4

Einstein Equations

4.1 General Form of Equations

It is natural to assume that the generally covariant equations of the gravitational field should be second-order differential equations, and that the energy-momentum tensor $T^{\mu\nu}$ should serve as a source in them. An additional assumption is that these equations should be linear in the Riemann tensor. Then their general structure is

$$aR^{\mu\nu} + bg^{\mu\nu}R + cg^{\mu\nu} = T^{\mu\nu}.$$

The condition $T^{\mu\nu}{}_{;\nu} = 0$ and identity (3.53) dictate that $b = -a/2$. In this way we arrive at the Einstein equations

$$R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R = 8\pi k T^{\mu\nu} + \Lambda g^{\mu\nu}. \tag{4.1}$$

The coefficient $8\pi k$ at $T^{\mu\nu}$ (k is the Newton constant) guarantees, as will be demonstrated below, the agreement with the common Newton law in the corresponding approximation. The so-called cosmological constant Λ is at any rate extremely small, according to experimental data; therefore, the last term in the left-hand side of equation (4.1) is usually omitted.

We note that if nevertheless $\Lambda \neq 0$, the cosmological term in (4.1) can be presented as an effective additional contribution

$$\tau^{\mu\nu} = \frac{\Lambda}{8\pi k}\,g^{\mu\nu}$$

to the energy-momentum tensor of the matter $T^{\mu\nu}$. This contribution is quite peculiar. As distinct from the energy-momentum tensor of particles with a rest mass, for $\tau^{\mu\nu}$ there is no reference frame where only the component τ^{00} differs from zero. As distinct from the energy-momentum tensor of massless particles, the trace of $\tau^{\mu\nu}$ does not vanish: $\tau^{\mu}_{\mu} = \Lambda/2\pi k$.

On the other hand, in the locally geodesic frame

$$\tau^{\mu\nu} = \frac{\Lambda}{8\pi k}\,\eta^{\mu\nu} = \frac{\Lambda}{8\pi k}\,\mathrm{diag}(1,-1,-1,-1)\,.$$

With this diagonal tensor $\tau^{\mu\nu}$, the corresponding effective energy density ρ_Λ and pressure p_Λ are as follows[1]:

$$\tau^{\mu\nu} = \mathrm{diag}(\rho_\Lambda, p_\Lambda, p_\Lambda, p_\Lambda)\,. \tag{4.2}$$

Clearly, such a peculiar "matter" has also quite a peculiar equation of state:

$$p_\Lambda = -\rho_\Lambda = -\tau_{00} = -\frac{\Lambda}{8\pi k}\,, \tag{4.3}$$

i.e. its pressure is negative! Modern data of the observational astronomy give serious reasons to believe that the cosmological term does not vanish. It is quite possible that, though being tiny on the usual scale, the cosmological term is very essential for the evolution of the Universe.

In the absence of matter $T^{\mu\nu} = 0$ and the Einstein equations (4.1) reduce to

$$R^{\mu\nu} = 0\,. \tag{4.4}$$

The spaces with metric satisfying condition (4.4) are called the Einstein spaces. Equation (4.1) (in the absence of the cosmological constant) can be rewritten as:

$$R^{\mu\nu} = 8\pi k\left(T^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}T^\lambda_\lambda\right). \tag{4.5}$$

The Einstein equations are in essence the content of general relativity.

Problem

4.1. Prove relation

$$R^{\alpha\beta\mu\nu}{}_{;\,\nu} = 8\pi k\left[T^{\mu\alpha;\,\beta} - T^{\mu\beta;\,\alpha} - \frac{1}{2}\left(g^{\mu\alpha}T^{\lambda;\,\beta}_\lambda - g^{\mu\beta}T^{\lambda;\,\alpha}_\lambda\right)\right]$$

(*A. Lichnerowicz*, 1960).

4.2 Linear Approximation

In the linear approximation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, the Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2}\left[\partial_\rho\partial_\nu h_{\mu\rho} + \partial_\mu\partial_\rho h_{\nu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h_{\rho\rho}\right].$$

[1] See, for instance, L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, §35, formula (35.1).