Gauss Elimination (GE) with Partial Pivoting

CS/SE 4X03

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Outline

Example

GE with partial pivoting

Scaled partial pivoting

Example

Example 1. Consider

$$10^{-5}x_1 + x_2 = 1$$
$$2x_1 + x_2 = 2$$

The solution is

$$x_1^* \approx 5.000\,025\,000\,125 \times 10^{-1} \approx 0.5$$
$$x_2^* \approx 9.999\,949\,999\,750 \times 10^{-1} \approx 1$$

Solve by Gauss elimination in $t = 5$ digit decimal floating-point arithmetic

Example cont.

Example 1. cont.

- Eliminate with the first row, also called pivot row
- 10^{-5} is the pivot
- Multiply the first row by $2/10^{-5} = 2 \times 10^5$:

$$2x_1 + 2 \times 10^5 x_2 = 2 \times 10^5$$

  and subtract from the second row:

$$(1 - 2 \times 10^5)x_2 = 2 - 2 \times 10^5$$

- $1 - 2 \times 10^5$ and $2 - 2 \times 10^5$ round to -2.0000×10^5
- The second equation becomes

$$-2.0000 \times 10^5 x_2 = -2.0000 \times 10^5$$

  from which we compute $\widetilde{x}_2 = 1.0000$

Example cont.

Example 1. cont.

- Using $10^{-5}x_1 + x_2 = 1$, compute

$$\widetilde{x}_1 = \frac{1 - \widetilde{x}_2}{10^{-5}} = \frac{0}{10^{-5}} = 0$$

- The error in $\widetilde{x}_2$ is

$$\begin{aligned} \widetilde{x}_2 - x_2^* &\approx 1 - 9.999\,949\,999\,75 \times 10^{-1} \\ &\approx 5 \times 10^{-6} \end{aligned}$$

- Hence

$$\widetilde{x}_2 \approx x_2^* + 5 \times 10^{-6}$$

Example cont.

Example 1. cont.

- We have

$$\widetilde{x}_1 = \frac{1-\widetilde{x}_2}{10^{-5}} = \frac{1-(x_2^* + 5\times 10^{-6})}{10^{-5}}$$
$$\approx \underbrace{\frac{1-x_2^*}{10^{-5}}}_{x_1^*} - \underbrace{5\times 10^{-6}}_{\text{error in } \widetilde{x}_2} \times \underbrace{\frac{1}{10^{-5}}}_{1/\text{pivot}}$$

$$= x_1^* - (\text{error in } \widetilde{x}_2) \times \frac{1}{\text{pivot}}$$
$$= x_1^* - 0.5$$

- The error in $\widetilde{x}_2$ is multiplied by $1/\text{pivot} = 10^5$
  Error in $\widetilde{x}_1$ is -0.5
- Avoid small pivots

Example cont.

Example 1. cont.

- Swap the equations

$$
\begin{aligned}
2x_1 + x_2 &= 2 \\
10^{-5}x_1 + x_2 &= 1
\end{aligned}
$$

- Pivot is 2
- Multiply the first row by $10^{-5}/2$

$$10^{-5}x_1 + 10^{-5}/2\, x_2 = 10^{-5}$$

and subtract from the second row

$$(1 - 10^{-5}/2)x_2 = 1 - 10^{-5}$$

- $1 - 10^{-5}/2$ and $1 - 10^{-5}$ round to 1

Example cont.

Example 1. cont.

- The second equation is $x_2 = 1$, find $\widetilde{x}_2 = 1$
- Using $2x_1 + x_2 = 2$, $\widetilde{x}_1 = \frac{2-\widetilde{x}_2}{2} = 0.5$
- Using $\widetilde{x}_2 \approx x_2^* + 5 \times 10^{-6}$

$$
\begin{aligned}
\widetilde{x}_1 &= \frac{2 - \widetilde{x}_2}{2} = \frac{2 - (x_2^* + 5 \times 10^{-6})}{2} \\
&= \underbrace{\frac{2 - x_2^*}{2}}_{x_1^*} - \underbrace{5 \times 10^{-6}}_{\text{error in } \widetilde{x}_2} \times \underbrace{\frac{1}{2}}_{1/\text{pivot}} \\
&= x_1^* - (\text{error in } \widetilde{x}_2) \times \frac{1}{\text{pivot}} \\
&= x_1^* - 2.5 \times 10^{-5}
\end{aligned}
$$

GE with partial pivoting

GE with partial pivoting

- Eliminate with the row with the largest (in magnitude) entry

Example 2. Solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 1.0001x_2 + 2x_3 &= 2 \\ x_1 + 2x_2 + 2x_3 &= 3 \end{aligned}$$

with partial pivoting and $t = 5$ decimal arithmetic

Can chose any row to eliminate x_1. Use first row:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0.0001x_2 + x_3 &= 1 \\ x_2 + x_3 &= 2 \end{aligned}$$

Now eliminate with third row:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 2 \\ 0.0001x_2 + x_3 &= 1 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 2 \\ (1 - 0.0001)x_3 &= 1 - 0.0002 \end{aligned}$$

Example 2. cont.

$$\begin{aligned} x_3 &= 9.9990 \times 10^{-1} \\ x_2 &= 2 - x_3 = 1.0001 \\ x_1 &= 1 - x_2 - x_3 = -1 \end{aligned} \tag{1}$$

Using MATLAB's backslash operator, `A\b` where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

we obtain

$$[-1, 1.000\,100\,010\,001, 9.998\,999\,899\,99 \times 10^{-1}]$$

The errors (in absolute value) in the computed x_1, x_2, x_3 are $\approx 0, 10^{-8}, 10^{-8}$, respectively.

