Problem 1

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5.6 *Gram–Schmidt algorithm.* Consider the list of n n-vectors

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad a_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

(The vector a_i has its first i entries equal to one, and the remaining entries zero.) Describe what happens when you run the Gram–Schmidt algorithm on this list of vectors, *i.e.*, say what $q_1, \ldots, q_n$ are. Is $a_1, \ldots, a_n$ a basis?

↳ turn this into an orthonormal basis

Problem 2

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Problem 2.2 (Computing projections in Euclidean space)
In this problem we use the notation $\text{Proj}_{\mathcal{S}}(x)$ to denote the projection of a vector x onto some set $\mathcal{S}$, which consists of vectors that are of same dimension as x. Consider the following vectors and subspaces.

$$x_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathcal{V}_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}, \quad \mathcal{V}_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$x_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathcal{V}_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Compute $\text{Proj}_{\mathcal{V}_i}(x_i)$ for $i = 1, 2, 3$.

(b) Consider the affine set $\mathcal{A}_i = \{v + b_i \mid v \in \mathcal{V}_i\}$. Compute $\text{Proj}_{\mathcal{A}_i}(x_i)$ for $i = 1, 2, 3$.

(c) On a 2-d map, sketch the subspace $\mathcal{V}_1$ (a line through the origin) and clearly indicate x_1 and $\text{Proj}_{\mathcal{V}_1}(x_1)$. What is the point on $\mathcal{V}_1$ that is the closest to x_1 in Euclidean sense? On the same axes, sketch $\mathcal{A}_1$ (a line shifted from the origin) and indicate $\text{Proj}_{\mathcal{A}_1}(x_1)$.

(d) Compute an orthonormal basis $\mathcal{B}_3$ for the subspace $\mathcal{V}_3$ via Graham-Schmidt. Recompute $\text{Proj}_{\mathcal{V}_3}(x_3)$ and $\text{Proj}_{\mathcal{A}_3}(x_3)$ using $\mathcal{B}_3$, and compare with your previous results.

a) $\text{Proj}_V x_1 = \dfrac{x_1 \cdot v_1}{v_1 \cdot v_1} v_1$

$$= \frac{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{3 - 2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$- \quad 5 \quad [2 \;\; 1]$$

$$= \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$\text{proj}_V \, x_1 = \frac{x_1 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x_1 \cdot v_2}{v_2 \cdot v_2} v_2$$

Problem 3

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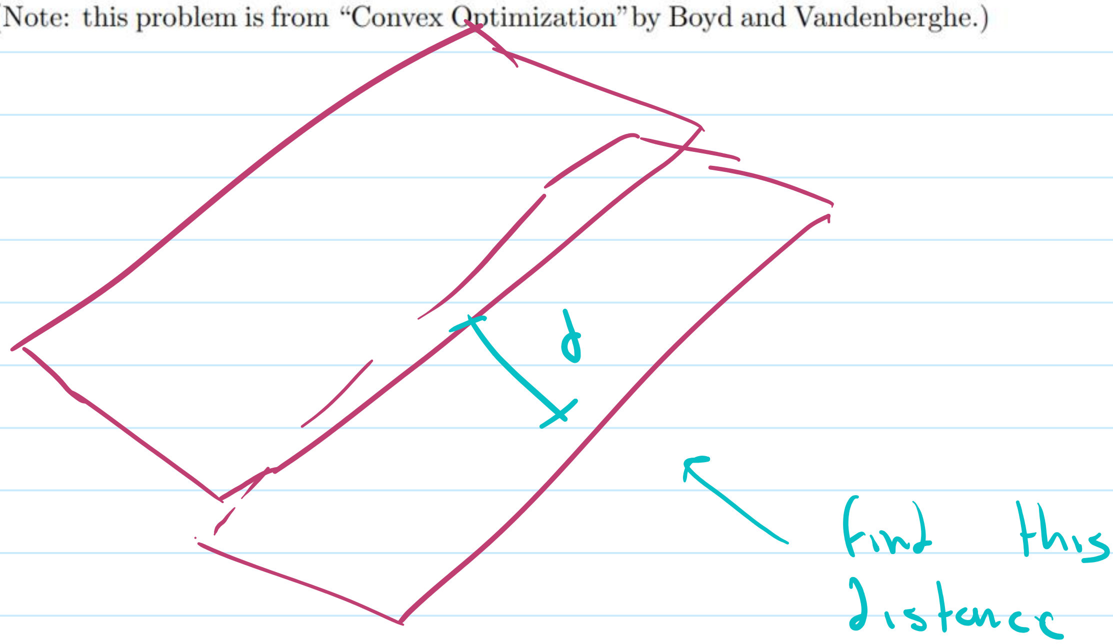
Problem 2.3 (Distance between a pair of parallel hyperplanes)
Find the distance between the two parallel hyperplanes $\mathcal{H}_i$, $i \in [2]$ where $\mathcal{H}_i = \{x \in \mathbb{R}^n | a^T x = b_i\}$. Your solution should be expressed in terms of the problem parameters, i.e., the vector $a \in \mathbb{R}^n$ and the scalars $b_i \in \mathbb{R}$. (Note: this problem is from "Convex Optimization" by Boyd and Vandenberghe.)



