BME 590

Non-Linear Control Design Project

Tracking Control of a Robot Manipulator with Uncertainties

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Section I: Introduction

The objective of this project is to design, simulate, and compare the performance of three non-linear controllers to perform a tracking task for a 2-link robot manipulator system which is subject to uncertainties.

The three non-linear controllers to be studied are: Exact Model Knowledge (EMK), Adaptive Control, and Sliding Mode Robust Control

This report comprises of non-linear control design and development of for each of the three mentioned controllers along with the stability analysis for each controller. Except for the EMK Controller, the controllers have been designed to account for uncertain system parameters. The assumptions made during the design phase have been mentioned in the report. Simulation results demonstrate the performance of each controller over time. All three controllers have been modeled and simulated in MATLAB SIMULINK.

The controllers have been compared based on their stability, control effect, and their robustness.

Section II: Dynamic Model

The non-linear robot dynamics are stated as below:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}p_1 + 2p_2\cos(q_2) & p_2 + p_3\cos(q_2)\\ p_2 + p_3\cos(q_2) & p_2\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix}-p_3\sin(q_2)\dot{q}_2 & -p_3\sin(q_2)(\dot{q}_1 + \dot{q}_2)\\ p_3\sin(q_2)\dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \begin{bmatrix}f_{d1} & 0\\ 0 & f_{d2}\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \begin{bmatrix}\tau d_1\\ \tau d_2\end{bmatrix}$$

Where,

$$p1 = 3.473 kg.m^2, p_2 = 0.196 kg.m^2, p_3 = 0.242 kg.m^2,$$
$$f_{d_1} = 5.3 N.m.sec, f_{d_2} = 1.1 N.m.sec$$

These equations are equivalent to:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{V}_{\boldsymbol{m}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}_{\boldsymbol{d}}\dot{\boldsymbol{q}} + \boldsymbol{\tau}_{\boldsymbol{d}}$$

The above equation describes the relationship between the motion of the robot links and the forces / torques causing them.

Where,

- $q \in R^2$ is a 2-dimensional vector representing joint positions
- $\dot{q} \in R^2$ is a 2-dimensional vector representing joint angular velocity
- $\ddot{q} \in R^2$ is a 2-dimensional vector representing joint angular acceleration
- τ is a 2-dimensional control input torque
- $M(q)$ is a symmetric inertia matrix and is positive definite for all $q \in R^2$
  $m_1||x||^2 \le x^T \mathrm{M}(\theta)\mathrm{x} \le m_2||x||^2$ $\quad \forall \mathrm{x} \in R^2,\ m_1, m_2 \in \mathrm{R}$ are positive constants
- $V_m(q, \dot{q})\dot{q}$ represents for centrifugal Coriolis forces
- $F_d\dot{q}$ represents the viscous damping, where F_d is a positive semi-definite symmetric matrix
- τ_d represents the disturbances acting on the system

- Control Objective

$$e = q_d - q$$

$e \to 0$ as $t \to \infty$

$q \to q_d$ as $t \to \infty$

The objective is to track a desired trajectory $q_d(t)$ to make the error equal to zero

- Auxiliary Signal

$$r = \dot{e} + \alpha e$$

where α is a positive scalar gain $\alpha > 0$

$$\dot{e} = \dot{q}_d - \dot{q}$$
$$\ddot{e} = \ddot{q}_d - \ddot{q}$$

- Open-Loop Error System

Taking the derivative of the auxiliary signal r

$$\begin{aligned} \dot{r} &= \ddot{e} + \alpha \dot{e} \\ M\dot{r} &= M\ddot{e} + \alpha M\dot{e} \\ &= M(\ddot{q}_d - \ddot{q}) + \alpha M\dot{e} \\ &= M\ddot{q}_d - M\ddot{q} + \alpha M\dot{e} \end{aligned}$$

Substituting robot dynamics,

$$\begin{aligned} M\dot{r} &= M\ddot{q}_d - (\tau - V_m\dot{q} - F_d\dot{q} - \tau_d) + \alpha M\dot{e} \\ &= M\ddot{q}_d - \tau + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e} \end{aligned}$$

The above equation gives the error dynamics for the system

Now, we design and implement EMK, Adaptive, and Robust controllers by designing the torque τ in equation above.

Section III: Exact Model Knowledge (EMK) Control

- Assumptions

- System parameters $p_1,\ p_2,\ p_3,\ f_{d_1}\ and\ f_{d_2}$ are known.
- Desired trajectories $q_d, \dot{q_d}\ and\ \ddot{q_d}$ are known and are bounded.
- Disturbances are not considered for EMK Controller i.e., τ_d is not considered

- Non-Linear Control Design

From the error dynamics of the system, we have:

$$\begin{aligned} M\dot{r} &= M\ddot{q_d} - (\tau - V_m\dot{q} - F_d\dot{q} - \tau_d) + \alpha M\dot{e} \\ &= M\ddot{q_d} - \tau + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e} \end{aligned}$$

The above equation can be modified by adding and then subtracting the term $V_m r$

$$M\dot{r} = M\ddot{q_d} - \tau + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e} + V_m r - V_m r$$

Considering $\tau_d = 0$

$$M\dot{r} = -V_m r - \tau + M\ddot{q_d} + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r$$

$$\boldsymbol{\tau = M\ddot{q}_d + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r + \square}$$

$$M\dot{r} = -V_m r - \square$$

- Stability Analysis

Consider the Lyapunov function

$$\boldsymbol{V = \frac{1}{2}r^{\mathrm{T}}Mr + \frac{1}{2}e^{T}e}$$

Differentiating,

$$\dot{V} = \frac{1}{2}r^{\mathrm{T}}\dot{M}\mathrm{r} + \mathrm{r}^{\mathrm{T}}M\dot{r} + e^{T}\dot{e}$$

Substituting error dynamics in the above equation,

$$\begin{aligned}\dot{V} &= \frac{1}{2}r^{\mathrm{T}}\dot{M}\mathrm{r} + r^{\mathrm{T}}(-V_m r - \square) + e^T\dot{e} \\ &= \frac{1}{2}r^{\mathrm{T}}\dot{M}\mathrm{r} - \mathrm{r}^{\mathrm{T}}V_m r - r^{\mathrm{T}}\square + e^T(r - \alpha e) \\ &= \frac{1}{2}r^{\mathrm{T}}(\dot{M} - V_m)r - r^{\mathrm{T}}\square + e^T r - e^T\alpha e\end{aligned}$$

By using skew-symmetric property,

$$\frac{1}{2}r^{\mathrm{T}}(\dot{M} - V_m)r = 0$$

$$\therefore \dot{\mathrm{V}} = -r^{\mathrm{T}}\square + e^T r - e^T\alpha e$$

Designing for the box term $\square$

Let $\square = e + Kr$ where K is a control gain matrix

K must be symmetric and positive definite.

$$\begin{aligned}\dot{V} &= -\mathrm{r}^{\mathrm{T}}\mathrm{e} - r^T K r + e^T r - e^T\alpha e \\ &= -e^T\alpha e - r^T K_1\mathrm{r} \\ &\leq -\alpha\|e\|^2 - \lambda_{min}(K_1)\|r\|^2 \\ \dot{V} &< 0\ (\mathrm{N.D.})\end{aligned}$$

As V is Negative Definite, this implies Global Asymptotic Stability (G.A.S)

Let $z = [e^T r^T]^T$

$$\begin{aligned}\dot{V} &\leq -\alpha\|e\|^2 - K\|r\|^2 \\ \dot{V} &\leq -[\alpha, \lambda_{min}(K)]_{min}(\|e\|^2 + \|r\|^2) \\ &\leq -[\alpha, \lambda_{min}(K)]_{min}\|z\|^2\end{aligned}$$

The Lyapunov function must be radially unbounded, positive definite and decrescent.

$$V_1 \leq V \leq V_2$$

$$V_1 = \left[\frac{\lambda_{min}(M)}{2}, \frac{1}{2}\right]_{min}\|z\|^2 = \lambda_1\|z\|^2$$

$$V_2 = \left[\frac{\lambda_{max}(M)}{2}, \frac{1}{2}\right]_{max}\|z\|^2 = \lambda_2\|z\|^2$$

Choosing $\beta < [\alpha, \lambda_{min}(K)]_{min}$

$-\beta > -[\alpha, \lambda_{min}(K)]_{min}$

$$\dot{V} \leq -\beta\|z\|^2$$

$$\lambda_1\|z\|^2 \leq V \leq \lambda_2\|z\|^2$$

$$\frac{\lambda_1}{\lambda_2}\|z\|^2 \leq \frac{V}{\lambda_2} \leq \|z\|^2$$

$$\frac{-\beta\lambda_1}{\lambda_2}\|z\|^2 \geq \frac{-\beta V}{\lambda_2} \geq -\beta\|z\|^2$$

$$\dot{V} \leq -\beta\|z\|^2$$

$$\dot{V} \leq \frac{-\beta V}{\lambda_2}$$

Integrating, we get:

$$V(t) \leq V(0)e^{\frac{-\beta t}{\lambda_2}}$$

This shows Global Exponential Stability (G.E.S)

