https://github.com/LizzieSparling/magnetic_pendulum

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CODE AT https://github.com/LizzieSparling/magnetic_pendulum

Euler's Method to plot pendulum trajectory:

Set height of the pendulum bob above the x-y plane to be 0.5cm. In order for our derived equations to hold, we need to assume that small angle approximations can be used, and thus we are making the assumption that the length of the pendulum is always very large in comparison to the radius of the magnets. This allows us to plot the trajectory by adopting the plan view of the system, projecting the bob's position onto the x-y plane.

Given initial point (0.9, -0.6). This point was chosen using the *numpy.random.uniform* method, selecting x and y from -1 to 1, increments of 0.1).

Initial velocity set to 0.0m/s, releasing the pendulum from rest.

Magnetic force (from each of the four magnets) is proportional to $1 / ((x_n - x)^2 + (y_n - y)^2 + h^2)$. Magnetic strength is kept at a constant such that the magnetic force constant of proportionality is 5.

Total magnetic force is the sum of the 4 magnetic forces provided by each magnet.

Gravitational restoring force acting on the pendulum is proportional to $-(x_0, y_0)$.

Damping force is proportional to chosen constant (b = 0.05) multiplied by the velocity of the pendulum at that point in time.

The total force acting on the pendulum is, therefore, given by the sum of these three force components.

Using Newton's Second Law, assuming the pendulum has unit mass, we equate the acceleration of the pendulum to the total force acting upon it.

Euler's Method gives us the following algorithm to compute the velocity of our pendulum with steps in time dt = 0.01 : $y_1 = y_0 + f(t_0, y_0)*(t - t_0) = y_0 + f(t_0, y_0)*(0.01)$

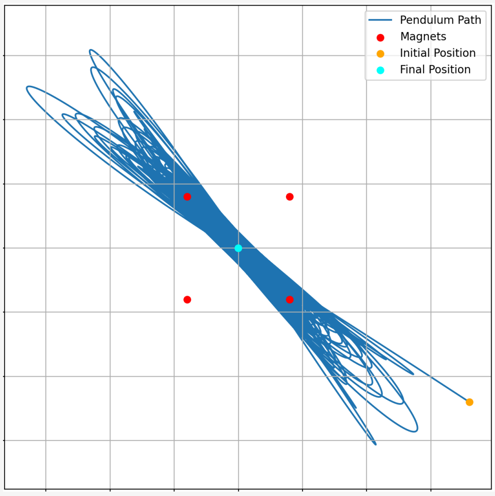
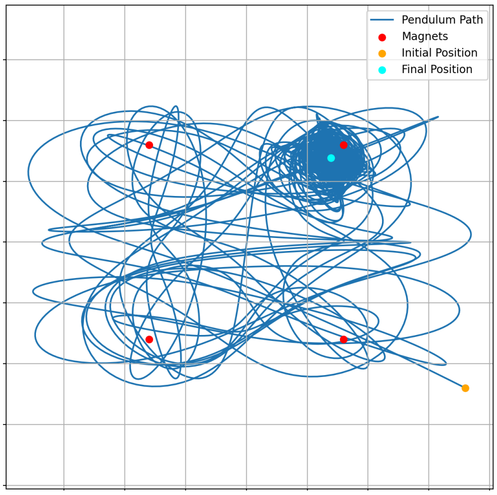
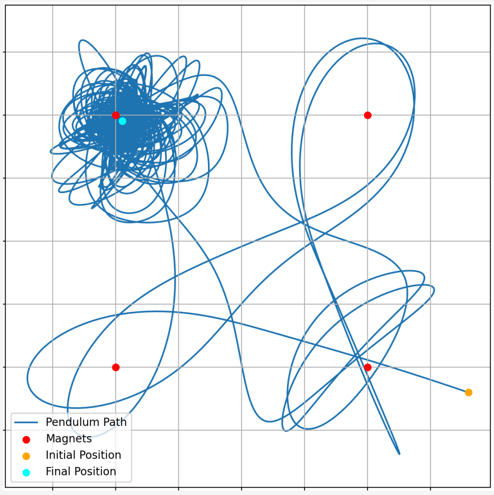
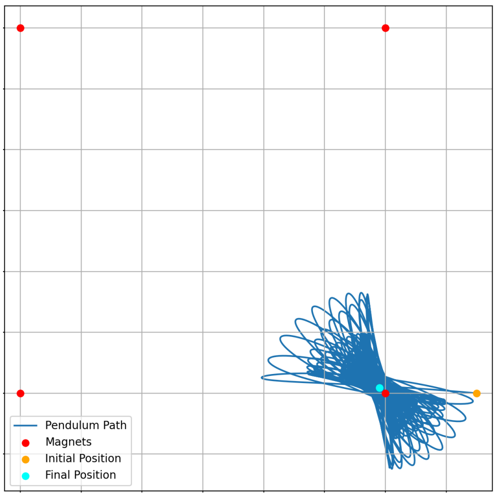
By calculating the current velocity of the pendulum (velocity += acceleration * dt), the new position of the pendulum can be found using Euler's method (position += velocity * dt).

