```python
#Assignment 5
#Tracking of a moving object which trajectory is disturbed by random acceleration
#Team 12
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#Skoltech, 2023

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

#Steps 1,2,3 - generate dataset and make preparations for filter creation
# Initial parameters
N = 200
M = 500
T = 1

#For plots
step = np.arange(N)
step1 = np.arange(N - 3)

sigma = 0.2**2
sigma1 = 20**2


# Initial position and velocity
x0 = np.array([[5], [1]])

# Transition Matrix
Fi = np.array([[1, T], [0, 1]])

# Input Matrix
G = np.array([[(T**2)/2], [T]])

# Observer Matrix
H = np.array([[1], [0]])

#build function for estimation model
def generate_Xt(x0, sigma, Fi, G, N):
    a = np.random.normal(0, np.sqrt(sigma), size=(1, 1, N))
    Xt = np.zeros((2, 1, N))
    Xt[:, :, 0] = x0

    for i in range(1, N):
        Xt[:, :, i] = np.dot(Fi, Xt[:, :, i-1]) + G * a[:, :, i-1]

    return Xt

#build function formeasurements model
def generate_Z(sigma1, Xt, H, N):
    nu = np.random.normal(0, np.sqrt(sigma1), size=(1, 1, N))
    Z = np.zeros((1, 1, N))
```

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    for i in range(N):
        Z[:, :, i] = np.dot(H.T, Xt[:, :, i]) + nu[:, :, i]

    return Z

#Step 4 - devrelop Kalman filter

# Initial filter estimates
X0 = np.array([[2], [0]])

# Initial filtration error covariance matrices
P0 = np.array([[10000, 0],
               [0, 10000]])

# Process noise matrix Q
Q = G * G.T * sigma

# Measurement noise
R = sigma1

def kalmanFilter(Fi, H, Q, R, X0, P0, Z, N):
    Xp = np.zeros((2, 1, N)) #Prediction of state vector at time i 
using  (i-1) measurements
    Pp = np.zeros((2, 2, N)) #Prediction error covariance matrix
    Xf = np.zeros((2, 1, N)) #Improved estimate by incorporating
    K = np.zeros((2, 1, N)) #Filter gain, weight of residual
    Pf = np.zeros((2, 2, N)) #Filtration error covariance matrix
    Ps = np.zeros((1, 1, N))

    Pf[:, :, 0] = P0
    Xf[:, :, 0] = X0
    for i in range(1, N):
        #Prediction
        Xp[:, :, i] = np.dot(Fi, Xf[:, :, i-1])
        Pp[:, :, i] = np.dot(np.dot(Fi, Pf[:, :, i-1]), Fi.T) + Q

        #Filtration Adjustment
        K[:, :, i] = np.dot(Pp[:, :, i], H)/(np.dot(np.dot(H.T, 
Pp[:, :, i]), H) + R)
        Xf[:, :, i] = Xp[:, :, i] + np.dot(K[:, :, i], Z[:, :, i] - 
np.dot(H.T, Xp[:, :, i])) # np.dot(H, Xp[:, i])
        Pf[:, :, i] = np.dot((np.eye(2) - np.dot(K[:, :, i], H.T)), 
Pp[:, :, i])
        # FOR TASK 6,11 square root of the first diagonal element of 
Pi,i
        #that corresponds to standard deviation of estimation error 
of coordinate xi
        Ps[:, :, i] = np.sqrt(np.abs(np.diag(Pf[:, :, i])[0]))