The control torque designed for EMK Controller:

$$\boxed{\boldsymbol{\tau = M\ddot{q}_d + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r + e + Kr}}$$

- Results

Parameters used for simulation:

a. Control gain matrix $K = 5{*}I_2$
b. Gain $\alpha = 5$
c. Simulation time, $t = 20\ sec$
d. Desired trajectory $q_d(t) = [cos(t)\ ; cos(2t)]$

SIMULINK Model:

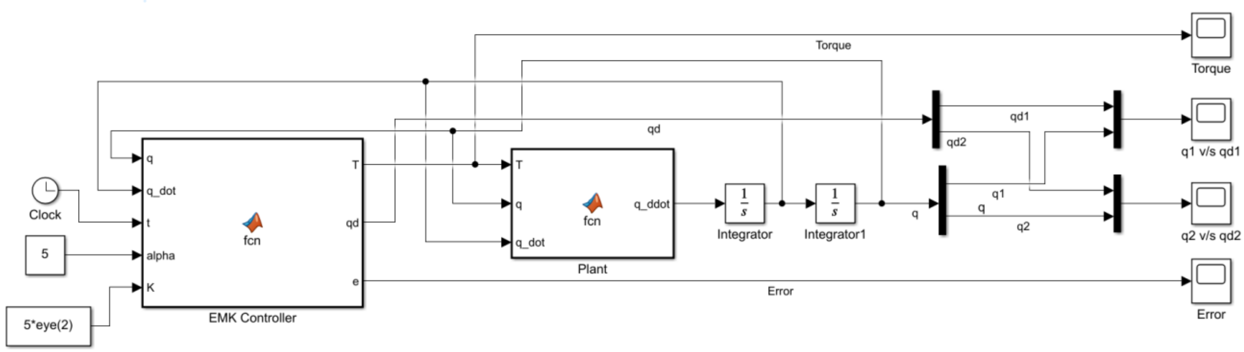

Figure 1: EMK Controller SIMULINK Model

Simulation Results:

<u>Error Plots:</u>

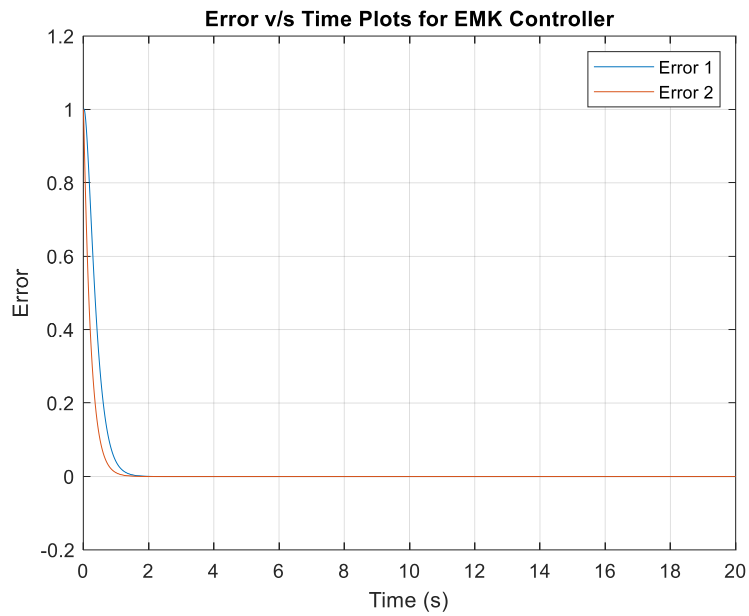

Figure 2: Error v/s Time plots for EMK Controller

From figure 2, we see that error converges to 0 after approximately 1 sec. This was proved in the stability analysis section.

<u>Angular Position Plots:</u>

q1 v/s qd1

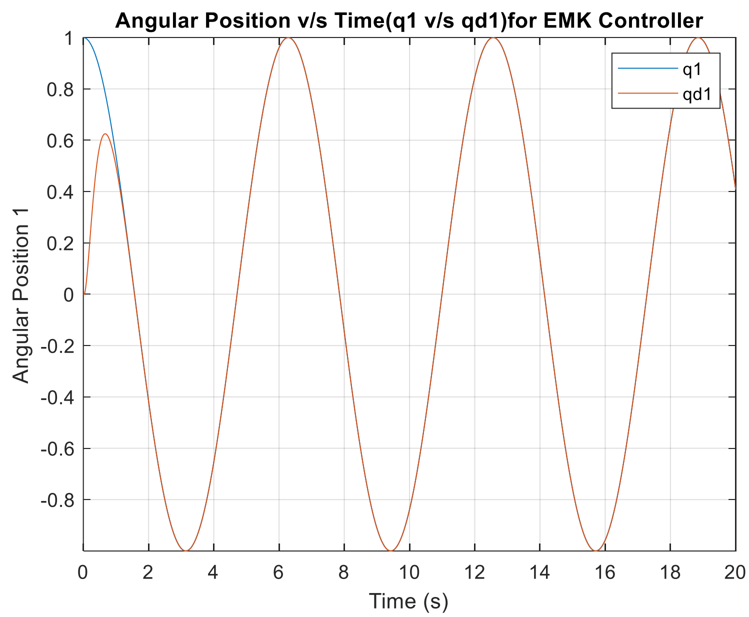

Figure 3: Angular Position Plots (q1 v/s qd1) for EMK Controller

q2 v/s qd2

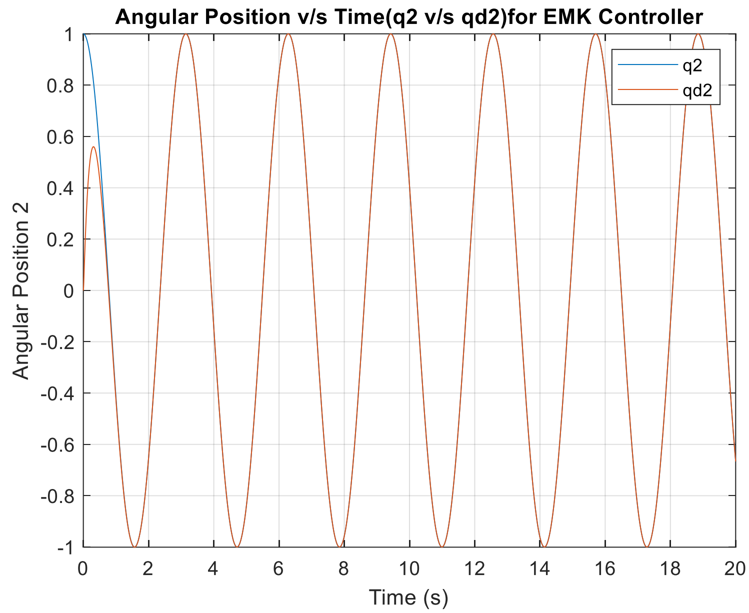

Figure 4: Angular Position Plots (q2 v/s qd2) for EMK Controller

Torque Plots:

