Probability and Stochastic Modelling 1

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Contents

1 Events and Probability

1.1 Events

Definition 1.1 (Event)**.** An event is a set of outcomes in a random experiment commonly denoted by a capital letter. Events can be simple (a single event) or compound (two or more simple events).

Definition 1.2 (Sample space)**.** The set of all possible outcomes of an experiment is known as the sample space for that experiment and is denoted Ω.

Definition 1.3 (Intersection)**.** An intersection between two events A and B describes the set of outcomes that occur in both A and B. The intersection can be represented using the set `AND` operator ($\cap$) — $A \cap B$ (or AB).

Definition 1.4 (Disjoint)**.** Disjoint (mutually exclusive) events are two events that cannot occur simultaneously, or have no common outcomes.

Theorem 1.1 (Intersection of disjoint events)**.** *The intersection of disjoint events results in the null set ($\varnothing$).*

Lemma 1.1.1. *Disjoint events are **dependent** events as the occurrence of one means the other cannot occur.*

Definition 1.5 (Union)**.** A union of two events A and B describes the set of outcomes in either A or B. The union is represented using the set `OR` operator ($\cup$) — $A \cup B$.

Definition 1.6 (Complement)**.** The complement of an event E is the set of all other outcomes in Ω. The complement of E is denoted $\overline{E}$.

Theorem 1.2 (Intersection of complement set)**.**

$$A\overline{A} = \varnothing$$

Theorem 1.3 (Union of complement set)**.**

$$A \cup \overline{A} = \Omega$$

Definition 1.7 (Subset)**.** A is a (non-strict) subset of B if all elements in A are also in B. This can be denoted as $A \subseteq B$.

Theorem 1.4. *All events E are subsets of Ω.*

Theorem 1.5. *Given $A \subseteq B$*

$$AB = A \qquad \textit{and} \qquad A \cup B = B$$

Corollary 1.5.1. *Given $\varnothing \subseteq E$*

$$\varnothing E = \varnothing \qquad \textit{and} \qquad \varnothing \cup E = E$$

Theorem 1.6 (Associative Identities)**.**

$$A\left(BC\right) = \left(AB\right)C$$
$$A \cup \left(B \cup C\right) = \left(A \cup B\right) \cup C$$

Theorem 1.7 (Distributive Identities)**.**

$$A\left(B \cup C\right) = AB \cup AC$$
$$A \cup BC = \left(A \cup B\right)\left(A \cup C\right)$$

1.2 Probability

Definition 1.8 (Probability)**.** Probability is a measure of the likeliness of an event occurring. The probability of an event E is denoted $\Pr(E)$ (sometimes $\mathrm{P}(E)$).

$$0 \le \Pr(E) \le 1$$

where a probability of 0 never happens, and 1 always happens.

Theorem 1.8 (Probability of Ω)**.**

$$\Pr(\Omega) = 1$$

Theorem 1.9 (Multiplicative rule)**.** *The probability of the intersection between two independent events A and B is given by*

$$\Pr(AB) = \Pr(A)\Pr(B)$$

Theorem 1.10 (Probability of disjoint events)**.** *The probability of disjoint events A and B is given by*

$$\begin{aligned} \Pr(AB) &= 0 \\ \Pr(\varnothing) &= 0. \end{aligned}$$

Disjoint events are dependent events, since the occurrence of one means the other cannot occur.

Theorem 1.11 (Addition rule)**.** *The probability of the union between two independent events A and B is given by*

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB).$$

If A and B are disjoint, then $\Pr(AB) = 0$, so that $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Corollary 1.11.1 (Addition rule for 3 event)**.** *The addition rule for 3 events is as follows*

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(B) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC).$$

Proof. If we write $D = A \cup B$ and apply the addition rule twice, we have

$$\begin{aligned} \Pr(A \cup B \cup C) &= \Pr(D \cup C) \\ &= \Pr(D) + \Pr(C) - \Pr(DC) \\ &= \Pr(A \cup B) + \Pr(C) - \Pr((A \cup B)C) \\ &= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - \Pr(AC \cup BC) \\ &= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - (\Pr(AC) + \Pr(BC) - \Pr(ACBC)) \\ &= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC) \end{aligned}$$

□

Theorem 1.12 (Complement rule)**.** *The probability of the complement of E is given by*

$$\Pr(\overline{E}) = 1 - \Pr(E)$$

Theorem 1.13 (Probability of subsets)**.** *If $A \subseteq B$ then $\Pr(A) \le \Pr(B)$. Also, $\Pr(AB) = \Pr(A)$ and $\Pr(A \cup B) = \Pr(B)$.*

Theorem 1.14 (Law of total probability)**.** *By writing the event A as $AB \cup A\overline{B}$, and noting that AB and $A\overline{B}$ are disjoint:*

$$\Pr(A) = \Pr(AB) + \Pr\left(A\overline{B}\right)$$

Theorem 1.15 (De Morgan's laws)**.** *Recall De Morgan's Laws:*

$$\begin{aligned} \overline{A \cup B} &= \overline{A}\ \overline{B} \\ \overline{AB} &= \overline{A} \cup \overline{B}. \end{aligned}$$

Taking the negation of both sides and applying the complement rule yields

$$\begin{aligned} \Pr(A \cup B) &= 1 - \Pr\left(\overline{A}\ \overline{B}\right) \\ \Pr(AB) &= 1 - \Pr\left(\overline{A} \cup \overline{B}\right) \end{aligned}$$

1.3 Circuits

A signal can pass through a circuit if there is a functional path from start to finish.

We can define a circuit where each component i functions with probability p, and is independent of other components.

Then W_i to be the event in which the associated component i functions, we can determine the event S in which the system functions, and probability $\Pr(S)$ that the system functions.

As the probability that any component functions is p, in other words

$$\Pr(W_i) = p,$$

$\Pr(S)$ will be a function of p defined $f : [0,\ 1] \to [0,\ 1]$.