Newton's Method for Nonlinear Equations

CS/SE 4X03

Ned Nedialkov

McMaster University

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Outline

Scalar case

- Given a scalar function f find a zero/root of f, i.e. an r such that $f(r) = 0$
- f may have no zeros, one, or many
- Let r be a root of f and let $x_n \approx r$
  From

$$
\begin{aligned}
0 &= f(r) = f(x_n) + f'(x_n)(r - x_n) + O(|r - x_n|^2) \\
0 &= f(r) \approx f(x_n) + f'(x_n)(r - x_n)
\end{aligned}
$$

we find x_{n+1} by solving

$$
f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0 \qquad (1)
$$

Newton in 1D

- That is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

- We start with an initial guess x_0 and compute $x_1, x_2, \ldots$
- How to choose x_0, does it converge to a root, when to stop iterating...?

Examples

Square root

- Given $a > 0$, compute $\sqrt{a}$
- Write $x = \sqrt{a}$, $f(x) = x^2 - a$
- Apply (2):

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{a}{2x_n} \\ &= 0.5\left(x_n + \frac{a}{x_n}\right) \end{aligned}$$

- Let $a = 2$ and $x_0 = 3$
- We compute

i	x_i	$\lvert x_i - \sqrt{2}\rvert$
1	1.833333333333333	4.19e-01
2	1.462121212121212	4.79e-02
3	1.414998429894803	7.85e-04
4	1.414213780047198	2.18e-07
5	1.414213562373112	1.67e-14
6	1.414213562373095	2.22e-16

Examples cont.

Dividing without division operation

- How to obtain a/b without division?
- $a/b = a * (1/b)$
- Find $1/b$. Write $f(x) = 1/x - b$ and apply (2)

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1/x_n - b}{-1/x_n^2} \\ &= x_n + x_n - b x_n^2 \\ &= x_n(2 - b x_n) \end{aligned}$$

Examples cont.

- With $b = 3$ and $x_0 = 0.3$, we compute

i	x_i	$\lvert x_i - 1/3 \rvert$
1	0.330000000000000	3.33e-03
2	0.333300000000000	3.33e-05
3	0.333333330000000	3.33e-09
4	0.333333333333333	5.55e-17

Newton for systems of equations

- Consider a system of n equations in n variables

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0\\
f_2(x_1, x_2, \ldots, x_n) &= 0\\
&\vdots\\
f_n(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
$$

- Denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ and $F = (f_1, f_2, \ldots, f_n)$
- Find $\mathbf{x}^*$ (if it exists) such that $F(\mathbf{x}^*) = 0$

Newton for systems of equations cont.

- Assume $\mathbf{x}^*$ is such that $F(\mathbf{x}^*) = 0$ and $\mathbf{x}^{(k)} \approx \mathbf{x}^*$
- From

$$0 = F(\mathbf{x}^*) \approx F(\mathbf{x}^{(k)}) + F'(\mathbf{x}^{(k)})(\mathbf{x}^* - \mathbf{x}^{(k)})$$

find $\mathbf{x}^{(k+1)}$ by solving (cf. (1))

$$F(\mathbf{x}^{(k)}) + F'(\mathbf{x}^{(k)})(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = 0 \tag{3}$$

- $F'(\mathbf{x}^{(k)})$ is the Jacobian of F at $\mathbf{x}^{(k)}$, an $n \times n$ matrix

Newton for systems of equations cont.

- Let $s = \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$
- Solve (assuming $F'(\mathbf{x}^{(k)})$ nonsingular) linear system

$$F'(\mathbf{x}^{(k)})s = -F(\mathbf{x}^{(k)}) \tag{4}$$

  and set

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + s \tag{5}$$

- (4,5) is basic Newton for systems of equations

Example

- Consider

$$0 = F(\mathbf{x}) = \begin{cases} x_1^2 + x_2^2 - 25 \\ x_1^2 - x_2 - 1 \end{cases}$$

- Jacobian is

$$F'(\mathbf{x}) = \begin{pmatrix} 2x_1 & 2x_2 \\ 2x_1 & -1 \end{pmatrix}$$

- Let $x_0 = (5, 1)^T$

- Then

$$F(\mathbf{x}^{(0)}) = (1, 23)^T$$
$$J(\mathbf{x}^{(0)}) = \begin{pmatrix} 10 & 2 \\ 10 & -1 \end{pmatrix}$$

- Solve $J(\mathbf{x}^{(0)})s = -F(\mathbf{x}^{(0)})$
- $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + s$ and so on
- We compute

i	x_1	x_2	$\|F(\mathbf{x})\|$
1	3.433333333333334	8.333333333333332	5.63e+01
2	2.632585333089088	5.289308176100628	9.93e+00
3	2.358810087435537	4.489032143454986	7.19e-01
4	2.329316858408983	4.424847176309882	5.06e-03
5	2.329040359270796	4.424428918660463	2.63e-07
6	2.329040339044829	4.424428900898053	7.11e-15