Probability and Stochastic Modelling 1

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Contents

1 Events and Probability

1.1 Events

Definition 1.1 (Event)**.** An event is a set of outcomes in a random experiment commonly denoted by a capital letter. Events can be simple (a single event) or compound (two or more simple events).

Definition 1.2 (Sample space)**.** The set of all possible outcomes of an experiment is known as the sample space for that experiment and is denoted Ω.

Definition 1.3 (Intersection)**.** An intersection between two events A and B describes the set of outcomes that occur in both A and B. The intersection can be represented using the set `AND` operator ($\cap$) — $A \cap B$ (or AB).

Definition 1.4 (Disjoint)**.** Disjoint (mutually exclusive) events are two events that cannot occur simultaneously, or have no common outcomes.

Theorem 1.1.1 (Intersection of disjoint events)**.** *The intersection of disjoint events results in the null set* ($\varnothing$)*.*

Lemma 1.1.1.1. *Disjoint events are* ***dependent*** *events as the occurrence of one means the other cannot occur.*

Definition 1.5 (Union)**.** A union of two events A and B describes the set of outcomes in either A or B. The union is represented using the set `OR` operator ($\cup$) — $A \cup B$.

Definition 1.6 (Complement)**.** The complement of an event E is the set of all other outcomes in Ω. The complement of E is denoted $\overline{E}$.

Theorem 1.1.2 (Intersection of complement set)**.**

$$A\overline{A} = \varnothing$$

Theorem 1.1.3 (Union of complement set)**.**

$$A \cup \overline{A} = \Omega$$

Definition 1.7 (Subset)**.** A is a (non-strict) subset of B if all elements in A are also in B. This can be denoted as $A \subseteq B$.

Theorem 1.1.4. *All events E are subsets of Ω.*

Theorem 1.1.5. *Given $A \subseteq B$*

$$AB = A \qquad \text{and} \qquad A \cup B = B$$

Corollary 1.1.5.1. *Given $\varnothing \subseteq E$*

$$\varnothing E = \varnothing \qquad \text{and} \qquad \varnothing \cup E = E$$

Theorem 1.1.6 (Associative Identities)**.**

$$A\,(BC) = (AB)\,C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

Theorem 1.1.7 (Distributive Identities)**.**

$$A\,(B \cup C) = AB \cup AC$$
$$A \cup BC = (A \cup B)\,(A \cup C)$$

1.2 Probability

Definition 1.8 (Probability)**.** Probability is a measure of the likeliness of an event occurring. The probability of an event E is denoted $\Pr(E)$ (sometimes $\mathrm{P}(E)$).

$$0 \leq \Pr(E) \leq 1$$

where a probability of 0 never happens, and 1 always happens.

Theorem 1.2.1 (Probability of Ω)**.**

$$\Pr(\Omega) = 1$$

Theorem 1.2.2 (Multiplication rule for independent events)**.** *The probability of the intersection between two independent events A and B is given by*

$$\Pr(AB) = \Pr(A)\Pr(B)$$

Theorem 1.2.3 (Probability of disjoint events)**.** *The probability of disjoint events A and B is given by*

$$\Pr(AB) = 0$$
$$\Pr(\varnothing) = 0.$$

Disjoint events are dependent events, since the occurrence of one means the other cannot occur.

Theorem 1.2.4 (Addition rule for independent events)**.** *The probability of the union between two independent events A and B is given by*

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB).$$

If A and B are disjoint, then $\Pr(AB) = 0$, so that $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Corollary 1.2.4.1 (Addition rule for 3 events)**.** *The addition rule for 3 events is as follows*

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(B) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC).$$

Proof. If we write $D = A \cup B$ and apply the addition rule twice, we have

$$\begin{aligned}
\Pr(A \cup B \cup C) &= \Pr(D \cup C) \\
&= \Pr(D) + \Pr(C) - \Pr(DC) \\
&= \Pr(A \cup B) + \Pr(C) - \Pr((A \cup B)C) \\
&= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - \Pr(AC \cup BC) \\
&= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - (\Pr(AC) + \Pr(BC) - \Pr(ACBC)) \\
&= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC)
\end{aligned}$$

□

Theorem 1.2.5 (Complement rule)**.** *The probability of the complement of E is given by*

$$\Pr(\overline{E}) = 1 - \Pr(E)$$

Theorem 1.2.6 (Probability of subsets)**.** *If* $A \subseteq B$ *then* $\Pr(A) \leq \Pr(B)$. *Also,* $\Pr(AB) = \Pr(A)$ *and* $\Pr(A \cup B) = \Pr(B)$.

