Problem 1.4

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Exercise 2.5 (Orthogonality) Let $x, y \in \mathbb{R}^n$ be two unit-norm vectors, that is, such that $\|x\|_2 = \|y\|_2 = 1$. Show that the vectors $x - y$ and $x + y$ are orthogonal. Use this to find an orthogonal basis for the subspace spanned by x and y.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$x - y = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix} \qquad x + y = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Orthogonal:

$$\langle u, v \rangle = 0$$

$$\langle x - y, x + y \rangle = \begin{bmatrix} x_1 - y_1 & \cdots & x_n - y_n \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$= (x_1 - y_1)(x_1 + y_1) + (x_2 - y_2)(x_2 - y_2) + \ldots + (x_n - y_n)(x_n + y_n)$$

Difference of Squares

$$= (x_1^2 - y_2^2) + (x_2^2 - y_2^2) + \ldots + (x_n^2 - y_n^2)$$

$$= x_1^2 + x_2^2 + x_3^2 \ldots + x_n^2 - (y_1^2 + y_2^2 + y_3^2 \ldots + y_n^2)$$

From Problem Statement:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = 1 \qquad \|y\|_2 = \sqrt{\sum_{i=1}^{n} y_i^2} = 1$$

$$\therefore \sum_{i=1}^{n} x_i^2 = 1 \qquad \therefore \sum_{i=1}^{n} y_i^2 = 1$$

$$= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2$$

$$= 1 - 1$$

$$= 0$$

$\therefore$ $\langle x - y, x + y \rangle = 0$ and $\overrightarrow{x-y}$ and $\overrightarrow{x+y}$ are orthogonal vectors to each other.

Basis definition:

1. Two orthogonal vectors ✓
2. Span entire subspace

$\text{span}\{x, y\} = a_1\vec{x} + a_2\vec{y}$ $\forall$ possible a_1 & a_2 values

$$\text{span}\{x-y, x+y\} = b_1\overrightarrow{(x-y)} + b_2(x+y) \quad \forall \text{ possible } b_1 \,\&\, b_2$$

$$= b_1\vec{x} - b_1\vec{y} + b_2\vec{x} + b_2\vec{y}$$

$$= b_1\vec{x} + b_2\vec{x} - b_1\vec{y} + b_2\vec{y}$$

$$= (b_1 + b_2)\vec{x} + (b_2 - b_1)\vec{y}$$

let $b_1 + b_2 = a_1$, $b_2 - b_1 = a_2$

$$= a_1\vec{x} - a_2\vec{y}$$

$$= \text{Span}\{x, y\}$$

$\therefore$ $\overrightarrow{x-y}$ & $\overrightarrow{x+y}$ span the entire subspace spanned by x and y. Therefore $\overrightarrow{x-y}$ & $\overrightarrow{x+y}$ are a basis that spans this subspace