Section 27: Prime and Maximal Ideals

Factoids: This section explores the connection that factor rings have to integral domains and to fields. Here are some interesting factoids:

- A factor ring of an integral domain may be a field. Example: $\mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}$.
- A factor ring of a ring may be an integral domain even though the original ring is not. Example: $(\mathbb{Z} \times \mathbb{Z})/N \simeq \mathbb{Z}$ where $N = \{(0, n)|n \in \mathbb{Z}\}$.
- If R is not even an integral domain, it is still possible for R/N to be a field. Example: $\mathbb{Z}_6/\{0, 3\} \simeq \mathbb{Z}_3$.
- A factor ring may also have a worse structure than the original ring. Example: $\mathbb{Z}$ is an integral domain, but $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_6$ is not.

Thm. If R is a ring with unity, and N is an ideal of R containing a unit, then $N = R$.

Corollary: A field contains no proper nontrivial ideals.

Note: This makes the factor rings of a field not very interesting. The factor ring will either be $\{0\}$ or the field itself.

Def: The *maximal ideal* of a ring R is an ideal M different from R such that there is no proper ideal N of R properly containing M.

Thm. Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if R/M is a field.

Proof Sketch: Suppose that M is a maximal ideal in R and that there is an element in R/M that does not have a multiplicative inverse. We can then construct an ideal of R that contains M, contradicting the original assumption. Therefore, every element in R/M needs to have a multiplicative inverse and is thus a field.

Example: $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$. Therefore $\mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}_p$ is a field.

Corollary: A commutative ring with unity is a field if and only if it has no proper nontrivial ideals.

Def: An ideal $N \neq R$ in a commutative ring R is a *prime ideal* if $ab \in N$ implies that either $a \in N$ or $b \in N$ for $a, b \in R$.

Example: $\{0\}$ is a prime ideal in any integral domain.

Note: Prime ideals are based on considering the zero divisors of factor rings. The definition is derived from the fact that

$$(a+N)(b+N)+N \implies a+N=N \text{ or } b+N=N$$

if we want our factor ring to be an integral domain. This is stated in the following theorem.

Thm. Let R be a commutative ring with unity, and let $N \neq R$ be an ideal in R. Then R/N is an integral domain if and only if N is a prime ideal in R.

Corollary: Every maximal ideal in a commutative ring with unity is a prime ideal.

Summary: Maximal and prime ideals are a very important concept to understand going forward. Here is a summary of the major results so far:

1. An ideal M of R is maximal iff R/M is a field.
2. An ideal N of R is prime iff R/N is an integral domain.
3. Every maximal ideal of R is a prime ideal.

Notice that these 3 statements form a hierarchy of ideals. Just like how a field is an integral domain with more requirements, a maximal ideal is a prime ideal with more requirements.