Runge-Kutta Methods

CS/SE 4X03

Ned Nedialkov

McMaster University

November 23, 2021

Outline

Implicit trapezoidal method

- Consider $y'(t) = f(t, y)$, $y(t_i) = y_i$
- From $y(t_{i+1}) = y(t_i) + \int_{t_i}^{t} f(s, y(s))ds$,

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s))ds$$

- Use the trapezoidal rule for the integral

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s))ds \\ &\approx y(t_i) + \frac{h}{2}[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))] \end{aligned}$$

- From

$$y(t_{i+1}) \approx y(t_i) + \frac{h}{2}[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))]$$

  write

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})]$$

  This is the implicit trapezoidal method

- We have to solve a nonlinear system in general for y_{i+1}

- Local truncation error is

$$d_i = \frac{y(t_{i+1}) - y(t_i)}{h} - \frac{1}{2}[f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))]$$

- $d_i = O(h^2)$

Explicit trapezoidal method

- In the implicit trapezoidal rule, we need to solve for y_{i+1}
- We can approximate $y(t_{i+1})$ first using forward Euler:

$$Y = y_i + hf(t_i, y_i)$$

- Then plug Y into the formula for the implicit trapezoidal method

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, Y)]$$

- This is a two-stage explicit Runge-Kutta method
- Local truncation error is

$$d_i = \frac{y(t_{i+1}) - y(t_i)}{h} - \frac{1}{2}[f(t_i, y(t_i)) + f(t_{i+1}, y(t_i) + hf(t_i, y(t_i)))]$$

$d_i = O(h^2)$, a bit involved to derive it

Implicit midpoint

- Use the midpoint quadrature rule:

$$\begin{aligned} y_{i+1} &= y_i + hf\left(t_{i+1/2}, y_{i+1/2}\right) \\ &= y_i + hf\left(t_i + h/2, (y_i + y_{i+1})/2\right) \end{aligned}$$

- That is, we solve for y_{i+1}
- Order is 2

Explicit midpoint method

- Take a step of size $h/2$ with forward Euler

$$Y = y_i + \frac{h}{2} f(t_i, y_i)$$

- Plug into the formula from the midpoint quadrature rule:

$$y_{i+1} = y_i + hf\left(t_i + h/2, Y\right),$$

- This is a two-stage explicit Runge-Kutta method
- Order is 2

Classical 4th order Runge-Kutta

- Based on Simpson's quadrature rule
- 4 stages
- Order 4, $O(h^4)$ accuracy

$$
\begin{aligned}
Y_1 &= y_i \\
Y_2 &= y_i + \frac{h}{2} f(t_i, Y_1) \\
Y_3 &= y_i + \frac{h}{2} f(t_i + h/2, Y_2) \\
Y_4 &= y_i + h f(t_i + h/2, Y_3) \\
y_{i+1} &= y_i + \frac{h}{6} \big[f(t_i, Y_1) + 2f(t_i + h/2, Y_2) + 2f(t_i + h/2, Y_3) + f(t_{i+1}, Y_4) \big]
\end{aligned}
$$

Stepsize control

- Estimate the error: Runge-Kutta pair (details omitted)
- Let h_i be the current stepsize
- Local error is of the form $e_i = ch_i^{q+1}$
- Assume an estimate for e_i is computed
- We require $e_i = ch_i^{q+1} \leq$ tol
- If the tolerance is satisfied, we accept the step and predict stepsize for the next step
- Otherwise, reject the step and repeat with smaller $\bar{h}_i$

- From $ch_i^{q+1} = e_i$,
$$c = \frac{e_i}{h_i^{q+1}}$$
and
$$ch_{i+1}^{q+1} = \frac{e_i}{h_i^{q+1}} h_{i+1}^{q+1} = \text{tol}$$
From which
$$h_{i+1} = h_i \left(\frac{\text{tol}}{e_i} \right)^{1/(q+1)}$$
- Since tol $\geq e_i$, $h_{i+1} \geq h_i$

- If $e_i > \text{tol}$, the stepsize is rejected
- Repeat the step with

$$\bar{h}_i = h_i \left(\frac{\text{tol}}{e_i} \right)^{1/(q+1)}$$

- For "safety", typically new stepsize is computed by

$$0.9 h_i \left(\frac{0.5\,\text{tol}}{e_i} \right)^{1/(q+1)}$$