hw07

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1/(a) p 的先验分布：$f(p)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot p^{\alpha-1}\cdot(1-p)^{\beta-1}$

似然函数：$L(p|x_1,\ldots,x_n)=\prod_{i=1}^{n}p(1-p)^{x_i-1}=p^n\cdot(1-p)^{\sum_{i=1}^{n}x_i-n}$

根据贝叶斯定理：$f(p|x_1,\ldots x_n)\propto f(p)\cdot L(p|x_1,\ldots,x_n)$

$\therefore f(p|x_1,\ldots,x_n)\propto p^n\cdot(1-p)^{\sum_{i=1}^{n}x_i-n}\cdot p^{\alpha-1}\cdot(1-p)^{\beta-1}$

$=p^{n+\alpha-1}\cdot(1-p)^{\beta+\sum_{i=1}^{n}x_i-n-1}$

$\sim Beta(\alpha+n,\ \beta+\sum_{i=1}^{n}x_i-n)$

即 p 的后验分布满足 β 分布

(b). σ^2 的先验分布：$f(\sigma^2)=\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\left(\frac{1}{\sigma^2}\right)^{\alpha+1}\cdot e^{-\frac{\beta}{\sigma^2}}$

似然函数：$L(\sigma^2|x_1,\ldots,x_n)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x_i-\mu_0)^2}{2\sigma^2}}$

$=(2\pi\sigma^2)^{-\frac{n}{2}}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}}$

根据贝叶斯定理 $f(\sigma^2|x_1,\ldots,x_n)\propto f(\sigma^2)\cdot L(\sigma^2|x_1,\ldots,x_n)$

$\therefore f(\sigma^2|x_1,\ldots,x_n)\propto\left(\frac{1}{\sigma^2}\right)^{\alpha+1}\cdot e^{-\frac{\beta}{\sigma^2}}\cdot(2\pi\sigma^2)^{-\frac{n}{2}}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}}$

$=\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\alpha+1}\cdot e^{\frac{2\beta+\sum_{i=1}^{n}(x_i-\mu_0)^2}{-2\sigma^2}}$

$\sim IG\left(\frac{n}{2}+\alpha,\ \beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)$

即 σ^2 的后验分布满足 IG 分布

1

2(a). μ的先验概率：$p(\mu) \sim N(\mu_0, \tau^2)$

似然函数：$p(x_1|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}$

∴ 后验分布 $p(\mu|x_1) \propto p(x_1|\mu) \cdot p(\mu)$

$$= e^{-\frac{(x_1-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(\mu-\mu_0)^2}{2\tau^2}}$$

$$= \exp\left\{-\frac{\tau^2+\sigma^2}{2\sigma^2\tau^2} \cdot \left[\mu - \left(\frac{\sigma^2\mu_0+\tau^2x_1}{\sigma^2+\tau^2}\right)^2\right]\right\}$$

$$\sim N\left(\frac{\sigma^2\mu_0+\tau^2x_1}{\sigma^2+\tau^2}, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$$

(b). $x_{new} \sim N(\mu, \sigma^2)$

$E(x_{new}) = E(\mu|x_1) = \mu$

$Var(x_{new}) = Var(\mu|x_1) + \sigma^2 = \sigma_1^2 + \sigma^2$

∴ 后验概率分布为 $N(\mu_1, \sigma_1^2+\sigma^2)$

(c). Frequentist prediction : $N(x_1, \sigma^2)$

Bayesian predictive : $N(\mu_1, \sigma_1^2+\sigma^2) = N\left(\frac{\sigma^2\mu_0+\tau^2x_1}{\sigma^2+\tau^2}, \sigma^2+\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$

在数据量较少或先验信息可靠时，贝叶斯方法通过结合先验知识能够提供更稳健的预测。此外，贝叶斯方法自然地给出了预测的不确定性，这对于决策制定特别有用。当样本量足够大时，贝叶斯预测与基于MLE的预测结果趋于一致，但贝叶斯方法仍然能提供额外的概率解释。