find orthogonal vectors. Measure magnitudes?

Problem 4

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Problem 2.4 (Taylor series expansion)
Consider the function $f(x) = -\sum_{l=1}^{m} \log(b_l - a_l^T x)$, where $x \in \mathbb{R}^n$, $b_l \in \mathbb{R}$ and $a_l \in \mathbb{R}^n$. Compute $\nabla f(x)$ and $\nabla^2 f(x)$. Write down the first three terms of the Taylor series expansion of $f(x)$ around some x_0.

$$\nabla f(x) = \nabla \left(- \sum_{i=1}^{m} \log(b_i - a_i^T x) \right)$$

$$= - \sum_{i=1}^{m} \nabla \left(\log(b_i - a_i^T x) \right)$$

$$= - \sum_{i=1}^{m} \frac{1}{(b_i - a_i^T \vec{x}) \ln(10)}$$

$$\nabla^2 f(x) =$$

Taylor Expansion:

$$f(x) \simeq f(\bar{x}) + \nabla f(x)^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x}) (x - \bar{x})$$

Problem 5

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Exercise 3.4 (Linear dynamical systems) Linear dynamical systems are a common way to (approximately) model the behavior of physical phenomena, via recurrence equations of the form[16]

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad t = 0, 1, 2, \ldots,$$

where t is the (discrete) time, $x(t) \in \mathbb{R}^n$ describes the state of the system at time t, $u(t) \in \mathbb{R}^p$ is the input vector, and $y(t) \in \mathbb{R}^m$ is the output vector. Here, matrices A, B, C, are given.

1. Assuming that the system has initial condition $x(0) = 0$, express the output vector at time T as a linear function of $u(0), \ldots, u(T-1)$; that is, determine a matrix H such that $y(T) = HU(T)$, where

$$U(T) \doteq \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}$$

contains all the inputs up to and including at time $T-1$.

2. What is the interpretation of the range of H?

$$y(T) = HU(T)$$

$$Cx(T) = H \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}$$

$$C\left[Ax(T-1) + Bu(T-1)\right] = H \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}$$

$$CAx(T-1) + CBu(T-1) = H_1 u(0) + H_2 u(1) + \ldots H_n (u(T-1))$$

This continues to recurse backwards until $t=0$

$$\therefore \quad H_n = CB$$

$$H_{n-1} = CAB$$

$$H_{n-2} = CAAB = CA^2B$$

$$\vdots$$

$$H_0 = CA^nB$$

$$\therefore \quad H = \begin{bmatrix} CA^{n}B \\ CA^{n-1}B \\ \vdots \\ CB \end{bmatrix}$$

b)