

https://github.com/LizzieSparling/magnetic_pendulum

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Euler's Method to plot pendulum trajectory:

Set height of the pendulum bob above the x-y plane to be 0.5cm. In order for our derived equations to hold, we need to assume that small angle approximations can be used, and thus we are making the assumption that the length of the pendulum is always very large in comparison to the radius of the magnets. This allows us to plot the trajectory by adopting the plan view of the system, projecting the bob's position onto the x-y plane.

Given initial point (0.9, -0.6). This point was chosen using the `numpy.random.uniform` method, selecting x and y from -1 to 1, increments of 0.1).

Initial velocity set to 0.0m/s, releasing the pendulum from rest.

Magnetic force (from each of the four magnets) is proportional to $1 / ((x_n - x)^2 + (y_n - y)^2 + h^2)$. Magnetic strength is kept at a constant such that the magnetic force constant of proportionality is 5.

Total magnetic force is the sum of the 4 magnetic forces provided by each magnet.

Gravitational restoring force acting on the pendulum is proportional to $-(x_0, y_0)$.

Damping force is proportional to chosen constant ($b = 0.05$) multiplied by the velocity of the pendulum at that point in time.

The total force acting on the pendulum is, therefore, given by the sum of these three force components.

Using Newton's Second Law, assuming the pendulum has unit mass, we equate the acceleration of the pendulum to the total force acting upon it.

Euler's Method gives us the following algorithm to compute the velocity of our pendulum with steps in time $dt = 0.01$: $y_1 = y_0 + f(t_0, y_0) * (t - t_0) = y_0 + f(t_0, y_0) * (0.01)$

By calculating the current velocity of the pendulum (velocity \pm acceleration * dt), the new position of the pendulum can be found using Euler's method (position \pm velocity * dt).

Implementing this method through python, we can use `matplotlib.pyplot`, to create plots of the pendulum trajectory, up until the point at which the pendulum reaches an arbitrarily small velocity ($< 1e-3$ m/s).

Our four magnets lie at the corner of a square centred at the origin, and thus by substituting in different values for the side length for this square.

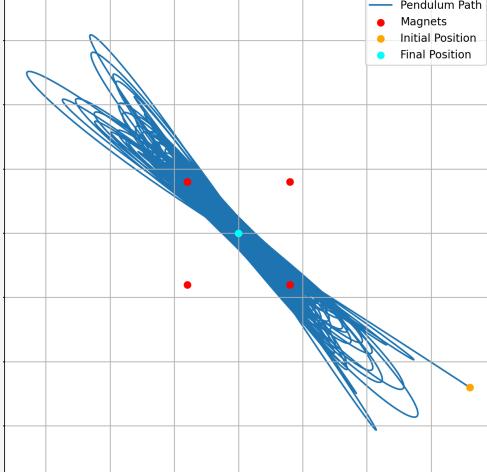
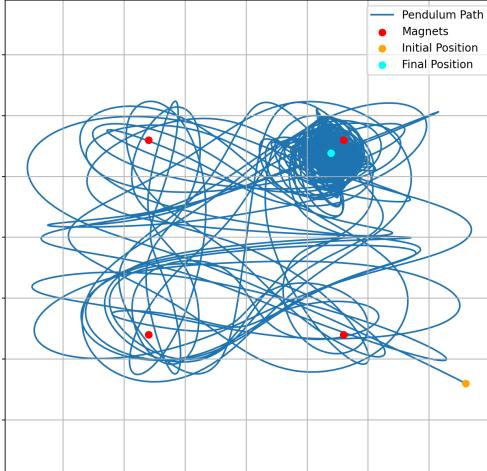
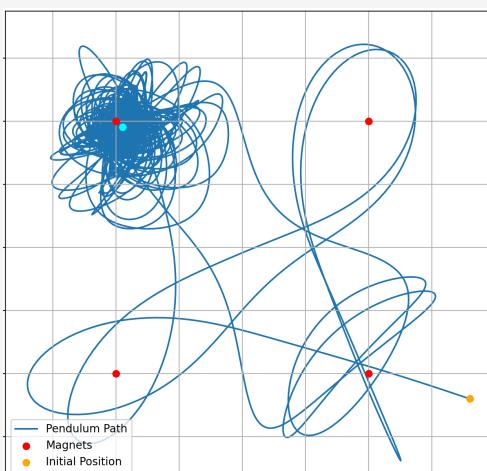
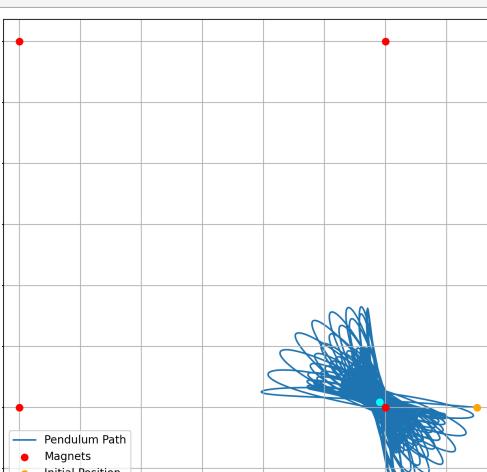
Randomly-chosen initial position : [0.9, -0.6]

