Problem Set 3

October 26, 2023 2:50 AM

Due Date: October 28

Overview:

Theory:

- Eigenvectors & Symmetric Matrices
- Quadratic Constraints
- Lower bound on rank
- Ellipses, eigenvalues, eigenvectors, rank
- SVD
- SVD with ellipsoids

Application:

Latent Semantic Indexing
↳ has theory bonus

} SVD

Eigenfaces and l_2 Projection

Problem 3.1

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Exercise 4.1 (Eigenvectors of a symmetric 2×2 matrix) Let $p, q \in \mathbb{R}^n$ be two linearly independent vectors, with unit norm ($\|p\|_2 = \|q\|_2 = 1$). Define the symmetric matrix $A \doteq pq^\top + qp^\top$. In your derivations, it may be useful to use the notation $c \doteq p^\top q$.

1. Show that $p + q$ and $p - q$ are eigenvectors of A, and determine the corresponding eigenvalues.
2. Determine the nullspace and rank of A.
3. Find an eigenvalue decomposition of A, in terms of p, q. *Hint:* use the previous two parts.
4. What is the answer to the previous part if p, q are not normalized?

1.

$$A = pq^T + qp^T$$

$Av = \lambda v \rightarrow$ should be

$$A(p+q) = (pq^T + qp^T)(p+q)$$

$$= pq^Tp + qp^Tp + pq^Tq + qp^Tq$$

$$= pq^Tp + q(1) + p(1) + qp^Tq$$

$$= p(1 + q^Tp) + q(1 + p^Tq)$$

Problem 3.2

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Exercise 4.2 (Quadratic constraints) For each of the following cases, determine the shape of the region generated by the quadratic constraint $x^\top Ax \leq 1$.

1. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$

3. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

Hint: use the eigenvalue decomposition of A, and discuss depending on the sign of the eigenvalues.

$Av = \lambda v$

Problem 3.3

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Exercise 4.9 (A lower bound on the rank) Let $A \in \mathbb{S}^n_+$ be a symmetric, positive semidefinite matrix.

1. Show that the trace, trace A, and the Frobenius norm, $\|A\|_F$, depend only on its eigenvalues, and express both in terms of the vector of eigenvalues.

2. Show that

$$(\text{trace } A)^2 \le \text{rank}(A)\|A\|_F^2.$$

3. Identify classes of matrices for which the corresponding lower bound on the rank is attained.

Problem 3.4

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Problem 3.4 (Ellipses, eigenvalues, eigenvectors, and volume)
Make neat and clearly-labelled *sketches* (i.e., draw by hand) of the ellipsoid $\mathcal{E} = \{x | (x-x_c)^T P^{-1}(x-x_c) = 1\}$ for the following sets of parameters:

(a) Center $x_c = [0\ 0]^T$ and $P = [1.5\ \ -0.5; -0.5\ 1.5]$.

(b) Center $x_c = [1\ \ -2]^T$ and $P = [3\ 0; 0\ 1]$.

(c) Center $x_c = [-2\ 1]^T$ and $P = [9\ \ -2; -2\ 6]$.

For each part (a)–(c) also compute each pair of eigenvalues and corresponding eigenvectors.

(d) Recall the geometrically meaningful property of the determinant of a square real matrix A: its magnitude $|\det A|$ is equal to the volume of the parallelepiped $\mathcal{P}$ formed by applying A to the unit cube $\mathcal{C} = \{x | 0 \leq x_i \leq 1, i \in [n]\}$. In other words, if $\mathcal{P} = \{Ax | x \in \mathcal{C}\}$ then $|\det(A)|$ is equal to the volume of $\mathcal{P}$. Furthermore, recall that the determinant of a matrix is zero if any of its eigenvalues are zero. Explain how to interpret this latter fact in terms of the interpretation of $|\det(A)|$ as the volume of $\mathcal{P}$. (This interpretation was mentioned in class so this is just a "I want to make sure you understand that comment" type of question.)

Problem 3.5

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Exercise 5.1 (SVD of an orthogonal matrix) Consider the matrix

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

1. Show that A is orthogonal.

2. Find a singular value decomposition of A.

$$\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = -2 - 2 + 4 = 0$$

$$\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$$

Problem 3.6

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Problem 3.6 (SVDs and ellipsoids)

This problem concerns the matrix

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix}.$$

(a) Find the SVD of A, $A = U\Sigma V^T$, specifying the matrix of normalized left-singular vectors U, the matrix of normalized right-singular vectors V, and the matrix of (non-negative) singular values Σ. (Since A is square, $\tilde{\Sigma} = \Sigma$.)

(b) Use your results from part (a) to write the SVD in outer-product form, as $A = \sigma_1 u^{(1)} (v^{(1)})^T + \sigma_2 u^{(2)} (v^{(2)})^T$.

Now consider the action of A on a unit vector x such that $\|x\|_2 = 1$ (or, in part (e), $\|x\|_2 \leq 1$). In answering parts (c)–(e) it may help to use the outer-product form from part (b) to write

$$Ax = U\Sigma V^T x = U\Sigma\bar{x} = U \begin{bmatrix} \sigma_1 \bar{x}_1 & 0 \\ 0 & \sigma_2 \bar{x}_2 \end{bmatrix} = u^{(1)} \sigma_1 \bar{x}_1 + u^{(2)} \sigma_2 \bar{x}_2, \tag{1}$$

where $\bar{x} = V^T x$ and $\|\bar{x}\|_2 = 1$ since $\|x\|_2 = 1$ and V is an orthogonal matrix.

(c) What is the unit input direction x, such that $\|x\|_2 = 1$, that leads to the greatest amplification (the largest $\|Ax\|_2$)? What is the amplification? What output direction Ax results from setting x equal to the input direction that yields the greatest amplification?

(d) What is the unit input direction x, such that $\|x\|_2 = 1$, that leads to the least amplification (the smallest $\|Ax\|_2$)? What is the amplification? What output direction Ax results from setting x equal to the input direction that yields the least amplification?

(e) Sketch the set $\{Ax \in \mathbb{R}^2 : \|x\|_2 \leq 1, x \in \mathbb{R}^2\}$. This set is the *image* of the unit ball under the linear map $f(x) = Ax$.

(f) Now, sketch the set $\{x \in \mathbb{R}^2 : \|Ax\|_2 \leq 1\}$. The logic is related to that in (c)-(e), but note that here you are asked to identify the inputs for which the corresponding *outputs* are constrained to the unit ball. This set is the *pre-image* of the unit ball under the linear map $f(x) = Ax$.