Implementing this method through python, we can use matplotlib.pyplot, to create plots of the pendulum trajectory, up until the point at which the pendulum reaches an arbitrarily small velocity (< 1e-3 m/s).

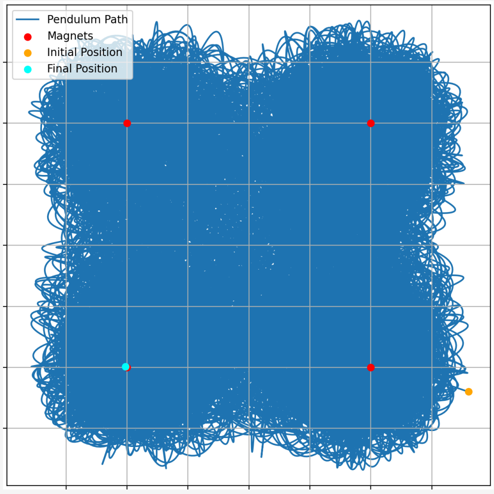
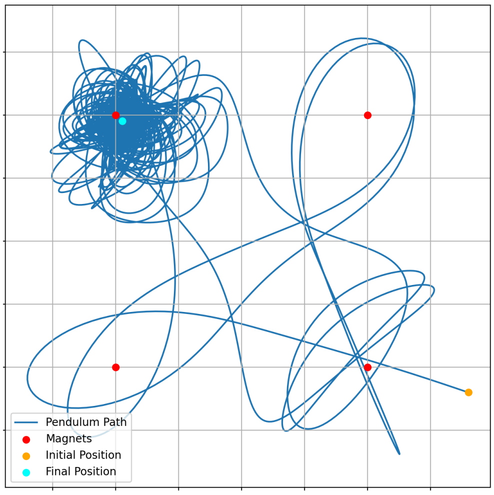
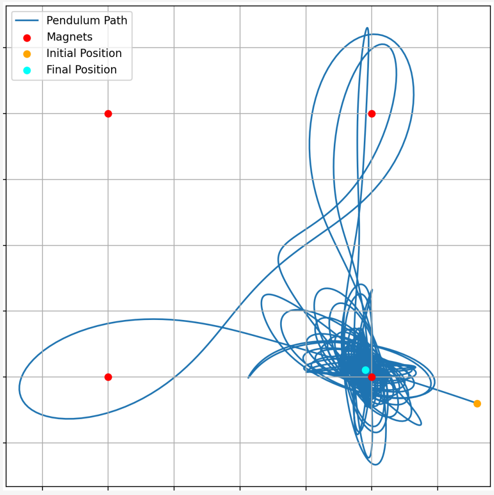
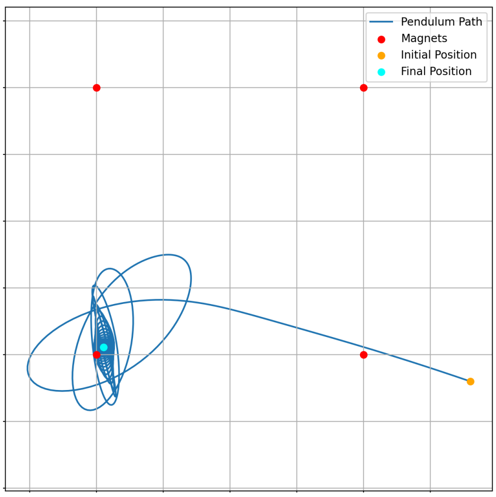
Our four magnets lie at the corner of a square centred at the origin, and thus by substituting in different values for the side length for this square.