Damping constant (b) = 0.05

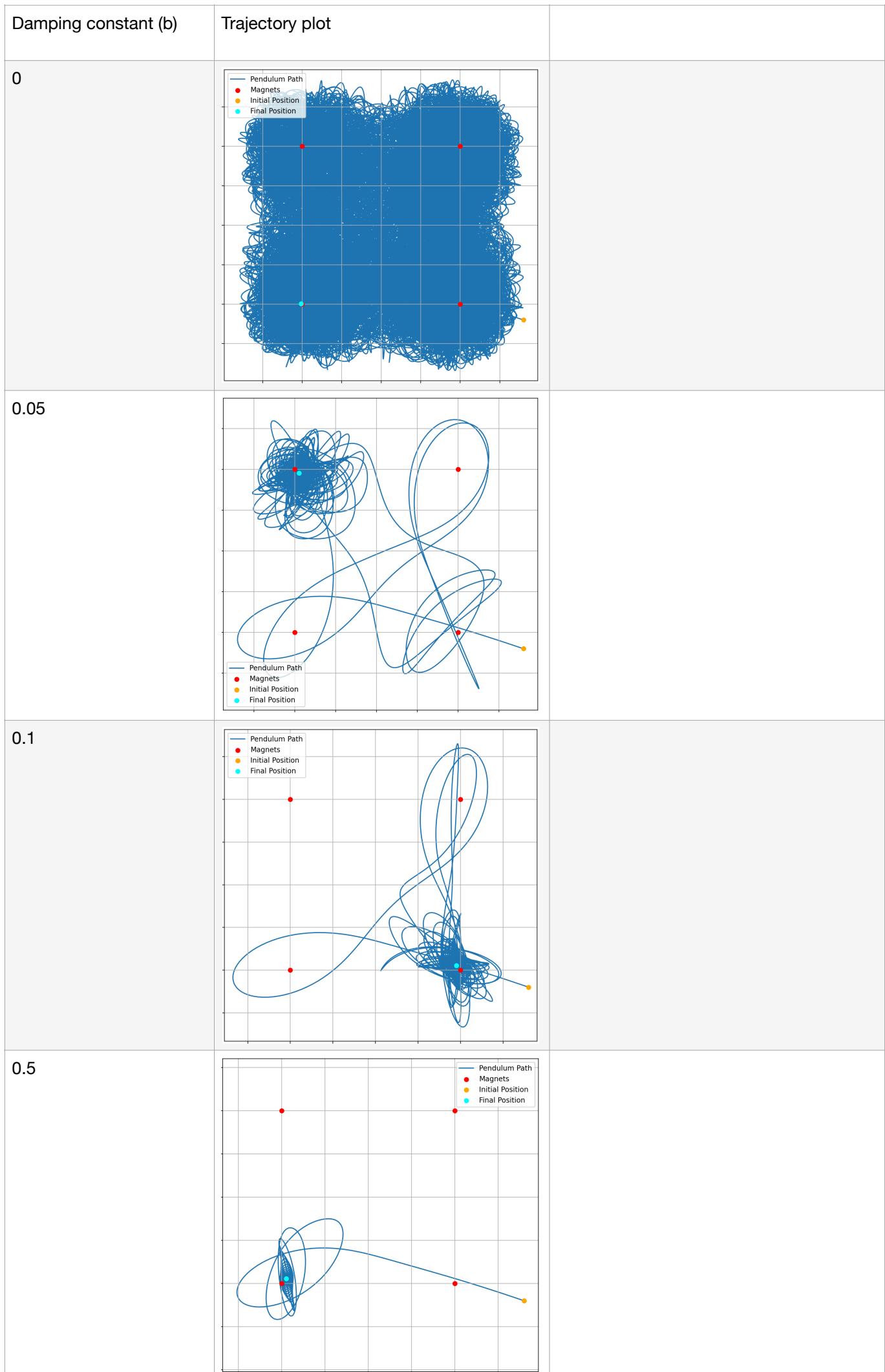
Height above x-y plane (h) = 0.5

Assuming pendulum length $\rightarrow \infty$

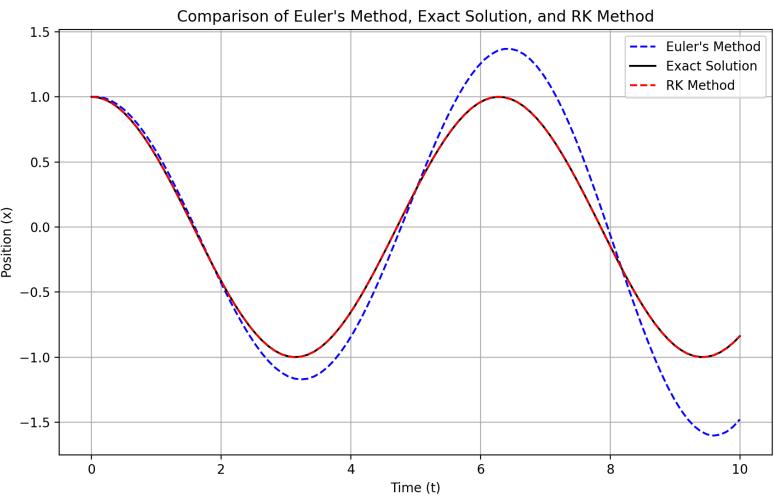
Changing magnet radius, height above x-y plane = 0.5cm

Side length of square / Magnet radius (cm)	Trajectory plot	
0.2	 <p>Pendulum Path Magnets Initial Position Final Position</p>	<p>Magnets are close together and close to the origin. Magnetic forces from all four magnets cause centralised net force, pulling pendulum towards centre of the square. Distance to all magnets is equal here and magnetic forces are equal in magnitude. Magnetic forces counteract each other. Damping force and equalised magnetic pull causes pendulum to eventually come to rest at the centre.</p>
0.4	 <p>Pendulum Path Magnets Initial Position Final Position</p>	<p>Complex interplay of forces results in chaotic behaviour. All magnetic forces are significant but equal and this not dominant enough to keep the magnet localised in their neighbourhood. Small variations in initial conditions can lead to dramatic changes in trajectory plot.</p>
