6.2

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]
x	-5	-4.000	-3.000	-2.000	-1.000	0.0	1.000	2.000	3.000	4.000	5.000
cdfs	0	-0.007	0.002	0.017	0.159	0.5	0.841	0.978	1.001	0.992	0.992
Phi	0	0.000	0.001	0.023	0.159	0.5	0.841	0.977	0.999	1.000	1.000

```
> print(var)
[1] 4.535192e-05
> print(CI)
[1] 0.9865767 1.0129755
```

6.3

```
> print(v_hat)
[1] 3.21047e-07
> print(v_star)
[1] 2.370681e-05
```

? 疑问求解答：现在 $Var(\hat{\theta}) < Var(\theta^*)$

但是根据重要性采样，θ^* 在采样时采用指数分布更接近原积分的分布，为什么误差反而更大？

6.6

$Var(e^u) = Var(e^{1-u})$

$= E(e^{2u}) - [E(e^u)]^2 = \frac{1}{2}(e^2-1) - (e-1)^2 = 0.242$

$Cov(e^u, e^{1-u}) = E(e^u e^{1-u}) - E(e^u)E(e^{1-u}) = e - (e-1)^2 = -0.234$

For Monte Carlo: $Var(\theta_1) = \frac{1}{m} \cdot Var(e^u) = \frac{0.242}{m}$

For Antithetic Variete: $Var(\theta_2) = \frac{1}{m^2} \sum_{i=1}^{\frac{m}{2}} [Var(e^{u_i} + e^{1-u_i})] = \frac{1}{2m} \cdot [Var(e^u) + Var(e^{1-u}) + 2Cov(e^u, e^{1-u})]$

$= \frac{1}{2m} \cdot [0.242 \times 2 - 2 \times 0.234]$

$= \frac{0.014}{2m}$

$\therefore \% = \frac{Var(\theta_1) - Var(\theta_2)}{Var(\theta_1)} = 0.942$

6.7

```
> print((v1-v2)/v1)
[1] 0.9693727
```

结果与6.6几乎一致.

6.8. 均匀分布是β对称分布的特例，因而直接证β对称分布情况下的

∵ 对称性

∴ $E[U] = E[1-U] = \frac{1}{2}$

令 $Var(U) = Var(1-U) = \sigma^2$

$E(U^2) = Var(U) + [E(U)]^2 = \sigma^2 + \frac{1}{4}$

而 $E(X) = E(X') = \frac{a}{2}$ $\quad Var(X) = Var(X') = a^2\sigma^2$

$E(X^2) = E(X'^2) = a^2\sigma^2 + \frac{1}{4}a^2$

$\therefore Cov(X, X') = a^2\{E[U(1-U)] - E(U)E(1-U)\}$

$= a^2\left[\frac{1}{2} - (\sigma^2 + \frac{1}{4}) - \frac{1}{4}\right]$

$= -a^2\sigma^2$

$\therefore \rho(X, X') = \frac{Cov(X, X')}{\sqrt{Var(X)Var(X')}} = \frac{-a^2\sigma^2}{a^2\sigma^2} = -1$

6.9 $F(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}$

根据逆变换法: $u = 1 - e^{-\frac{x^2}{2\sigma^2}}$

$\therefore F^{-1}(u) = \sigma \cdot (-2\log(1-u))^{\frac{1}{2}}$

```
> print(var_reduction)
[1] 94.72956
>
```

6.10

```
> print((v1-v2)/v1)
[1] 0.966296
```

6.11

$$\hat{\theta}_c = c\,\hat{\theta}_1 + (1-c)\,\hat{\theta}_2$$

$$\begin{aligned} Var(\hat{\theta}_c) &= c^2\, Var(\hat{\theta}_1) + (1-c)^2\, Var(\hat{\theta}_2) + 2c(1-c)\, Cov(\hat{\theta}_1, \hat{\theta}_2) \\ &= \left[Var(\hat{\theta}_1) + Var(\hat{\theta}_2) - 2Cov(\hat{\theta}_1, \hat{\theta}_2)\right] c^2 - \left[2Var(\hat{\theta}_2) - 2Cov(\hat{\theta}_1, \hat{\theta}_2)\right] c + Var(\hat{\theta}_2) \end{aligned}$$

要使 $Var(\hat{\theta}_c)$ 最小，则 $c^* = \dfrac{Cov(\hat{\theta}_1, \hat{\theta}_2) - Var(\hat{\theta}_2)}{Var(\hat{\theta}_1) + Var(\hat{\theta}_2) - 2Cov(\hat{\theta}_1, \hat{\theta}_2)}$

2/(1) 证明见：

https://en.wikipedia.org/wiki/Volume_of_an_n-ball?spm=5176.28103460.0.0.6eed5d27ErLQ8R#The_volume_is_proportional_to_the_nth_power_of_the_radius

$$V_d(R) = \frac{\pi^{\frac{d}{2}} R^d}{\Gamma\left(\frac{d}{2}+1\right)}$$

(2) 使用MC近似π.

```
> print(rbind(dimensions,ns))
           [,1]  [,2]  [,3] [,4] [,5]    [,6]  [,7] [,8]     [,9]
dimensions    2     3     4    5    6       7     8    9       10
ns         9753 19746 10077  997 6440 6715578 15390 4191 16820432
```