

https://github.com/LizzieSparling/magnetic_pendulum

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Euler's Method to plot pendulum trajectory:

Set height of the pendulum bob above the x-y plane to be 0.5cm. In order for our derived equations to hold, we need to assume that small angle approximations can be used, and thus we are making the assumption that the length of the pendulum is always very large in comparison to the radius of the magnets. This allows us to plot the trajectory by adopting the plan view of the system, projecting the bob's position onto the x-y plane.

Given initial point (0.9, -0.6). This point was chosen using the *numpy.random.uniform* method, selecting x and y from -1 to 1, increments of 0.1).

Initial velocity set to 0.0m/s, releasing the pendulum from rest.

Magnetic force (from each of the four magnets) is proportional to $1 / ((x_n - x)^2 + (y_n - y)^2 + h^2)$. Magnetic strength is kept at a constant such that the magnetic force constant of proportionality is 5.

Total magnetic force is the sum of the 4 magnetic forces provided by each magnet.

Gravitational restoring force acting on the pendulum is proportional to $-(x_0, y_0)$.

Damping force is proportional to chosen constant ($b = 0.05$) multiplied by the velocity of the pendulum at that point in time.

The total force acting on the pendulum is, therefore, given by the sum of these three force components.

Using Newton's Second Law, assuming the pendulum has unit mass, we equate the acceleration of the pendulum to the total force acting upon it.

Euler's Method gives us the following algorithm to compute the velocity of our pendulum with steps in time $dt = 0.01$: $y_1 = y_0 + f(t_0, y_0) * (t - t_0) = y_0 + f(t_0, y_0) * (0.01)$

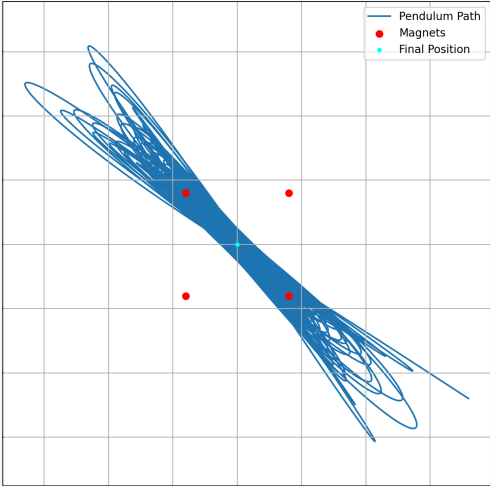
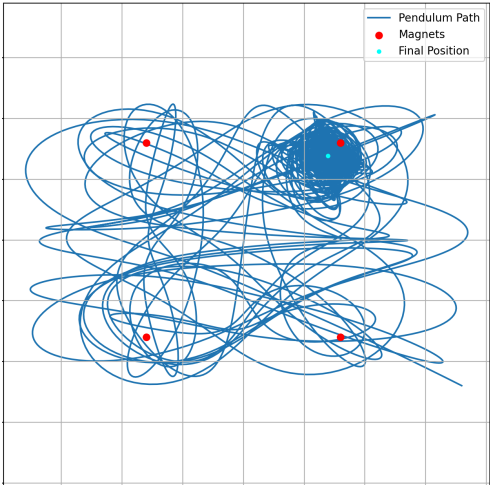
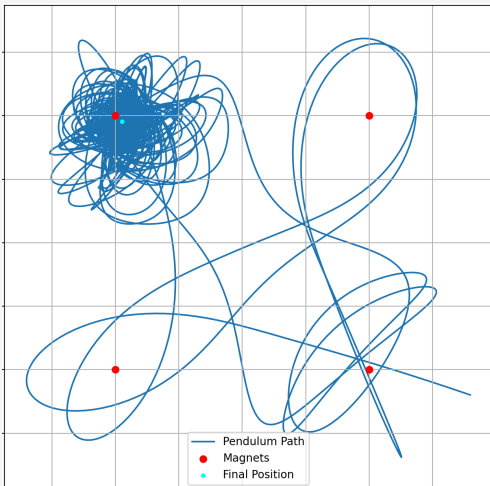
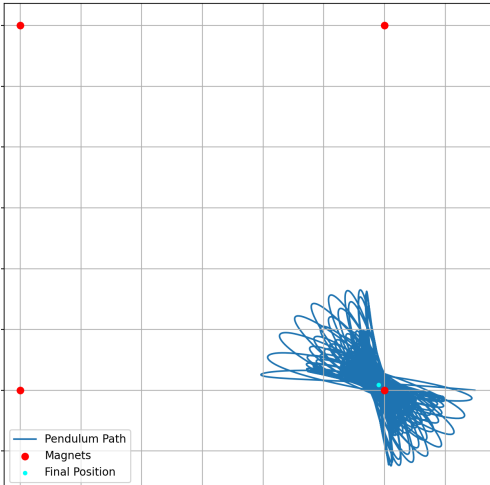
By calculating the current velocity of the pendulum ($velocity += acceleration * dt$), the new position of the pendulum can be found using Euler's method ($position += velocity * dt$).

