Section 26: Homomorphisms and Factor Rings

Def: A map ϕ of a ring R into a ring R' is a *homomorphism* if

$$\phi(a+b) = \phi(a) + \phi(b)$$

and

$$\phi(ab) = \phi(a)\phi(b)$$

for all elements $a, b \in R$.

Example: *The Projection Homomorphisms:* Let $R_1, R_2, \ldots, R_n$ be rings. For each i, the map $\pi_i : R_1 \times R_2 \times \cdots \times R_n \to R_i$ defined by $\pi_i(r_1, r_2, \ldots, r_n) = r_i$ is a homomorphism. This is called the "projection onto the ith component of R_i." It is very easy to see that this meets all the requirements of the homomorphism.

A number of key results easily transfer over from group homomorphisms.

Thm: Let ϕ be a homomorphism of a ring R into a ring R'. The following properties hold:

1. If 0 is the additive identity in R, then $\phi(0) = 0'$ is the additive identity in R'.
2. If $a \in R$, then $\phi(-a) = -\phi(a)$.
3. If S is a subring of R, then $\phi(S)$ is a subring of R'.
4. If S' is a subring of R', then $\phi^{-1}(S')$ is a subring of R.
5. If R has unity 1, then $\phi(1)$ is unity for $\phi(R)$.

Def: Let a map $\phi : R \to R'$ be a ring homomorphism. The subring

$$\phi^{-1}(0') = \{r \in R | \phi(r) = 0'\}$$

is the *kernel* of ϕ, denoted $Ker(\phi)$.

Factor(Quotient) Rings: Now that we have transferred what we know about group homomorphisms to ring homomorphisms, we can extend the concept of a factor group to get a factor ring.

Thm: Let $\phi : R \to R'$ be a ring homomorphism with kernel H. Then the additive cosets of H form a ring R/H whose binary operations are defined by

$$(a+H) + (b+H) = (a+b) + Hb$$

, and

$$(a+H)(b+H) = (ab) + H$$

.

Note: The map $\mu : R/H \to \phi(R)$ defined by $\mu(a+H) = \phi(a)$ is an isomorphism.

Example: This theorem allows us to prove that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$.

Thm: Let H be a subring of R. The multiplication of additive cosets of H is well-defined if and only if $ah \in H$ and $hb \in h$ for all $a, b \in R$ and $h \in H$.

Def: An adative subgroup N of a ring R satisfying the properties:

- $aN \subseteq N$

- $Nb \subseteq N$

for all $a, b \in R$ is an *ideal*.

Note: Ideals are the ring equivalent of normal subgroups.

Example: $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$. $n\mathbb{Z}$ is a subring and $s(mn) = (nm)s = n(ms) \in n\mathbb{Z}$ for all $s \in \mathbb{Z}$. What we did was to show that both left and right cosets are a subset of $n\mathbb{Z}$

Def: Let N be an ideal of a ring R. The additive cosets of N form a ring R/N with the binary operations by defined as

$$(a+N) + (b+N) = (a+b) + N$$

and

$$(a+N)(b+N) = ab + N$$

. This ring is the *factor ring* of *quotient ring* of R by N.

Thm: Let N be and ideal of a ring R. Then $\gamma : R \to R/N$ given by $\gamma(y) = x + N$ is a ring homomorphism with $Ker(\gamma) = N$.

Fundamental Homomorphism Theorem: Let $\phi : R \to R'$ be a ring homomorphism with kernel N. Then $\phi(R)$ is a ring, and the map $\mu : R/N \to \phi(R)$ given by $\mu(x+N) = \phi(x)$ is an isomorphism. If $\gamma : R \to R/N$ is the homomorphism given by $\gamma(x) = x + N$, then for each $x \in R$, we have $\phi(x) = \mu(\gamma(x))$.

Note: This theorem is basically saying that we can consider a homomorphism in two pieces. The ring is first mapped to a factor ring and then another mapping goes from the factor ring to $\phi(R)$. The book says it this way, *"every ring homomorphism with domain R gives rise to a factor ring R/N, and every factor ring R/N gives rise to a homomorphism mapping R into R/N."*

Selected Exercises:

1. Describe all ring homomorphisms of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$. Note that if ϕ is such a homomorphism, then $\phi((1,0)) = \phi((1,0))\phi((1,0))$ and $\phi((0,1)) = \phi((0,1))\phi((0,1))$. Consider also $\phi((1,0)(0,1))$.

$\phi(1,0) = (1,0)$ and $\phi(0,1) = (0,1)$ or $(0,0)$