Probability and Stochastic Modelling 1

Semester 1, 2022

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Contents

1 Events and Probability

1.1 Events

Definition 1.1 (Event)**.** An event is a set of outcomes in a random experiment commonly denoted by a capital letter. Events can be simple (a single event) or compound (two or more simple events).

Definition 1.2 (Sample space)**.** The set of all possible outcomes of an experiment is known as the sample space for that experiment and is denoted Ω.

Definition 1.3 (Intersection)**.** An intersection between two events A and B describes the set of outcomes that occur in both A and B. The intersection can be represented using the set `AND` operator ($\cap$) — $A \cap B$ (or AB).

Definition 1.4 (Disjoint)**.** Disjoint (mutually exclusive) events are two events that cannot occur simultaneously, or have no common outcomes.

Theorem 1.1.1 (Intersection of disjoint events)**.** *The intersection of disjoint events results in the null set ($\varnothing$).*

Lemma 1.1.1.1. *Disjoint events are **dependent** events as the occurrence of one means the other cannot occur.*

Definition 1.5 (Union)**.** A union of two events A and B describes the set of outcomes in either A or B. The union is represented using the set `OR` operator ($\cup$) — $A \cup B$.

Definition 1.6 (Complement)**.** The complement of an event E is the set of all other outcomes in Ω. The complement of E is denoted $\overline{E}$.

Theorem 1.1.2 (Intersection of complement set)**.**

$$A\overline{A} = \varnothing$$

Theorem 1.1.3 (Union of complement set)**.**

$$A \cup \overline{A} = \Omega$$

Definition 1.7 (Subset)**.** A is a (non-strict) subset of B if all elements in A are also in B. This can be denoted as $A \subset B$.

Theorem 1.1.4. *All events E are subsets of Ω.*

Theorem 1.1.5. *Given $A \subset B$*

$$AB = A \qquad \text{and} \qquad A \cup B = B$$

Corollary 1.1.5.1. *Given $\varnothing \subset E$*

$$\varnothing E = \varnothing \qquad \text{and} \qquad \varnothing \cup E = E$$

Theorem 1.1.6 (Associative Identities)**.**

$$\begin{aligned} A\left(BC\right) &= \left(AB\right)C \\ A \cup \left(B \cup C\right) &= \left(A \cup B\right) \cup C \end{aligned}$$

Theorem 1.1.7 (Distributive Identities)**.**

$$\begin{aligned} A\left(B \cup C\right) &= AB \cup AC \\ A \cup BC &= \left(A \cup B\right)\left(A \cup C\right) \end{aligned}$$

1.2 Probability

Definition 1.8 (Probability)**.** Probability is a measure of the likeliness of an event occurring. The probability of an event E is denoted $\Pr(E)$ (sometimes $\mathrm{P}(E)$).

$$0 \leq \Pr(E) \leq 1$$

where a probability of 0 never happens, and 1 always happens.

Theorem 1.2.1 (Probability of Ω)**.**

$$\Pr(\Omega) = 1$$

Theorem 1.2.2 (Complement rule)**.** *The probability of the complement of E is given by*

$$\Pr\left(\overline{E}\right) = 1 - \Pr(E)$$

Theorem 1.2.3 (Multiplication rule for independent events)**.** *The probability of the intersection between two independent events A and B is given by*

$$\Pr(AB) = \Pr(A)\Pr(B)$$

Theorem 1.2.4 (Addition rule for independent events)**.** *The probability of the union between two independent events A and B is given by*

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB).$$

If A and B are disjoint, then $\Pr(AB) = 0$, so that $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Corollary 1.2.4.1 (Addition rule for 3 events)**.** *The addition rule for 3 events is as follows*

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC).$$

Proof. If we write $D = A \cup B$ and apply the addition rule twice, we have

$$\begin{aligned}
\Pr(A \cup B \cup C) &= \Pr(D \cup C) \\
&= \Pr(D) + \Pr(C) - \Pr(DC) \\
&= \Pr(A \cup B) + \Pr(C) - \Pr((A \cup B)C) \\
&= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - \Pr(AC \cup BC) \\
&= \Pr(A) + \Pr(B) - \Pr(AB) + \Pr(C) - (\Pr(AC) + \Pr(BC) - \Pr(ACBC)) \\
&= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(AC) - \Pr(BC) + \Pr(ABC)
\end{aligned}$$

□

Theorem 1.2.5 (De Morgan's laws)**.** *Recall De Morgan's Laws:*

$$\begin{aligned}
\overline{A \cup B} &= \overline{A}\ \overline{B} \\
\overline{AB} &= \overline{A} \cup \overline{B}.
\end{aligned}$$