Randomly-chosen initial position : [0.9, -0.6]
Damping constant (b) = 0.05
Height above x-y plane (h) = 0.5
Assuming pendulum length → ∞

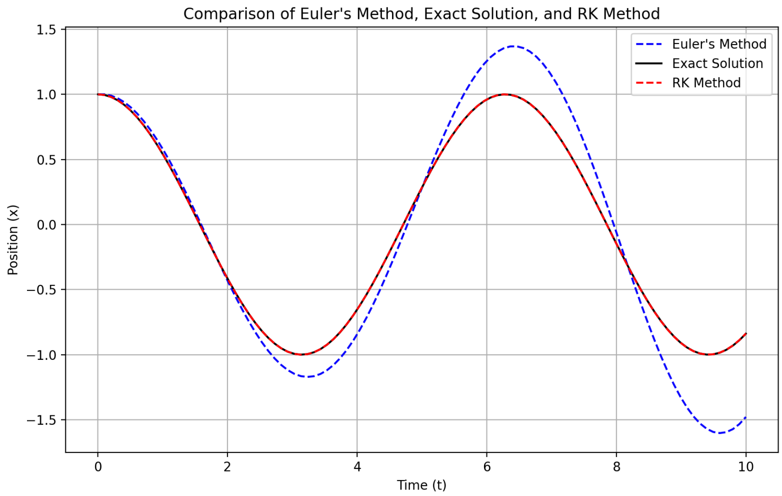
Changing magnet radius, height above x-y plane = 0.5cm

<table>
<tr><td>Side length of square / Magnet radius (cm)</td><td>Trajectory plot</td><td></td></tr>
<tr><td>0.2</td><td>
</td><td>Magnets are close together and close to the origin. Magnetic forces from all four magnets cause centralised net force, pulling pendulum towards centre of the square. Distance to all magnets is equal here and magnetic forces are equal in magnitude. Magnetic forces counteract each other. Damping force and equalised magnetic pull causes pendulum to eventually come to rest at the centre., as potential wells created by the magnets overlap at the centre point.</td></tr>
<tr><td>0.4</td><td>
</td><td>Complex interplay of forces results in chaotic behaviour. All magnetic forces are significant but equal and this not dominant enough to keep the magnet localised in their neighbourhood. Small variations in initial conditions can lead to dramatic changes in trajectory plot. The potential wells of the magnets interact in a way that creates multiple attractive forces, leading to chaotic trajectories.</td></tr>
<tr><td>0.5</td><td>
</td><td></td></tr>
<tr><td>0.6</td><td>
</td><td>Magnets further apart so more likely to be influenced by one magnet at a time. Pendulum falls into gravitational well of one magnet and magnetic pull from other magnets is significantly weaker since they are a further distance away. Magnetic potential wells are isolated. Pendulum oscillates around one magnet with little influence from others, due to magnetic force decreasing rapidly with distance. Eventually, due to damping, pendulum comes to rest above this magnet.</td></tr>
</table>

