Problem 1.6

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3.24 *Distance versus angle nearest neighbor.* Suppose $z_1, \ldots, z_m$ is a collection of n-vectors, and x is another n-vector. The vector z_j is the (distance) nearest neighbor of x (among the given vectors) if

$$\|x - z_j\| \leq \|x - z_i\|, \quad i = 1, \ldots, m,$$

i.e., x has smallest distance to z_j. We say that z_j is the *angle nearest neighbor* of x if

$$\angle(x, z_j) \leq \angle(x, z_i), \quad i = 1, \ldots, m,$$

i.e., x has smallest angle to z_j.

(a) Give a simple specific numerical example where the (distance) nearest neighbor is not the same as the angle nearest neighbor.

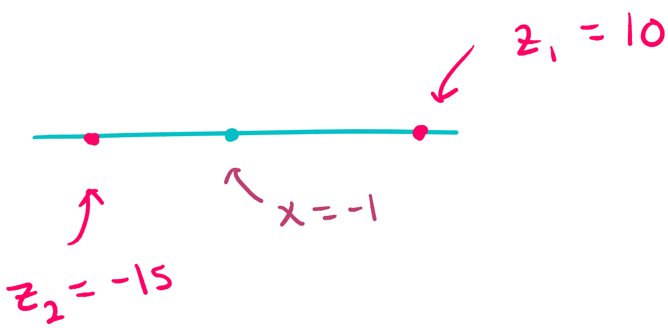
(b) Now suppose that the vectors $z_1, \ldots, z_m$ are normalized, which means that $\|z_i\| = 1$, $i = 1, \ldots, m$. Show that in this case the distance nearest neighbor and the angle nearest neighbor are always the same. *Hint.* You can use the fact that arccos is a decreasing function, *i.e.*, for any u and v with $-1 \leq u < v \leq 1$, we have $\arccos(u) > \arccos(v)$.

Definition of angle between Vectors:

$$a^T b = \|a\| \|b\| \cos\theta$$

$$\therefore \theta = \arccos\left(\frac{a^T b}{\|a\| \|b\|}\right)$$

a) 1D example:



Distance nearest neighbor:

$$\|x - z_1\| = |-1 - 10| = 11$$

$$\|x - z_2\| = |-1 - (-15)| = 14$$

$\therefore z_1$ is the distance nearest neighbor

$\therefore$ z_1 is the distance nearest neighbor

Angle nearest neighbor

$$\theta_{z_1} = \arccos\left(\frac{x^T z_1}{\|x\| \|z_1\|}\right)$$

$$= \arccos\left(\frac{(-1)(10}{|-1||10|}\right)$$

$$= \arccos(-1)$$

$$= \pi$$

$$\theta_{z_2} = \arccos\left(\frac{x^T z_2}{\|x\|\|z_2\|}\right)$$

$$= \arccos\left(\frac{(-1)(-15)}{|-1| \, |-15|}\right)$$

$$= \arccos(1)$$

$$= 0$$

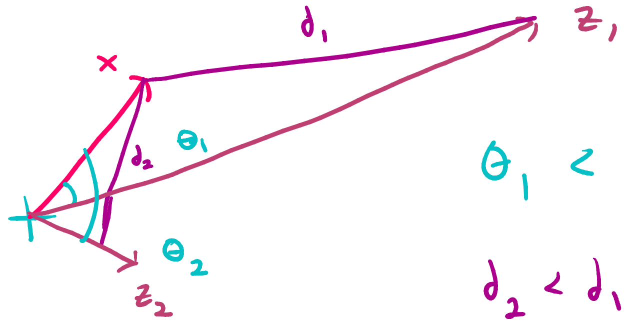
$\therefore$ z_2 is the angle nearest neighbor.

In this example the distance nearest neighbor was different than the angle nearest neighbor

b) 2D example:

For non-normalized vectors, the distance and angle nearest neighbors will be different when the magnitude of the angle nearest vector is less than another z vector

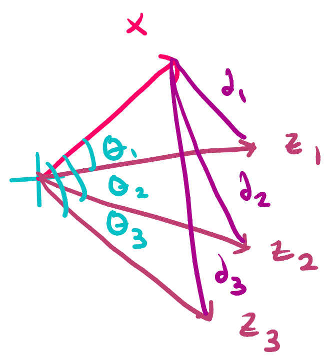
i.e.



$\theta_1 < \theta_2 \rightarrow z_1$ is angle nearest neighbor

$d_2 < d_1 \rightarrow z_2$ is distance nearest neighbor

But if all the z_i vectors are normalized



$d_1 < d_2$ & d_3 &

$\theta_1 < \theta_2 < \theta_3$

So z_1 is both the angle and distance nearest neighbor

Hint Property:

If $-1 \leq u < v \leq 1$, then,

$$\arccos u \geq \arccos(v)$$

If $\|z_i\| = 1$ for $\forall\, i \ldots m$ then

$$\theta_i = \arccos\left(\frac{x^T z_i}{\|x\|\|z_i\|}\right)$$

$$= \arccos\left(\frac{x^T z_i}{\|x\|}\right)$$

$$\theta_1 = \arccos\left(\frac{x^T z_1}{\|x\|}\right) \quad \text{and} \quad z_2 = \arccos\left(\frac{x^T z_2}{\|x\|}\right)$$

the only difference being the value of $x^T z_i$

If $\theta_1 < \theta_2 \rightarrow$ so z_1 is the angle nearest neighbor

$$\arccos\left(\frac{x^T z_1}{\|x\|}\right) < \arccos\left(\frac{x^T z_2}{\|x\|}\right)$$

which based on the property above means

$$-1 \leq \frac{x^T z_2}{\|x\|} < \frac{x^T z_1}{\|x\|} \leq 1$$

which means $z_2 < z_1$

$$z_2 < z_1$$

$$-z_2 > -z_1$$

inequality laws with multiplying or dividing by negative #s

$$\|-z_2 + x\| > \|-z_1 + x\|$$

$$\|x - z_2\| > \|x - z_1\|$$

$\therefore$ z_1 is also the distance nearest neighbor.