Taking the negation of both sides and applying the complement rule yields

$$\begin{aligned}
\Pr(A \cup B) &= 1 - \Pr\left(\overline{A}\ \overline{B}\right) \\
\Pr(AB) &= 1 - \Pr\left(\overline{A} \cup \overline{B}\right)
\end{aligned}$$

1.3 Circuits

A signal can pass through a circuit if there is a functional path from start to finish.
We can define a circuit where each component i functions with probability p, and is independent of other components.
Then W_i to be the event in which the associated component i functions, we can determine the event S in which the system functions, and probability $\Pr(S)$ that the system functions.
As the probability that any component functions is p, in other words

$$\Pr(W_i) = p,$$

$\Pr(S)$ will be a function of p defined $f : [0,\ 1] \to [0,\ 1]$.

2 Independence

Definition 2.1 (Conditional probability)**.** When discussing multiple events, it is possible that the occurrence of one event changes the probability that another will occur. This can be denoted using a vertical bar, and is read as "the probability of event A given B":

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}.$$

Definition 2.2 (Multiplication rule)**.** For events A and B, the general multiplication rule states that

$$\Pr(AB) = \Pr(A \mid B) \Pr(B)$$

Theorem 2.0.1 (Independent events)**.** *If A and B are independent events then*

$$\Pr(A \mid B) = \Pr(A)$$
$$\Pr(B \mid A) = \Pr(B)$$

Theorem 2.0.2 (Complement of independent events)**.** *If A and B are independent, all complement pairs are also independent. Given*

$$\Pr(A \mid B) = \Pr(A)$$
$$\Pr(B \mid A) = \Pr(B)$$

the following statements are also true

$$\Pr\left(A \mid \overline{B}\right) = \Pr(A) \qquad \Pr\left(B \mid \overline{A}\right) = \Pr(B)$$
$$\Pr\left(\overline{A} \mid B\right) = \Pr\left(\overline{A}\right) \qquad \Pr\left(\overline{B} \mid A\right) = \Pr\left(\overline{B}\right)$$
$$\Pr\left(\overline{A} \mid \overline{B}\right) = \Pr\left(\overline{A}\right) \qquad \Pr\left(\overline{B} \mid \overline{A}\right) = \Pr\left(\overline{B}\right)$$

2.1 Probability Rules with Conditional

ALl probability rules hold when conditioning on some event C.

Theorem 2.1.1 (Complement rule with condition)**.**

$$\Pr\left(\overline{A} \mid C\right) = 1 - \Pr\left(A \mid C\right)$$

Theorem 2.1.2 (Addition rule with condition)**.**

$$\Pr\left(A \cup B \mid C\right) = \Pr\left(A \mid C\right) + \Pr\left(B \mid C\right) - \Pr\left(AB \mid C\right)$$

Theorem 2.1.3 (Multiplication rule with condition)**.**

$$\Pr\left(AB \mid C\right) = \Pr\left(A \mid BC\right) \Pr\left(B \mid C\right)$$

In the above examples, all probabilities are conditional on the sample space, hence we are effectively changing the sample space.

2.2 Conditional Independence

Definition 2.3 (Conditional independence)**.** Suppose events A and B are not independent, i.e.,

$$\Pr\left(A \mid B\right) \neq \Pr\left(A\right)$$

but they become independent when conditioned with another event C, i.e.,

$$\Pr\left(A \mid BC\right) = \Pr\left(A \mid C\right)$$

Here we say that A and B are **conditionally independent** given C. Furthermore

$$\Pr\left(AB \mid C\right) = \Pr\left(A \mid C\right) \Pr\left(B \mid C\right)$$

Conversely, events A and B may be conditionally dependent but unconditionally independent, i.e.,

$$\begin{aligned}
\Pr\left(A \mid B\right) &= \Pr\left(A\right) \\
\Pr\left(A \mid BC\right) &\neq \Pr\left(A \mid C\right) \\
\Pr\left(AB \mid C\right) &= \Pr\left(A \mid BC\right) \Pr\left(B \mid C\right)
\end{aligned}$$

Theorem 2.2.1. *Given events A, B, and C. Pairwise independence does not imply mutual independence. I.e.,*

$$\begin{cases}
\Pr\left(AB\right) = \Pr\left(A\right) \Pr\left(B\right) \\
\Pr\left(AC\right) = \Pr\left(A\right) \Pr\left(C\right) \\
\Pr\left(BC\right) = \Pr\left(B\right) \Pr\left(C\right)
\end{cases}$$

does not imply

$$\Pr\left(ABC\right) = \Pr\left(A\right) \Pr\left(B\right) \Pr\left(C\right).$$

In summary, independence should not be assumed unless explicitly stated.