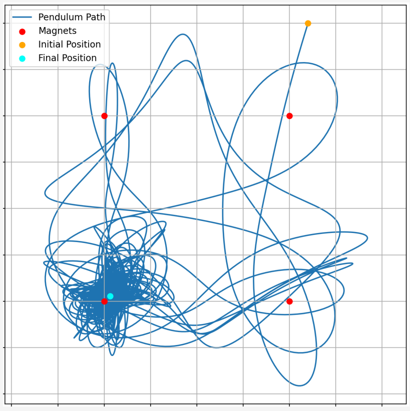
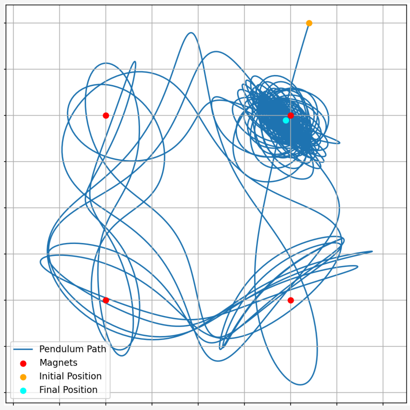
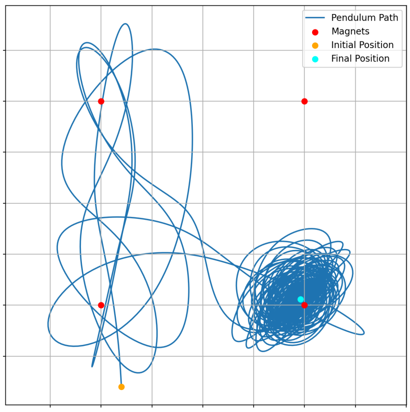
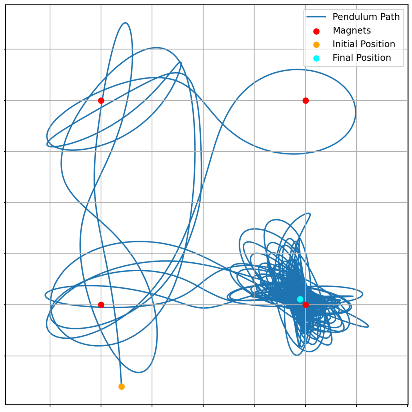
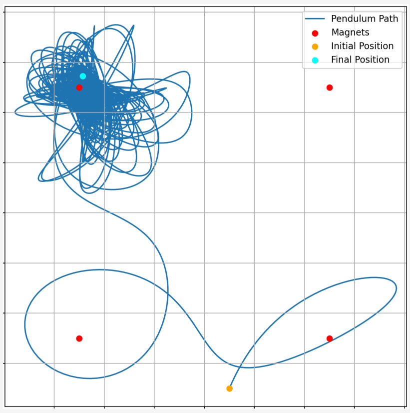
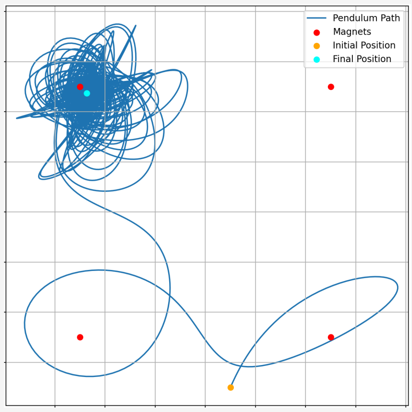
Magnet radius = 0.5cm , height above x-y plane = 0.5cm, changing damping

Damping constant (b)	Trajectory plot	
0		Pendulum retains most of its kinetic energy and thus it takes a long time for the pendulum to eventually come to rest. The pendulum moves through a large number of positions on the plane, causing extended trajectories which cover a significant portion of the plane. System exhibits chaotic behaviour, interacting with all magnets over a long time period before eventually coming to rest due to very gradual loss of kinetic energy.
0.05		The pendulum loses enegery at a rate which allows it to exhibit chaotic behaviour as it explores the plane, however movement is not indefinite. Eventually comes to rest as a result of moderate loss of kinetic energy. The equations of motion for the pendulum all involve a velocity term with coefficient -b. We see that the total mechanical energy of the system (kinetic + potential energy) decreases over time due to ramping. The magnitude of the constant, b, corresponds to how quickly the energy of the system decreases.
0.1		
0.5		High damping forces causes the pendulum to lose kinetic energy very quickly. Since motion is heavily damped, the pendulum settles into a relatively simple trajectory. The trajectory is quickly confined to the neighbourhood of a single magnet, oscillating around this magnet for a short period before eventually coming to rest directly above it.

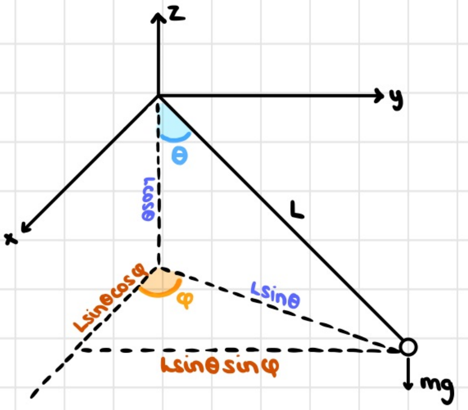
Euler VS Runge Kutta VS exact



b=0.05, R=0.5, h=0.5

Start point	Euler Trajectory	Runge Kutta Trajectory
0.6, 1		
-0.4, -0.9		
0.1, -0.7		

