus avoided further discussion. Consequently, I decided to explore this topic as the focus of my final assignment.

## 2 Regular Conditional Probability

In our in-class discussion, we observed that conditional probability

satisfies

for , pairwise disjoint

almost surely.

However, the null exceptional set depends on all of measurable set , and there exists no universal null set that makes the conditional probability be a probability measure. This motivates us to explore a more refined approach to characterize in a stronger sense.

**Definition 1 (Probability Kernel)** Let and be a pair of measurable spaces. A function satisfy:

* is an -measurable map for all ;
* is a probability measure on for all .

Then we call it **Probability Kernel**.

To obtain the desired “conditional probability”, we need to work within a better-behaved space, where such constructions can be rigorously defined.

**Definition 2 (Standard Borel Space)** A measurable space is called **standard Borel space**, if there exists a complete metric on , such that is separable and the Borel -algebra is generated by the topology on .

**Proposition 1 (Regular Conditional Probability)** cf.[1, Proposition 5.4.3]

Let be a standard Borel space, a probability measure on . Then for each -algebra , there exists a probability kernel from to , such that for all , it holds that:

where the exceptional set might depend on .

If is another probability kernel from to with the same property, then there exists a -zero set , such that for every and every , it holds that:

The probability is called regular conditional probability given by .

## 3 The Existence of Regular Conditional Probability

Before proceeding with the proof, we first state and prove some lemmas and theorems that will be essential for the following arguments.

**Lemma 1 (Doob-Dynkin lemma)** Let be a probability space, and let be a measurable function into a measurable space . If is -measurable, then there exists a measurable function such that almost surely.

*Proof* (Proof of [Lemma 1](#lem-doob-dynkin)). Let be -measurable, where is measurable.

#### Step 1: Structure of

The -algebra generated by is defined as:

Since is -measurable, for every Borel set , we have:

Thus, there exists a set such that:

This defines a mapping from Borel sets in to . To ensure consistency, this mapping must preserve set operations (e.g., unions, intersections, complements), which follows from the fact that and are both -homomorphisms.

#### Step 2: Constructing via rational intervals

We construct as follows. For each rational , define:

where corresponds to the set in such that .

For , define:

This infimum is well-defined because:

* For any , since for all , there exists some such that , hence .
* The set is bounded below (by, say, ).

#### Step 3: Verifying almost surely

Let . We claim that except on a -null set.

* **For with :**
  + If , then , so , implying .
  + Conversely, if , then , so .
* **For with :**
  + If , then , so , implying .

Thus, and hold for all outside a null set where may not equal . However, since and are measurable, the set is measurable and has measure zero.

#### Step 4: Measurability of

To show is -measurable, we verify that for any , the set belongs to .

* **For rational :**
* This follows from the definition of as an infimum over rationals.
* **For general :** Approximate by a decreasing sequence of rationals . Then:
* Since each , the countable intersection is also in .

Hence, is -measurable.

#### Step 5: Uniqueness up to null sets

If is another measurable function satisfying almost surely, then for -almost every . Since is measurable, the pushforward measure ensures that almost surely with respect to .

We have constructed a measurable function such that almost surely, completing the proof.

*Remark*. The **Doob-Dynkin lemma** establishes that any -measurable function can be expressed as for some measurable . In the context of regular conditional probabilities, it ensures that the conditional expectation , being -measurable, can be represented as a function of a generating random variable (e.g., ). This allows the construction of the kernel as a measurable function of , satisfying the required properties.

**Theorem 1 (Carathéodory extension theorem)** Let be an algebra of subsets of a set , and let be a countably additive pre-measure. Then there exists a measure on the -algebra such that . Moreover, this extension is unique if is -finite.

*Proof* (Proof of [Theorem 1](#thm-Caratheodory-extension-theorem)). Let be an algebra of subsets of , and a countably additive pre-measure.

#### Step 1: Outer measure construction

Define the outer measure on all subsets by

We verify that is an outer measure:

* **Monotonicity**: If , then any cover of is also a cover of , so .
* **Countable subadditivity**: For any sequence , we construct covers of with . The union covers , and . Letting gives .
* **Empty set**: since and .