2.3 Disjoint Events

Theorem 2.3.1 (Probability of disjoint events). *The probability of disjoint events A and B is given by*

$$\Pr(AB) = 0$$
$$\Pr(\varnothing) = 0.$$

Disjoint events are highly dependent events, since the occurrence of one means the other cannot occur. This implies

$$\Pr(A \mid B) = 0$$

2.4 Subsets

Theorem 2.4.1 (Probability of subsets). *If $A \subset B$ then $\Pr(A) \leq \Pr(B)$. We also know that $\Pr(AB) = \Pr(A)$ and $\Pr(A \cup B) = \Pr(B)$.*
Here, if A happens, then B definitely happens.

$$\Pr(B \mid A) = 1$$

Given $\Pr(AB) = \Pr(A)$

$$\Pr(A \mid B) = \frac{\Pr(A)}{\Pr(B)}$$

These events are also highly dependent.

3 Total Probability

Definition 3.1 (Marginal probability). Marginal probability is the probability of an event irrespective of the outcome of another variable.

Theorem 3.0.1 (Total probability for complements). *By writing the event A as $AB \cup A\overline{B}$, and noting that AB and $A\overline{B}$ are disjoint, the marginal probability of A is given by*

$$\Pr(A) = \Pr(AB) + \Pr\left(A\overline{B}\right).$$

By applying the multiplication rule to each joint probability:

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr\left(A \mid \overline{B}\right)\Pr\left(\overline{B}\right)$$

Theorem 3.0.2 (Law of total probability). *The previous theorem partitioned Ω into disjoint events B and $\overline{B}$.*
By partitioning Ω into a collection of disjoint events $B_1,\ B_2,\ \dots,\ B_n$, such that $\bigcup_{i=1}^{n} B_i = \Omega$, we have

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \mid B_i)\Pr(B_i)$$

Theorem 3.0.3 (Bayes' Theorem). *Given the probability for A given B, the probability of the reverse direction is given by*

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$$

4 Combinatorics

Definition 4.1 (Number of outcomes)**.** Let $|A|$ denote the number of outcomes in an event A.

Theorem 4.0.1 (Addition principle)**.** *Given a sample space S with k disjoint events $\{S_1, \ldots, S_k\}$, where the ith event has n_i possible outcomes, the number of possible samples from any event is given by*

$$\left|\bigcup_{i=0}^{k} S_i\right| = \sum_{i=1}^{k} n_i$$

Theorem 4.0.2 (Multiplication principle)**.** *Given a sample space S with k events $\{S_1, \ldots, S_k\}$, where the ith event has n_i possible outcomes, the number of possible samples from every event is given by*

$$\left|\bigcap_{i=0}^{k} S_i\right| = \prod_{i=1}^{k} n_i$$

Theorem 4.0.3 (Counting probability)**.** *Given a sample space S with equally likely outcomes, the probability of an event $S_i \subset S$ is given by*

$$\Pr\left(S_i\right) = \frac{|S_i|}{|S|}$$

4.1 Ordered Sampling with Replacement

When ordering is important and repetition is allowed, the total number of ways to choose k objects from a set with n elements is

$$n^k$$

4.2 Ordered Sampling without Replacement

When ordering is important and repetition is not allowed, the total number of ways to arrange k objects from a set of n elements is known as a k-permutation of n-elements denoted ${}^{n}P_k$

$$\begin{aligned} {}^{n}P_k &= n \times (n-1) \times \cdots \times (n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

for $0 \leq k \leq n$.

Definition 4.2 (Permutation of n elements)**.** An n-permutation of n elements is the permutation of those elements. In this case, $k = n$, so that

$$\begin{aligned} {}^{n}P_n &= n \times (n-1) \times \cdots \times (n-n+1) \\ &= n! \end{aligned}$$

4.3 Unordered Sampling without Replacement

When ordering is not important and repetition is not allowed, the total number of ways to choose k objects from a set of n elements is known as a k-combination of n-elements denoted ${}^{n}C_{k}$ or $\binom{n}{k}$

$$\begin{aligned} {}^{n}C_{k} &= \frac{{}^{n}P_{k}}{k!} \\ &= \frac{n!}{k!\,(n-k)!} \end{aligned}$$

for $0 \leq k \leq n$. We divide by $k!$ because any k-element subset of n-elements can be ordered in $k!$ ways.

4.4 Unordered Sampling with Replacement

When ordering is not important and repetition is allowed, the total number of ways to choose k objects from a set with n elements is

$$n^{k}$$