2

2(d)

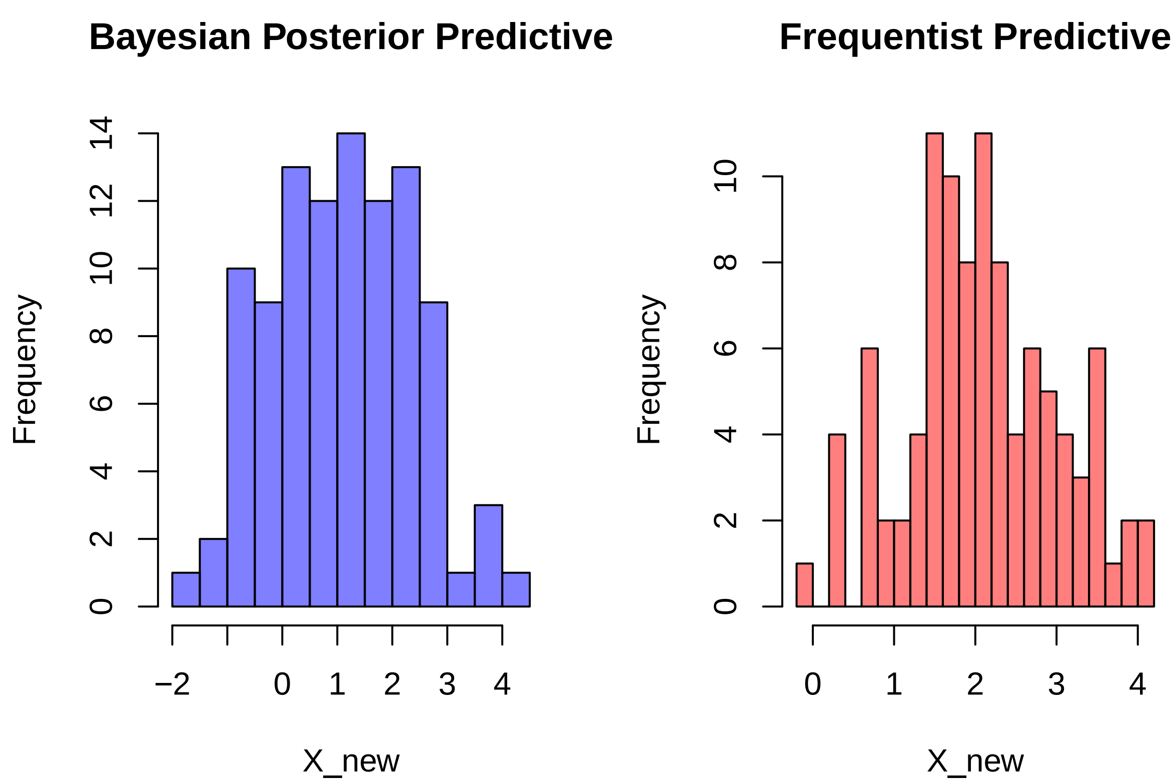
```
sigma2 <- 1   # 方差
mu0 <- 0      # 先验均值
tau2 <- 1     # 先验方差
X1 <- 2       # 观测值

# 后验分布参数
mu1 <- (sigma2 * mu0 + tau2 * X1) / (sigma2 + tau2)
tau12 <- (sigma2 * tau2) / (sigma2 + tau2)
```