For K = 5

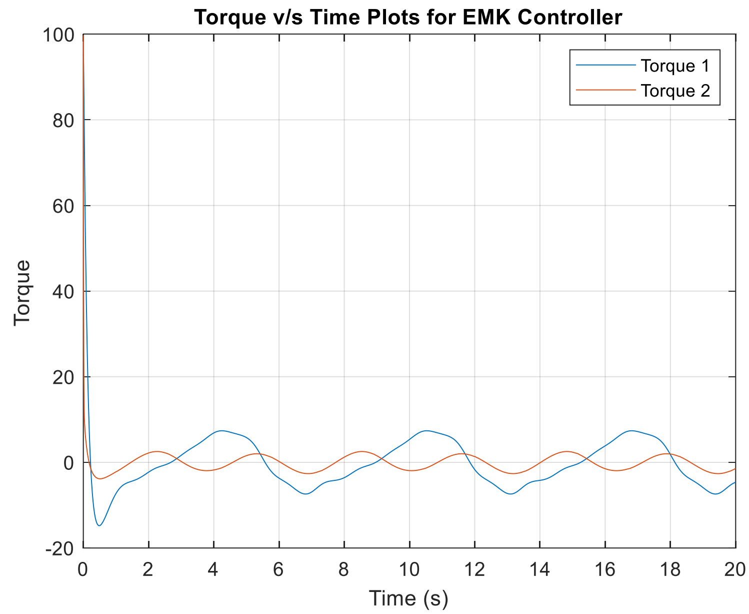

Figure 5: Torque Plots for EMK Controller with K = 5

For K = 100

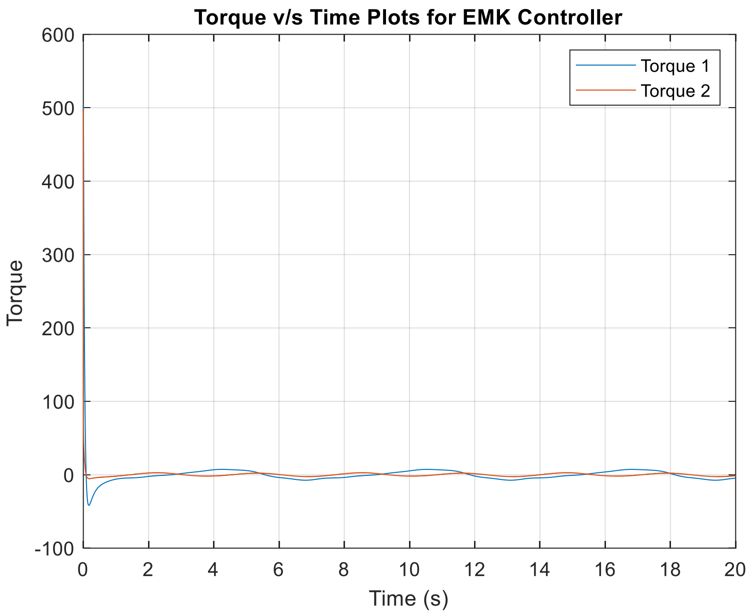

Figure 6: Torque Plots for EMK Controller with K = 100

From Figure 6 we can see that we get stable output torque with high K gain value. However, due to the high gain, there is a spike in the initial value of the torque

Section IV: Adaptive Control

- Assumptions

- System parameters $p_1,\ p_2,\ p_3,\ f_{d_1}\ and\ f_{d_2}$ are known.
- Desired trajectories $q_d, \dot{q_d}\ and\ \ddot{q_d}$ are known and are bounded.
- Disturbances τ_d are considered for Adaptive Controller. The controller is designed to consider the bounded disturbances in the system.

- Non-Linear Control Design

Design T using adaptive control law

$$
\begin{aligned}
&M\dot{r} = -V_m r - \tau\ +\ M\ddot{q_d} + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e} + V_m r \\
&= -V_m r - \tau\ +\ Y(e, \dot{e}, \dot{q_d}, \ddot{q_d})\theta + \tau_d \\
&= -V_m r - \tau\ +\ Y\theta + \tau_d \\
&\therefore Y\left(e, \dot{e}, \dot{q}_d, \ddot{q}_d\right)\theta = M\ddot{q}_d + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r
\end{aligned}
$$

Where,

Y is the known Linear Regression Matrix and θ is a vector of unknown constants

$$
Y = \begin{bmatrix} \ddot{q_{d1}} + \alpha\dot{e_1} & \ddot{q_{d2}} + \alpha\dot{e_2} & 2\cos(q_2)\left(\ddot{q_{d_1}} + \alpha\dot{e_1}\right) + \cos(q_2)\left(\ddot{q_{d_2}} + \alpha\dot{e_2}\right) - \sin(q_2)\,\dot{q_2}\left(\dot{q_{d_1}} + \alpha e_1\right) - \sin(q_2)\,(\dot{q_1} + \dot{q_2})\left(\dot{q_{d_1}} + \alpha e_1\right) & \dot{q_1} & 0 \\ 0 & \ddot{q_{d1}} + \alpha\dot{e_1} + \ddot{q_{d2}} + \alpha\dot{e_2} & \cos(q_2)\left(\ddot{q_{d_1}} + \alpha\dot{e_1}\right) + \sin(q_2)\,\dot{q_1}\left(\dot{q_{d_1}} + \alpha e_1\right) & 0 & \dot{q_2} \end{bmatrix}
$$

$$
\theta = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ f_{d_1} \\ f_{d_2} \end{bmatrix}
$$

$$
\tau = Y\hat{\theta} + \square
$$

Error Dynamics

$$
\begin{aligned}
M\dot{r} &= -V_m r - Y\hat{\theta}\ - \square + \ Y\theta + \tau_d \\
&= -V_m r + Y\tilde{\theta} - \square + \tau_d \qquad \text{where } \tilde{\theta} = \theta - \hat{\theta}
\end{aligned}
$$

- Stability Analysis

Consider Lyapunov function: $V = \frac{1}{2}r^{T}Mr + \frac{1}{2}\tilde{\theta}^{T}\gamma\tilde{\theta}$

Differentiating,

$$\dot{V} = \frac{1}{2} r^{\mathrm{T}} \dot{M} \mathrm{r} + r^{\mathrm{T}} M \dot{r} + \tilde{\theta}^T \gamma \dot{\tilde{\theta}}$$

Substitute error dynamics in the above equation

$$\begin{aligned}\dot{V} &= \frac{1}{2} r^{\mathrm{T}} \dot{M} \mathrm{r} + r^{\mathrm{T}} \left(-V_m r \ + Y\tilde{\theta} - \square + \tau_d\right) + \tilde{\theta}^T \gamma \dot{\tilde{\theta}} \\ &= \frac{1}{2} r^{\mathrm{T}} \left(\dot{M} - V_m\right) r \ + r^{\mathrm{T}} Y \tilde{\theta} - r^{\mathrm{T}} \square + r^{\mathrm{T}} \tau_d - \tilde{\theta}^T \gamma \dot{\hat{\theta}}\end{aligned}$$

where $\dot{\tilde{\theta}} = \dot{\theta} - \dot{\hat{\theta}} = -\dot{\hat{\theta}}$

Using skew-symmetric property,

$$\dot{V} = r^{\mathrm{T}} Y \tilde{\theta} - r^{\mathrm{T}} \square + r^{\mathrm{T}} \tau_d - \tilde{\theta}^T \gamma \dot{\hat{\theta}}$$

$$\square = \boldsymbol{Kr}$$

Where, K is symmetric and positive definite control gain matrix

$$\therefore \dot{V} = r^{\mathrm{T}} Y \tilde{\theta} - r^{\mathrm{T}} \mathrm{Kr} \ + r^{\mathrm{T}} \tau_d - \tilde{\theta}^T \gamma \dot{\hat{\theta}}$$

Designing $\dot{\hat{\theta}}$

$$\boxed{\dot{\hat{\theta}} = \gamma^{-1} Y^T r}$$

$$\begin{aligned}\dot{V} &= r^{\mathrm{T}} Y \tilde{\theta} - r^{\mathrm{T}} K r - \tilde{\theta}^T Y^T r + r^{\mathrm{T}} \tau_d \\ &= -r^{\mathrm{T}} K r + r^{\mathrm{T}} \tau_d\end{aligned}$$

When $\boldsymbol{\tau_d \ = \ 0}$,

$$\begin{aligned}&\dot{V} \leq -\lambda_{min}(K) \|r\|^2 \\ &\dot{V} \leq 0 \qquad\qquad (N.S.D.)\end{aligned}$$

Integrating,

$$V(t) \leq V(0)$$

$$\therefore V \ \in \mathcal{L}_\infty \because V \ \in \mathcal{L}_\infty, r \in \mathcal{L}_\infty \ and \ \tilde{\theta} \in \mathcal{L}_\infty$$

$$r = \dot{e} + \alpha e \ \because r \in \mathcal{L}_\infty, e \in \mathcal{L}_\infty \ and \ \dot{e} \in \mathcal{L}_\infty$$
$$\tau = Y\hat{\theta} + Kr \text{ and } \tilde{\theta} = \theta - \hat{\theta} \because \tilde{\theta} \in \mathcal{L}_\infty, \hat{\theta} \in \mathcal{L}_\infty$$

Y is a function of $e, \dot{e}, \dot{q_d}, \ddot{q_d}$

$\because e, \dot{e}, \dot{q_d}, \ddot{q_d} \in \mathcal{L}_\infty, Y \in \mathcal{L}_\infty$ and $\because Y, \hat{\theta}, r \in \mathcal{L}_\infty, T \in \mathcal{L}_\infty$

$\therefore$ The control torque is bounded.

$$\dot{V} \leq -\lambda_{min}(K)\|r\|^2$$

Differentiating $\dot{V}$, then $\ddot{V}_L$will give r and $\dot{r}$.

$\because r, \dot{r} \in \mathcal{L}_\infty, \ddot{V} \in \mathcal{L}_\infty \quad \therefore \ddot{V}$ is bounded and $\because \ddot{V} \in \mathcal{L}_\infty, \dot{V}$ is uniformly continuous.

