

Partial Pivoting

CS/SE 4X03

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Outline

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Small pivots

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Partial pivoting (PP)

Examples

- The matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is nonsingular, but does not have LU factorization

Gauss elimination breaks down on this matrix since the multiplier is $1/0$

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is singular and has the LU factorization

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = LU$$

Small pivots

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

- Multiply the first row by $1/\epsilon$ and subtract from the second

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix}$$

- When ϵ small, in floating-point arithmetic,

$$U \approx \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix}$$

as $1 - \frac{1}{\epsilon} \approx -\frac{1}{\epsilon}$. Take e.g. $\epsilon = 10^{-16}$ in double precision

$$LU \approx \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = A$$

- Loss of accuracy

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

- Permute the rows

$$\overline{A} = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$$

- Multiple first row by ϵ and subtract from second row

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 - \epsilon \end{bmatrix}$$

$$\overline{L} = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix}, \quad \overline{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 - \epsilon \end{bmatrix}$$

- Permuting the rows of A is PA , where P is permutation matrix

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$$

Example

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + 2x_3 = 2$$

$$x_1 + 2x_2 - 16x_3 = 1$$

Eliminating with the first row:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & -16 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -17 & 0 \end{array} \right]$$

GE breaks down, as the next multiplier is $1/0$

Swap second and third rows and eliminate:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Example: Replace $a_{22} = 1$ by 1.0001 and consider

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + 1.0001x_2 + 2x_3 = 2$$

$$x_1 + 2x_2 - 16x_3 = 1$$

Gauss elimination gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1.0001 & 2 & 2 \\ 1 & 2 & -16 & 1 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.0001 & 1 & 1 \\ 0 & 1 & -17 & 0 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.0001 & 1 & 1 \\ 0 & 0 & -10017 & -10000 \end{array} \right]$$

Applying backward substitution in 5-digit decimal arithmetic:

$$\begin{aligned}\tilde{x}_3 &= -10000/(-10017) \\ &= 0.9983, \quad \text{roundoff} \approx 2.8851 \times 10^{-6}\end{aligned}$$

$$0.0001x_2 + x_3 = 1 :$$

$$\begin{aligned}\tilde{x}_2 &= (1 - \tilde{x}_3)/0.0001 = 0.0017/0.0001 \\ &= 17\end{aligned}$$

$$x_1 + x_2 + x_3 = 1 :$$

$$\begin{aligned}\tilde{x}_1 &= 1 - \tilde{x}_2 - \tilde{x}_3 = 1 - 17 - 0.9983 \\ &= -16.998\end{aligned}$$

Exact solution to 5 digits is $(-16.969, 16.971, 0.9983)^T$

Why \tilde{x}_2 is not accurate?

- $\tilde{x}_3 = x_3 + \epsilon$, ϵ is the roundoff error
- Then

$$\begin{aligned}\tilde{x}_2 &= (1 - \tilde{x}_3)/0.0001 \\ &= (1 - x_3 - \epsilon)/0.0001 \\ &= (1 - x_3)/0.0001 - \epsilon/0.0001 \\ &= x_2 - \epsilon \times 10^4\end{aligned}$$

- The roundoff error in \tilde{x}_3 is multiplied by 10^4 : $\epsilon \times 10^4 \approx 0.029$
- $\tilde{x}_2 - x_2 = 17 - 16.971 = 0.029$

Before eliminating with second row, swap second and third row and then eliminate

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -17 & 0 \\ 0 & 0.0001 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1.0017 & 1 \end{array} \right]$$

We have

$$\hat{x}_3 = 1/1.0017 = 0.9983, \quad \text{roundoff} \approx 2.8851 \times 10^{-6}$$

$$\hat{x}_2 = 0 + 17\hat{x}_3 = 16.971, \quad \text{roundoff multiplied by 17}$$

$$\hat{x}_1 = 1 - \hat{x}_2 - \hat{x}_3 = -16.969$$

Exact solution to 5 digits is $(-16.969, 16.971, 0.9983)^T$

Partial pivoting (PP)

- If a pivot is small, then $1/(\text{pivot})$ is large
- Roundoff errors are multiplied

Partial pivoting

- at step $k = 1 : n - 1$ chose the row q for which $|a_{qk}|$ is the largest
- eliminate with row q
now we divide by the largest element in column k