Section 50: Splitting Fields

Intro: If an algebraic extension E of a filed F is such that all its isomorphic mappings onto a subfield $\bar{F}$ leaving F fixed are actually automorphisms of E, then for every $\alpha \in E$, all conjugates of α over F must be in E also.

Def: Let F be a field with algebraic closure $\bar{F}$. Let $\{f_i(x)|i \in I\}$ be a collection of polynomials in $F[x]$. A field $E \leq \bar{F}$ is the *splitting field* of $\{f_i(x)|i \in I\}$ over F if E is the smallest subfield of $\bar{F}$ containing F and all the zeros in $\bar{F}$ of each of the $f_i(x)$. A field $K \leq \bar{F}$ is a *splitting field over* F if it is the splitting field of some set of polynomials in $F[x]$.

Thm. A field E, where $F \leq E \leq \bar{F}$, is a splitting field over F if and only if every automorphism of $\bar{F}$ leaving F fixed maps E onto itself and thus induces an automorphism of E leaving E fixed.

Def: Let E be an extension field of a field F. A polynomial $f(x) \in F[x]$ *splits* in E if it factors into a polynomial of linear factors in $E[x]$.

Corollary: If $E \leq \bar{F}$ is a splitting field over F, then every irreducible polynomial in $F[x]$ having a zero in E splits in E.

Corollary: If $E \leq \bar{F}$ is a splitting field over F, then every isomorphic mapping of E onto a subfield of $\bar{F}$ and leaving F fixed is actually an automorphism of E. In particular, if E is a splitting field of finite degree over F, then

$$\{E : F\} = |G(E/F)|$$

Selected Exercises

17. Show that if a finite extension E of a field F is a splitting field over F, then E is a splitting field of one polynomial in $F[x]$.

18. Show that if $[E : F] = 2$, then E is a splitting field over F.

20. Show that $\mathbb{Q}(\sqrt[3]{2})$ has only the identity automorphism.