Invoking Barbalat's Lemma,

$\dot{V}_L \rightarrow 0$ as $t \rightarrow \infty \Rightarrow r \rightarrow 0$ as $t \rightarrow \infty \Rightarrow e \rightarrow 0$ as $t \rightarrow \infty$ (G.A.S)

When $\boldsymbol{\tau_d} \neq \mathbf{0}$,

The disturbance is bounded $\|\tau_d\| \leq$ d

a $< \lambda_{min}(K)$
$-a > -\lambda_{min}(K)$

$$\begin{aligned}
\dot{V} &= -r^{\mathrm{T}}Kr + r^{\mathrm{T}}\tau_d \\
\dot{V} &\leq -\lambda_{min}(K)\|r\|^2 + \|r\|\|\tau_d\| \\
&\leq -\|r\|\{\lambda_{min}(K)\|r\| - \|\tau_d\|\} \\
&\leq -\|r\|\{a\|r\| - \|\tau_d\|\} \\
&\leq -a\|r\|\{\|r\| - \frac{d}{a}\} \\
\dot{V} &\leq 0 \quad (N.S.D.) \quad if \quad \|r\| > \frac{d}{a}
\end{aligned}$$

We know that the control input is bounded.

The condition $\|r\| > \frac{d}{a}$ must be satisfied for the error to converge to zero as $t \rightarrow \infty$, So we can say that the solution is Global Uniformly Ultimately Bounded (G.U.U.B).

Control Torque:

$$\boxed{\boldsymbol{\tau = Y\hat{\theta} + Kr}}$$

Update Law:

$$\boxed{\boldsymbol{\dot{\hat{\theta}} = \gamma^{-1} Y^T r}}$$

- Results

Parameters used for simulation:

a. Control gain matrix $K = 10 * I_2$
b. Gain $\alpha = 5$
c. Simulation time, $t = 20\ sec$
d. Desired trajectory $q_d(t) = [cos(t)\,; cos(2t)]$
e. Control gain matrix, $\gamma = 2I_5$
f. Disturbance, $\tau_d = [sin(t)\,; sin(t)]$

SIMULINK Model:

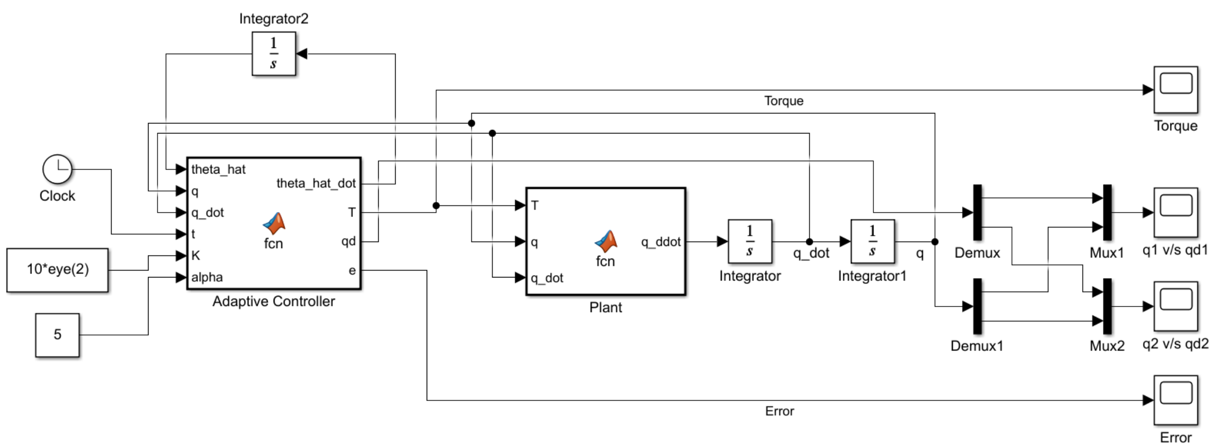

Figure 7: Adaptive Controller SIMULINK Model

Simulation Results:

1. **Without Disturbance**

Error Plots:

For t = 20 sec

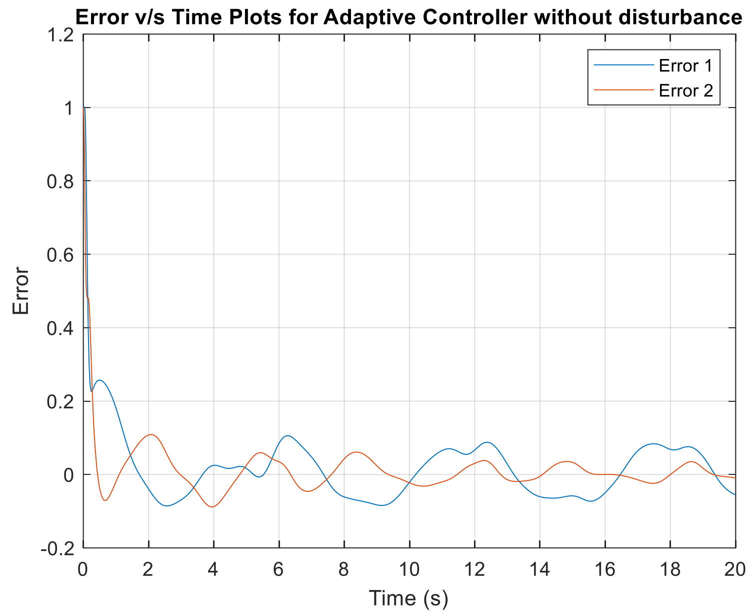

Figure 8: Error v/s Time Plots for Adaptive Controller without disturbance at t = 20s

For t = 400 sec

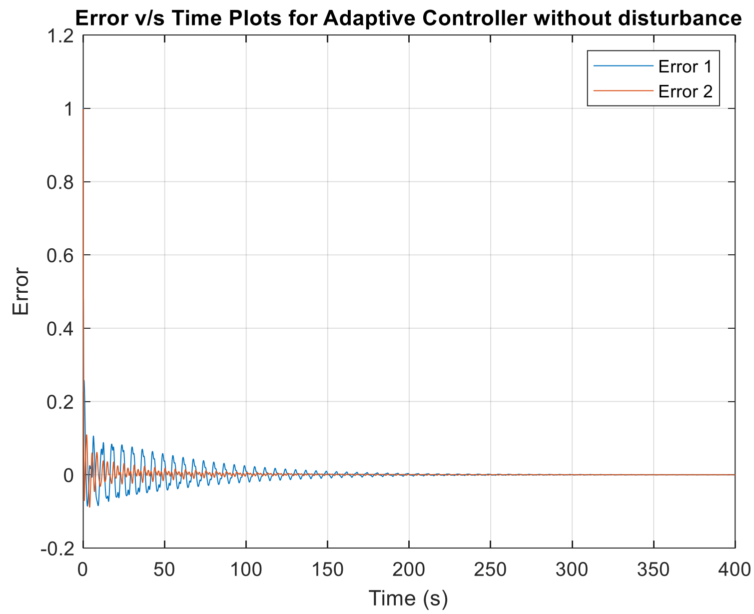

Figure 9: Error v/s Time Plots for Adaptive Controller without disturbance at t = 400s

Figure 9 shows that it takes approximately 400 seconds for the error to converge to 0. This can be shortened by increasing control gains, however doing this also increases the Torque.

Angular Position Plots:

q1 v/s qd1

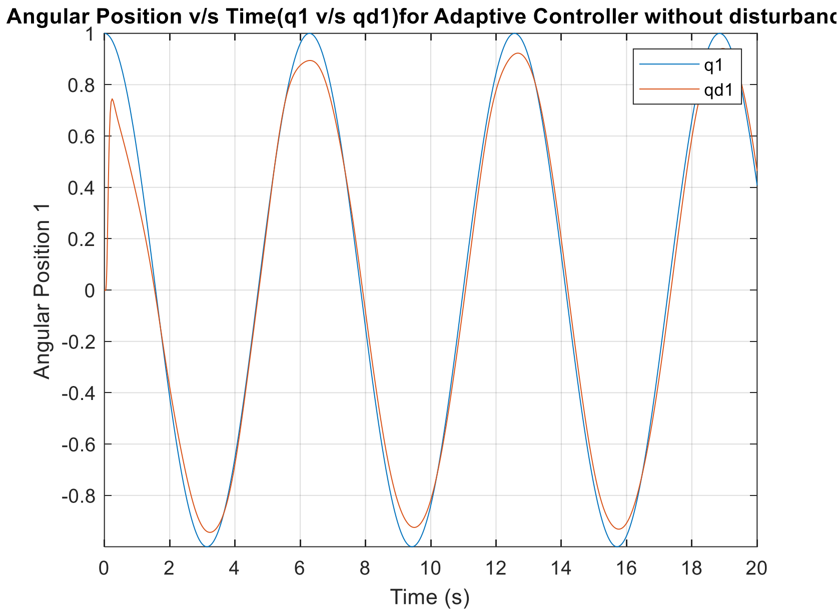

Figure 10: Angular Position (q1 v/s qd1) Plots for Adaptive Controller without disturbance

