Section 51: Separable Extensions

Def: Let $f(x) \in F[x]$. An element $\alpha \in \bar{F}$ such that $f(\alpha) = 0$ is a *zero of multiplicity* v, if v is the greatest integer such that $(x - \alpha)^v$ is a factor of $f(x)$ in $\bar{F}[x]$.

Thm. Let $f(x)$ be irreducible in $F[x]$. Then all the zeros of $f(x)$ in $\bar{F}$ have the same multiplicity.

Corollary: If $f(x)$ is irreducible in $F[x]$, then $f(x)$ has a factorization in $\bar{F}[x]$ of the form

$$a \prod_i (x - \alpha_i)^v$$

where the α_i are the distinct zeros of $f(x)$ in $\bar{F}$ and $a \in F$.

Note: $\{F(\alpha) : F\}$ is the number of distinct zeros of $irr(\alpha, F)$.

Thm. If E is a finite extension of F, then $\{E : F\}$ divides $[E : F]$.

Def: A finite extension E of F is a *separable extension* of F if $\{E : F\} = [E : F]$. An element α of $\bar{F}$ is *separable* over F if $F(\alpha)$ is a separable extension of F. An irreducible polynomial of $f(x) \in F[x]$ is *separable over* F if every zero of $f(x)$ in $\bar{F}$ is separable over F.

Thm. If K is a finite extension of E and E is a finite extension of F, that is, $F \leq E \leq K$, then K is separable over F if and only if K is separable over E and E is separable over F.

Corollary: If E is a finite extension of F, then E is separable over F if and only if each α in E is separable over F.

Lemma: Let $\bar{F}$ be an algebraic closure of F, and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be any monic polynomial in $\bar{F}[x]$. If $(f(x))^m \in F[x]$ and $m \cdot 1 \neq 0$ in F, then $f(x) \in F[x]$, that is, all $a_i \in F$.

Def: A field is *perfect* if every finite extension is a separable extension.

Thm. Every field of characteristic zero is perfect.

Thm. Every finite field is perfect.

Primitive Element Theorem: Let E be a finite separable extension of a field F. Then there exists $\alpha \in E$ such that $E = F(\alpha)$. α is called a *primitive element.* That is, a finite separable extension of a field is a simple extension.

Corollary: A finite extension of a field of characteristic zero is a simple extension.

Selected Exercises

11. Prove that if E is an algebraic extension of a perfect field F, then E is perfect.

12. A possibly infinite extension algebraic extension E of a field F is a *separable extension* of F if for every αin, $F(\alpha)$ is a separable extension of F, in the sense defined. Show that that if E is a separable extension of F and K is a separable extension of E, then K is a separable extension of F.

13. Let E be an algebraic extension of a field F. Show that the set of all elements in E that are separable over F forms a subfield of E, the *separable closure* of F in E.