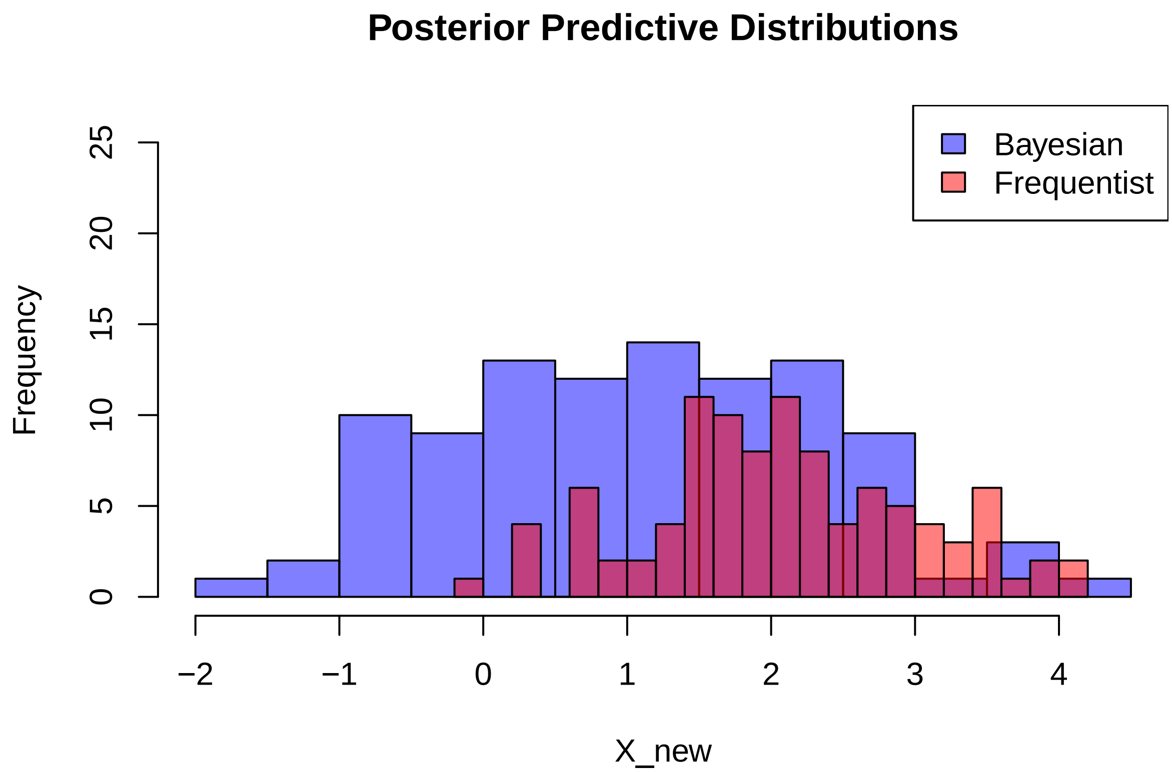
```r
# 贝叶斯后验预测分布
n_samples <- 100
mu_samples <- rnorm(n_samples, mu1, sqrt(tau12))
X_new_bayesian <- rnorm(n_samples, mu_samples, sqrt(sigma2))

# 频率学派预测分布
X_new_frequentist <- rnorm(n_samples, X1, sqrt(sigma2))

# 绘制结果
par(mfrow=c(1,2))
hist(X_new_bayesian, breaks=20, col=rgb(0,0,1,0.5), main="Bayesian Posterior Predictive", xlab="X_new")
hist(X_new_frequentist, breaks=20, col=rgb(1,0,0,0.5), main="Frequentist Predictive", xlab="X_new")
```




```r
# 合并两个直方图
par(mfrow=c(1,1))
hist(X_new_bayesian, breaks=20, col=rgb(0,0,1,0.5), main="Posterior Predictive Distributions", xlab="X_
hist(X_new_frequentist, breaks=20, col=rgb(1,0,0,0.5), add=TRUE)
legend("topright", legend=c("Bayesian", "Frequentist"), fill=c(rgb(0,0,1,0.5), rgb(1,0,0,0.5)))
```



Posterior Predictive Distributions
Bayesian
Frequentist
Frequency
0
5
10
15
20
25
−2
−1
0
1
2
3
4
X_new