Polynomial Interpolation
Newton's Form

CS/SE 4X03

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Outline

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Basis

- Basis functions are

$$\phi_j(x) = \prod_{i=0}^{j-1}(x-x_i) = (x-x_0)(x-x_1)\cdots(x-x_{j-1}), \quad j = 0 : n$$

- Example: for a cubic interpolant, we have

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x - x_0 \\
\phi_2(x) &= (x - x_0)(x - x_1) \\
\phi_3(x) &= (x - x_0)(x - x_1)(x - x_2)
\end{aligned}$$

Computing coefficients

Let $y_i = f(x_i)$. From

$$\begin{aligned} p_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \cdots \\ + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \\ p_n(x_i) = c_0 + c_1(x_i - x_0) + c_2(x_i - x_0)(x_i - x_1) + \cdots \\ + c_n(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{n-1}) = f(x_i) \end{aligned}$$

at $x = x_0$, we have

$$\begin{aligned} p_n(x_0) = c_0 + c_1(x_0 - x_0) + c_2(x_0 - x_0)(x_0 - x_1) + \cdots \\ + c_n(x_0 - x_0)(x_0 - x_1)\cdots(x_0 - x_{n-1}) = f(x_0) \\ c_0 = f(x_0) \end{aligned}$$

Computing coefficients

At x_1,

$$p_n(x_1) = c_0 + c_1(x_1 - x_0) + c_2(x_1 - x_0)(x_1 - x_1) + \cdots$$
$$+ c_n(x_1 - x_0)(x_1 - x_1) \cdots (x_1 - x_{n-1}) = f(x_1)$$

$$\begin{aligned} c_0 + c_1(x_1 - x_0) &= f(x_1) \\ c_1 &= \tfrac{f(x_1) - c_0}{x_1 - x_0} \\ &= \tfrac{f(x_1) - f(x_0)}{x_1 - x_0} \end{aligned}$$

Computing coefficients

At x_2,

$$\begin{aligned} p_n(x_2) = c_0 + c_1(x_2 - x_0) &+ c_2(x_2 - x_0)(x_2 - x_1) \\ &+ c_3(x_2 - x_0)(x_2 - x_1)(x_2 - x_2) + \cdots \\ &+ c_n(x_1 - x_0)(x_1 - x_1)\cdots(x_1 - x_{n-1}) = f(x_1) \end{aligned}$$

Then

$$c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

$$c_2 = \frac{f(x_2) - c_0 - c_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Exercise: verify the last equality

Divided differences

Given $x_0, x_1, \ldots, x_n$, where $0 \le i < j \le n$, define

$$f[x_i] = f(x_i)$$
$$f[x_i, \ldots, x_j] = \frac{f[x_{i+1}, \ldots, x_j] - f[x_i, \ldots, x_{j-1}]}{x_j - x_i}$$

$f[x_i, \ldots, x_j]$ are divided differences over $x_i, \ldots, x_j$

Divided differences

$$
\begin{aligned}
c_0 &= f(x_0) = f[x_0] \\
c_1 &= \frac{f(x_1)-f(x_0)}{x_1-x_0} = f[x_0,x_1] \\
c_2 &= \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = f[x_0,x_1,x_2] \\
&\vdots \\
c_n &= \frac{f[x_1,\ldots,x_n]-f[x_0,\ldots,x_{n-1}]}{x_n-x_0} = f[x_0,x_1,\ldots,x_n]
\end{aligned}
$$

$$
\begin{aligned}
p_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\
+ f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})
\end{aligned}
$$

Example

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$
0	1	1		
1	2	3	2	
2	4	3	0	$-\frac{2}{3}$

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 1 + 2(x - 1) - \tfrac{2}{3}(x - 1)(x - 2) \end{aligned}$$

Example

Suppose we add a new point $(3, 5)$

Then

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	1	1			
1	2	3	2		
2	4	3	0	$-\frac{2}{3}$	
3	3	5	-2	-2	$-\frac{2}{3}$

$$p_3(x) = 1 + 2(x-1) - \tfrac{2}{3}(x-1)(x-2) - \tfrac{2}{3}(x-1)(x-2)(x-4)$$