Polynomial Interpolation

CS/SE 4X03

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Outline

The problem

Given data points $\{(x_i, y_i)\}_{i=0}^{n}$ find a function $v(x)$ that fits the data such that

$$v(x_i) = y_i, \qquad i = 0, \ldots, n$$

Some applications

- Approximating functions. For a complicated function $f(x)$ find a simpler $v(x)$ that approximates $f(x)$. Usually it is less expensive to work with $v(x)$ than with $f(x)$
- We can use $v(x)$ to approximate $f(x)$ at some $x^* \neq x_0, x_1, \ldots x_n$
- We may need derivatives or an integral of f, and we can differentiate/integrate v

Representation

$$v(x) = \sum_{j=0}^{n} c_j \phi_j(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x)$$

- The c_j are unknown coefficients
- The ϕ_j are given basis functions
  They must be linearly independent
  If $v(x) = 0$ for all x then $c_j = 0$ for all j

Representation cont.

From

$$v(x_i) = c_0\phi_0(x_i) + c_1\phi_1(x_i) + \cdots + c_n\phi_n(x_i) = y_i, \quad i = 0, \ldots, n$$

we have the linear system of $(n+1)$ equations for the c_i

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Basis functions

- Monomial basis

$$\phi_j(x) = x^j, \quad j = 0, 1, \ldots, n$$
$$v(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

- Trigonometric functions, e.g.

$$\phi_j(x) = \cos(jx), \quad j = 0, 1, \ldots, n$$

  Useful in signal processing, for wave and other periodic behavior

- Piecewise interpolation: linear, quadratic, cubic, splines

Monomial interpolation

The polynomial is of the form $p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

Example 1. Interpolate

x_i	1	2	4
y_i	1	3	3

using a polynomial of degree 2. We seek the coefficients of $p_2(x) = c_0 + c_1 x + c_2 x^2$

From

$$\begin{aligned} p_2(1) &= c_0 + c_1 + 1c_2 = 1 \\ p_2(2) &= c_0 + 2c_1 + 4c_2 = 3 \\ p_2(4) &= c_0 + 4c_1 + 16c_2 = 3 \end{aligned}$$

Solve this linear system to obtain

$$p_2(x) = -\frac{7}{3} + 4x - \frac{2}{3}x^2$$

Uniqueness of the interpolating polynomial

From

$$p_n(x_i) = c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_n x_i^n = y_i$$

we have the linear system

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

- The coefficient matrix is a Vandermonde matrix
  Denote it by X
- $\det(X) = \prod_{i=0}^{n-1} \left[\prod_{j=i+1}^{n} (x_j - x_i) \right]$

Uniqueness of the interpolating polynomial cont.

If all x_i are distinct then

- $\det(X) \neq 0$
- X is nonsingular
- this system has a unique solution
- there is a unique polynomial of degree $\leq n$ that interpolates the data

However,

- this system can be poorly conditioned
- work is $O(n^3)$
- difficult to add new points

Lagrange interpolation

- Lagrange basis functions

$$L_j(x_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

- Lagrange polynomial $p_n(x) = \sum_{j=0}^{n} y_j L_j(x)$

Then

$$\begin{aligned} p_n(x_i) &= \sum_{j=0}^{n} y_j L_j(x_i) \\ &= \sum_{j=0}^{i-1} y_j \underbrace{L_j(x_i)}_{=0} + y_i \underbrace{L_i(x_i)}_{=1} + \sum_{j=i+1}^{n} y_j \underbrace{L_j(x_i)}_{=0} \\ &= y_i \end{aligned}$$

Lagrange interpolation cont.

$$
\begin{aligned}
L_j(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)} \\
&= \prod_{i=0, i\neq j}^{n} \frac{x-x_i}{x_j-x_i}
\end{aligned}
$$

Example: write the Lagrange polynomial for $(1,1)$, $(2,3)$, $(4,3)$