0.5	 <p>Pendulum Path Magnets Initial Position Final Position</p>	
0.6	 <p>Pendulum Path Magnets Initial Position Final Position</p>	<p>Magnets further apart so more likely to be influenced by one magnet at a time. Pendulum falls into gravitational well of one magnet and magnetic pull from other magnets is significantly weaker since they are a further distance away. Pendulum oscillates around one magnet with little influence from others, due to magnetic force decreasing rapidly with distance. Eventually, due to damping, pendulum comes to rest above this magnet.</p>

Magnet radius = 0.5cm , height above x-y plane = 0.5cm, changing damping



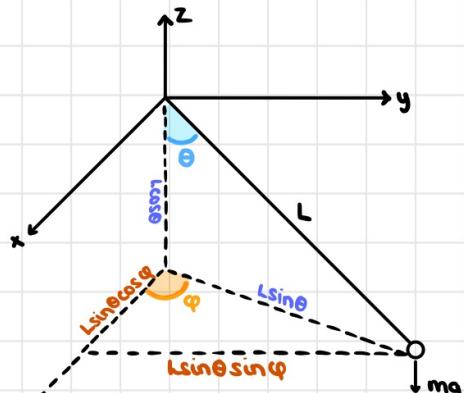
Euler VS Runge Kutta VS exact



$b=0.05, R=0.5, h=0.5$

Start point	Euler Trajectory	Runge Kutta Trajectory
0.6, 1		
-0.4, -0.9		
0.1, -0.7		