    return Xf, Pf, K, Ps


# Generate true trajectories
Xt = generate_Xt(x0, sigma, Fi, G, N)
```

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# Generate measurement trajectories
Z = generate_Z(sigma1, Xt, H, N)

# Calculate the estimation for true trajectories using Kalman filter
Xf, Pf, K, Ps = kalmanFilter(Fi, H, Q, R, X0, P0, Z, N)

#Step 5 - plot results for true trajectory, measurements and
filtered estimates of state vector Xi
y1 = Xt[0][0]
y2 = Z[0][0]
y3 = Xf[0][0]
x = step
plt.plot(x,y1,label = 'True trajectory')
plt.plot(x,y2,label = 'Measurements')
plt.plot(x,y3,label = 'Filtered estimates of statate vector Xi')
plt.xlabel('step')
plt.ylabel('X')
plt.suptitle('Coordinate')
plt.legend()
plt.grid(True)
```
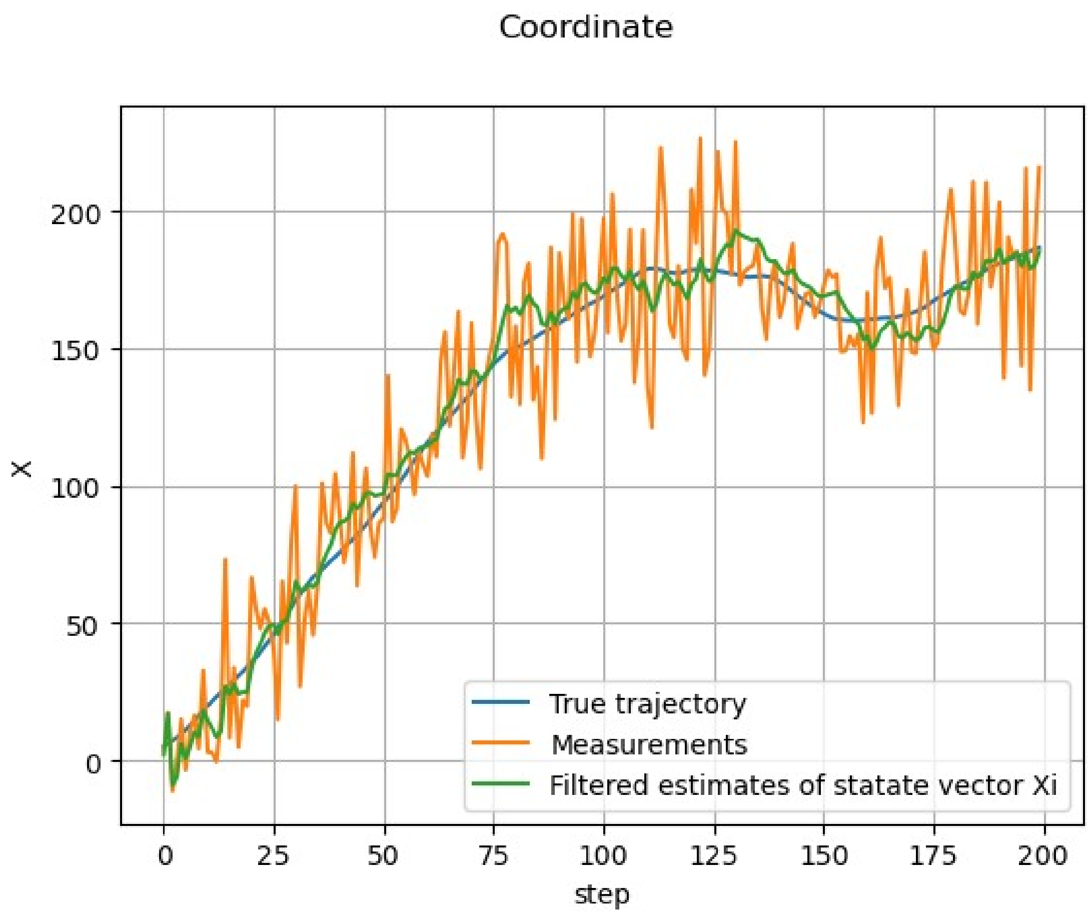
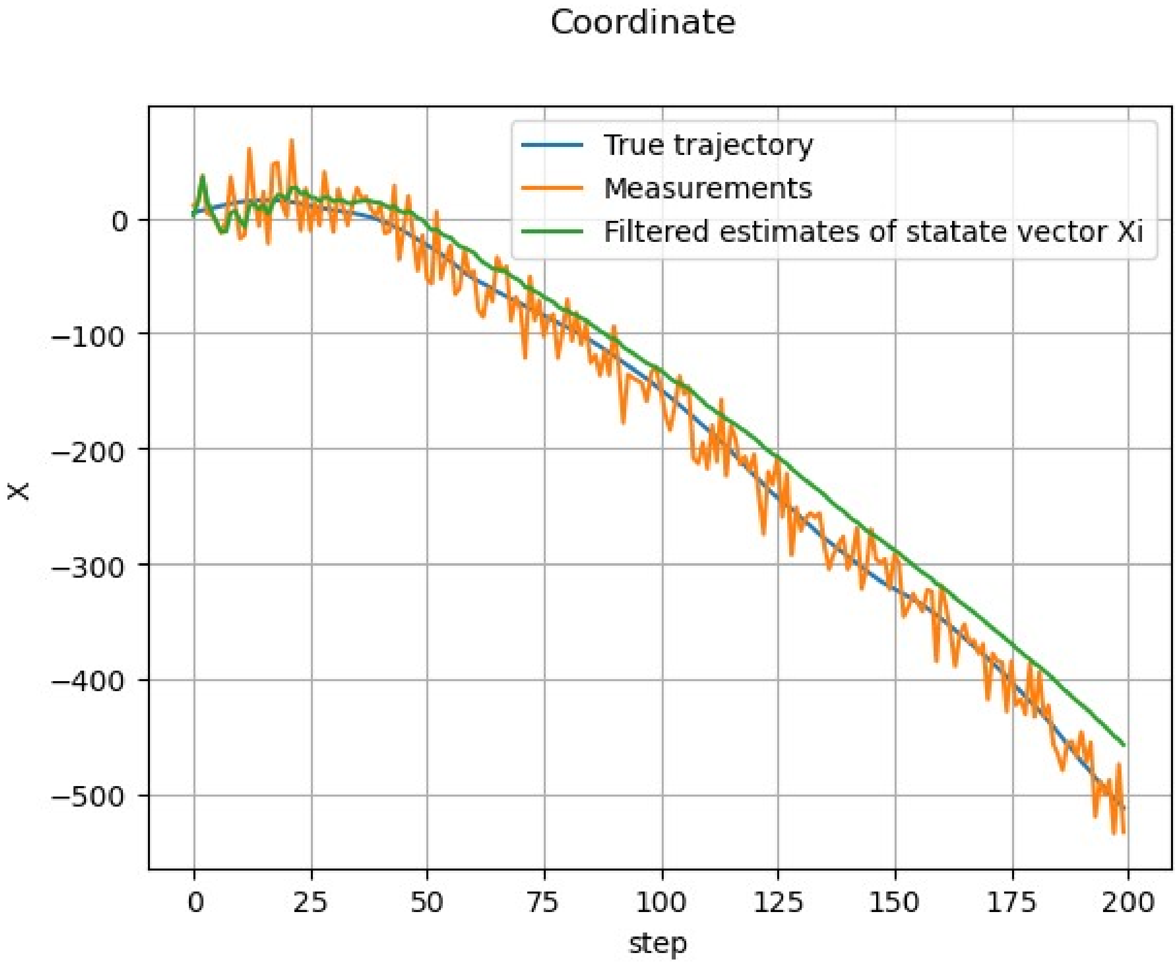
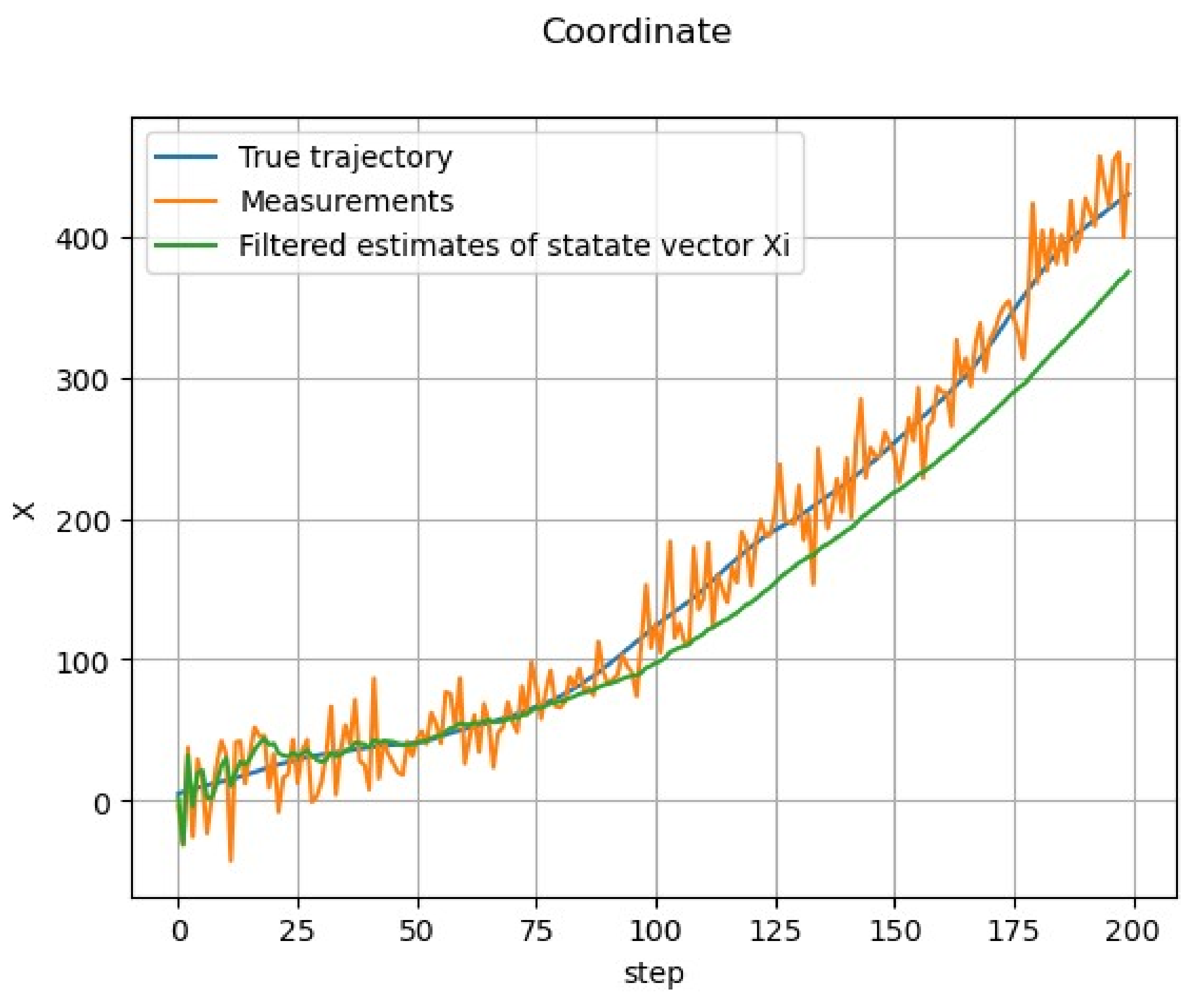
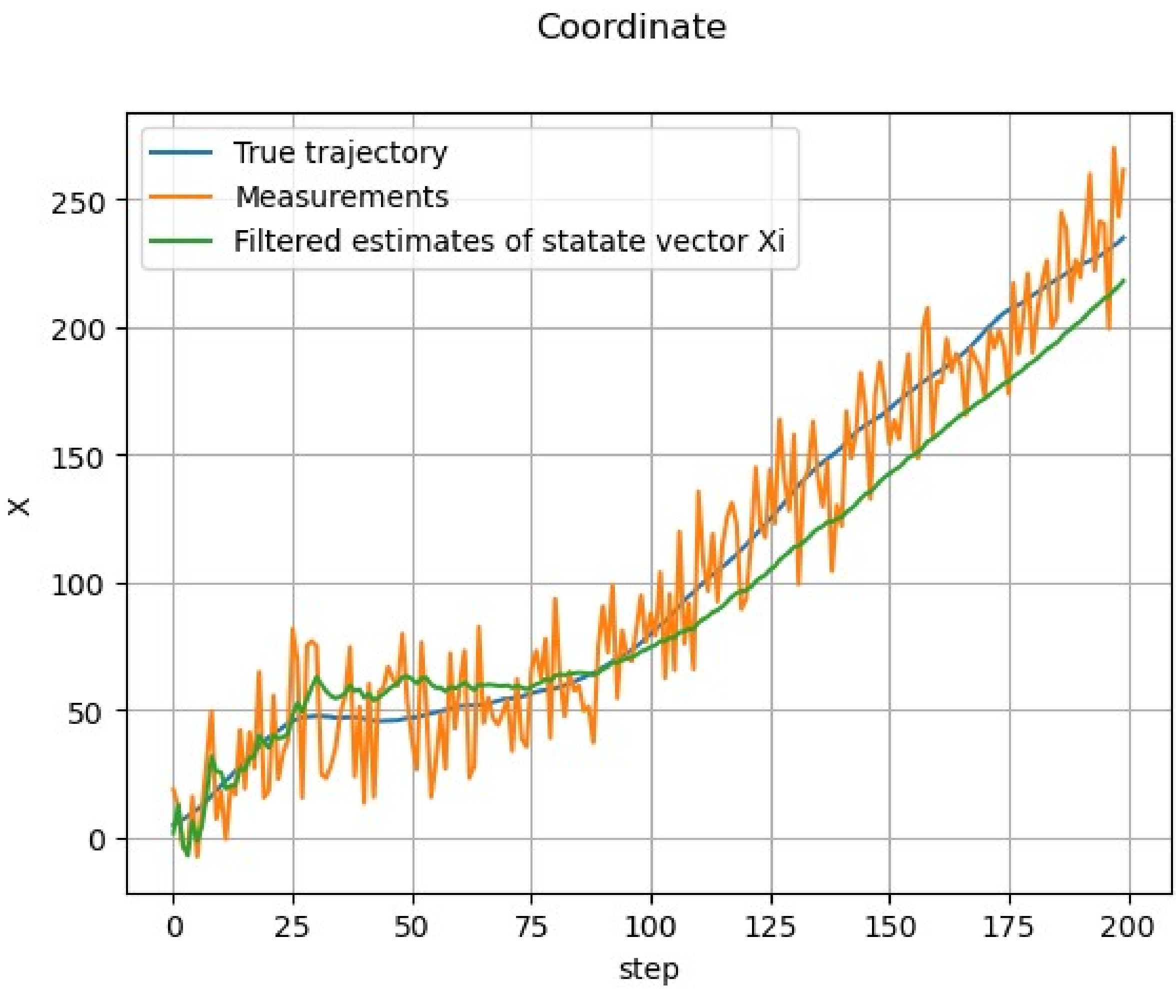




```
#Conclusion: we clearly see how effective Kalman filter is in
prosessing of noisy measurements, but that is not only thing it can
do
```

Run filter several times to see that estimation results are different with every new trajectory.

