Numerical Integration
Composite Rules

CS/SE 4X03

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Outline

How to increase the accuracy of a rule

- We can increase the degree of the polynomial, but the error might be large
- Apply a basic rule over small subintervals
  - subdivide $[a, b]$ into r subintervals
  - $h = \frac{b-a}{r}$ length of each subinterval
  - $t_i = a + ih$, $i = 0, 1, \ldots, r$
    $t_0 = a$, $t_r = b$

$$\int_a^b f(x)dx = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} f(x)dx$$

Composite trapezoidal rule

From the basic rule on $[t_{i-1}, t_i]$, $i = 1, \dots, r$

$$\int_{t_{i-1}}^{t_i} f(x)dx \approx \frac{t_i - t_{i-1}}{2}\left[f(t_{i-1}) + f(t_i)\right] = \frac{h}{2}\left[f(t_{i-1}) + f(t_i)\right]$$

we derive

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} f(x)dx \approx \frac{h}{2}\sum_{i=1}^{r}\left[f(t_{i-1}) + f(t_i)\right] \\
&= \frac{h}{2}\left(\sum_{i=1}^{r} f(t_{i-1}) + \sum_{i=1}^{r} f(t_i)\right) \\
&= \frac{h}{2}\left(f(t_0) + f(t_1) + \cdots + f(t_{r-1})\right) \\
&+ \frac{h}{2}\left(\qquad f(t_1) + \cdots + f(t_{r-1}) + f(t_r)\right) \\
&= \frac{h}{2}\left[f(a) + f(b)\right] + h\sum_{i=1}^{r-1} f(t_i)
\end{aligned}$$

Error of composite trapezoidal rule

From

$$\int_{t_{i-1}}^{t_i} f(x)dx = \frac{h}{2}\left[f(t_{i-1}) + f(t_i)\right] - \frac{f''(\eta_i)}{12}h^3$$

we have

$$\int_a^b f(x)dx = \underbrace{\sum_{i=1}^{r} \frac{h}{2}\left[f(t_{i-1}) + f(t_i)\right]}_{\text{composite}} \underbrace{- \sum_{i=1}^{r} \frac{f''(\eta_i)}{12}h^3}_{\text{error}}$$

Assuming $f''(x)$ continuous on $[a, b]$,

$$\min_{x\in[a,b]} f''(x) \le f''(\eta_i) \le \max_{x\in[a,b]} f''(x)$$

Then

$$\min_{x\in[a,b]} f''(x) \le \frac{1}{r}\sum_{i=1}^{r} f''(\eta_i) \le \max_{x\in[a,b]} f''(x)$$

Error of composite trapezoidal rule cont.

From the Intermediate Value Theorem, there exists μ, such that

$$f''(\mu) = \frac{1}{r}\sum_{i=1}^{r} f''(\eta_i)$$

Then the error is

$$\begin{aligned} -\sum_{i=1}^{r} \frac{f''(\eta_i)}{12} h^3 &= -\frac{1}{12}\left[\frac{1}{r}\sum_{i=1}^{r} f''(\eta_i)\right] r \cdot h \cdot h^2 \\ &= -\frac{f''(\mu)}{12}(b-a)h^2, \end{aligned}$$

$h = (b-a)/r$, and $r \cdot h = b - a$

Composite Simpson & midpoint rules

Simpson:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 2\sum_{i=1}^{r/2-1} f(t_{2i}) + 4\sum_{i=1}^{r/2} f(t_{2i-1}) + f(b)\right]$$

Error

$$-\frac{f^{(4)}(\zeta)}{180}(b-a)h^4$$

Midpoint:

$$\int_a^b f(x)dx \approx h\sum_{i=1}^{r} f\left(a + (i-1/2)h\right)$$

Error

$$\frac{f''(\xi)}{24}(b-a)h^2$$