Modelling system with spherical coordinates:



$$\text{POSITION VECTOR: } \mathbf{x} = \begin{bmatrix} L\sin\theta\cos\varphi \\ L\sin\theta\sin\varphi \\ -L\cos\theta \end{bmatrix}$$

$$\text{VELOCITY: } \dot{\mathbf{x}} = L \begin{bmatrix} \dot{\theta}\cos\theta\cos\varphi - L\dot{\varphi}\sin\theta\sin\varphi \\ \dot{\theta}\cos\theta\sin\varphi + L\dot{\varphi}\sin\theta\cos\varphi \\ \dot{\theta}\sin\theta \end{bmatrix}$$

$$\text{KINETIC ENERGY: } E = \frac{m}{2} |\dot{\mathbf{x}}| \cdot |\dot{\mathbf{x}}| = \frac{mL^2}{2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)$$

$$\text{POTENTIAL ENERGY} = U_{\text{grav}} + V_{\text{mag}}$$

$$U_{\text{grav}} = -mgL\cos\theta$$

$$V_{\text{mag}} = \sum_{i=1}^4 \frac{-p}{|x-x_i|^4} \quad \text{magnitude of magnetic force predicted to vary as } 1/\text{dist}^4$$

$$\begin{aligned} \text{LAGRANGIAN: } L &= KE - U_{\text{grav}} - V_{\text{mag}} \\ &= \frac{1}{2} mL^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + mgL\cos\theta + \sum_{i=1}^4 \frac{p}{|x-x_i|^4} \end{aligned}$$

Linear damping proportional to velocity, use Rayleigh dissipation function $F = \frac{1}{2} b \dot{x}^2$
where b = constant of proportionality $\Rightarrow F = \frac{1}{2} bL^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)$

$$\begin{aligned} \text{EULER-LAGRANGE: } \textcircled{1} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= - \frac{\partial F}{\partial \dot{\theta}} \\ \textcircled{2} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} &= - \frac{\partial F}{\partial \dot{\varphi}} \end{aligned}$$

$$\textcircled{1} \quad mL^2 \ddot{\theta} - mL^2 \dot{\varphi}^2 \sin\theta \cos\theta + mgL\sin\theta + \frac{\partial V_{\text{mag}}}{\partial \theta} = -bL^2 \dot{\theta}$$

$$\frac{\partial V_{\text{mag}}}{\partial \theta} = \sum_{i=1}^4 \frac{4p}{|x-x_i|^5} \times \frac{1}{|x-x_i|} \times [L\cos\theta\cos\varphi (L\sin\theta\cos\varphi - x_i) + L\cos\theta\sin\varphi (L\sin\theta\sin\varphi - y_i) + L\sin\theta ((L+h) - L\cos\theta)]$$

$$\textcircled{2} \quad mL^2 \ddot{\varphi} \sin^2\theta + 2mL^2 \dot{\theta} \dot{\varphi} \sin\theta \cos\theta + \frac{\partial V_{\text{mag}}}{\partial \varphi} = -bL^2 \dot{\varphi} \sin^2\theta$$

$$\frac{\partial V_{\text{mag}}}{\partial \varphi} = \sum_{i=1}^4 \frac{4p}{|x-x_i|^5} \times \frac{1}{|x-x_i|} \times [-L\sin\theta\sin\varphi (L\sin\theta\cos\varphi - x_i) + L\sin\theta\cos\varphi (L\sin\theta\sin\varphi - y_i)]$$

Code uses Runge Kutta method for integration (scipy.integrate.RK45)

In the forward Euler method, we used the information on the slope or the derivative of y at the given time step to extrapolate the solution to the next time-step. [...] Runge-Kutta methods are a class of methods which judiciously uses the information on the 'slope' at more than one point to extrapolate the solution to the future time step. - Michael Zeltkevic

https://en.wikipedia.org/wiki/Runge–Kutta_methods

side and plan view of trajectory beginning at $\theta = \phi = \pi/3$