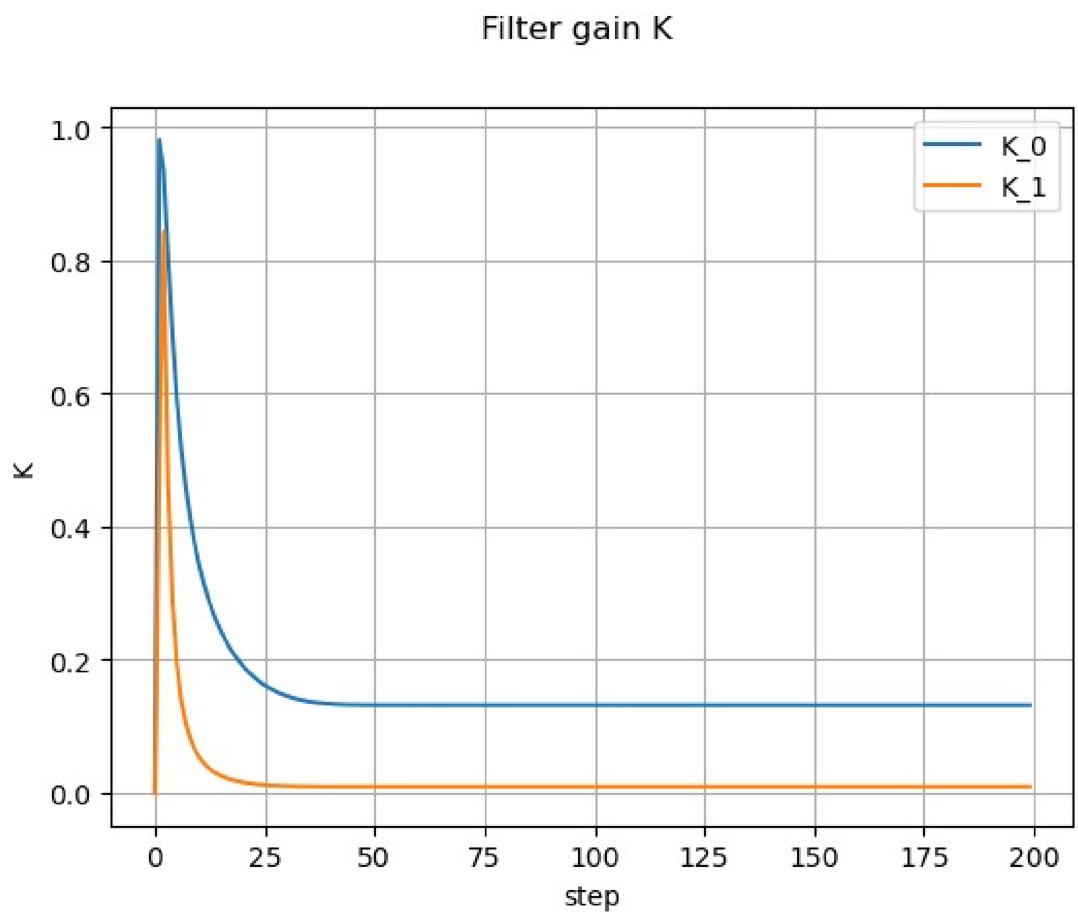



Coordinate
True trajectory
Measurements
Filtered estimates of statate vector Xi
X
400
300
200
100
0
step
0
25
50
75
100
125
150
175
200

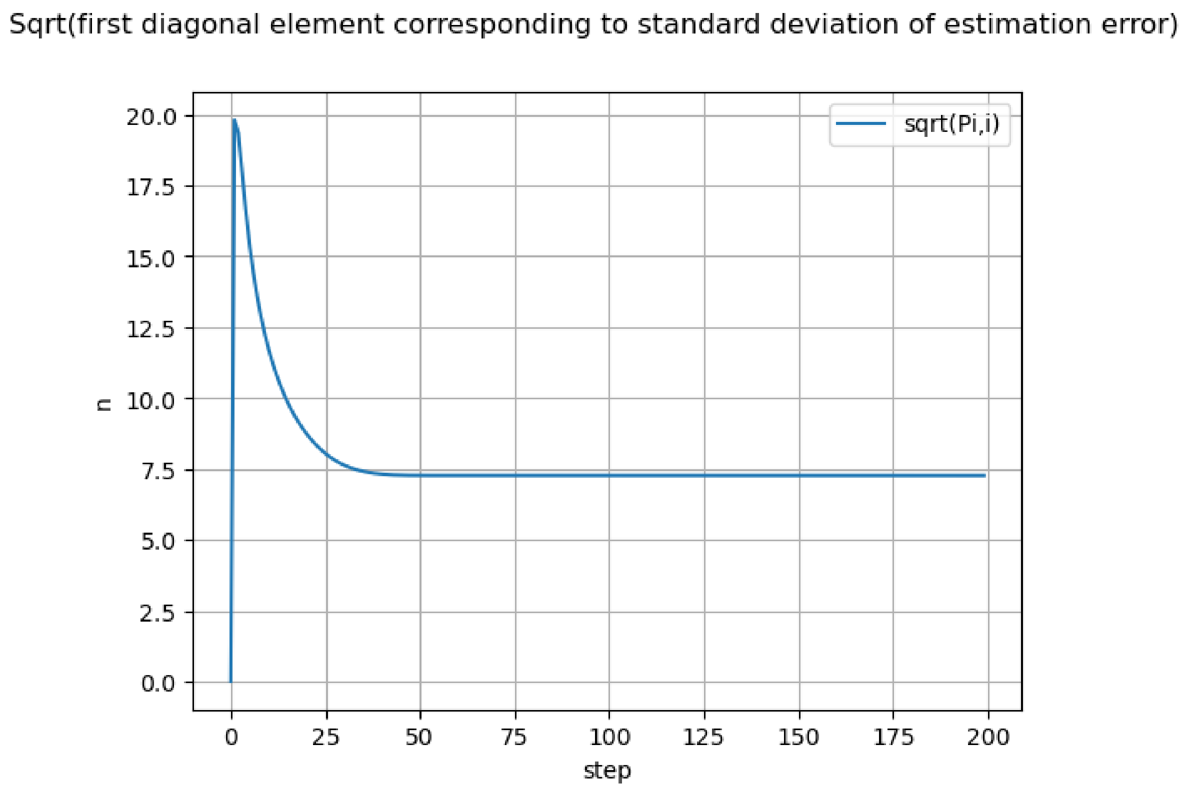


Conclusion: Every time we get new estimation for every new trajectory

```
#Step 6 - plot filter gain K over the whole  filtration interval
y1 = K[0][0]
y2 = K[1][0]
x = step
plt.plot(x,y1,label = 'K_0')
plt.plot(x,y2,label = 'K_1')
plt.xlabel('step')
plt.ylabel('K')
plt.suptitle('Filter gain K')
plt.legend()
plt.grid(True)
```




```
#Conclusion: the bigger the K is rhe less steps it takes to
stabilise

#square root of its first diagonal element corresponding to standard
deviation of estimation error of xi
y1 = Ps[0][0]
x = step
plt.plot(x,y1,label = 'sqrt(Pi,i)')
plt.xlabel('step')
plt.ylabel('n')
plt.suptitle('Sqrt(first diagonal element corresponding to standard
deviation of estimation error)')
plt.legend()
plt.grid(True)
```




```
#Verify whether filter gain K and filtration error covariance matrix
become constant very
#quickly. It means that in conditions of a trajectory disturbed by
random noise we cannot
#estimate more than established limit of accuracy due to
uncertainty.

#Conclusion: Yes, the establishment occurs approximately in 40-50
iterations

#Stepn 7 - Add to the code extrapolation on m = 7 steps ahead on
every time step
def kalman_mSteps(m, Fi, N, Xf):
    Xim = np.zeros((2, 1, N))
    Fim = np.linalg.matrix_power(Fi, m-1)

    for i in range(m-1):
        Xim[:, :, i] = Xf[:, :, i]

    for i in range(m-1, N-m):
        Xim[:, :, i+7] = np.dot(Fim, Xf[:, :, i])

    return Xim

# calculating m extraposion of kalman filter
m = 7
Xpm = kalman_mSteps(m,Fi,N,Xf)

#plot the results
y1 = Xt[0][0]
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y2 = Z[0][0]
y3 = Xf[0][0]
y4 = Xpm[0][0]
x = step
plt.plot(x,y1,label = 'True trajectory')
plt.plot(x,y2,label = 'Measurements')
plt.plot(x,y3,label = 'Filtered estimates')
plt.plot(x,y4,label = 'Filtered estimates m steps ahead')
plt.xlabel('step')
plt.ylabel('X')
plt.suptitle('Coordinate')
plt.legend()
plt.grid(True)
```
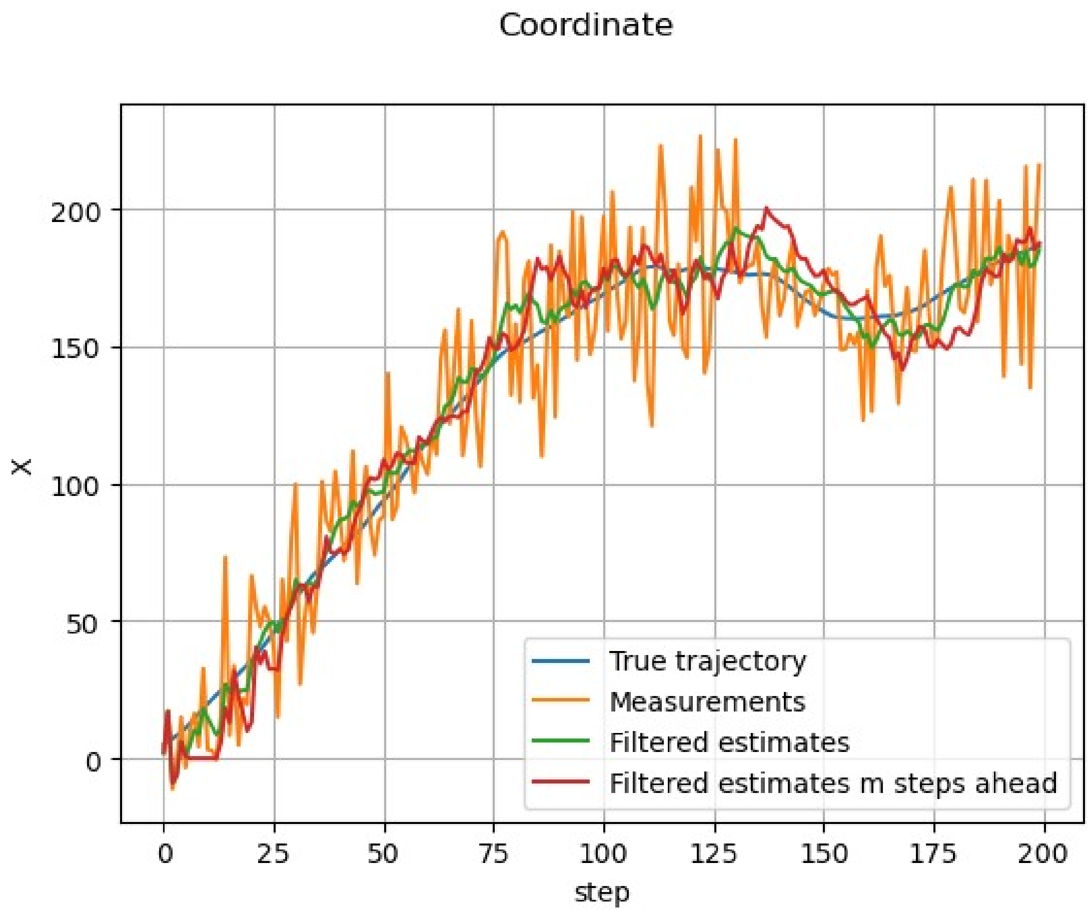
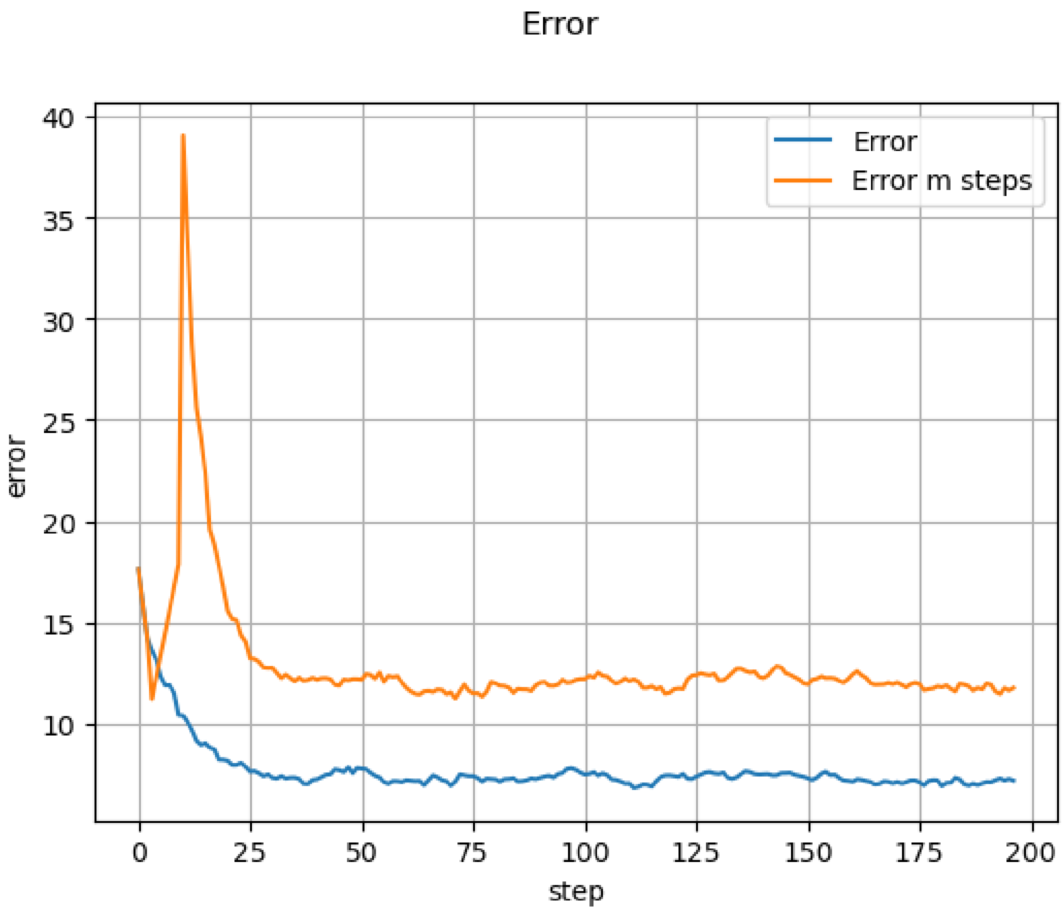