Modelling system with spherical coordinates:



- pendulum attached to fixed point $(0,0,0)$
- pendulum length $= L$
- magnet positions $\begin{pmatrix} 0 \\ R \\ -(L+h) \end{pmatrix}, \begin{pmatrix} 0 \\ R \\ -(L+h) \end{pmatrix}, \begin{pmatrix} R \\ 0 \\ -(L+h) \end{pmatrix}, \begin{pmatrix} R \\ 0 \\ -(L+h) \end{pmatrix}$

SPERICAL COORDINATES:

- more accurate representation of pendulum motion
- fully incorporate 3D dynamics (dependence on both azimuthal and polar angles)

POSITION VECTOR: $x = \begin{bmatrix} L\sin\theta\cos\varphi \\ L\sin\theta\sin\varphi \\ -L\cos\theta \end{bmatrix}$

VELOCITY: $\dot{x} = L \begin{bmatrix} \dot{\theta}\cos\theta\cos\varphi - L\dot{\varphi}\sin\theta\sin\varphi \\ \dot{\theta}\cos\theta\sin\varphi + L\dot{\varphi}\sin\theta\cos\varphi \\ \dot{\theta}\sin\theta \end{bmatrix}$

KINETIC ENERGY: $E = m/2 \; |\dot{x}| \cdot |\dot{x}| = \frac{mL^2}{2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)$

POTENTIAL ENEGY $= U_{grav} + V_{mag}$

$U_{grav} = -mgL\cos\theta$

$V_{mag} = \sum_{i=1}^{4} \frac{-p}{|x - x_i|^4}$ magnitude of magnetic force predicted to vary as $1/\text{dist}^4$

LAGRANGIAN: $L = KE - U_{grav} - V_{mag}$

$= \frac{1}{2} mL^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + mgL\cos\theta + \sum^4 \frac{p}{|x - x_i|^4}$

Linear damping proportional to velocity, use Rayleigh dissipation function $F = \frac{1}{2} b\dot{x}^2$ where $b =$ constant of proportionality $\Rightarrow$ $F = \frac{1}{2} bL^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)$

EULER LAGRANGE:

① $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\partial F}{\partial \dot{\theta}}$

② $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = -\frac{\partial F}{\partial \dot{\varphi}}$

① $mL^2 \ddot{\theta} - mL^2 \dot{\varphi}^2 \sin\theta\cos\theta + mgL\sin\theta + \frac{\partial V_{mag}}{\partial \theta} = -bL^2 \dot{\theta}$

$\frac{\partial V_{mag}}{\partial \theta} = \sum_{i=1}^{4} \frac{4p}{|x - x_i|^5} \times \frac{1}{|x - x_i|} \times [L\cos\theta\cos\varphi(L\sin\theta\cos\varphi - x_i) + L\cos\theta\sin\varphi(L\sin\theta\sin\varphi - y_i) + L\sin\theta((L+h) - L\cos\theta)]$

② $mL^2 \ddot{\varphi} \sin^2\theta + 2mL^2 \dot{\theta}\dot{\varphi}\sin\theta\cos\theta + \frac{\partial V_{mag}}{\partial \varphi} = -bL^2 \dot{\varphi}\sin^2\theta$

$\frac{\partial V_{mag}}{\partial \theta} = \sum_{i=1}^{4} \frac{4p}{|x - x_i|^5} \times \frac{1}{|x - x_i|} \times [-L\sin\theta\sin\varphi(L\sin\theta\cos\varphi - x_i) + L\sin\theta\cos\varphi(L\sin\theta\sin\varphi - y_i)]

Code uses Runge Kutta method for integration (scipy.integrate.RK45)

In the forward Euler method, we used the information on the slope or the derivative of y at the given time step to extrapolate the solution to the next time-step. […] Runge-Kutta methods are a class of methods which judiciously uses the information on the 'slope' at more than one point to extrapolate the solution to the future time step. - Michael Zeltkevic

https://en.wikipedia.org/wiki/Runge–Kutta_methods

side and plan view of trajectory beginning at theta=phi=pi/3