q2 v/s qd2

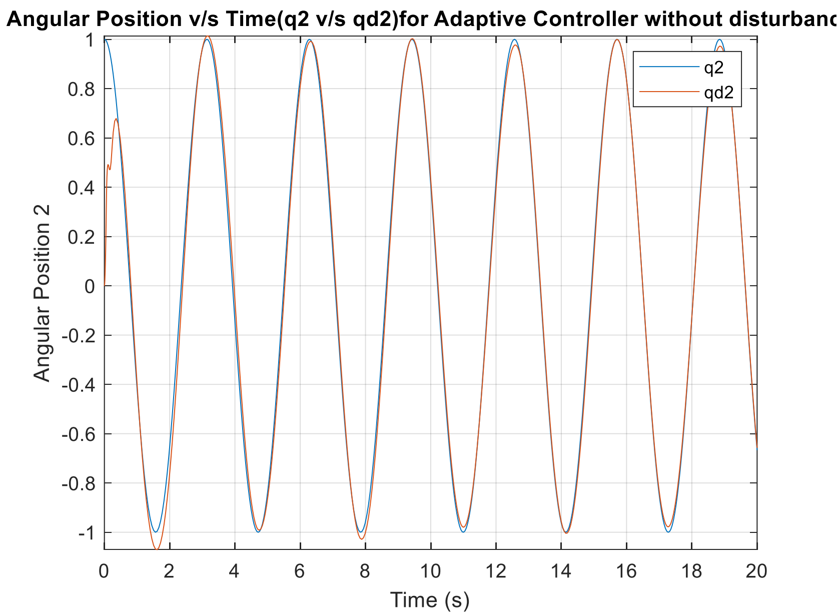

Figure 11: Angular Position (q2 v/s qd2) Plots for Adaptive Controller without disturbance

Torque Plots:

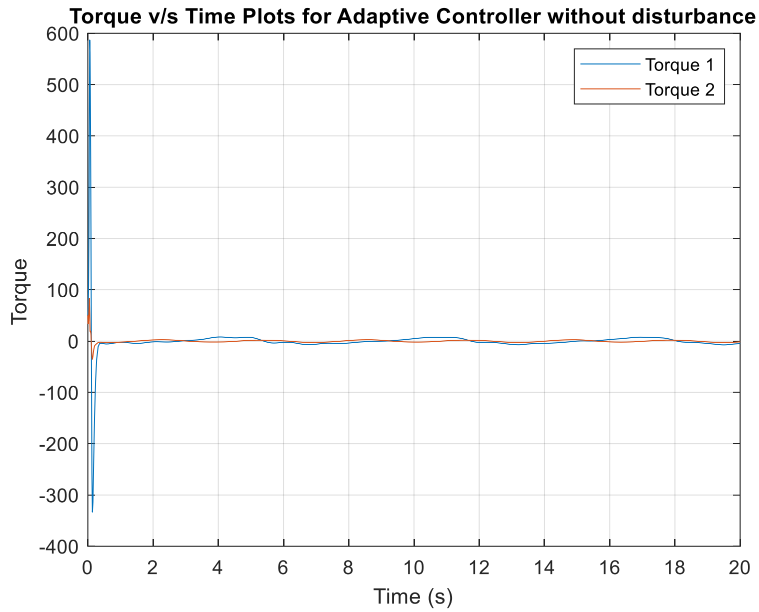

Figure 12: Torque Plots for Adaptive Controller without disturbance

From Figure 12, we can see that Torque initially spikes and then decreases to a negative value. This happens due to the control gains which can be tuned as required. The error in Figures 8 and 9 can converge to zero faster if the control gains K and α are increased. However, this also causes a spike in the initial value of the Torque.

1. **With Disturbance**

Error Plots:

For t = 20 sec

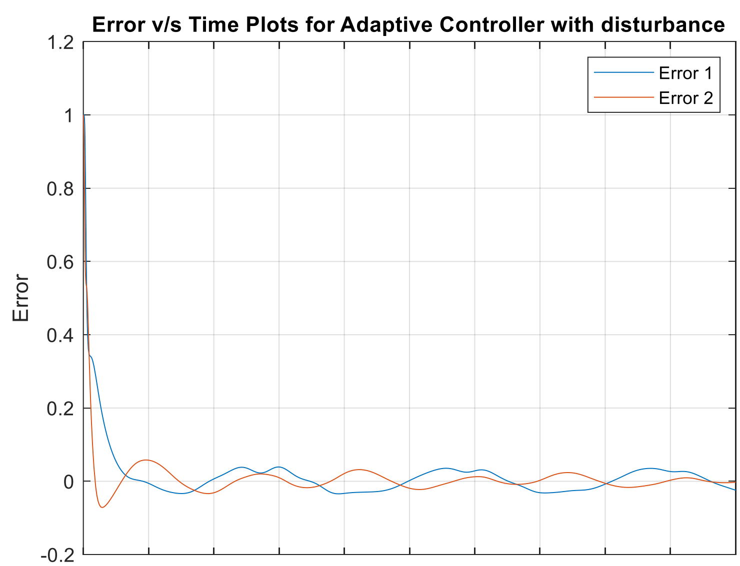

Figure 13: Error v/s Time Plots for Adaptive Controller with disturbance at t = 20s

For t = 400 sec

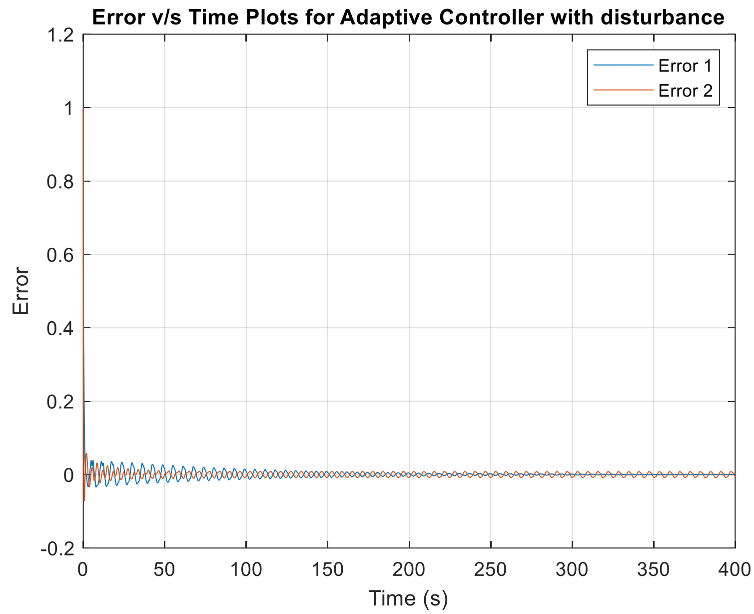

Figure 14: Error v/s Time Plots for Adaptive Controller with disturbance at t = 400s

From Figure 14, we can see that the adaptive controller shows ripples in the error plot and does not perform that well for systems with disturbance.

<u>Angular Position Plots:</u>

q1 v/s qd1

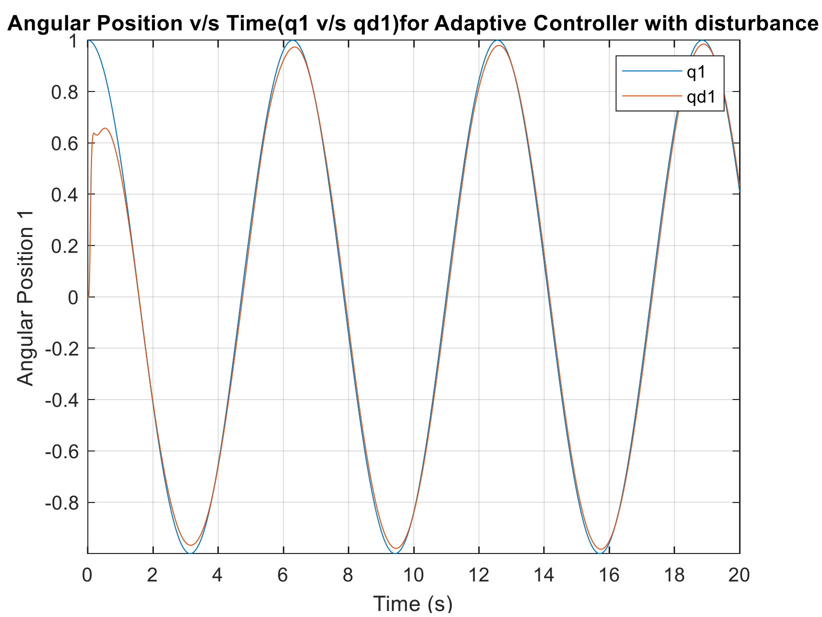

Figure 15: Angular Position (q1 v/s qd1) Plots for Adaptive Controller with disturbance