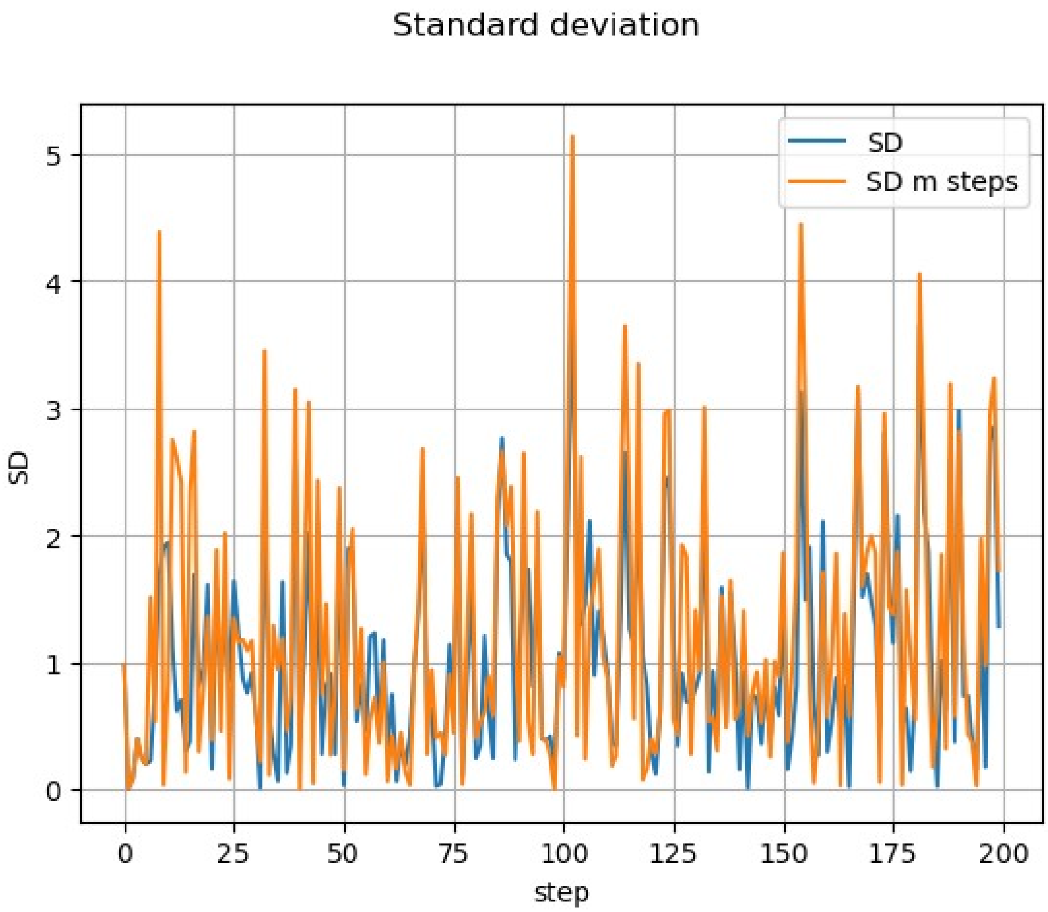

```
#Conclusion: new estimations based on 7 steps ahead predictions are
more noisy then original one step estimation, but still  very close
to true trajectory

#Step 8 - estimate dynamics of mean-squared error of estimation over
observation interval
#set function for mean-squared error
def calculate_MSE(m, N, M, x0, sigma, sigma1, Fi, G, H, Q, R, X0,
P0):
    Error_run = np.zeros((2, M, N))
    final_Error = np.zeros((2, 1, N-3))
```

```
    Error_run1 = np.zeros((2, M, N))
    final_Error1 = np.zeros((2, 1, N-3))

    for i in range(M):
        Xt = generate_Xt(x0, sigma, Fi, G, N)
        Z = generate_Z(sigma1, Xt, H, N)
        Xf, Pf, K, Ps = kalmanFilter(Fi, H, Q, R, X0, P0, Z, N)
        Xpm = kalman_mSteps(m, Fi, N, Xf)

        for j in range(N):
            Error_run[:, i, j] = ((Xt[:, :, j] - Xf[:, :,
j])**2).T[0]
            Error_run1[:, i, j] = ((Xt[:, :, j] - Xpm[:, :,
j])**2).T[0]

    for i in range(N-3):
        for k in range(M):
            final_Error[:, :, i] += Error_run[:, k, i+3].T[0]
            final_Error1[:, :, i] += Error_run1[:, k, i+3].T[0]

        final_Error[:, :, i] = np.sqrt(final_Error[:, :, i] / (M -
1))
        final_Error1[:, :, i] = np.sqrt(final_Error1[:, :, i] / (M -
1))

    return final_Error, final_Error1

#Calculating the mean sequare error with P0 = np.array([[1000, 0],
[0, 1000]])
MSE1, MSE2 = calculate_MSE(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X0,P0)

#Plot final error and check when it becomes almost constant and
estimation accuracy doesn’t
#increase anymore. At this moment filter becomes stationary and in
practice this constant
#filter gain can be used in the algorithm instead of calculating
filter gain at every ti

y1 = MSE1[0][0]
y2 = MSE2[0][0]
x = step1
plt.plot(x,y1,label = 'Error')
plt.plot(x,y2,label = 'Error m steps')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```



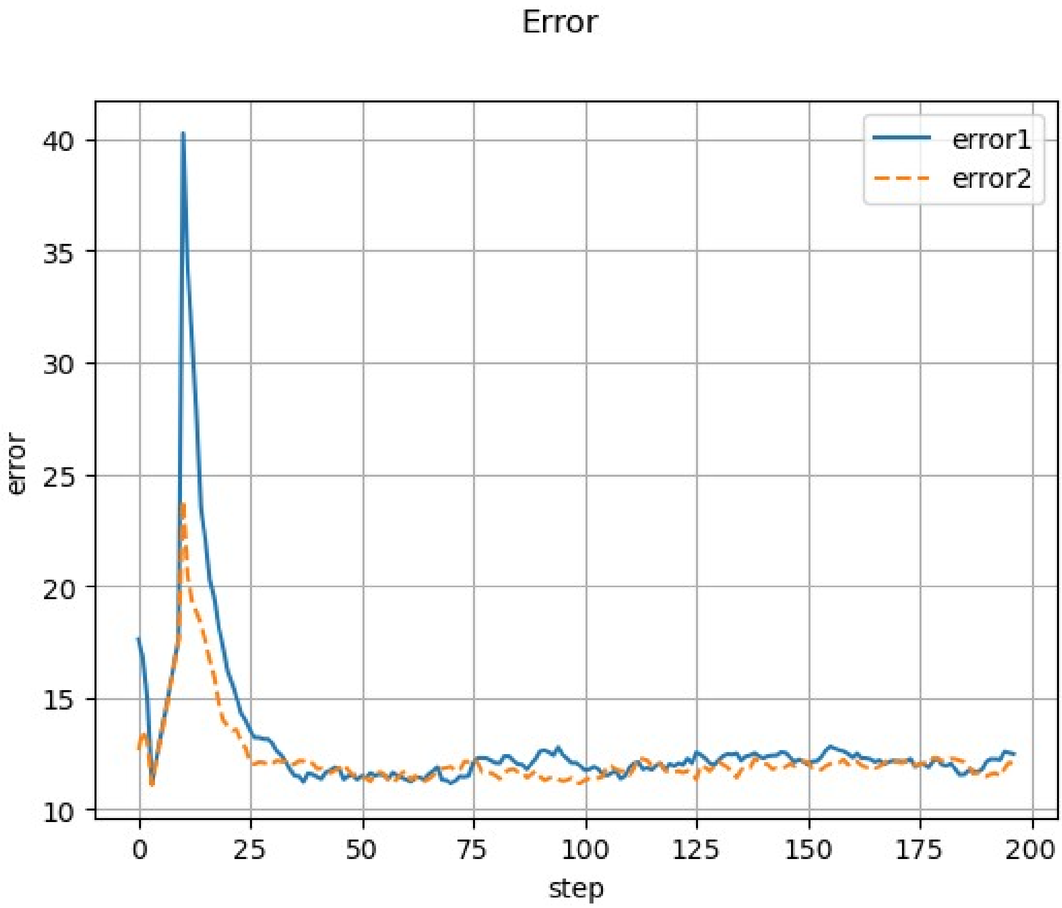

