Problem 1.2

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Exercise 2.6 (norm - inequalities)

1. Show the following inequalities hold for any vector x:

$$\underbrace{\frac{1}{\sqrt{n}}\|x\|_2 \le \overbrace{\|x\|_\infty}^{②}}_{①} \le \underbrace{\|x\|_2 \le \overbrace{\|x\|_1}}_{③} \overbrace{\le \sqrt{n}\,\|x\|_2}^{④} \underbrace{\le n\|x\|_\infty}_{⑤}$$

① $\frac{1}{\sqrt{n}}\|x\|_2$

$$= \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n} x^2} = \sqrt{\frac{\sum_{i=1}^{n} x^2}{n}} = \underbrace{\sqrt{\frac{(x_1^2 + \dots x_n^2)}{n}}}_{\text{mean of values in } \vec{x}}$$

$$\|x\|_\infty = \underbrace{\max_{i\dots n}|x_i|}_{\text{max \# in values of } \vec{x}}$$

$$\text{mean}(\vec{x}) \le \max(\vec{x})$$

$$\therefore \frac{1}{\sqrt{n}}\|x\|_2 \le \|x\|_\infty$$

② $\|x\|_\infty = \max\limits_{i\dots n}|x_i| = x_j$ → max of $\vec{x}$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$= \sqrt{(x_1^2 + x_2^2 \dots x_n^2)}$$

$$= \sqrt{(x_1^2 + x_2^2 \dots x_n^2)}$$

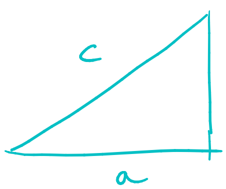
$$x_j \leq \sqrt{x_1^2 + x_2^2 \dots x_n^2}$$

$$\therefore \; \|x\|_\infty \leq \|x\|_2$$

③ $$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$= \sqrt{(x_1^2 + x_2^2 + \dots x_n^2)}$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = (|x_1| + |x_2| + \dots |x_n|)$$



$$\rightarrow \quad c \leq a + b \; , \; c^2 = a^2 + b^2$$

$$\therefore \sqrt{a^2 + b^2} \leq a + b$$

Generalizing to multiple dimensions:

$$\sqrt{x_1^2 + x_2^2 \dots + x_n^2} \leq x_1 + x_2 + \dots x_n \leq |x_1| + |x_2| + \dots + |x_n|$$

$$\sqrt{x_1^2 + x_2^2 \dots x_n^2} \leq |x_1| + |x_2| + \dots |x_n|$$

$$\sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i|$$

$$\therefore \; \|x\|_2 \leq \|x\|_1$$

④ $$\|x\|_1 = \sum^{n} |x_i| = |x_1| + |x_2| + \dots |x_n|$$

(4) $\|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \dots |x_n|$

Cauchy-Schwartz inequality

$|\langle u, v \rangle| \leq \|u\|_2 \|v\|_2$

$|u^T v| \leq \|u\|_2 \|v\|_2$

let $u = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix}$

$|u^T v| = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} |x_i| = \|x\|_1$

$\|u\|_2 = \sqrt{\sum_{i=1}^{n} 1^2} = \sqrt{n}$

$\|v\| = \sqrt{\sum_{i}^{n} |x_i^2|} = \|x\|_2$

$\therefore \|x\|_1 \leq \sqrt{n}\|x\|_2$

(5) $\sqrt{n}\|x\|_2 \leq n\|x\|_\infty$

$\sqrt{n}\|x\|_2 = \sqrt{nx_1^2 + nx_2^2 + \dots + nx_n^2}$

$n\|x\|_\infty = \max_{i \dots n} |x_i| \cdot n$

let $x_j = \max_{i \ldots n} |x_i| \quad \therefore \quad x_i \leq x_j$

$n\|x\|_\infty = n \cdot x_j$

$\sqrt{n x_1^2 + n x_2^2 \ldots n x_n^2} \leq \sqrt{n} x_1 + \sqrt{n} x_2 \ldots \sqrt{n} x_n = \sqrt{n} \sum_{i=1}^{n} |x_i| \leq n \cdot x_j$