6.12

令 $\frac{f(x)}{g(x)}$ 被 B 限制

$$\hat{\theta}_f^{IS} = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)}{f(X_i)}$$

我们要证 $Var(\hat{\theta}_f^{IS})$ 有界，即证 $Var\left(\frac{g(x)}{f(x)}\right)$ 有界.

$$Var\left(\frac{g(x)}{f(x)}\right) = E_f\left(\frac{g(x)}{f(x)}\right)^2 - \left[E_f\left(\frac{g(x)}{f(x)}\right)\right]^2$$

$$= \int_A \frac{g^2(x)}{f^2(x)}\cdot f(x)\,dx - \theta^2$$

$$= \int_A \frac{g^2(x)}{f(x)}\,dx - \theta^2$$

$\because \frac{g(x)}{f(x)} \le B \quad \therefore \frac{g^2(x)}{f(x)} \le B\,g(x)$

$\therefore Var\left(\frac{g(x)}{f(x)}\right) \le \underbrace{B\int_A g(x)\,dx - \theta^2}$ 是一个有界的常数.

$\therefore Var(\hat{\theta}_f^{IS})$ 有界

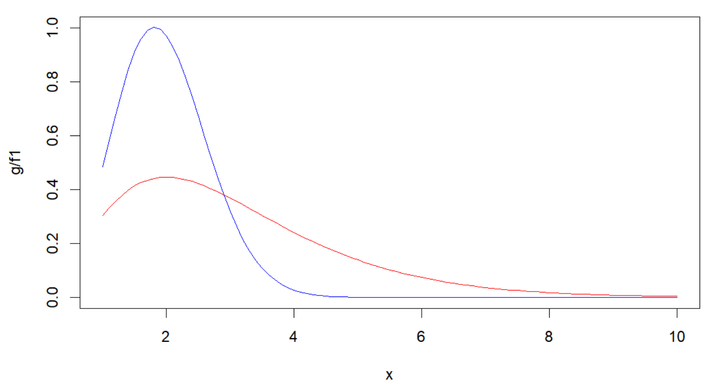
6.13 & 6.14.

取: $f_1(x) = \frac{2}{(x+1)^2}$

$$f_2(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-1)^2}{2}}$$

```r
# 生成x序列
x <- seq(1, 10, by = 0.1)
# 定义函数
g <- x^2 * exp(-x^2 / 2) / sqrt(2 * pi)
f1 <- 2 / (x + 1)^2
f2 <- 2 * dnorm(x, mean = 1, sd = 1)
# 绘制g/f1 和 g/f2 的比值图像
plot(x, g / f1, type = 'l', col = 'blue')
lines(x, g / f2, col = 'red' )

# 初始化
set.seed(123)
m <- 10000
theta.hat <- variance <- numeric(2)
# 定义g函数
g <- function(x) {
  x^2 * exp(-x^2 / 2) / sqrt(2 * pi) * (x > 1)
}
# 使用f1进行估计
u <- runif(m)
x <- 2 / (1 - u) - 1  # 逆变换法
fg <- g(x) / (2 / (x + 1)^2)
theta.hat[1] <- mean(fg)
variance[1] <- var(fg)
# 使用f2进行估计
x <- numeric(m)
i <- 1
while (i <= m) {
  temp <- rnorm(1, mean = 1, sd = 1)
  if (temp > 1) {
    x[i] <- temp
    i <- i + 1
  }
}
fg <- g(x) / (2 * dnorm(x, mean = 1, sd = 1))
theta.hat[2] <- mean(fg)
variance[2] <- var(fg)
# 结果汇总
t <- rbind(theta.hat, variance)
colnames(t) <- c('f1', 'f2')
print(t)
```



可以看出 $\frac{g(x)}{f_2(x)}$ 更接近一个常量，更为平坦，因而选 $f_2(x)$

下面结果也表明用 f_2 采样方差会更小.

```
                 f1          f2
theta.hat 0.4033674 0.400543024
variance  0.1571396 0.001973264
```

E2. 根据条件方差：$Var(Y|X) = E[(Y-E(Y|X))^2|X]$

$$= E(Y^2|X) - [E(Y|X)]^2$$

$$\therefore E(Y^2|X) = Var(Y|X) + [E(Y|X)]^2$$

两边取期望，$E(Y^2) = E[Var(Y|X)] + E[E(Y|X)^2]$ ①

与此同时 $Var(E(Y|X)) = E[[E(Y|X)]^2] - [E(E(Y|X))]^2$

$$= E[[E(Y|X)]^2] - [E(Y)]^2 \quad ②$$

$$\because Var(Y) = E(Y^2) - [E(Y)]^2$$

将①、②中的 $E(Y^2)$ 和 $[E(Y)]^2$ 代入

$$\therefore Var(Y) = E(Var(Y|X)) + E[(E(Y|X))^2] - \{E[[E(Y|X)]^2] - Var(E(Y|X))\}$$

$$= E(Var(Y|X)) + Var(E(Y|X))$$

E3.