```
#Conclusion: time of stabilisation for both estimation tactics is
almost the same after time of 50 steps

#Step 9 - compare mean-squared error of filtered estimate of
coordinate x with standard deviation of measurement
errors.-----------№9------------
def calculate_SD(Z, Xf, N):
    SD = np.zeros((1, 2, N))

    for j in range(N):
        SD[:, :, j] += ((Z[:, :, j] - Xf[:, :, j])**2).T[0]

    for i in range(N):
        SD[:, :, i] = np.sqrt(SD[:, :, i]/(N - 1))

    return SD

SD_error = calculate_SD(Z,Xf,N)
SD_error_mSteps = calculate_SD(Z,Xpm,N)

y1 = SD_error[0][0]
y2 = SD_error_mSteps[0][0]
x = step
plt.plot(x,y1,label = 'SD')
plt.plot(x,y2,label = 'SD m steps')
plt.xlabel('step')
```

```
plt.ylabel('SD')
plt.suptitle('Standard deviation')
plt.legend()
plt.grid(True)
```



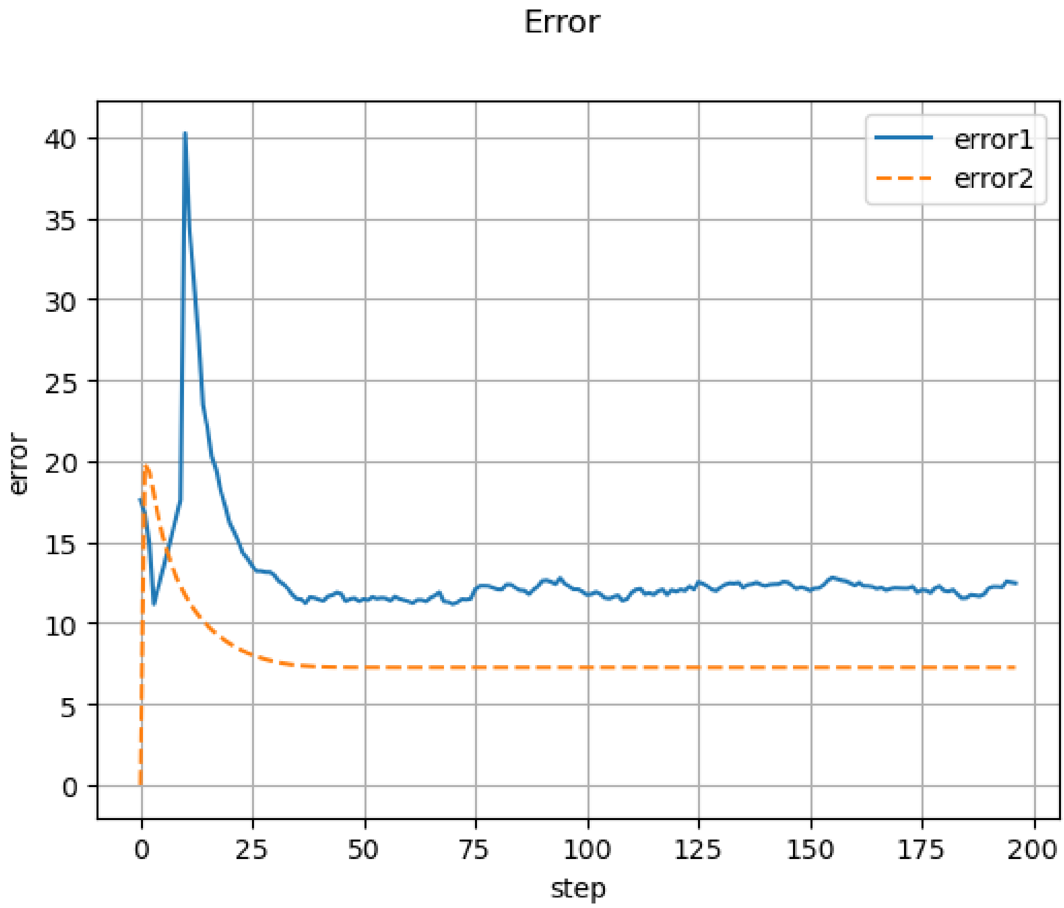

```
#Conclusion: effectivness of filtration is high as both plots are
very close to each other

#Step 10 -  analyze how the accuracy of initial conditions P affects
the estimation results
P01 = np.array([[100, 0],
                [0, 100]])

Xf10, Pf10, K10, Ps10 = kalmanFilter(Fi, H, Q, R, X0, P01, Z, N)
#Calculating the mean sequare error  with P01 = np.array([[100, 0],
[0, 100]])
MSE3, MSE4 = calculate_MSE(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X0,P01)

y1 = MSE2[0][0]
y2 = MSE4[0][0]
x = step1
plt.plot(x,y1,label = 'error1')
plt.plot(x,y2,label = 'error2',linestyle = '--')
plt.xlabel('step')
plt.ylabel('error')
```

```
plt.suptitle('Error')
plt.legend()
plt.grid(True)

#When the choice of initial conditions doesn’t affect the estimation results?
```



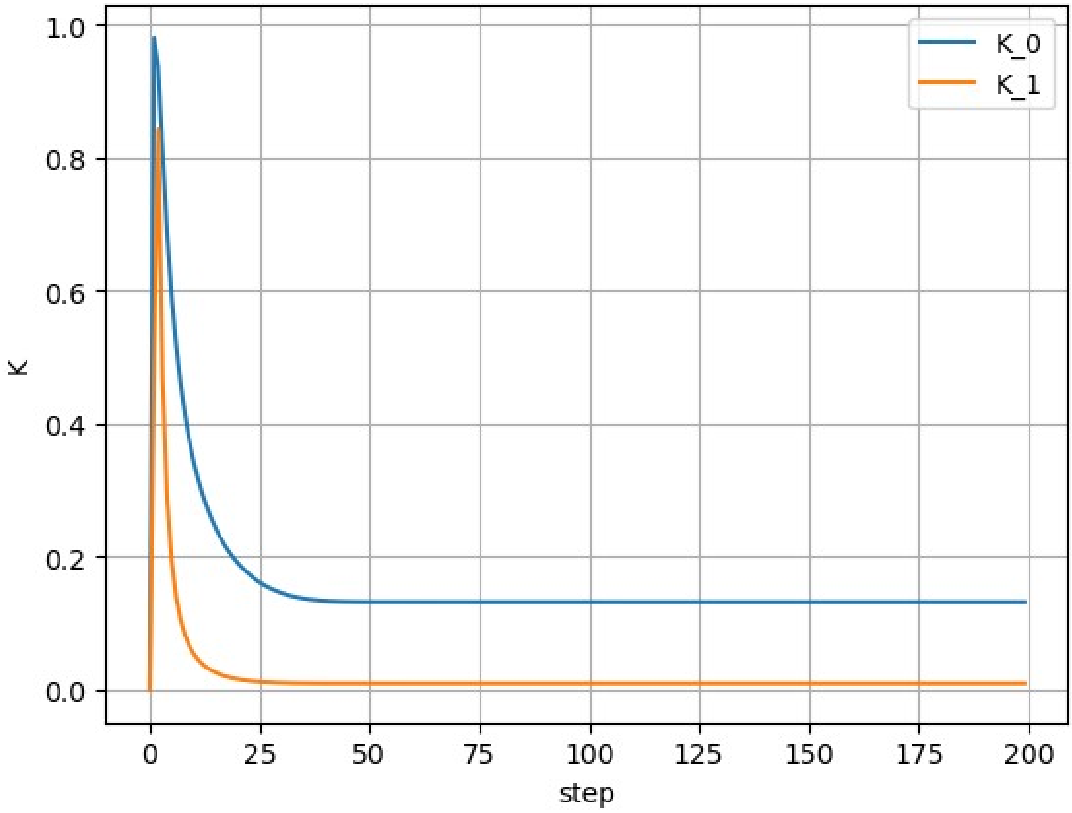

```
#Conclusion: analisys of the plot shows us that accuracy of conditions P
#affects the max size and prolongation of stabilisation period: more precise conditions stabilise better.
#But after stabilisation period initial conditions stop playing cricial role.

