Problem 4.5

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Problem 4.5 (Optimal control of a unit mass)
Consider a unit mass with position $x(t)$ and velocity $\dot{x}(t)$ subject to force $f(t)$, where the force is piecewise constant over intervals of duration one second, i.e., $f(t) = p_n$ for $n-1 < t \leq n$, $n = 1, \cdots, 10$; we consider the system for 10 seconds in total. Ignore friction. Assume the mass has zero initial position and velocity, i.e., $x(0) = \dot{x}(0) = 0$.

(a) Derive the "state-space" equations that describe a discrete-time version of the dynamics of this system. In particular, derive relationships between $x(n)$ and $\dot{x}(n)$ in terms of $x(n-1)$, $\dot{x}(n-1)$, and the driving force p_n, for each $n \in \{1, 2, \ldots, 10\}$. (You will need to use your knowledge of basic physics to determine these relationships.) The two equations you derive should be

$$x(n) = x(n-1) + \dot{x}(n-1) + (1/2)p_n,$$
$$\dot{x}(n) = \dot{x}(n-1) + p_n,$$

which you can then stack up in vector form to form the state-space equations

$$\begin{bmatrix} x(n) \\ \dot{x}(n) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x(n-1) \\ \dot{x}(n-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}_{b} p_n. \tag{1}$$

We note that this continuous time system can only be discretized exactly because in this system the control effort $f(t)$ is held constant over each time interval. (At this point we recommend you remind yourself of the discussion of state-space models of linear dynamical systems in OptM Exercise 3.4.)

(b) Find the p_n, $n \in [10]$ that minimizes

$$\sum_{n=1}^{10} p_n^2$$

subject to the following specifications: $x(10) = 1$, $\dot{x}(10) = 0$. Plot the optimal f, the resulting x and $\dot{x}$. (I.e., plot $f(t)$, $x(t)$, and $\dot{x}(t)$ for $0 \leq t \leq 10$.) Give a short intuitive explanation of what you see.

(c) Suppose that we add one more specification $x(5) = 0$, i.e., we require the mass to be at position 0 at time 5. Plot the optimal f, the resulting x and $\dot{x}$. Give a short intuitive explanation of what you see.

a) $f = ma \qquad m = 1$

$\hookrightarrow \quad a = \frac{f}{m}$

$$\dot{x}(t) = \dot{x}(0) + at = \dot{x}(0) + \frac{f}{m}(t)$$

$$x(t) = x(0) + \dot{x}(0) \cdot t + \frac{1}{2}\left(\frac{f}{m}\right)(t^2)$$

discrete time:

$$x[n+1] = x[n] + \dot{x}[n] + \frac{1}{2}p_n$$

$$\dot{x}[n+1] = \dot{x}[n] + p_n$$

b) From lecture

$$p^* = C^T(CC^T)^{-1}(x_{des} - A^{10}x_{init})$$

$$C = [A^9B \quad A^8B \quad A^7B \quad A^6B \quad A^5B \quad A^4B \quad A^3B \quad A^2B \quad A^1B \quad B]$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ? \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1 \end{bmatrix}$$

x