Algebraic Extensions

Def: An extension field E of a field F is an *algebraic extension* of F if very element is algebraic over F.

Def: If an extension field E of a field F is of finite dimension n as a vector space over F, then E is a *finite extension* of degree n over F. We shall denote $[E : F]$ to be the degree n of E over F.

Note: $[E : F] = 1$ if and only if $E = F$.

Thm. A finite extension field E of a field F is an algebraic extension of F.

Thm. If E is a finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and

$$[K : F] = [K : E][E : F]$$

Note: If if $\{\alpha_i | i = 1, \dots, n\}$ is a basis for E over F and $\{\beta_j | j = 1, \dots, m\}$ is a basis for K over E, then the set $\{\alpha_i \beta_i\}$ of mn products is a basis for K over F.

Corollary: If F_i is a field for $i = 1, \dots, r$ and F_{i+1} is a finite extension of F_i, then F_r is a finite extension of F_1, and

$$[F_r : F_1] = [F_r : F_{r-1}][F_{r-1} : F_{r-2}] \dots [F_2 : F_1]$$

Corollary: If E is an extension field of F, $\alpha \in E$ is algebraic over F, and $\beta \in F(\alpha)$, then $deg(\beta, F)$ divides $deg(\alpha, F)$.

Notation: We denote the adjoining of α_2 to $F(\alpha_1)$, i.e. $(F(\alpha_1))(\alpha_2)$, as $F(\alpha_1, \alpha_2)$.

Thm. Let E be an algebraic extension of a field F. Then there exist a finite number of elements $\alpha_1, \dots, \alpha_n$ in E such that $E = F(\alpha_1, \dots, \alpha_n)$ if and only if E is a finite dimensional vector space over F, that is, if and only if E is a finite extension of F.

Note: A previous theorem stated that an finite extension of field is always an algebraic extension of the field. This last theorem basically says that not all algebraic extensions need to be finite, but when a finite number of elements of the extension can be adjoined to the original field to get the extension field, then the extension is finite.

Thm. Let E be an extension field of F. Then

$$\bar{F}_E = \{\alpha | \alpha \text{ is algebraic over } F\}$$

is a subfield of E called the *algebraic closure* of F in E.

Corollary: The set of all algebraic numbers forms a field.

Def: A field F is *algebraically closed* if every nonconstant polynomial in $F[x]$ has a zero in F.

Thm. A field F is algebraically closed if and only if every nonconstant polynomial in $F[x]$ factors in $F[x]$ into linear factors or is a linear factor itself.

Corollary: An algebraically closed field F has no proper algebraic extensions, that is, F itself is its only algebraic extension.

Thm. Every field F has an *algebraic closure*, that is, an algebraic extension $\bar{F}$ that is algebraically closed.

Fundamental Theorem of Algebra: The field $\mathbb{C}$ of complex numbers is an algebraically closed field.

Def: A *partial ordering* of a set S is given by a relation $\leq$ defined for certain ordered pairs of elements of S such that the following conditions are satisfied:

1. $a \leq a$ for all $a \in S$.
2. If $a \leq b$ and $b \leq a$, then $a = b$.
3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

Note: In a partially ordered set, not all elements need to be comparable.

Def: A subset T of a a partially ordered set S is a *chain* if every two elements in T are comparable.

Def: An element of a set $u \in S$ is an *upper bound* if $s \leq u$ for all $s \in S$.

Def: An element m of a set S is *maximal* if there is no $s \in S$ such that $m < s$.

Zorn's Lemma If S is a partially ordered set such that every chain in S has an upper bound in S, then S has at least one maximal element.

Note: Zorn's Lemma is equivalent to the Axiom of Choice.

Note: We can use Zorn's Lemma to show that every field has an algebraic closure. We do this by making chains of bigger and bigger extensions of a field until we hit an extension that contains every possible algebraic extension of F. The application of Zorn's Lemma tells us that there is a maximal algebraic extension $\bar{F}$. Since this is a maximal element in terms of Zorn's Lemma, there is no way to go any further and make an algebraic extension of $\bar{F}$.

Selected Exercises:

23. Show that if E is a finite extension field of a field F and $[E : F]$ is a prime number, then E is a simple extension of F and $E = F(\alpha)$ for every $\alpha \in E$ not in F.

32. Let E be an extension field of a field F. Prove that every $\alpha \in E$ that is not in the algebraic closure $\bar{F}_E$ of F in E is transcendental over $\bar{F}_E$.

38. Use Zorn's Lemma to show that every proper ideal of a ring R with unity is contained in some maximal ideal.