DATA130062: Homework 3

Due via eLearning at 23:59 on October 22, 2024

1. Rizzo book (2nd edition) Exercises 6.12, 6.13, 6.14.

2. Given two random variables X and Y, prove the law of total variance

$$\mathrm{var}(Y) = \mathrm{E}\{\mathrm{var}(Y \mid X)\} + \mathrm{var}\{\mathrm{E}(Y \mid X)\}.$$

3. Define $\theta = \int_A g(x)\,dx$, where A is a bounded set and $g \in \mathcal{L}_2(A)$. Let f be an importance function which is a density function supported on the set A.

   (a) Show that the variance of the importance sampling estimator $\hat{\theta}_n$ is

$$\mathrm{var}(\hat{\theta}_n) = \frac{1}{n}\left\{\int_A \frac{g^2(x)}{f(x)}\,dx - \theta^2\right\}$$

   (b) Show that the *optimal* importance function f^*, i.e., the minimizer of $\mathrm{var}(\hat{\theta}_n)$, is

$$f^*(x) = \frac{|g(x)|}{\int_A |g(x)|\,dx},$$

   and derive the theoretical lower bound of $\mathrm{var}(\hat{\theta}_n)$.

4. Consider estimating the rare event probability $P(Z > 10)$ where $Z \sim N(0, 1)$.

   (a) Apply the simple Monte Carlo method, record your results when the sample size is $10^3, 10^4, \ldots$

   (b) Apply the change of measure using $Y \sim N(10, 1)$, record your results when the sample size is $10^3, 10^4, \ldots$

5. Read Section 6.8 of Rizzo book (2nd edition) on Stratified Importance Sampling. Compare the formulation and the proof of Proposition 6.3 on page 177. Answer the following two questions:

(a) On page 176, the bottom 9th line. The stratified importance sampling estimator is defined as $\hat{\theta}^{SI} = \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}_j$. Is it a typo that there is $1/k$ factor? Compare with the standard stratified sampling estimator discussed in class.

(b) Explain why X^* and Y^* from the two-stage experiment have the same distribution as X and Y/k, respectively.