Algebraic Extensions

Def: An extension field E of a field F is an *algebraic extension* of F if very element if algebraic over F.

Def: If an extension field E of a field F is of finite dimension n as a vector space over F, then E is a *finite extension* of degree n over F. We shall denote $[E : F]$ to be the degree n of E over F.

Note: $[E : F] = 1$ if and only if $E = F$.

Thm. A finite extension field E of a field F is an algebraic extension of F.

Thm. If E is a finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and

$$[K : F] = [K : E][E : F]$$

Note: If if $\{\alpha_i | i = 1, \ldots, n\}$ is a basis for E over F and $\{\beta_j | j = 1, \ldots, m\}$ is a basis for K over E, then the set $\{\alpha_i \beta_i\}$ of mn products is a basis for K over F.

Corollary: If F_i is a field for $i = 1, \ldots, r$ and F_{i+1} is a finite extension of F_i, then F_r is a finite extension of F_1, and

$$[F_r : F_1] = [F_r : F_{r-1}][F_{r-1} : F_{r-2}] \ldots [F_2 : F_1]$$

Corollary: If E is an extension field of F, $\alpha \in E$ is algebraic over F, and $\beta \in F(\alpha)$, then $deg(\beta, F)$ divides $deg(\alpha, F)$.

Notation: We denote the adjoining of α_2 to $F(\alpha_1)$, i.e. $(F(\alpha_1))(\alpha_2)$, as $F(\alpha_1, \alpha_2)$.

Thm. Let E be an algebraic extension of a field F. Then there exist a finite number of elements $\alpha_1, \ldots, \alpha_n$ in E such that $E = F(\alpha_1, \ldots, \alpha_n)$ if and only if E is a finite dimensional vector space over F, that is, if and only if E is a finite extension of F.

Thm. Let E be an extension field of F. Then

$$\bar{F}_E = \{\alpha$$