Section 49: The Isomorphic Extension Theorem

Isomorphism Extension Theorem: Let E be an algebraic extension of a field F. Let σ be an isomorphism of F onto a field F'. Let $\bar{F}'$ be an algebraic closure of F'. Then σ can be extended to an isomorphism τ of E onto a subfield of $\bar{F}'$ such that $\tau(a) = \sigma(a)$ for all $a \in F$.

Note: This theorem allows us to relate extensions of isomorphic fields.

Corollary: If $E \leq \bar{F}$ is an algebraic extension of F and $\alpha, \beta \in E$ are conjugate over F, then the conjugation isomorphism $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$, can be extended to an isomorphism of E onto a subfield of $\bar{F}$.

Corollary: Let $\bar{F}$ and $\bar{F}'$ be two algebraic closures of F. Then $\bar{F}$ is isomorphic to $\bar{F}'$ under an isomorphism leaving each element of F fixed.

Note: Algebraic closure of a field is unique up to isomorphism.

Thm. Let E be a finite extension of a field F. Let σ be an isomorphism of F onto a field F', and let $\bar{F}'$ be an algebraic closure of F'. Then the nnumber of extensions of σ to an isomorphism τ of E onto a subfield of $\bar{F}'$ is finite, and independent of F', $\bar{F}'$, and σ. That is, the number of extensions is completely determined by the fields E and F.

Def: Let E be a finite extension of a field F. The number of isomorphisms of E onto a subfield of $\bar{F}$ leaving F fixed is the *index* $\{E : F\}$ of E over F.

Corollary: If $F \leq E \leq K$, where K is a finite extension field of the field F, then $\{K : F\} = \{K : E\}\{E : F\}$.

Selected Exercises

10. Let E be an algebraic extension of a field F. Show that every isomorphism of E onto a subfield of $\bar{F}$ leaving F fixed can be extended to an automorphism of $\bar{F}$.

11. Prove that if E is an algebraic extension of a field F, then two algebraic closures $\bar{F}$ and $\bar{E}$ of F and E, respectively, are isomorphic.

12. Prove that the algebraic closure of $\mathbb{Q}(\sqrt{\pi})$ in $\mathbb{C}$ is isomorphic to any algebraic closure of $\bar{\mathbb{Q}}(x)$, where $\bar{\mathbb{Q}}$ is the field of algebraic numbers and x is an indeterminate.