Section 26: Homomorphisms and Factor Rings

Def: A map ϕ of a ring R into a ring R' is a *homomorphism* if

$$\phi(a+b) = \phi(a) + \phi(b)$$

and

$$\phi(ab) = \phi(a)\phi(b)$$

for all elements $a, b \in R$.

Example: *The Projection Homomorphisms:* Let $R_1, R_2, \ldots, R_n$ be rings. For each i, the map $\pi_i : R_1 \times R_2 \times \cdots \times R_n \to R_i$ defined by $\pi_i(r_1, r_2, \ldots, r_n) = r_i$ is a homomorphism. This is called the "projection onto the ith component of R_i." It is very easy to see that this meets all the requirements of the homomorphism.

A number of key results easily transfer over from group homomorphisms.

Thm: Let ϕ be a homomorphism of a ring R into a ring R'. The following properties hold:

1. If 0 is the additive identity in R, then $\phi(0) = 0'$ is the additive identity in R'.

2. If $a \in R$, then $\phi(-a) = -\phi(a)$.

3. If S is a subring of R, then $\phi(S)$ is a subring of R'.

4. If S' is a subring of R', then $\phi^{-1}(S')$ is a subring of R.

5. If R has unity 1, then $\phi(1)$ is unity for $\phi(R)$.