#### Step 2: Carathéodory measurability

A set is called -measurable if for all ,

Let be the collection of all -measurable sets. We show is a -algebra:

* **Closed under complements**: If , then satisfies the same condition by symmetry.
* **Closed under countable unions**: First, prove closure under finite unions by induction. For countable unions , use induction to show finite unions , then apply the definition of -measurability to approximate by finite unions and take .

#### Step 3: Restriction to is a measure

The restriction is a measure. To verify countable additivity:

* Let be pairwise disjoint. By countable subadditivity, .
* For the reverse inequality, fix and apply the Carathéodory condition iteratively to and , showing . Taking and using monotonicity gives .

#### Step 4: Extension property

For , we prove is -measurable and :

* **Measurability**: For any , let be a cover of . Then and belong to (since is an algebra), and . Summing over gives . Taking infima over covers yields .
* **Equality**: By definition, . For the reverse inequality, suppose . Then (by countable subadditivity of ) . Taking infima gives .

#### Step 5: Uniqueness

If is -finite, the extension is unique. Let be another measure on with :

* Apply the **- theorem**:
* is a -system (closed under finite intersections).
  + The set is a -system.
  + Since and agree on , they agree on .
* **-finiteness**: Write with . For each , , so .

*Remark*. The **Carathéodory extension theorem** ensures that a pre-measure defined on an algebra can be uniquely extended to a measure on the generated -algebra, provided the pre-measure is -finite. In the proof of the existence of regular conditional probabilities, it guarantees that the finitely additive map , defined on a countable generator , extends uniquely to a probability measure on . This step is critical for constructing the regular conditional probability kernel rigorously.

**Theorem 2** [4, Theorem 6.3]

For any Borel space and measurable space , let and be random elements in and , respectively. Then there exists a probability kernel from to satisfying

and is unique a.e. .

*Proof* (Proof of [Theorem 2](#thm-k)). We may assume that . For every we may choose some measurable function such that

Let be the set of all such that is nondecreasing in with limits 1 and 0 at . Since is specified by countably many measurable conditions, each of which holds a.e. at , we have and a.e. Now define

and note that is a distribution function on for every . Hence, by Proposition~ there exist some probability measures on with

The function is clearly measurable in for each , and by a monotone class argument it follows that is a kernel from to .

By [Equation 1](#eq-1) and the monotone convergence property of , we have

Using a monotone class argument based on the a.e. monotone convergence property, we may extend the last relation to

In particular, we get a.e., and so [Equation 2](#eq-2) remains true on with replaced by the kernel

where is arbitrary. If is another kernel with the stated property, then

and a monotone class argument yields a.e.

*Proof* (Proof of Existence of Regular Conditional Probability). Let be a standard Borel space and a probability measure on . Let be a sub--algebra.

#### Step 1: Countable Generator and Conditional Expectation

Since is standard Borel, there exists a Polish topology on such that is the Borel -algebra. Standard Borel spaces have the property that every probability measure admits a regular conditional probability with respect to any sub--algebra.

Let be a countable -system generating . For each , the conditional expectation exists as an -measurable function, unique up to -null sets (by the definition of conditional expectation). By the Doob-Dynkin lemma (see [Lemma 1](#lem-doob-dynkin)), for each , there exists a measurable function such that:

where is a measurable function generating (e.g., ).

#### Step 2: Construction of the Kernel via Extension

For fixed , define for . The map is:

* **Finitely additive**: For disjoint , .
* **Non-negative**: .
* **Normalized**: .

To extend to a probability measure on , we apply the Carathéodory extension theorem (see [Theorem 1](#thm-Caratheodory-extension-theorem)). However, since is a -system, the extension is unique if is countably additive on . This follows from the dominated convergence theorem and the fact that generates .

Thus, for -almost every , there exists a unique probability measure on such that:

#### Step 3: Measurability of the Kernel

For each , the map must be -measurable. Since generates , we use the - theorem: - Let . - is a -system containing the -system , hence .