#Step 11 - Compare calculation errors of estimation P with rue estimation errors
Ps_plot = np.zeros((N-3))
for i in range(N-3):
    Ps_plot[i] = Ps[0][0][i]

y1 = MSE2[0][0]
y2 = Ps_plot
x = step1
plt.plot(x,y1,label = 'error1')
plt.plot(x,y2,label = 'error2',linestyle = '--')
```

```
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)

#Verify if calculation errors of estimation correspond to true
estimatimation
```

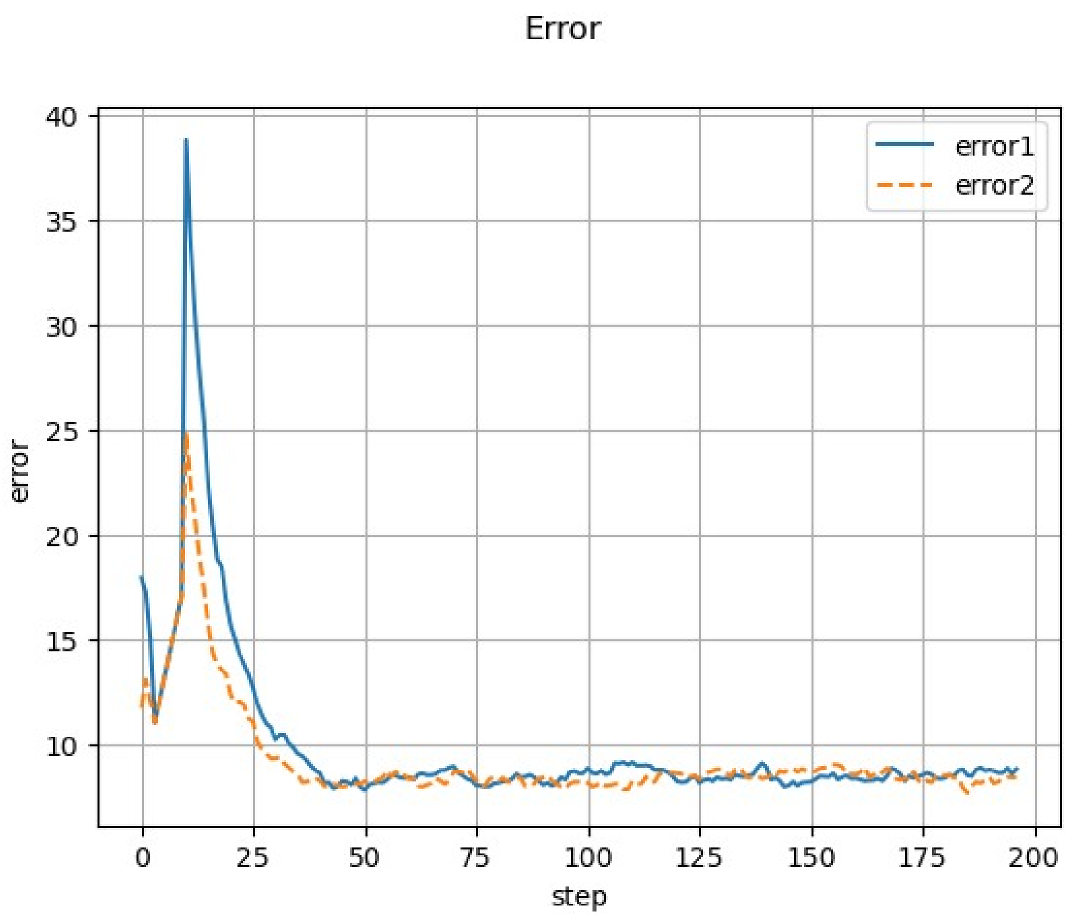



```
#Conclusion: true estimation error staibylise faster  and of  course
does not have noise as it
#estiamtion error is a bit higher but  nevertheless is pretty close

#Step 12
sigma_0 = 0
Xt_0 = generate_Xt(x0, sigma_0, Fi, G, N)
Z_0 = generate_Z(sigma1, Xt_0, H, N)
Xf_0, Pf_0, K_0, Ps_0 = kalmanFilter(Fi, H, Q, R, X0, P0, Z_0, N)
Xpm_0 = kalman_mSteps(m,Fi,N,Xf_0)

#gain K over the whole  filtration interval
y1 = K[0][0]
y2 = K[1][0]
x = step
plt.plot(x,y1,label = 'K_0')
```

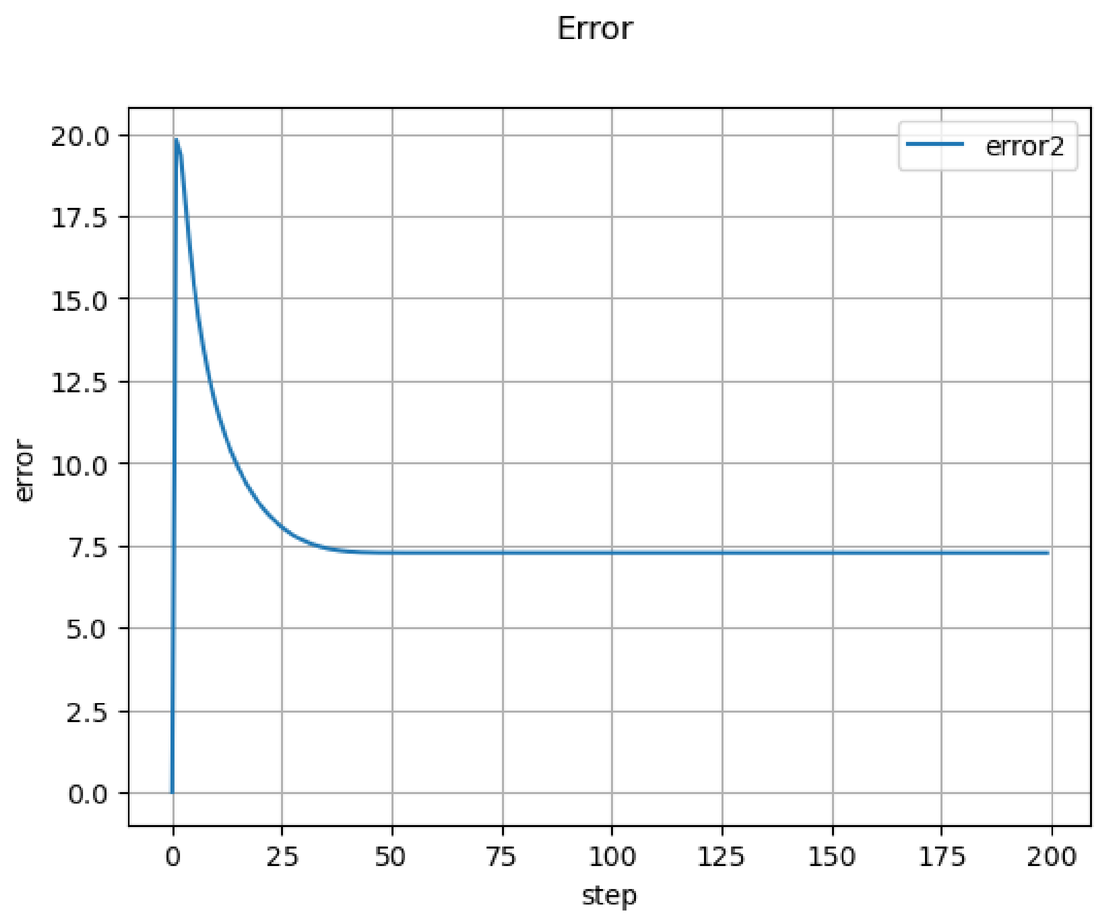
```
plt.plot(x,y2,label = 'K_1')
plt.xlabel('step')
plt.ylabel('K')
plt.suptitle('')
plt.legend()