Implementing this method through python, we can use matplotlib.pyplot, to create plots of the pendulum trajectory, up until the point at which the pendulum reaches an arbitrarily small velocity ($< 1e-3$ m/s).

Our four magnets lie at the corner of a square centred at the origin, and thus by substituting in different values for the side length for this square.

**Randomly-chosen initial position : [0.9, -0.6]
Damping constant (b) = 0.05**

Changing magnet radius, height above x-y plane = 0.5cm

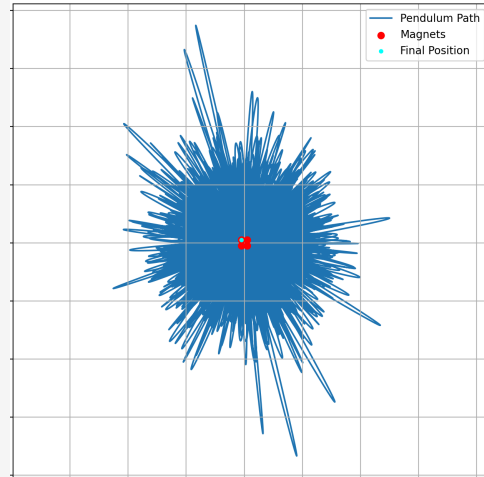
Side length of square / Magnet radius (cm)	Trajectory plot
0.2	 <p>A plot showing the trajectory of a pendulum in a square region with side length 0.2 cm. The plot features a dense, elongated blue line representing the 'Pendulum Path', which is oriented diagonally from the top-left to the bottom-right. Four red dots, labeled 'Magnets', are positioned at the corners of the square. A single cyan dot, labeled 'Final Position', is located near the center of the path. The legend in the top right corner identifies the blue line as 'Pendulum Path', the red dots as 'Magnets', and the cyan dot as 'Final Position'.</p>
0.4	 <p>A plot showing the trajectory of a pendulum in a square region with side length 0.4 cm. The blue 'Pendulum Path' is highly complex and chaotic, filling a large portion of the square. Four red dots representing 'Magnets' are at the corners. A cyan dot for 'Final Position' is located near the center of the chaotic region. The legend is in the top right corner.</p>
0.5	 <p>A plot showing the trajectory of a pendulum in a square region with side length 0.5 cm. The blue 'Pendulum Path' consists of several large, distinct loops and a dense central region. Four red dots representing 'Magnets' are at the corners. A cyan dot for 'Final Position' is located within the dense central region. The legend is in the bottom right corner.</p>
0.6	 <p>A plot showing the trajectory of a pendulum in a square region with side length 0.6 cm. The blue 'Pendulum Path' is concentrated in the bottom-right corner, forming a dense, fan-like shape. Four red dots representing 'Magnets' are at the corners. A cyan dot for 'Final Position' is located within the dense region in the bottom-right corner. The legend is in the bottom left corner.</p>

