CS/SE 4X03 Final Examination

DAY CLASS
DURATION OF EXAMINATION: 2 hours
MCMASTER UNIVERSITY FINAL EXAMINATION

Dr. N. Nedialkov
14 December, 2021, 16:00–18:00

Special Instructions

1. You must not communicate with anybody during this exam.
2. You must not use the Internet.
3. Matlab and similar programs are not allowed. You can use a calculator.
4. Textbooks are allowed.
5. Write your solutions in the space provided in this PDF, or on separate paper.
6. Sign the next page and you must submit it. Electronic signature, or signing a printed copy and scanning it, or signing the PDF using a tablet is fine.

Submission. You will have 10 minutes extra time for submission. This time is not for writing the exam. There will be 3 drop boxes.

Dropbox 1 will be open from 16:00 to 18:09:59

Dropbox 2 will be open from 18:10 to 18:24:59. A submission in it will have a 20% penalty.

Dropbox 3 will be open from 18:25 to 18:40. A submission in it will have a 40% penalty.

Only exams on Avenue will be marked.

If you have a SAS accommodation, please email me your exam within the time indicated in your SAS letter + 10min.

COURSE: CS/SE4X03

EXAM DATE: 14 December, 2021

McMaster University Statement on Academic Integrity:
You are expected to exhibit honesty and use ethical behaviour in all aspects of the learning process. Academic credentials you earn are rooted in principles of honesty and academic integrity.

Academic dishonesty is to knowingly act or fail to act in a way that results or could result in unearned academic credit or advantage. This behaviour can result in serious consequences, e.g. the grade of zero on an assignment, loss of credit with a notation on the transcript (notation reads: "Grade of F assigned for academic dishonesty") and/or suspension of expulsion from the university.

"By signing this document I agree to follow the McMaster University Policy on Academic Integrity. My signature below confirms that the work submitted for this exam is my own and did not involve the use of unauthorized aids."

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STUDENT ID: 400118526

STUDENT SIGNATURE: Lizhiyuan Lin

Problem 1:

Problem 1 [3 points] Suppose you need to generate $n+1$ equally spaced points in the interval $[a,b]$ with spacing $h=(b-a)/n$, $n>1$. You can generate them using

$$x_0 = a, \quad x_i = x_{i-1} + h, \quad \text{or} \tag{1}$$
$$x_i = a + ih, \tag{2}$$

for $i = 1, \ldots, n-1$, $x_0 = a$ and $x_n = b$. Which of (1) and (2) would be more accurate and why?

(2) would be more efficient.

Because when x_i and x_{i-1} gets larger, h will become too small to add to x_{i-1}.

(2) will first accumulate all h's needed, and then add to a, which avoids this error.

For example: if x_{i-1} is 5999999 and h is something like 1e-15, adding h to x_{i-1} will still result in x_{i-1}.

But if i is say 1e15, adding ih to a will give a better result.

Problem 2:

CS/SE Eng 4X03

Problem 2 [5 points] You need to evaluate $\log(x - \sqrt{x^2-1})$ in floating-point arithmetic.

a. [2 points] What numerical inaccuracies can occur and for what values of x?

b. [2 points] Obtain an equivalent formula to avoid such inaccuracies.

a) Cancellation can happen when $x \approx \sqrt{x^2-1}$. Because if they are very close, they share lots of digits. If two numbers share lots of digits, they will waste accuracy because the size of floats in machine is limited, which is cancellation.

b) $\log(x-\sqrt{x^2-1})$

$$= \log\left(\frac{(x-\sqrt{x^2-1}) \times (x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}}\right)$$

$$= \log\left(\frac{x^2 - x^2+1}{x+\sqrt{x^2-1}}\right)$$

$$= \log\left(\frac{1}{x+\sqrt{x^2-1}}\right)$$

In this case, cancellation is avoided when $x \approx \sqrt{x^2-1}$ because we are doing addition instead of subtraction. Cancellation only happens to subtraction

Problem 3:

Problem 3 [3 points] Given a scalar R, describe a method for computing $1/R^2$ using only addition (or subtraction) and multiplication operations.

~~Let f(x) =~~

~~Let~~ $\cancel{f(x) = \frac{1}{x^2} - R}$

~~and apply newton's method~~

Let $f(x) = \frac{1}{\sqrt{x}} - R$, $x = \frac{1}{R^2}$

and apply Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{\sqrt{x_n}} - R}{-\frac{1}{2}x_n^{-\frac{3}{2}}}$$

$$= 3x_n - 2Rx_n^{\frac{3}{2}}$$

Then give an guess x_0 and apply this formula. Stop when You get satisfactory results.

Problem 4 [3 points] Consider Newton's method applied to the system of nonlinear equations

$$x_1^2 - x_2^2 = 0$$
$$2x_1x_2 = 1$$

Is there an initial guess x_0 for which Newton would break down? If so, give such a guess.

It will break down when $F'(x^{(k)})$ is not singular, meaning $\det(F'(x)) = 0$

$$F'(x) = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$$

$$\det(F'(x)) = (2x_1)^2 + (2x_2)^2 = 0$$

If $x_1 = x_2 = 0$, Newton would break down.