q2 v/s qd2

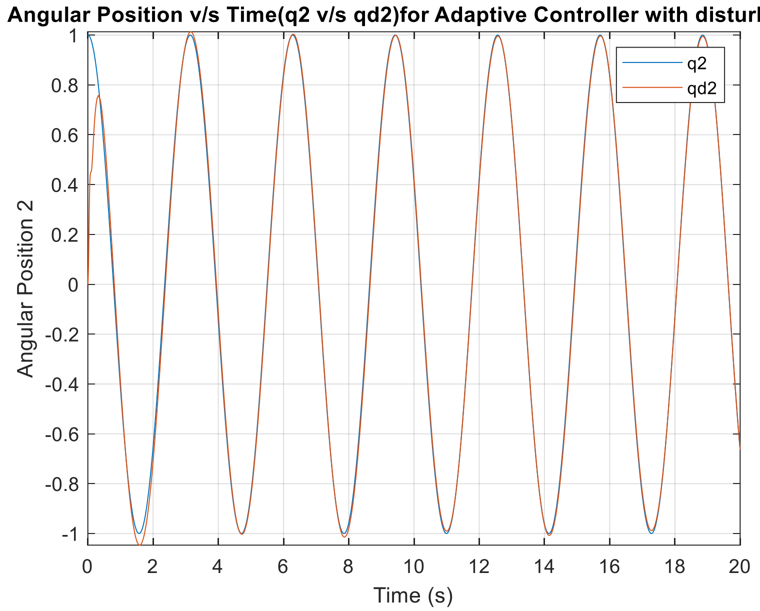

Figure 16: Angular Position (q2 v/s qd2) Plots for Adaptive Controller with disturbance

From Figures 15 and 16, we can say that the tracking is not very accurate for Adaptive Controller in the presence of disturbances.

<u>Torque Plots:</u>

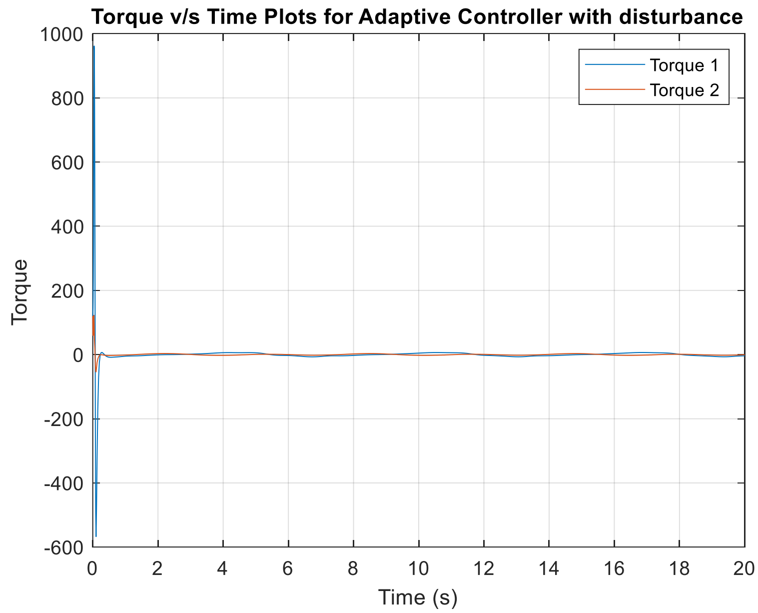

Figure 17: Torque Plots for Adaptive Controller with disturbance

Section V: Robust Control: Sliding Mode Control

- Assumptions

- System parameters $p_1,\ p_2,\ p_3,\ f_{d_1}\ and\ f_{d_2}$ are known.
- Desired trajectories $q_d, \dot{q_d}\ and\ \ddot{q_d}$ are known and are bounded.
- Disturbances τ_d are considered for Robust Controller. The controller is designed to consider the bounded disturbances in the system.
- Sliding Mode Robust Controller has been used here.

- Non-Linear Control Design

Designing τ using robust control

$$
\begin{aligned}
&M\dot{r} = -V_m r - \tau + M\ddot{q_d} + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e} + V_m r \\
&= -V_m r - \tau + Y(e, \dot{e}, \dot{q_d}, \ddot{q_d})\theta + \tau_d \\
&= -V_m r - \tau + Y\theta + \tau_d; \qquad\qquad \|Y\theta\| \le \rho(t) \\
&\boxed{\tau = \rho sgn(r) + Kr}
\end{aligned}
$$

Where, K is a symmetric and positive definite control gain matrix

Error Dynamics,

$$M\dot{r} = -V_m r - \rho sgn(r) - Kr + Y\theta + \tau_d$$

- Stability Analysis

Consider Lyapunov function: $V = \frac{1}{2}r^{\mathrm{T}}Mr$

Differentiating,

$$\dot{V} = \frac{1}{2}r^{\mathrm{T}}\dot{M}r + r^{\mathrm{T}}M\dot{r}$$

Substituting error dynamics in the above equation

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}r^{\mathrm{T}}\dot{M}r + r^{\mathrm{T}}(-V_m r - \rho sgn(r) - Kr + Y\theta + \tau_d) \\
&= \frac{1}{2}r^{\mathrm{T}}\left(\dot{M} - V_m\right)r - r^{\mathrm{T}}\rho sgn(r) - r^{\mathrm{T}}Kr + r^{\mathrm{T}}Y\theta + r^{\mathrm{T}}\tau_d
\end{aligned}
$$

Using skew-symmetric property,

$$\begin{aligned}
\dot{V} &= -\rho r^{\mathrm{T}} sgn(r) - r^{\mathrm{T}} Kr + r^{\mathrm{T}} Y\theta + r^{\mathrm{T}}\tau_d && r^{\mathrm{T}} sgn(r) = \|r\| \\
&\leq -r^{\mathrm{T}} Kr + \|r\|\|Y\theta\| - \rho\|r\| + r^{\mathrm{T}}\tau_d && \|Y\theta\| \leq \rho(t) \\
&\leq -r^{\mathrm{T}} Kr + \|r\|\rho - \rho\|r\| + r^{\mathrm{T}}\tau_d \\
\dot{\mathrm{V}} &\leq -r^{\mathrm{T}} Kr + r^{\mathrm{T}}\tau_d
\end{aligned}$$

When $\boldsymbol{\tau_d} = 0$,

$$\dot{V} \leq -\lambda_{min}(K)\|r\|^2$$
$$\dot{V} < 0 \text{ (N.D.)}$$

V is Negative Definite and so this implies Global Asymptotic Stability (G.A.S)

Consider,

$$V_1 \leq V \leq V_2$$

$$V_1 = \frac{1}{2}\lambda_{min}(M)\|z\|^2 = \lambda_3\|z\|^2$$
$$V_2 = \frac{1}{2}\lambda_{max}(M)\|z\|^2 = \lambda_4\|z\|^2$$

Choosing $\beta < \lambda_{min}(K)$, $-\beta > -\lambda_{min}(K_3)$

$$\dot{V} \leq -\beta\|z\|^2$$

$$\lambda_3\|z\|^2 \leq V \leq \lambda_4\|z\|^2$$

$$\frac{\lambda_3}{\lambda_4}\|z\|^2 \leq \frac{V}{\lambda_4} \leq \|z\|^2$$

$$\frac{-\beta\lambda_3}{\lambda_4}\|z\|^2 \geq \frac{-\beta V}{\lambda_4} \geq -\beta\|z\|^2$$

$$\dot{V} \leq -\beta\|z\|^2$$

$$\dot{V} \leq \frac{-\beta V}{\lambda_4}$$

Integrating,

$$V(t) \leq V(0)e^{\frac{-\beta t}{\lambda_4}} \qquad \text{(G.E.S)}$$

When $\boldsymbol{\tau_d} \neq 0$

The disturbance is bounded $\|\tau_d\| \leq$ d

$\beta < \lambda_{min}(K), -\beta > -\lambda_{min}(K)$

$$
\begin{aligned}
\dot{V} &\leq -r^{\mathrm{T}}Kr + r^{\mathrm{T}}\tau_d \\
&\leq -\lambda_{min}(K)\|r\|^2 + \|r\|\|\tau_d\| \\
&\leq -\|r\|\{\lambda_{min}(K)\|r\| - \|\tau_d\|\} \\
&\leq -\|r\|\{\beta\|r\| - \|\tau_d\|\} \\
&\leq -\beta\|r\|\{\|r\| - \frac{d}{\beta}\}
\end{aligned}
$$

$$
\dot{V} < 0 \quad (N.D.) \quad if \quad \|r\| > \frac{d}{\beta}
$$

Using theorem 4.18, we can say that the solution is Global Uniformly Ultimately Bounded (G.U.U.B)