plt.grid(True)
```



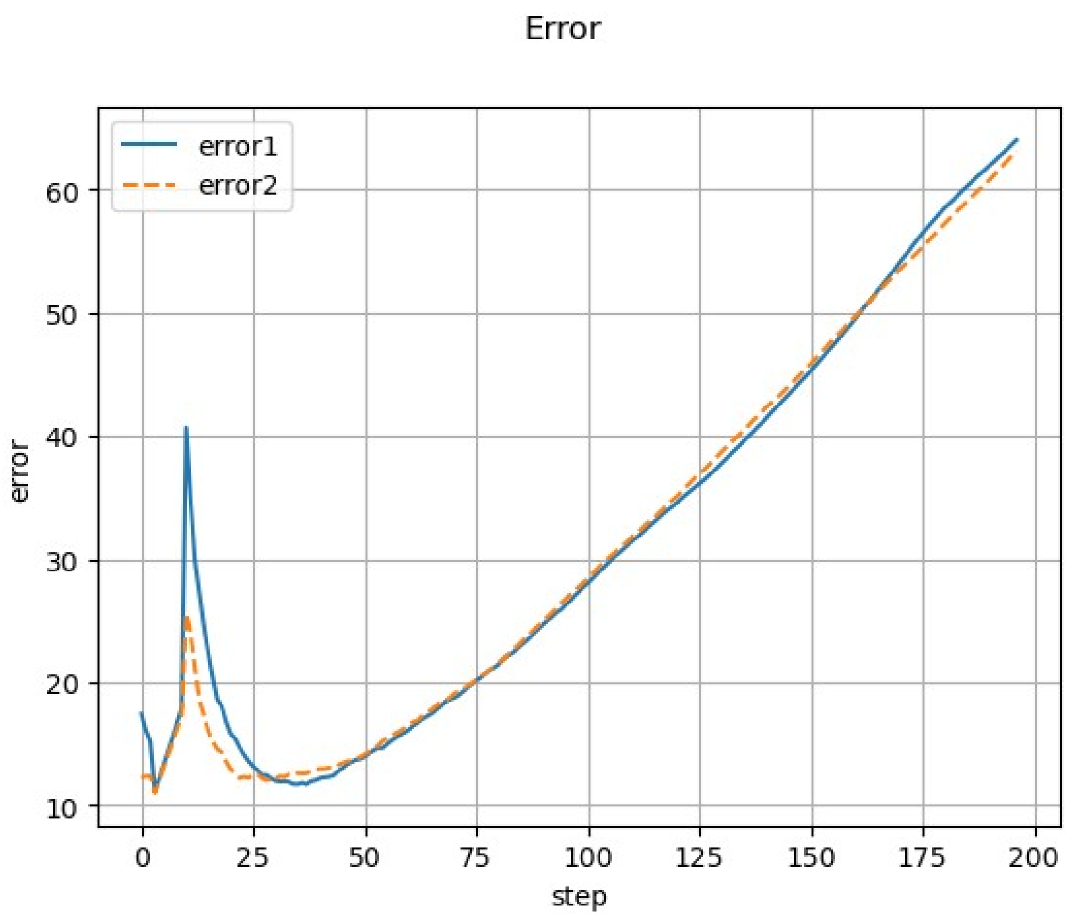

```
#Conclusion: filter gain K approaches to zero

MSE1_0, MSE2_0 =
calculate_MSE(m,N,M,x0,sigma_0,sigma1,Fi,G,H,Q,R,X0,P0)
MSE3_0, MSE4_0 =
calculate_MSE(m,N,M,x0,sigma_0,sigma1,Fi,G,H,Q,R,X0,P01)

y1 = MSE2_0[0][0]
y2 = MSE4_0[0][0]
x = step1
plt.plot(x,y1,label = 'error1')
plt.plot(x,y2,label = 'error2',linestyle = '--')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```



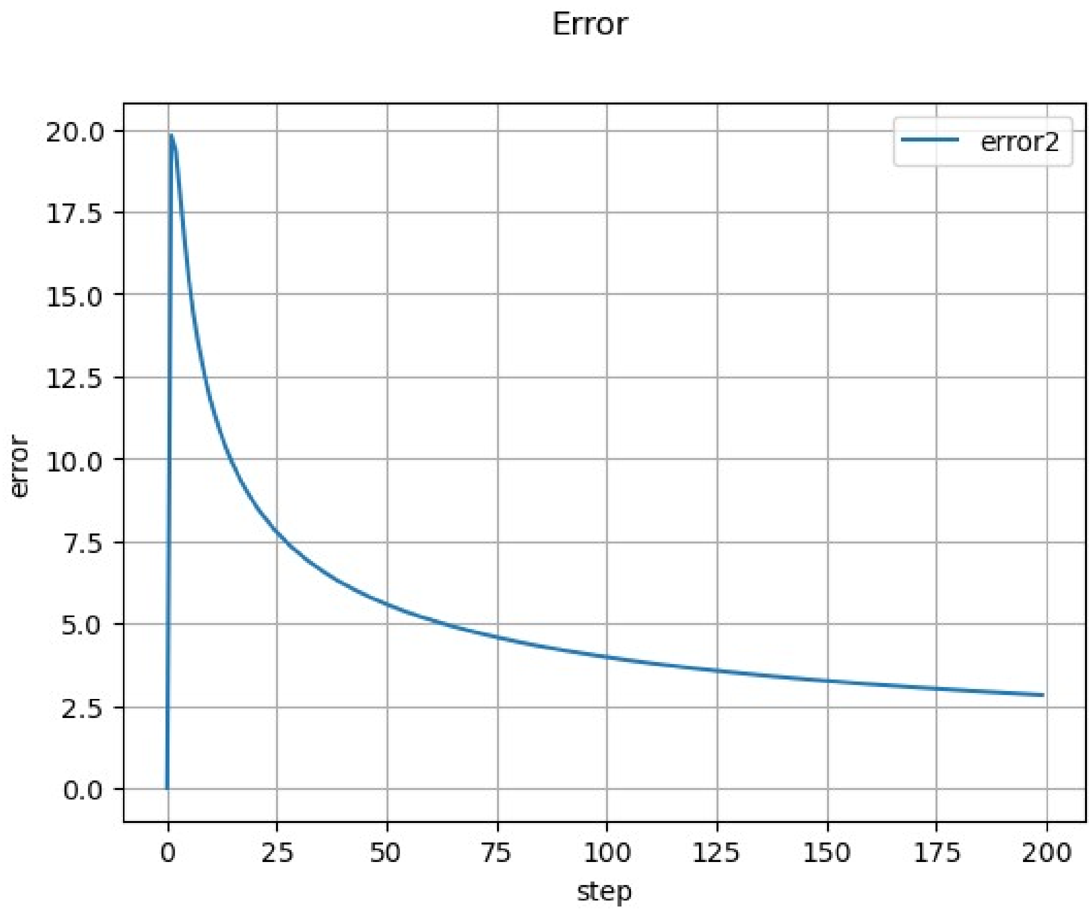

```
#Conclusion: both true estimation errors and calculation errors
approach to zero.
#Filter switches off from measurements (new measurements almost do
not adjust estimates).

y2 = Ps_0[0][0]
x = step
plt.plot(x,y2,label = 'error2')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```




```
#Step 13 - Verify what happens if you use deterministic model of
motion
Q = 0
Xt_1 = generate_Xt(x0, sigma, Fi, G, N)
Z_1 = generate_Z(sigma1, Xt_1, H, N)
Xf_1, Pf_1, K_1, Ps_1 = kalmanFilter(Fi, H, Q, R, X0, P0, Z_1, N)
Xpm_1 = kalman_mSteps(m,Fi,N,Xf_1)

MSE1_1, MSE2_1 =
calculate_MSE(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X0,P0)
MSE3_1, MSE4_1 =
calculate_MSE(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X0,P01)

y1 = MSE2_1[0][0]
y2 = MSE4_1[0][0]
x = step1
plt.plot(x,y1,label = 'error1')
plt.plot(x,y2,label = 'error2',linestyle = '--')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```

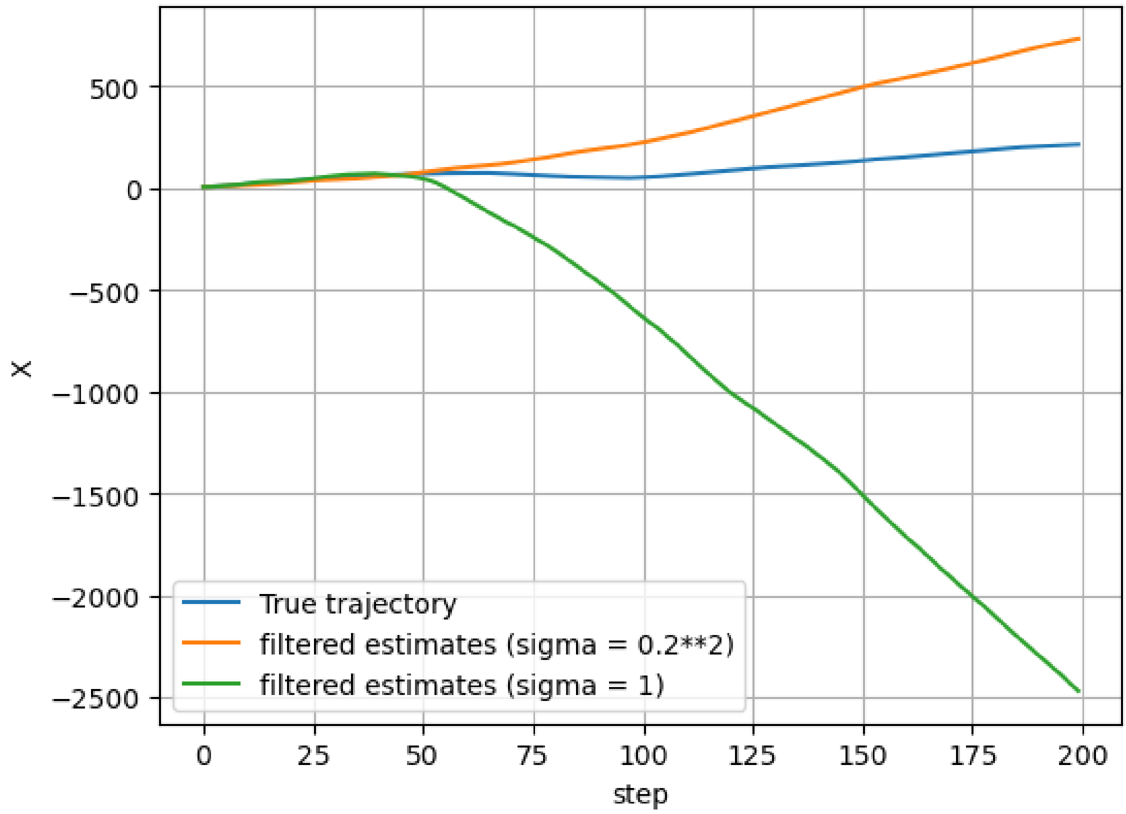



```
#Conclusion: usage of deterministic model only slightly shifts the
error stabilisation period in the beginning

y2 = Ps_1[0][0]
x = step
plt.plot(x,y2,label = 'error2')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```