Problem 5 [3 points] Given the data

x	0	1	2	4
$f(x)$	1	9	23	93

find an approximation for $f(2.5)$ by evaluating the polynomial interpolating these data points.

Since we have 4 points, $n=3$.

$P_n(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3$

$P(0) = C_0 + 0 + 0 + 0 = 1$

$P(1) = C_0 + C_1 + C_2 + C_3 = 9$

$P(2) = C_0 + 2C_1 + 4C_2 + 8C_3 = 23$

$P(3) = C_0 + 4C_1 + 16C_2 + 64C_3 = 93$

Solving the above system against $b = \begin{bmatrix} 1 \\ 9 \\ 23 \\ 93 \end{bmatrix}$

we have $x \begin{cases} C_0 = 1 \\ C_1 = 7 \\ C_2 = 0 \\ C_3 = 1 \end{cases}$

$\therefore P_n(x) = 1 + 7x + x^3$

$\therefore f(2.5) = 1 + 7 \times 2.5 + 2.5^3 = 34.125$

Problem 6 [3 points] Consider $f(x) = e^x$ over $[0, \pi]$. Suppose we approximate $f(x)$ by a trigonometric polynomial of the form $p(x) = a + b\cos(x) + c\sin(x)$. What is the linear system to be solved for determining the least squares fit of p to f?

$\therefore$ we have 3 terms in $P(x)$

$\therefore$ $n = 2$

$\therefore$ need 3 data points

$\therefore$ $f(0)$, $f(\frac{\pi}{2})$, $f(\pi)$ is used.

~~$\therefore$ we have the system~~

$\therefore$ $b = \begin{bmatrix} 1 \\ e^{\frac{\pi}{2}} \\ e^{\pi} \end{bmatrix}$

$P(0) = a + b$, $P(\frac{\pi}{2}) = a + c$, $P(\pi) = a - b$

$\therefore$ we want to solve:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\frac{\pi}{2}} \\ e^{\pi} \end{bmatrix}$$

Problem 7 [4 points] Let A, B, and C be $n \times n$ matrices with A nonsingular, and let d be an n-vector.

A. [2 points] How would you efficiently compute the product $A^{-1}(B+C)d$?

B. [2 points] Derive the complexity of your approach in terms of big-O notation.

A. ① Calculate $(B+C)$

② Find A^{-1} by applying LU decomp.

③ Calculate ②×①

④ Calculate ③×d.

B. step ① $O(n^2)$

step ② $O(n^3)$

Step ③ $O(n^3)$

Step ④ $O(n^2)$

∴ Overall, complexity is $O(n^3)$.

Problem 10 [4 points] Consider the following method

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))]$$

for integrating the ODE $y' = f(t, y)$. What is the condition for h so this method is stable when applied to $y' = \lambda y$ with $\lambda < 0$.

$$y_{i+1} = y_i + \frac{h}{2}(\lambda y_i + \lambda(y_i + h\lambda y_i))$$

$$= y_i + \frac{h}{2}(2\lambda y_i + h\lambda^2 y_i)$$

$$= y_i + \lambda h y_i + \frac{h^2\lambda^2}{2} y_i$$

$$= y_i(1 + \lambda h + \frac{h^2\lambda^2}{2})$$

$\therefore -1 \leq 1 + \lambda h + \frac{h^2\lambda^2}{2} \leq 1$ should hold.

Solving the above inequality, we get: $0 \leq h \leq \frac{2}{-\lambda}$.

So, $0 \leq h \leq \frac{2}{-\lambda}$ should hold for this to be stable.

Problem 9 [4 points] What is the smallest number of points that are needed to compute $\int_0^\pi \sin x\,dx$ with accuracy 10^{-6} using the following methods with equally spaced points:

(a) trapezoid composite rule

(b) Simpson's composite rule

a/ $a=0,\ b=\pi,\ f''(x) = -\sin(x),\ f''(\mu) = -1$

$$10^{-6} = \frac{1}{12}(\pi-0)\left(\frac{\pi-0}{r}\right)^2$$

$\therefore r = 1607.43781$

$\therefore$ r is at least 1608

$\therefore$ At least $1608+1 = 1609$ Points.

b) $a=0,\ b=\pi,\ f^{(4)}(x) = \sin(x),\ f''(\mu) = 1$

$$10^{-6} = \left|-\frac{1}{180}\times\pi\times\left(\frac{\pi-0}{r}\right)^4\right|$$

$\therefore r = 36.10931$

$\therefore$ r at least 37

$\therefore$ At least $37+1 = 38$ Points.

3

ints] Under what circumstances does a small residual vector $r = b - Ax$ imply that x is an
to the linear system $Ax = b$?

x is exact, $\tilde{x}$ is approx, $r = b - A\tilde{x}$

$\because$ $Cond(A) = ||A|| \cdot ||A^{-1}||$

$\therefore$ Error bound is $\frac{||x-\tilde{x}||}{||x||} \le Cond(A)\frac{||r||}{||b||}$

$\because$ Cond(A) is unchanged, $||b||$ is also not changing.