Theorem 1.2.7 (Law of total probability)**.** *By writing the event* A *as* $AB \cup A\overline{B}$*, and noting that* AB *and* $A\overline{B}$ *are disjoint:*

$$\Pr(A) = \Pr(AB) + \Pr\left(A\overline{B}\right)$$

Theorem 1.2.8 (De Morgan's laws)**.** *Recall De Morgan's Laws:*

$$\begin{aligned}\overline{A \cup B} &= \overline{A}\ \overline{B} \\ \overline{AB} &= \overline{A} \cup \overline{B}.\end{aligned}$$

Taking the negation of both sides and applying the complement rule yields

$$\begin{aligned}\Pr(A \cup B) &= 1 - \Pr\left(\overline{A}\ \overline{B}\right) \\ \Pr(AB) &= 1 - \Pr\left(\overline{A} \cup \overline{B}\right)\end{aligned}$$

1.3 Circuits

A signal can pass through a circuit if there is a functional path from start to finish.

We can define a circuit where each component i functions with probability p, and is independent of other components.

Then W_i to be the event in which the associated component i functions, we can determine the event S in which the system functions, and probability $\Pr(S)$ that the system functions.

As the probability that any component functions is p, in other words

$$\Pr(W_i) = p,$$

$\Pr(S)$ will be a function of p defined $f : [0,\ 1] \to [0,\ 1]$.

2 Independence

Definition 2.1 (Conditional probability)**.** When discussing multiple events, it is possible that the occurrence of one event changes the probability that another will occur. This can be denoted using a vertical bar, and is read as "the probability of event A given B":

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}.$$

Definition 2.2 (Multiplication rule)**.** For events A and B, the general multiplication rule states that

$$\Pr(AB) = \Pr(A \mid B)\Pr(B)$$

Theorem 2.0.1 (Independent events)**.** *If* A *and* B *are independent events then*

$$\begin{aligned}\Pr(A \mid B) &= \Pr(A) \\ \Pr(B \mid A) &= \Pr(B)\end{aligned}$$

Theorem 2.0.2 (Complement of independent events)**.** *If A and B are independent, all complement pairs are also independent. Given*

$$\begin{aligned}\Pr(A \mid B) &= \Pr(A)\\ \Pr(B \mid A) &= \Pr(B)\end{aligned}$$

the following statements are also true

$$\begin{aligned}\Pr\left(A \mid \overline{B}\right) &= \Pr(A) & \Pr\left(B \mid \overline{A}\right) &= \Pr(B)\\ \Pr\left(\overline{A} \mid B\right) &= \Pr\left(\overline{A}\right) & \Pr\left(\overline{B} \mid A\right) &= \Pr\left(\overline{B}\right)\\ \Pr\left(\overline{A} \mid \overline{B}\right) &= \Pr\left(\overline{A}\right) & \Pr\left(\overline{B} \mid \overline{A}\right) &= \Pr\left(\overline{B}\right)\end{aligned}$$

2.1 Probability Rules with Conditional

ALl probability rules hold when conditioning on some event C.

Theorem 2.1.1 (Complement rule with condition)**.**

$$\Pr\left(\overline{A} \mid C\right) = 1 - \Pr(A \mid C)$$

Theorem 2.1.2 (Addition rule with condition)**.**

$$\Pr(A \cup B \mid C) = \Pr(A \mid C) + \Pr(B \mid C) - \Pr(AB \mid C)$$

Theorem 2.1.3 (Multiplication rule with condition)**.**

$$\Pr(AB \mid C) = \Pr(A \mid BC)\Pr(B \mid C)$$

In the above examples, all probabilities are conditional on the sample space, hence we are effectively changing the sample space.

2.2 Conditional Independence

Definition 2.3 (Conditional independence)**.** Suppose events A and B are not independent, i.e.,

$$\Pr(A \mid B) \neq \Pr(A)$$

but they become independent when conditioned with another event C, i.e.,

$$\Pr(A \mid BC) = \Pr(A \mid C)$$

Here we say that A and B are **conditionally independent** given C. Furthermore

$$\Pr(AB \mid C) = \Pr(A \mid C)\Pr(B \mid C)$$

Conversely, events A and B may be conditionally dependent but unconditionally independent, i.e.,

$$\begin{aligned}\Pr(A \mid B) &= \Pr(A)\\ \Pr(A \mid BC) &\neq \Pr(A \mid C)\\ \Pr(AB \mid C) &= \Pr(A \mid BC)\Pr(B \mid C)\end{aligned}$$

Theorem 2.2.1. *Given events A, B, and C. Pairwise independence does not imply mutual independence. I.e.,*

$$\begin{cases} \Pr(AB) = \Pr(A)\Pr(B) \\ \Pr(AC) = \Pr(A)\Pr(C) \\ \Pr(BC) = \Pr(B)\Pr(C) \end{cases}$$

does not imply

$$\Pr(ABC) = \Pr(A)\Pr(B)\Pr(C).$$