Changing height above plane, magnet radius = 0.5cm

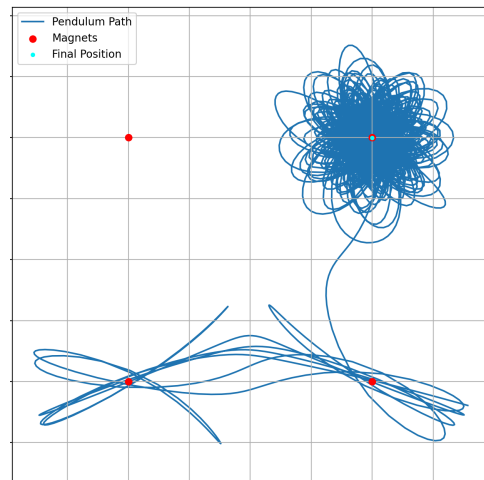
Height above x-y plane

Trajectory plot

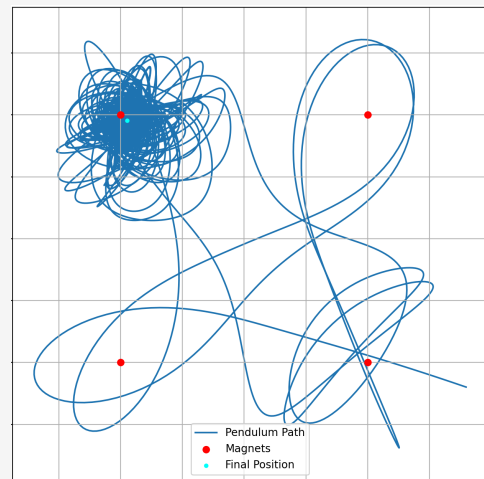
0.125



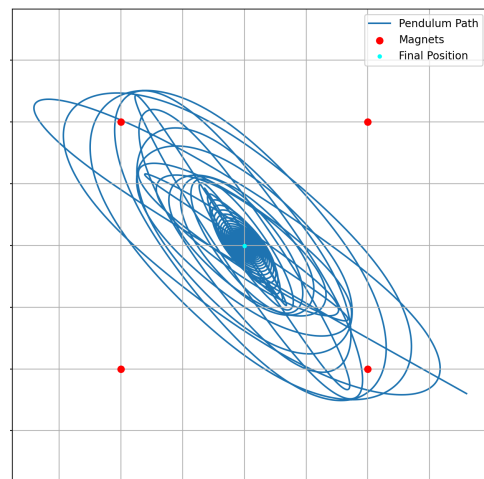
0.25



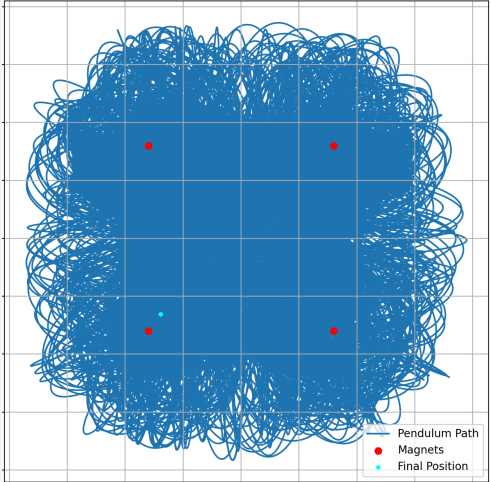
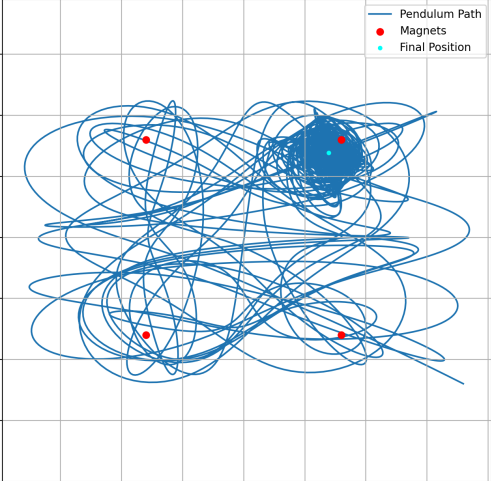
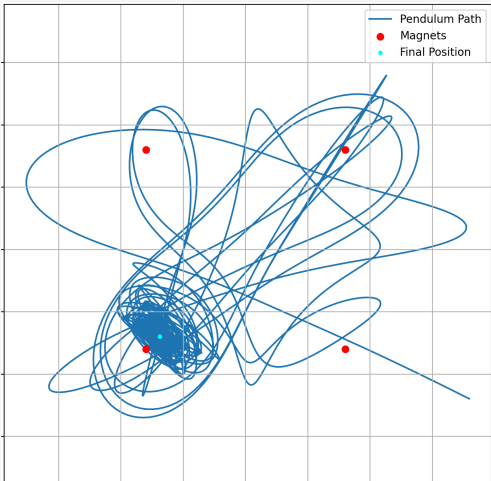
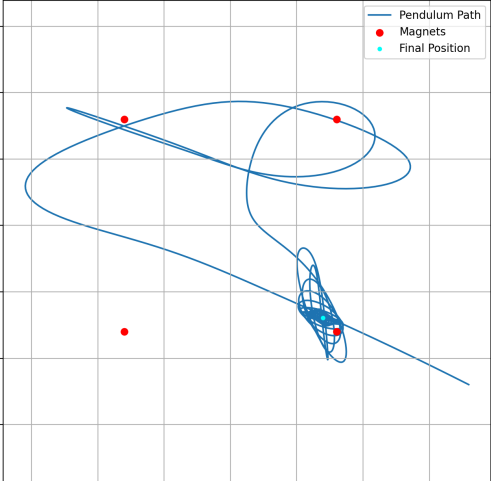
0.5



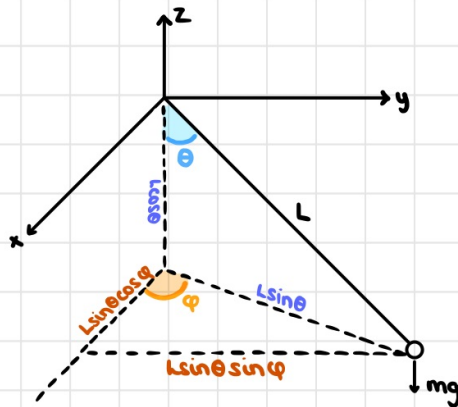
1.0



Magnet radius = 0.4cm , height above x-y plane = 0.5cm, changing damping

Damping constant (b)	Trajectory plot
0	 <p>The plot shows a highly dense, circular blue trajectory representing the pendulum path. Four red dots represent the magnet positions, and a single cyan dot represents the final position at the center of the circular path.</p>
0.05	 <p>The plot shows a moderately dense, irregular blue trajectory. Four red dots represent the magnet positions, and a cyan dot represents the final position near the center of the trajectory.</p>
0.1	 <p>The plot shows a sparse, irregular blue trajectory. Four red dots represent the magnet positions, and a cyan dot represents the final position near the center of the trajectory.</p>
0.5	 <p>The plot shows a very sparse, irregular blue trajectory. Four red dots represent the magnet positions, and a cyan dot represents the final position near the center of the trajectory.</p>

Modelling system with spherical coordinates:



- pendulum attached to fixed point $(0,0,0)$
- pendulum length = L

- magnet positions $\begin{pmatrix} 0 \\ R \\ -(L+h) \end{pmatrix}, \begin{pmatrix} 0 \\ R \\ -(L+h) \end{pmatrix}, \begin{pmatrix} R \\ 0 \\ -(L+h) \end{pmatrix}, \begin{pmatrix} R \\ 0 \\ -(L+h) \end{pmatrix}$

