Section 48: Automorphisms of Fields

Def: Let E be an algebraic extension of a field F. Two elements $\alpha, \beta \in E$ are *conjugate* over F if $irr(\alpha, F) = irr(\beta, F)$, that is, if α and β are zeros of the same irreducible polynomial over F.

Example: Conjugate complex numbers $a+bi$ and $a-bi$ are roots of the same polynomial.

Conjugation Isomorphisms: Let F be a field, and let α and β be algebraic over F with $deg(\alpha, F) = n$. The map $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ defined by

$$\psi_{\alpha,\beta}(c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1}) = c_0 + c_1\beta + \cdots + c_{n-1}\beta^{n-1}$$

for $c_i \in F$ is an isomorphism of $F(\alpha)$ onto $F(\beta)$ if and only if α and β are conjugate over F.

Corollary: Let α be algebraic over a field F. Every isomorphism ψ mapping $F(\alpha)$ onto a subfield of $\bar{F}$ such that $\psi(a) = a$ for $a \in F$ maps α onto a conjugate β of α over F. Conversely, for each conjugate β of α over F, there exists exactly one isomorphism $\psi_{\alpha,\beta}$ of $F(\alpha)$ onto a subfield of $\bar{F}$ mapping α onto β and mapping each $a \in F$ onto itself.

Corollary: Let $f(x) \in \mathbb{R}[x]$. If $f(a+bi) = 0$ for $a+bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$, then $f(a-bi) = 0$ also.

Def: An isomorphism of a field onto itself is an *automorphism* of the field.

Def: If σ is an isomorphism of a field E onto some field, then an element a of E is *left fixed* by σ if $\sigma(a) = a$. A collection S of isomorphisms of E *leaves* a subfield F of E *fixed* if each $a \in F$ is left fixed by every $\sigma \in S$. If $\{\sigma\}$ leaves F fixed, then σ leaves F fixed.

Thm. Let $\{\sigma_i | i \in I\}$ be a collection of automorphisms of a field E. Then the set $E_{\{\sigma_i\}}$ of all $a \in E$ left fixed by every σ_i for $i \in I$ forms a subfield of E.

Def: The field $E_{\{\sigma_i\}}$ is the *fixed field* of $\{\sigma_i | i \in I\}$. For a single automorphism σ, we shall refer to $E_{\{\sigma\}}$ as the *fixed subfield* of σ.

Thm. The set of all automorphisms of a field E is a group under function composition.

Note: These automorphisms are basically permutation groups of the field.

Thm. Let E be a field and let F be a subfield of E. Then the set $G(E/F)$ of all automorphisms of E leaving F fixed forms a subgroup of the group of all automorphisms of E. Furthermore, $F \leq E_{G(E/F)}$.

Def: The group $G(E/F)$ of the preceding theorem is the *group of automorphisms* of E leaving F fixed, or, the *group of* E *over* F.

Note: The notation $G(E/F)$ is a little misleading since it is not useful to think of this group as a quotient space. Instead think of it as referring to that E is an extension field of F.

Thm: Let F be a finite field of characteristic p. Then the map $\sigma_p : F \to F$ via $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism called the *Frobenius automorphism* of F. Also, $F_{\{\sigma_p\}} \simeq \mathbb{Z}_p$.

Note: The Frobenius automorphism is important because it is the generator of the group of automorphisms on a field.

Selected Exercises:

39a. Prove that and automorphism of a field E carries elements that are squares of elements in E onto elements that are squares that are elements of E.

39b. Prove that an automorphism of the field $\mathbb{R}$ carries positive numbers onto positive numbers.

39c. Prove that if σ is an automorphism of $\mathbb{R}$ and $a < b$, where $a, b \in \mathbb{R}$, then $\sigma(a) < \sigma(b)$.

39d. Finally, prove that the only automorphism of $\mathbb{R}$ is the identity automorphism.