Section 36: Sylow Theorems

Def: Let X be a set and G a group. An *action* of G on X is a map $* : G \times X \to X$ such that

1. $ex = x$ for all $x \in X$.
2. $(g_1 g_2)(x) = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

Under these conditions, X is a *G-set.*

Orbit Equation: The major results in this section come from counting the number of elements in a finite G-set. We can get the size of the set X with the equation

$$|X| = |X_G| + \sum_{i=s+1}^{r} |Gx_i|$$

where $X_G = \{x \in X | gx = x, \forall g \in G\}$ is the union of the one-element orbits in X and the Gx_i's are the orbits with more than one element.

Thm. Let G be a group of order p^n and let X be a finite G-set. Then $|X| \equiv |X_g| \mod p$.

Def: Let p be a prime. A group G is a *p-group* if every element in G has order a power of the prime p. A subgroup of a group G is a *p-subgroup* of G if the subgroup is itself a p-group.

Cauchy's Theorem: Let p be a prime. Let G be a finite group and let p divide $|G|$. Then G has an element of order p and, consequently, a subgroup of order p.

Corollary: Let G be a finite group. Then G is a p-group if and only if $|G|$ is a power of p.

Def: The subgroup $G_H = \{g \in G | gHg^{-1} = H\}$ where H is a subgroup of G is the *normalizer* of H in G and will be denoted $N[H]$.

Lemma: Let H be a p-subgroup of a finite group G. Then

$$(N[H] : H) \equiv (G : H) \mod p$$

Corollary: Let H be a p subgroup of a finite group G. If p divides $(G : H)$, then $N[H] \neq H$.

First Sylow Theorem: Let G be a finite group and let $|G| = p^n m$ where $m \geq 1$ and where p does not divide m, then

1. G contains a subgroup of order p^i for each i where $1 \leq i \leq n$.
2. Every subgroup H of G of order p^i is a normal subgroup of a subgroup of order p^{i+1} for $1 \leq i < n$.

Def: A *Sylow p-subgroup* of a group G is a maximal p-subgroup of G, that is, a p-subgroup contained in no larger p-subgroup.

Second Sylow Theorem: Let P_1 and P_2 be Sylow p-subgroups of a finite group G. Then P_1 and P_2 are conjugate subgroups of G.

Third Sylow Theorem: If G is a finite group and p divides $|G|$, then the number of Sylow p-groups is congruent to 1 modulo p and divides $|G|$.

Selected Exercises:

11. Let H be a subgroup of a group G. Show that $G_H = \{g \in G | gHg^{-1} = H\}$ is a subgroup of G.

14. Prove that if G is a finite group, then G is a p-group if and only if $|G|$ is a power of p.

22. Let G be a finite group and let P be a normal p-subgroup of G. Show that P is contained in every Sylow p-subgroup of G.