```
#Steep 14 - analyze how the relationship between state and
measurement noise affect time when filter gain become almost
constant
sigma_1 = 1
sigma = 0.2**2
# Generate true trajectories
X02 = generate_Xt(x0, sigma, Fi, G, N)

# Generate measurement trajectories
Z02 = generate_Z(sigma1, X02, H, N)

# Calculate the estimation for true trajectories using Kalman filter
Xf, Pf, K, Ps = kalmanFilter(Fi, H, Q, R, X0, P0, Z02, N)

X1 = generate_Xt(x0, sigma_1, Fi, G, N)

# Generate measurement trajectories
Z1 = generate_Z(sigma1, X1, H, N)

# Calculate the estimation for true trajectories using Kalman filter
Xf1, Pf1, K1, Ps1 = kalmanFilter(Fi, H, Q, R, X0, P0, Z1, N)

y1 = Xt[0][0]
y2 = X02[0][0]
y3 = X1[0][0]
x = step
```

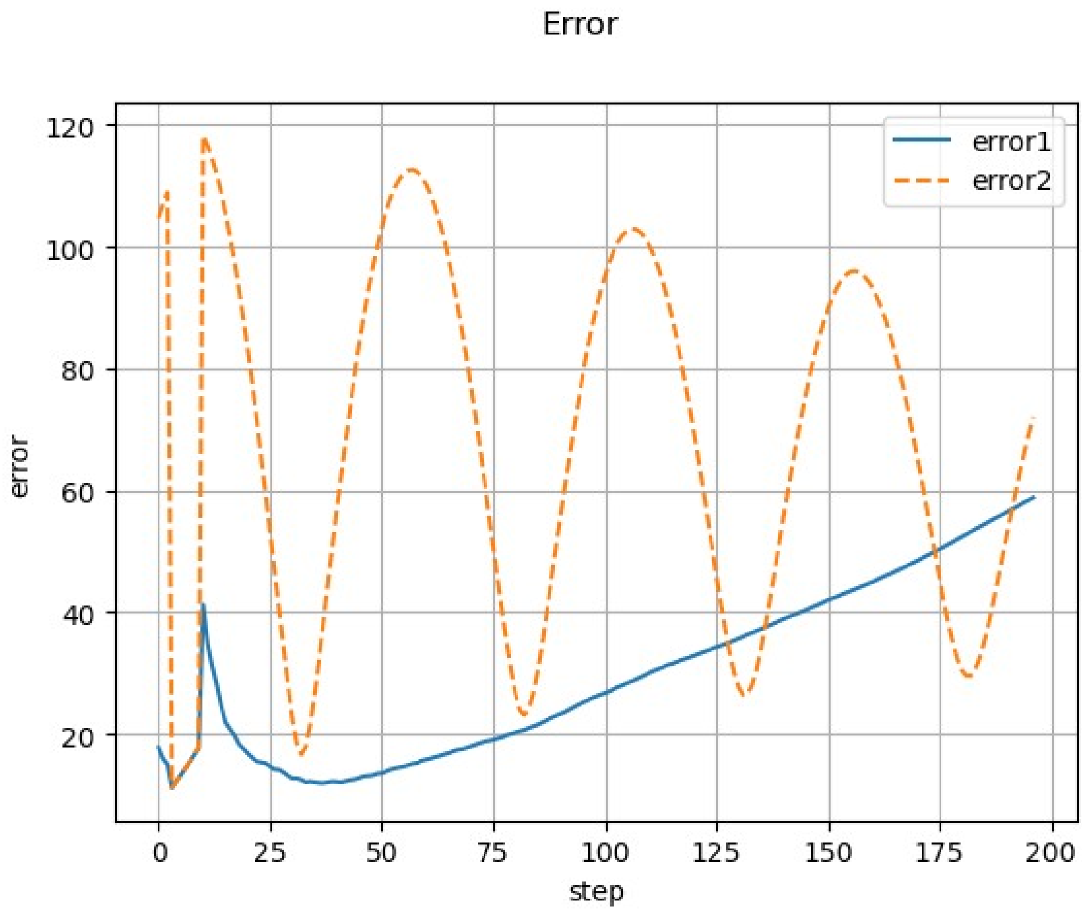
```
plt.plot(x,y1,label = 'True trajectory')
plt.plot(x,y2,label = 'filtered estimates (sigma = 0.2**2)')
plt.plot(x,y3,label = 'filtered estimates (sigma = 1)')
plt.xlabel('step')
plt.ylabel('X')
plt.suptitle('')
plt.legend()
plt.grid(True)
```




```
#Conclusion: for bigger sigma the  differernce from true trajectory
is bigger and the number of steps for stabilisation smaller

#Step 15
#a) calculate optimal filter gain according to Kalman filter
equations and calculate mean-squared error
sigma = 0.2**2
X1 = np.array([[100], [5]]) # Initial filter estimates


# Generate true trajectories
X = generate_Xt(x0, sigma, Fi, G, N)

# Generate measurement trajectories
Z = generate_Z(sigma1, X, H, N)

# Calculate the estimation for true trajectories using Kalman filter
Xf, Pf, K, Ps = kalmanFilter(Fi, H, Q, R, X1, P0, Z, N)
```

```
MSE15_1, MSE15_2 =
calculate_MSE(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X1,P0)

def kalmanFilter15(Fi, H, Q, R, X0, P0, Z, N,K_underestimated):
    Xp = np.zeros((2, 1, N)) #Prediction of state vector at time i
using  (i-1) measurements
    Pp = np.zeros((2, 2, N)) #Prediction error covariance matrix
    Xf = np.zeros((2, 1, N)) #Improved estimate by incorporating
    K = np.zeros((2, 1, N)) #Filter gain, weight of residual
    Pf = np.zeros((2, 2, N)) #Filtration error covariance matrix
    Ps = np.zeros((1, 1, N))

    Pf[:, :, 0] = P0
    Xf[:, :, 0] = X0
    for i in range(1, N):
        Xp[:, :, i] = np.dot(Fi, Xf[:, :, i-1])
        Pp[:, :, i] = np.dot(np.dot(Fi, Pf[:, :, i-1]), Fi.T) + Q
        Xf[:, :, i] = Xp[:, :, i] + np.dot(K_underestimated, Z[:, :,
i] - np.dot(H.T, Xp[:, :, i])) # np.dot(H, Xp[:, i])
        Pf[:, :, i] = np.dot((np.eye(2) - K_underestimated)*H.T,
Pp[:, :, i])
        # FOR TASK 6,11 square root of the first diagonal element of
Pi,i
        #that corresponds to standard deviation of estimation error
of coordinate xi
        Ps[:, :, i] = np.sqrt(np.abs(np.diag(Pf[:, :, i])[0]))

    return Xf, Pf, K, Ps


#b) run filter with underestimated filter gain K
K_underestimated = K[0][0][199]/5

Xf15, Pf15, K15, Ps15 = kalmanFilter15(Fi, H, Q, R, X0, P0, Z,
N,K_underestimated)

def calculate_MSE15(m, N, M, x0, sigma, sigma1, Fi, G, H, Q, R, X0,
P0,K_underestimated):
    Error_run = np.zeros((2, M, N))
    final_Error = np.zeros((2, 1, N-3))
    Error_run1 = np.zeros((2, M, N))
    final_Error1 = np.zeros((2, 1, N-3))

    for i in range(M):
        Xt = generate_Xt(x0, sigma, Fi, G, N)
        Z = generate_Z(sigma1, Xt, H, N)
        Xf, Pf, K, Ps = kalmanFilter15(Fi, H, Q, R, X0, P0, Z,
N,K_underestimated)
        Xpm = kalman_mSteps(m, Fi, N, Xf)

        for j in range(N):
            Error_run[:, i, j] = ((Xt[:, :, j] - Xf[:, :,
j])**2).T[0]
            Error_run1[:, i, j] = ((Xt[:, :, j] - Xpm[:, :,
```

```
j])**2).T[0]

    for i in range(N-3):
        for k in range(M):
            final_Error[:, :, i] += Error_run[:, k, i+3].T[0]
            final_Error1[:, :, i] += Error_run1[:, k, i+3].T[0]

        final_Error[:, :, i] = np.sqrt(final_Error[:, :, i] / (M -
1))
        final_Error1[:, :, i] = np.sqrt(final_Error1[:, :, i] / (M -
1))

    return final_Error, final_Error1

MSE15_3, MSE15_4 =
calculate_MSE15(m,N,M,x0,sigma,sigma1,Fi,G,H,Q,R,X1,P0,K_underestima
ted)

y1 = MSE15_2[0][0]
y2 = MSE15_4[0][0]
x = step1
plt.plot(x,y1,label = 'error1')
plt.plot(x,y2,label = 'error2',linestyle = '--')
plt.xlabel('step')
plt.ylabel('error')
plt.suptitle('Error')
plt.legend()
plt.grid(True)
```




```
#Conclusion: the period of stabilisatioin for underestimated filter
gain K (error 2)  is much bigger than number of steps and its
behavior differs a  lot from optimal filter gain (error1)
```

Learning log: Yaroslav: first time Kalman filter in action, besides high quality of filtration, its mait goal is accurate estimation. Deffinately learned new powerful instrument for my engineering work and amased how besides its first sight complexity, it uses linear equations for work Lisa: I learned how to implement Kalman filter for data analisys with usage of popular libraries. No doubt this srategy of analisys will be useful for my education process in Robotics Selamawit: this big exersise tought me that Kalman filter isnt scary gaint from high level math, but the tool i want to use. But to master the art of adjusting it in the right way takes time , so I am looking forward to implement the new knowlege.