Section 29: Introduction to Extension Fields

Goal: The results in this section will allow us to prove that every nonconstant polynomial has a zero.

Def: A field E is an *extension field* of a field F if $F \leq E$.

Kronecker's Theorem: Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then there exists an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$.

Proof:
$f(x)$ has a factorization in $F[x]$ into polynomials that are irreducible over F. Let $p(x)$ be an irreducible polynomial in such a factorization. It is E.T.S that there exists an extension field of F that contains a root of $p(x)$. We can get our required extension field by noting that, since $\langle p(x) \rangle$ is a maximal ideal in $F[x]$, $F[x]/\langle p(x) \rangle$ must be a field. To show that F is a subfield of $E = F[x]/\langle p(x) \rangle$, we construct the following map, $\psi : F \to E$ via

$$\psi(a) = a + \langle p(x) \rangle$$

for $a \in F$. ψ is 1-1 since $\psi(a) = \psi(b)$ implies that $a - b \in \langle p(x) \rangle$ and since $a, b \in F$, $a - b = 0$ and $a = b$.