Chaos in the magnetic pendulum:

Explanation of set up:

- Model set-up
- Parameters
- Assumptions
- Simplifications

Permutations:

- Number of magnets
- Polarity
- Ratio of string length to magnet radius

Derivation of equations

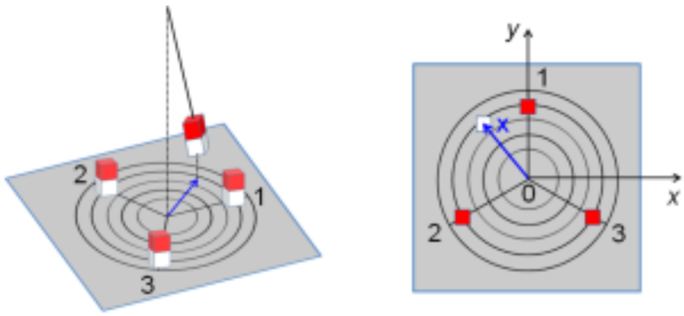
Finding equilibrium points

Explanation of set-up:

- Magnetic bob at the end of a string, hangs freely from a fixed point.
- Similar magnets (same strength and size) all arranged with attractive poles face-up placed on the vertices of a regular polygon with the geometric centre of this polygon aligned beneath the straight-down position.
- Pull the bob back in any arbitrary direction (fixing an initial starting point for the oscillation) release and observe subsequent motion.
- The bob is pulled simultaneously toward the base-plane magnets (invisible tug-of-war).
- For each initial position, the trajectory of the pendulum eventually stabilises around one of the plane magnets.

3 magnets:

The trajectory of the magnetic bob in the three- dimensional space is projected onto the horizontal base plane. The magnets are subsequently located at vertices $X1$, $X2$, and $X3$, all of which lie along the circumference of a circle (equivalently, they all lie at the corner of an equilateral-triangle) with radius taken to be the unit length.



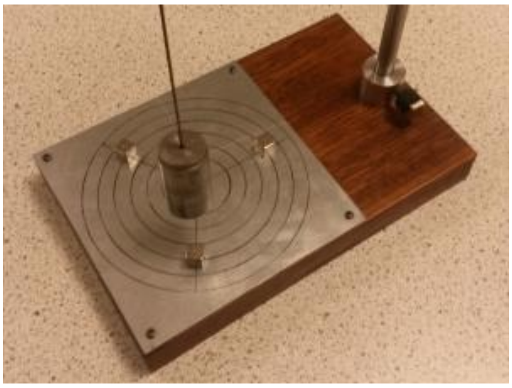
Parameters:

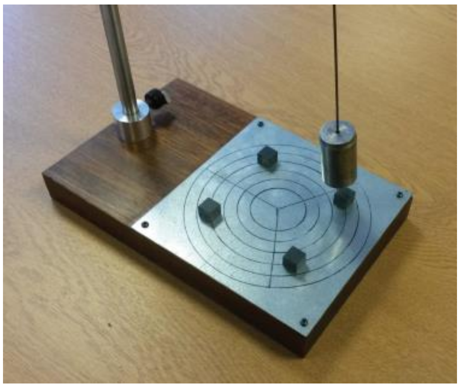
- (x, y): The Cartesian coordinates of the bob projected onto the two-dimensional plane.
- (xi, yi): The Cartesian coordinates of the magnets.
- h: The vertical distance from the bob to the x-y plane.
- b; The damping coefficient.

Assumptions:
- Pendulum bob is confined to plane, not sphere (two-dimensional simplification). Plan-view of the system allows us to denote the projection of the bob's position, at time t, as vector x(t) = (x(t), y(t)).
- All magnets are positioned with opposite (attracting pole face-up).
- Gravity supplies a torque that pulls the bob towards the straight-down position above the centre of the circle (x=0).
- All magnets have the same strength (p1 = p2 = p3 = +1).
- Gravitational restoring force is modelled by F(grav) proportional to –x. Choose units so that the constant of proportionality may be set to one. Lumped losses are accommodated by anticipating that damping is proportional to speed and hence Flosses = $-b\dot{x}$, where $\dot{x} = dx/dt$ (b > 0, the dimensionless loss coefficient, eg, due to air-resistance).
- The magnitude of the force between magnetic dipoles is predicted to vary as $1/\text{distance}^4$ (rather than $1/\text{distance}^2$, Coulomb's Law).

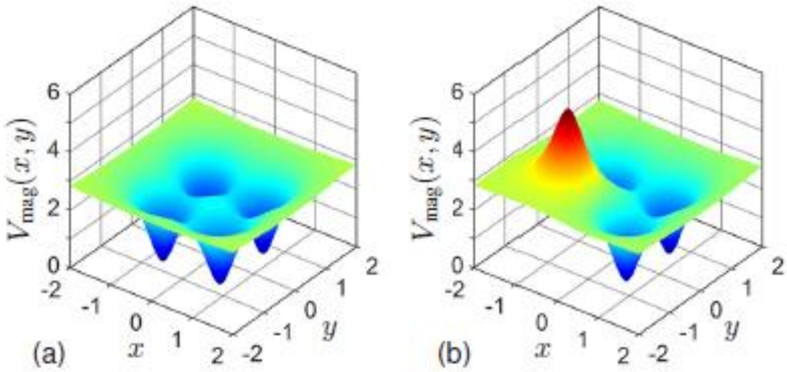
Permutations:

- Number of magnets

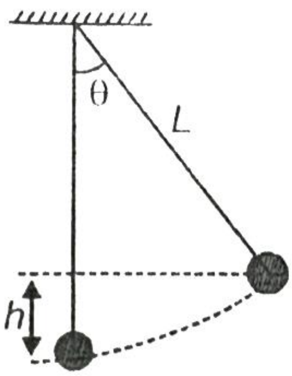




- Arrangement of magnets regarding their polarity



- Ratio of pendulum string length to radial displacement of magnets



Equations:

Distance between pendulum bob and magnet n:

$\sqrt{((x_n - x)^2 + (y_n - y)^2 + h^2)}$

Magnetic force is proportional to:

$1 / ((x_n - x)^2 + (y_n - y)^2 + h^2)$

Gravitational force pulls bob back to origin (straight-down) position.

Friction force acts in opposite direction to direction of motion and is proportional to velocity.

Using Newton's Law (equate sum of magnetic forces, gravitational forces, and damping to acceleration of mass):

$$\frac{d^2\mathbf{x}}{dt^2} + b\frac{d\mathbf{x}}{dt} + \mathbf{x} = \sum_{n=1}^{3} \frac{\mathbf{X}_n - \mathbf{x}}{\left(|\mathbf{X}_n - \mathbf{x}|^2 + h^2\right)^{5/2}}.$$

Equilibrium Points:

The equilibrium points are defined to be the points at which $x = x_{eq}$ (unchanging in time). The velocity and acceleration of the bob must be zero at those points.

Expect pendulum to come to rest as $t \to \infty$.

$$\mathbf{x}_{eq} = \sum_{n=1}^{3} \frac{\mathbf{X}_n - \mathbf{x}_{eq}}{\left(|\mathbf{X}_n - \mathbf{x}_{eq}|^2 + h^2\right)^{5/2}}.$$

Repeat the process and observe the final position of the pendulum, where the bob tends to stop, with $x_{eq} = 0$ being the origin.

Non-trivial roots of the above equation do not occur at $x_{eq} = x_n$ as one might suspect. Insted, they lie at the same angular positions as x_n (as symmetry demands) but at a radial distance $| x_{eq} |$ that is slightly less than unit length. At these positions, the force from gravity and the force from magnetism are perfectly balanced.