Control Torque:

$$
\boxed{\boldsymbol{\tau = \rho sgn(r) + Kr}}
$$

- Results

Parameters used for simulation:

a. Control gain matrix K = 10*I_2
b. Gain $\alpha = 5$
c. Simulation time, $t = 20\ sec$
d. Desired trajectory $q_d(t) = [cos(t)\,; cos(2t)]$
e. Control gain matrix, $\gamma = 2I_5$
f. Disturbance, $\tau_d = [sin(t)\,; sin(t)]$

SIMULINK Model

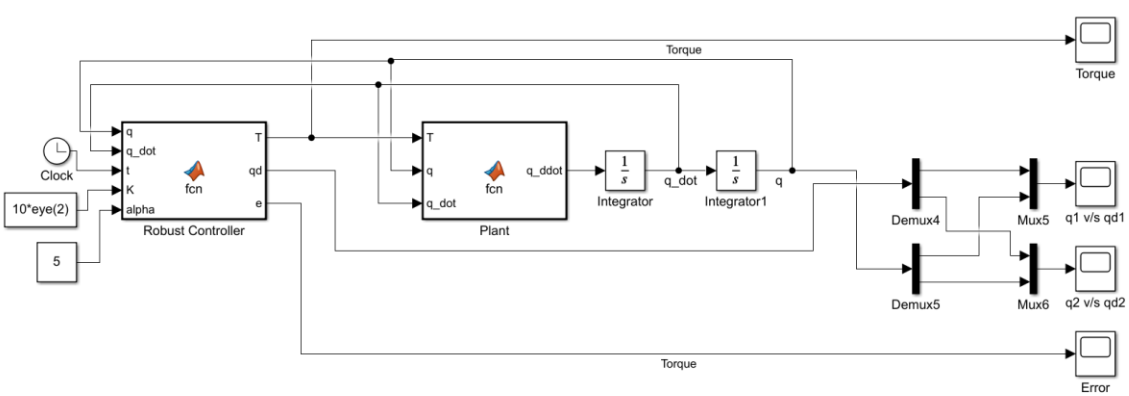

Figure 18: Robust Controller SIMULINK Model

Simulation Results:

1. Without Disturbance

Error Plots:

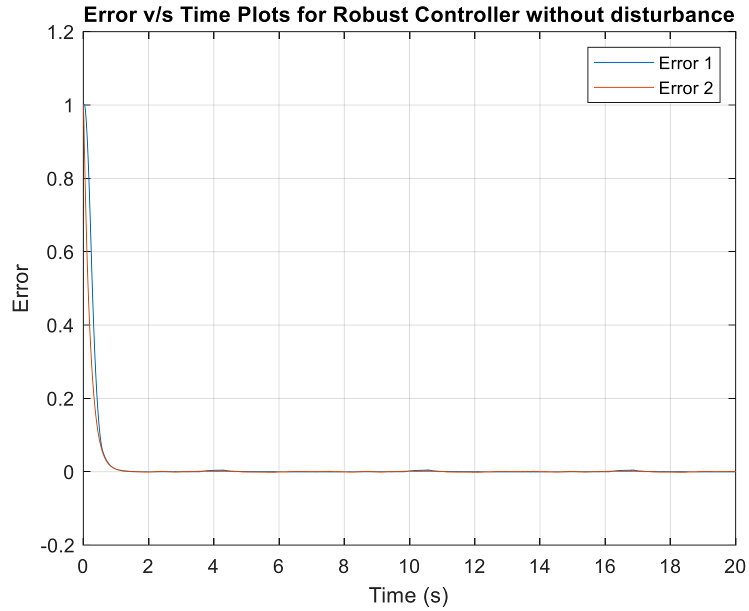

Figure 19: Error v/s Time Plots for Robust Controller without Disturbance

From Figure 19, we can see that the error converges to zero very fast in < 1 sec.

Angular Position Plots:

q1 v/s qd1

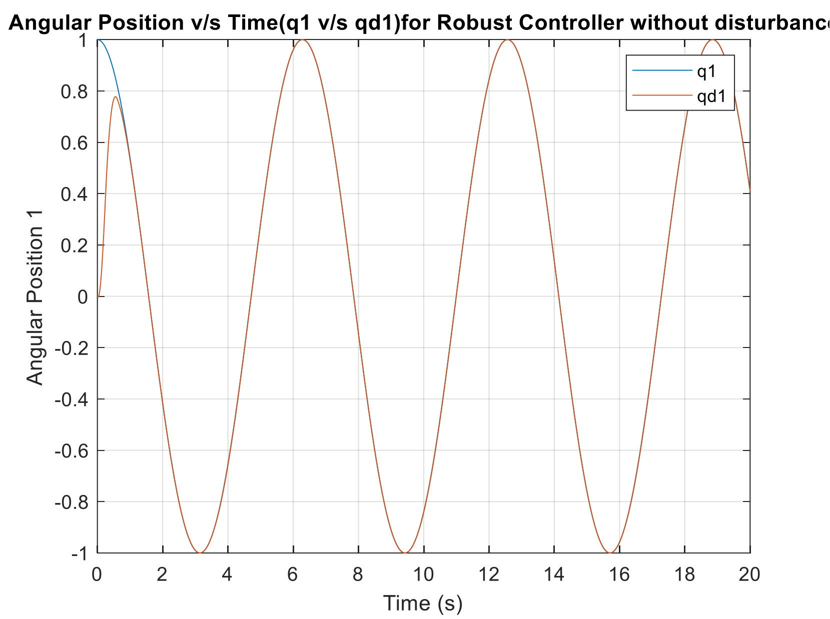

Figure 20: Angular Position (q1 v/s qd1) plots for Robust Controller without disturbance

q2 v/s qd2

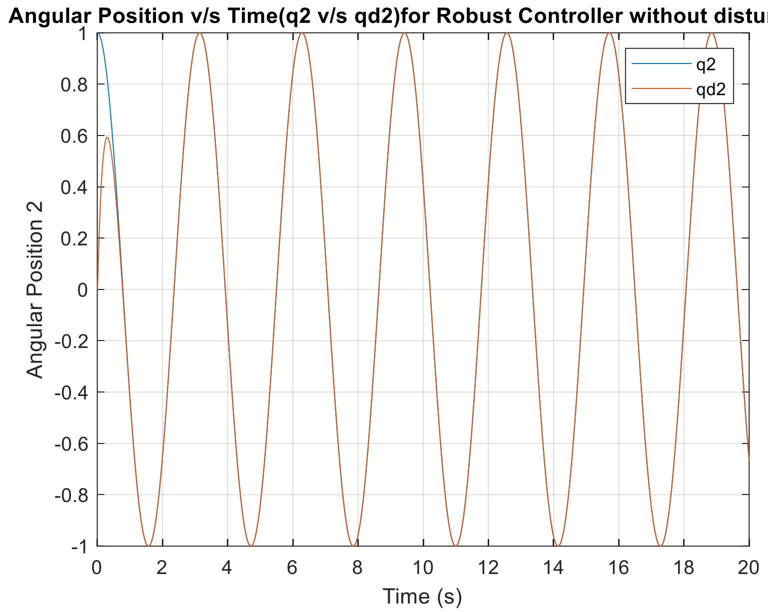

Figure 21: Angular Position (q2 v/s qd2) plots for Robust Controller without disturbance

From Figures 20 and 21 we can see that Robust Controller gives the best tracking.