#### Step 4: Joint Measurability

By [Theorem 2](#thm-k), there exists a probability kernel from to such that:

where and are random elements in . Here, for -almost every . The joint measurability of follows from the construction using a countable generator and the uniqueness of a.e. .

#### Step 5: Verification of Conditional Probability

For all , satisfies:

* **Measurability**: is -measurable.
* **Integration**: For any ,
* This holds for by construction and extends to all via the - theorem.

Thus, is a regular conditional probability kernel.

cf.[5] and [4]

Thus, a regular conditional probability exists for any sub--algebra in a standard Borel space.

*Remark*. The standard Borel space assumption is essential. For general measurable spaces, regular conditional probabilities may not exist.

## 4 The Uniqueness of Regular Conditional Probability

Regular conditional probabilities are unique up to -null sets. That is, if and are both regular conditional probabilities with respect to , then there exists a -null set such that for all and all ,

This means the regular conditional probability is essentially unique: any two versions agree outside a set of probability zero.

*Proof* (Proof of Uniqueness of Regular Conditional Probability). Let and be two regular conditional probabilities with respect to . For each , define

#### Step 1: Null Sets for Countable Generator

By the definition of regular conditional probability, and are both versions of , hence they are equal -almost surely. Thus, for each .

Since is standard Borel, let be a countable -system generating . Define the union of null sets:

As is countable, is a countable union of -null sets, so .

#### Step 2: Extension to the Entire -Algebra via - Theorem

Fix . For this , define the collection of sets:

We show that is a -system containing :

* **Contains** : , so .
* **Closed under disjoint unions**: If are pairwise disjoint, then:
* **Closed under complements**: If , then:

Since (by ) and is a -system, the - theorem implies . Thus, for all , for every .

The set is -null, and for all , the kernels and agree on . Hence, regular conditional probabilities are unique up to -null sets.

*Remark*. This uniqueness property ensures that, although the regular conditional probability may not be defined uniquely everywhere, any two versions coincide almost surely.

## 5 Representation of Conditional Expectation as an Integral

By now, we have established that conditional probability admits a refined version (the regular conditional probability), which qualifies as a probability measure. This allows us to define an integral with respect to it.

In the following, we aim to demonstrate that this integral operation on the regular conditional probability precisely coincides with the conditional expectation .

**Theorem 3 (Representation of Conditional Expectation as an Integral)** Let be a probability space, a sub--algebra, and a regular conditional probability kernel such that a.e. for all . Then for any integrable random variable , the conditional expectation satisfies:

*Proof* (Proof of [Theorem 3](#thm-cond-exp-as-integral)).

#### Step 1: Indicator Functions

Let for . By definition of the regular conditional probability:

#### Step 2: Simple Functions

Let with and . By linearity of conditional expectation and integration:

#### Step 3: Non-Negative Measurable Functions

Let be measurable. Take an increasing sequence of simple functions . By the monotone convergence theorem (MCT):

* a.e.
* .

Thus:

#### Step 4: General Integrable Functions

For arbitrary integrable , decompose with . By Step 3:

#### Step 5: Measurability and Uniqueness

* **Measurability**: The integral is -measurable by construction of the kernel .
* **Uniqueness**: Regular conditional probability kernels agree a.e. , ensuring the integral representation is unique a.e.

## 6 Conclusion

Through this assignment, we have systematically clarified the construction logic of Regular Conditional Probability and rigorously proved its existence and uniqueness in standard Borel spaces. This result demonstrates that, in measure spaces with well-behaved topological structures, conditional probability can be elevated to a probability kernel dependent on sample points ω , thereby resolving the “null set selection problem” inherent in classical definitions of conditional probability (where properties depend on specific events A ). This conclusion provides a rigorous mathematical foundation for the integral representation of conditional expectation.

However, the construction of regular conditional probability heavily relies on the structural properties of the underlying space. If extended to general measurable spaces, the existence of such kernels may fail.

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