(a). $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}\frac{g(x_i)}{f(x_i)}$

$$Var(\hat{\theta}_n) = E[\hat{\theta}_n^2] - E(\hat{\theta}_n)^2 = E[\hat{\theta}_n^2] - \theta^2 \quad ①$$

$$E(\hat{\theta}_n^2) = E\left(\left(\frac{1}{n}\sum_{i=1}^{n}\frac{g(x_i)}{f(x_i)}\right)^2\right) = \frac{1}{n^2}\cdot E\left(\sum_{i=1}^{n}\frac{g^2(x_i)}{f^2(x_i)} + 2\sum_{i\neq j}\frac{g(x_i)g(x_j)}{f(x_i)g(x_j)}\right)$$

因为 x_i 相互独立 $\therefore$ 对于 $i\neq j$ 的部分，其期望值为 θ^2

$$\therefore E(\hat{\theta}_n^2) = \frac{1}{n^2}\left[\sum_{i=1}^{n}E\left(\frac{g^2(x_i)}{f^2(x_i)}\right) + 2\cdot\frac{n(n-1)}{2}\cdot\theta^2\right]$$

$$= \frac{1}{n^2}\cdot\left[\int_A \frac{g^2(x)}{f^2(x)}\cdot f(x)\,dx\cdot n + n(n-1)\theta^2\right]$$

$$= \frac{1}{n}\cdot\left[\int_A \frac{g^2(x)}{f(x)}\,dx + (n-1)\theta^2\right]$$ ，代入 ①

$$\therefore Var(\hat{\theta}_n) = \frac{1}{n}\left\{\int_A \frac{g^2(x)}{f(x)}\,dx - \theta^2\right\}$$

(b). 要使 $\mathrm{Var}(\hat{\theta}_n)$ 最小，即最小化 $\int_A \frac{g^2(x)}{f(x)}\,dx$，变量为 $f(x)$

构造一个 Lagrange 函数. $L(f) = \int_A \frac{g^2(x)}{f(x)}\,dx + \lambda\left(\int_A f(x)\,dx - 1\right)$

对 $f(x)$ 求偏导 $\frac{\partial L(f)}{\partial f(x)} = -\frac{g^2(x)}{f(x)^2} + \lambda = 0$

$$\therefore -\frac{g^2(x)}{f(x)^2} = -\lambda$$

$$\therefore f^*(x) = \sqrt{\frac{g^2(x)}{\lambda}}$$

又 $\because \int_A f(x)\,dx = 1$

即 $f^*(x) = \frac{|g(x)|}{\sqrt{\lambda}}$

$$\therefore \int_A \frac{|g(x)|}{\sqrt{\lambda}}\,dx = 1$$

$$\therefore \sqrt{\lambda} = \int_A |g(x)|\,dx$$

代入

$$\therefore f^*(x) = \frac{|g(x)|}{\int_A |g(x)|\,dx}$$

E4.

```r
# 定义函数来计算概率
estimate_p <- function(n, method="normal") {
  if(method == "normal") {
    # 使用标准正态分布
    z <- rnorm(n)
    sum(z > 10)/n
  } else if(method == "change_of_measure") {
    # 使用改变度量的方法
    y <- rnorm(n, mean=10, sd=1)
    sum(y > 10)/n
  }
}

# 蒙特卡洛模拟
for(i in c(3, 4)) {
  print(paste("Sample size:", 10^i))

  for(j in c(1e3, 1e4, 1e5, 1e6)) {
    estimate <- replicate(100, estimate_p(j, method="normal"))
    print(paste("Normal distribution:", round(mean(estimate), 15)))

    estimate <- replicate(100, estimate_p(j, method="change_of_measure"))
    print(paste("Change of measure:", round(mean(estimate), 15)))
  }
}
```

25:2 (Top Level)

Console Terminal Background Jobs

R 4.4.1 · ~/

```
1] "Sample size: 1000"
1] "Normal distribution: 0"
1] "Change of measure: 0.49917"
1] "Normal distribution: 0"
1] "Change of measure: 0.500389"
1] "Normal distribution: 0"
1] "Change of measure: 0.4999983"
1] "Normal distribution: 0"
1] "Change of measure: 0.4999683"
1] "Sample size: 10000"
1] "Normal distribution: 0"
1] "Change of measure: 0.50102"
1] "Normal distribution: 0"
1] "Change of measure: 0.500203"
1] "Normal distribution: 0"
1] "Change of measure: 0.4999166"
1] "Normal distribution: 0"
1] "Change of measure: 0.50007619"
```

E5 (a) $\sigma^2 - k\sum_{j=1}^{k}\sigma_j^2 \geq 0$，当且仅当 $\theta_1 = \dots = \theta_k$

相比于不分层，分层具有更小方差，因而包含 $\frac{1}{k}$ 因子是合理的

(b) 在两阶段实验中，首先抽一个编号 J ($J \in [1,k]$)，在 $J=j$ 的条件下，在 $f_j(x)$ 中生成随机变量 X^*，$Y^* = \frac{g_j(x)}{k f_j(x)}$

$f_j(x) = \frac{f(x)}{k}$ 在子区间 I_j 内，有：

$$Y^* = \frac{g_j(x)}{k \cdot f_j(x)} = \frac{k}{k} \cdot \frac{g_j(x)}{f(x)} = \frac{g(x)}{f(x)}$$

$\therefore$ Y^* 与 Y/k 有相同分布

同样，X^* 是从 I_j 上的 $f_j(x)$ 生成，与 X 也有相同分布