<u>Torque Plots:</u>

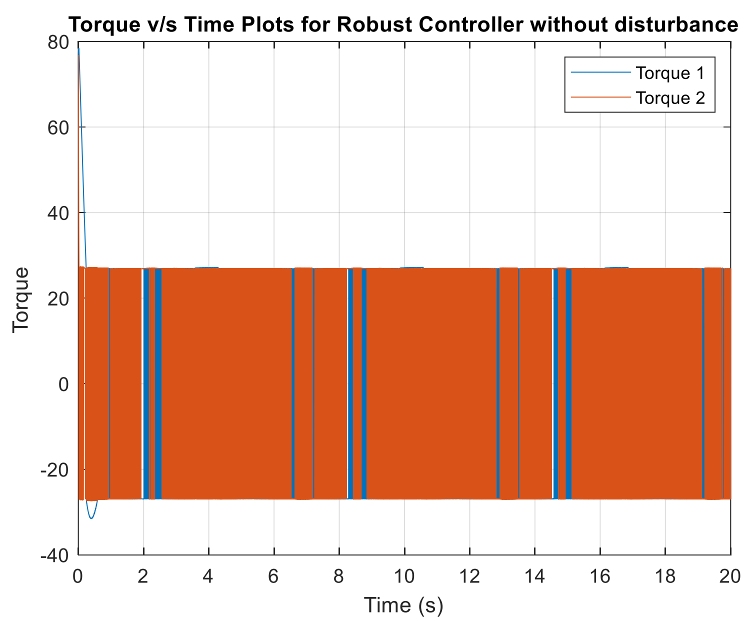

Figure 22: Torque plots for Robust Controller without disturbance

From Figure 22, we can a lot of switching in the Torque. Although, its good performance in tracking and fast convergence of error to zero, the rapid switching in Torque makes the Robust Controller unsuitable for physical systems.

1. With Disturbance

Error Plots:

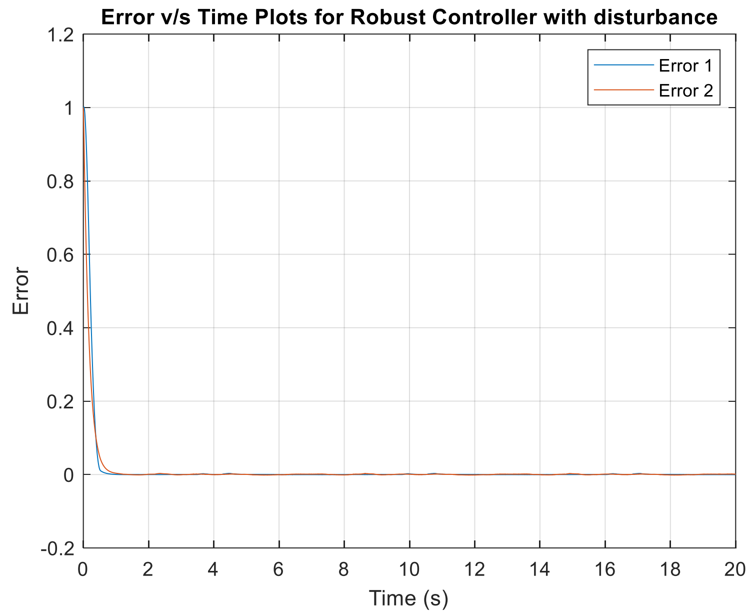

Figure 23: Error v/s Time Plots for Robust Controller with Disturbance

From Figure 23, we can see that, even with disturbances present, the error converges to zero very quickly.

Angular Position Plots:

q1 v/s qd1

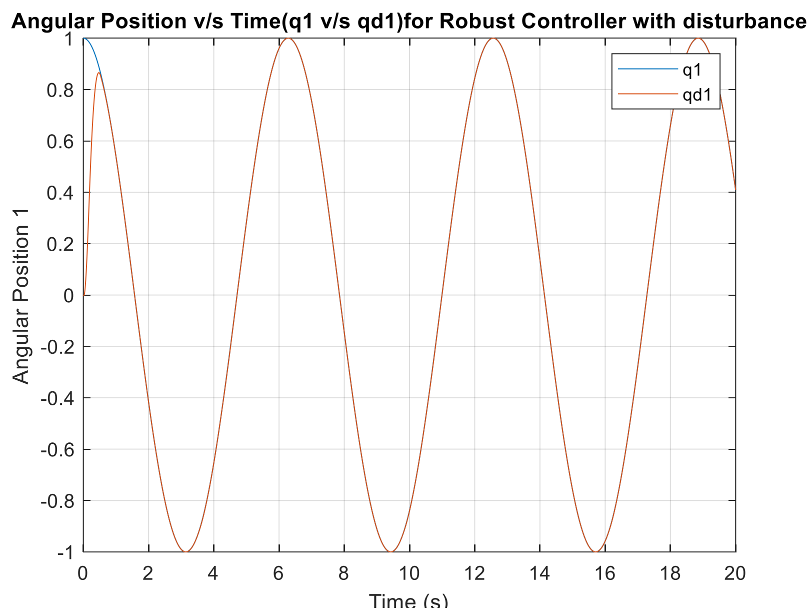

Figure 24: Angular Position (q1 v/s qd1) plots for Robust Controller with disturbance

q2 v/s qd2

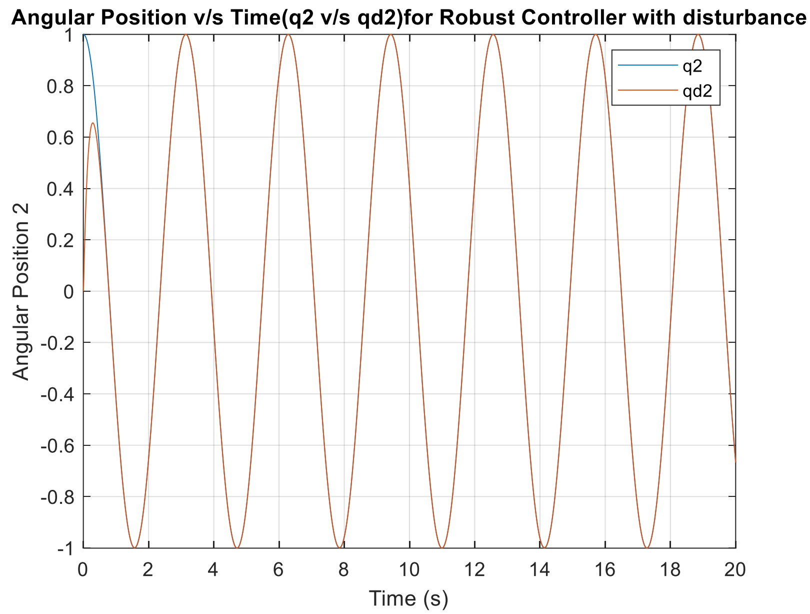

Figure 25: Angular Position (q2 v/s qd2) plots for Robust Controller with disturbance

From Figures 24 and 25, we can see that the tracking remains very good even in the presence of disturbances.

<u>Torque Plots:</u>

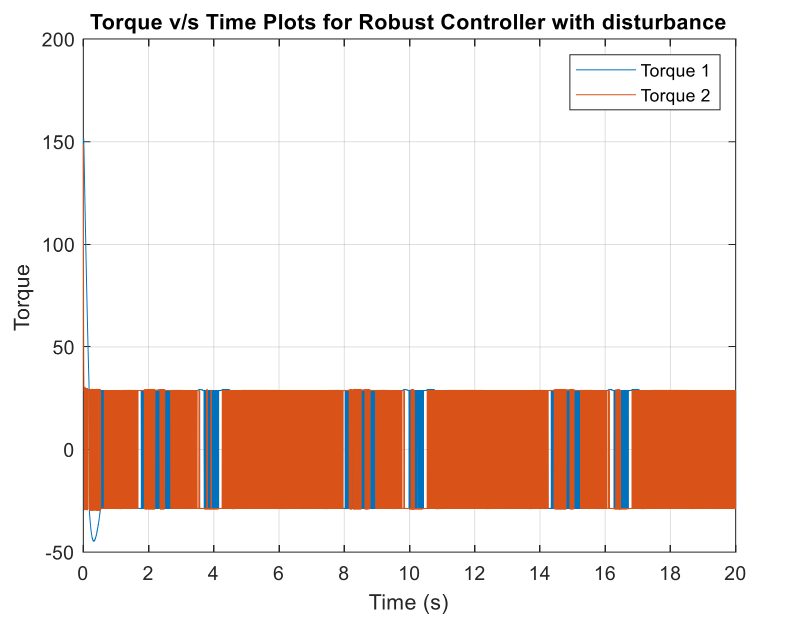

Figure 26: Torque plots for Robust Controller with disturbance

Section VI: Conclusion

Sr. No.	Controller	Control Effect	Stability		Robustness
			With Disturbance	Without Disturbance	
1	**Exact Model Knowledge Controller**	The error converges to zero and tracking performance is good.	G.U.U.B	G.E.S	Not Robust, cannot handle disturbances as it only works when we have the exact knowledge of model parameters
2	**Adaptive Controller**	The error converges to zero and tracking performance is good.	G.U.U.B	G.A.S	Controller is robust provided parametric uncertainty is linear.
3	**Sliding Mode Robust Controller**	Rapid switching in the Torque, unsuitable for physical systems. Control effect is not good	G.E.S	G.E.S	Controller is highly robust, can handle bounded disturbances easily with smooth trajectory tracking

Table 1: Conclusion