Correction

December 2, 2023 11:00 PM

4.1

$Y = ax + b \longrightarrow$ solve for a & x

$$\min_x \sum (Y_i - a x_i - b_i)^2 \longleftrightarrow \min_x \Big\| \underbrace{\begin{bmatrix} \vdots \\ y \\ \vdots \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} \vdots & \vdots \\ x & 1 \\ \vdots & \vdots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{X} \Big\|_2^2$$

Based on leasts squares:

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = (A^T A)^{-1} A^T Y$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad X = \begin{bmatrix} a^* \\ b^* \end{bmatrix}$$

$$\begin{bmatrix} a^* \\ b^* \end{bmatrix} = \left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

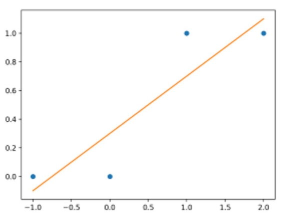
$$= \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \end{bmatrix}$$

$\therefore\ a = \frac{4}{10},\ b = \frac{3}{10}$



4.2

Exercise 6.3 (Lotka's law and least squares) Lotka's law describes the frequency of publication by authors in a given field. It states that $X^a Y = b$, where X is the number of publications, Y the relative frequency of authors with X publications, and a and b are constants

Exercise 6.3 (Lotka's law and least squares) Lotka's law describes the frequency of publication by authors in a given field. It states that $X^a Y = b$, where X is the number of publications, Y the relative frequency of authors with X publications, and a and b are constants (with $b > 0$) that depend on the specific field. Assume that we have data points (X_i, Y_i), $i = 1, \ldots, m$, and seek to estimate the constants a and b.

1. Show how to find the values of a, b according to a linear least-squares criterion. Make sure to define the least-squares problem involved precisely.

2. Is the solution always unique? Formulate a condition on the data points that guarantees unicity.

$$X^a Y = b$$

leasts square: $\min \|\vec{b} - A\vec{x}\|_2^2$

I want $X^a y = b$ in the form $\vec{b} - Ax$ with x relating to a & b

$$\vec{x} = \begin{bmatrix} b \\ X^a \end{bmatrix}$$

$$X^a y = b$$

$$X_2 Y = X_1$$

$$0 = X_1 - X_2 Y \rightarrow \begin{bmatrix} 1 & Y \end{bmatrix} \begin{bmatrix} X_1 \\ -X_2 \end{bmatrix}$$

$$\min \left\| \begin{bmatrix} 1 & Y \end{bmatrix} \begin{bmatrix} X_1 \\ -X_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_2^2$$

$$x^* = (A^T A)^{-1} A^T b$$

$$= \left(\begin{bmatrix} 1 & Y_i \\ \vdots & \vdots \\ 1 & Y_m \end{bmatrix}^T \begin{bmatrix} 1 & Y_i \\ \vdots & \vdots \\ 1 & Y_m \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & Y_i \\ \vdots & \vdots \\ 1 & Y_m \end{bmatrix}^T \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

2.

4.3

1. $$y = \frac{\beta_1 x}{\beta_2 + x}$$

$$\frac{1}{y} = \frac{\beta_2 + x}{\beta_1 x} = \frac{\frac{1}{x}(\beta_2 + x)}{\frac{1}{x}\beta_1 x} = \frac{\beta_2\left(\frac{1}{x}\right) + 1}{\beta_1}$$

$$\frac{1}{y} = \frac{\beta_2}{\beta_1}\left(\frac{1}{x}\right) + \frac{1}{\beta_1}$$

2. $$\min \sum_{i=1}^{m} \left(\frac{1}{y} - \frac{\beta_2}{\beta_1}\frac{1}{x} - \frac{1}{\beta_1} \right)^2$$

$$\min \left\| \underbrace{\begin{bmatrix} \frac{1}{Y_1} \\ \vdots \\ \frac{1}{Y_m} \end{bmatrix}}_{b} - \underbrace{\begin{bmatrix} \frac{1}{x} & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{\beta_2}{\beta_1} \\ \beta_1 \end{bmatrix}}_{x} \right\|_2^2$$

$$\Big\| \underbrace{\begin{bmatrix} \frac{1}{y_m} \end{bmatrix}}_{b} \; \underbrace{\begin{bmatrix} \cdots \end{bmatrix}}_{A} \; \underbrace{\begin{bmatrix} \beta_1 \end{bmatrix}}_{x} \Big\|_2$$

$$x^* = \begin{bmatrix} \frac{\beta_2}{\beta_1} \\ \frac{1}{\beta_1} \end{bmatrix} = \left(\begin{bmatrix} \frac{1}{x} & 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{x} & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{x} & 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{y_1} \\ \vdots \\ \frac{1}{y_m} \end{bmatrix}$$

3. If y or x are close to zero, then you'll get really weird results since its dividing by $\simeq 0$

4.4

$x^* = A^+ y = V_r \Sigma^{-1} U_r^T = (A^T A)^{-1} A^T$

must satisfy $A^T A x^* = A^T y$

$A^+ = (A^T A)^{-1} A^T$

$A^T A (A^T A)^{-1} A^T y$

$= I A^T y$

$= A^T y$

$(U_r \Sigma V_r^T)^T \cancel{(U_r \Sigma V_r^T)} \cancel{(V_r \Sigma^{-1} U_r^T)} y$

$= (U_r \Sigma V_r^T)^T y$

$= A^T y$

b) i) $x^* = A^+ y = (V_r \Sigma^{-1} U_r^T) y = (AA^T) A^T$

Since A^+ is based on A

$A^+ \subset R(A)$

Since $A^+ \subset R(A)$,

$A^+ y \subset R(A)$

$x^* \subset R(A)$ always

ii) Does x^* satisfy $Ax^* = y$

$Ax^* = A(AA^T)^{-1} A^T y$

$= AA^T (AA^T)^{-1} y$

$= y$

this depends on if the matrix A is full rank, or else you cannot simply commute A^T to the other side

c) $x^* = A^+ y$ where $\text{Rank}(A) = r < m < n$

4.5

4.5

a) $f = ma \qquad m=1$

$\hookrightarrow a = \frac{f}{m}$

$$\dot{x}(t) = \dot{x}(0) + at = \dot{x}(0) + \frac{f}{m}(t)$$

$$x(t) = x(0) + \dot{x}(0)\cdot t + \frac{1}{2}\left(\frac{f}{m}\right)(t^2)$$

Discrete time:

$$x[n+1] = x[n] + \dot{x}[n] + \frac{1}{2}p_n$$

$$\dot{x}[n+1] = \dot{x}[n] + p_n$$

b) From lecture

$$p^* = C^T(CC^T)^{-1}(x_{des} - A^{10}x_{init})$$

$$C = [A^9B \quad A^8B \quad A^7B \quad A^6B \quad A^5B \quad A^4B \quad A^3B \quad A^2B^6 \quad A^1B \quad B]$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1 \end{bmatrix}$$

Missing Code