Example 2. cont.

If we eliminate with the second row

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0.0001x_2 + x_3 &= 1 \\ x_2 + x_3 &= 2 \end{aligned} \qquad \rightarrow \qquad \begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0.0001x_2 + x_3 &= 1 \\ -9.9990 \times 10^3 x_3 &= -9.9980 \times 10^3 \end{aligned}$$

$$\begin{aligned} x_3 &= 9.9990 \times 10^{-1} \\ x_2 &= \frac{1 - x_3}{0.0001} = 1.0000 \\ x_1 &= -9.9990 \times 10^{-1} \end{aligned}$$

The errors now are $\approx 10^{-4}, 10^{-4}, 10^{-8}$

Note: the errors in x_3 are the same, but the error in x_2 changed form 10^{-8} to 10^{-4}, similarly for x_1

Scaled partial pivoting

Example 3. Consider

$$2x_1 + 2cx_2 = 2c$$
$$x_1 + x_2 = 2$$

$c > 1$ is a constant

- Partial pivoting: first row as pivot row $(2 > 1)$
- GE gives

$$2x_1 + 2cx_2 = 2c$$
$$(1 - c)x_2 = 2 - c$$

- For c sufficiently large, $1 - c \approx -c$, $2 - c \approx -c$

Scaled partial pivoting cont.

Example 3. cont.

- Backward substitution gives

$$\widetilde{x}_2 \approx 1, \qquad \widetilde{x}_1 = \frac{2c - 2c\widetilde{x}_2}{2} \approx 0$$

If $\delta = \widetilde{x}_2 - x_2$,

$$\widetilde{x}_1 = \frac{2c - 2c\widetilde{x}_2}{2} = c - c(x_2 + \delta) = c - cx_2 - c\delta = x_1 - c\delta$$

- Error is multiplied by c
- When c is sufficiently large,

$$x_2 = \frac{c-2}{c-1} \approx 1, \qquad x_1 = \frac{c}{c-1} \approx 1$$

Scaled partial pivoting cont.

Example 3. cont.

Chose the row with the largest entry with respect to the entries in this row

$$\begin{aligned} 2x_1 + 2cx_2 &= 2c \\ 1x_1 + 1x_2 &= 1 \end{aligned}$$

- Scale vector $s = (2c, 1)$, $2c$ largest in first row, 1 largest in second row
- Ratio vector
$$r = \left(\frac{2}{2c}, \frac{1}{1} \right)$$
- Chose row with largest ratio as pivot row
- Eliminate with second row

Scaled partial pivoting cont.

Example 3. cont.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2cx_2 &= 2c \end{aligned}$$

- GE gives

$$\begin{aligned} x_1 + x_2 &= 2 \\ (2c - 2)x_2 &= 2c - 4 \end{aligned}$$

- Backward substitution (when c sufficiently large)

$$\begin{aligned} \widehat{x}_2 &\approx 1 \\ \widehat{x}_1 &\approx 1 \end{aligned}$$

Scaled partial pivoting cont.

Example 4.

$$Ax = \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ 34 \\ 16 \\ 26 \end{bmatrix} = b$$

- Scale vector $s = (13, 18, 6, 12)$
  $s_i = \max\{|a_{ij}| \mid j = 1, 2, 3, 4\}$
- Ratio vector
  $$r = \left(\frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12}\right)$$
- Select index in r with largest ratio: 3 or 4
- Pick 3 and eliminate with row 3

Scaled partial pivoting cont.

Example 4. cont.

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 50 \\ 16 \\ -6 \end{bmatrix}$$

- Scale vector $s = (13, 18, 6, 12)$
- Ratio vector
$$r = \left(\frac{12}{13}, \frac{2}{18}, -, \frac{4}{12} \right)$$
  $-$ means entry does not matter
- Select index from 1,2,4 with largest ratio: 1
- Eliminate with row 1

Scaled partial pivoting cont.

Example 4. cont.

With rounding to 4 decimal places

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 4.3333 & -13.8333 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -0.6667 & 1.6667 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 45.5 \\ 16 \\ 3 \end{bmatrix}$$

- Scale vector $s = (13, 18, 6, 12)$
- Ratio vector
$$r = \left(-, \frac{4.3333}{18}, -, \frac{0.6667}{12} \right)$$
- Select index from 2,4 with largest ratio: 2
- Eliminate with row 2

Example 4. cont.

$$\begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 4.3333 & -13.8333 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -0 & -0.4615 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ 45.5 \\ 16 \\ 10 \end{bmatrix}$$

$$\begin{aligned} x_4 &= -21.6667 \\ x_3 &= (45.5 - (-13.8333) * (-21.6667))/(4.3333) \\ &= -58.6671 \\ x_2 &= (-27 - 8 * (-58.6671) - 1 * (-21.6667))/(-12) \\ &= -38.6667 \\ x_1 &= (16 - (-2) * (-38.6667) - 2 * (-58.6671) - 4 * (-21.6667))/6 \\ &= 23.7779 \end{aligned}$$