SPERICAL COORDINATES:

- more accurate representation of pendulum motion
- fully incorporate 3D dynamics (dependence on both azimuthal and polar angles)

POSITION VECTOR: $\mathbf{x} = \begin{bmatrix} L \sin \theta \cos \varphi \\ L \sin \theta \sin \varphi \\ -L \cos \theta \end{bmatrix}$

VELOCITY: $\dot{\mathbf{x}} = L \begin{bmatrix} \dot{\theta} \cos \theta \cos \varphi - L \dot{\varphi} \sin \theta \sin \varphi \\ \dot{\theta} \cos \theta \sin \varphi + L \dot{\varphi} \sin \theta \cos \varphi \\ \dot{\theta} \sin \theta \end{bmatrix}$

KINETIC ENERGY: $E = m/2 |\dot{\mathbf{x}}| \cdot |\dot{\mathbf{x}}| = \frac{mL^2}{2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)$

POTENTIAL ENERGY = $U_{\text{grav}} + V_{\text{mag}}$

$U_{\text{grav}} = -mgL \cos \theta$

$V_{\text{mag}} = \sum_{i=1}^4 \frac{-P}{|x-x_i|^4}$ magnitude of magnetic force predicted to vary as $1/\text{dist}^4$

LAGRANGIAN: $L = KE - U_{\text{grav}} - V_{\text{mag}}$
 $= \frac{1}{2} mL^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + mgL \cos \theta + \sum \frac{P}{|x-x_i|^4}$

Linear damping proportional to velocity, use Rayleigh dissipation function $F = \frac{1}{2} b \dot{x}^2$
 where $b = \text{constant of proportionality} \Rightarrow F = \frac{1}{2} b L^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)$

EULER LAGRANGE: ① $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = - \frac{\partial F}{\partial \dot{\theta}}$
 ② $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = - \frac{\partial F}{\partial \dot{\varphi}}$

① $mL^2 \ddot{\theta} - mL^2 \dot{\varphi}^2 \sin \theta \cos \theta + mgL \sin \theta + \frac{\partial V_{\text{mag}}}{\partial \theta} = -bL^2 \dot{\theta}$
 $\frac{\partial V_{\text{mag}}}{\partial \theta} = \sum_{i=1}^4 \frac{4P}{|x-x_i|^5} \times \frac{1}{|x-x_i|} \times [L \cos \theta \cos \varphi (L \sin \theta \cos \varphi - x_i) + L \cos \theta \sin \varphi (L \sin \theta \sin \varphi - y_i) + L \sin \theta ((L+h) - L \cos \theta)]$

② $mL^2 \ddot{\varphi} \sin^2 \theta + 2mL^2 \dot{\theta} \dot{\varphi} \sin \theta \cos \theta + \frac{\partial V_{\text{mag}}}{\partial \varphi} = -bL^2 \dot{\varphi} \sin^2 \theta$
 $\frac{\partial V_{\text{mag}}}{\partial \varphi} = \sum_{i=1}^4 \frac{4P}{|x-x_i|^5} \times \frac{1}{|x-x_i|} \times [-L \sin \theta \sin \varphi (L \sin \theta \cos \varphi - x_i) + L \sin \theta \cos \varphi (L \sin \theta \sin \varphi - y_i)]$

Code uses Runge Kutta method for integration (scipy.integrate.RK45)

In the forward Euler method, we used the information on the slope or the derivative of y at the given time step to extrapolate the solution to the next time-step. [...]

Runge-Kutta methods are a class of methods which judiciously uses the information on the 'slope' at more than one point to extrapolate the solution to the future time step. - Michael Zeltkevic

https://en.wikipedia.org/wiki/Runge-Kutta_methods

side and plan view of trajectory beginning at $\theta = \phi = \pi/3$

