Section 30: Vector Spaces

Def: Let F be a field. A *vector space* over F consists of an abelian group V under addition together with an operation of scalar multiplication of each element of V by each element of F on the left, such that for all $a, b \in F$ and $\alpha, \beta \in V$, the following conditions are satisfied:

1. $a\alpha \in V$
2. $a(b\alpha) = (ab)\alpha$
3. $(a+b)\alpha = (a\alpha) + (b\alpha)$
4. $a(\alpha + \beta) = (a\alpha) + (a\beta)$
5. $1\alpha = \alpha$

The elements of V are *vectors* and the elements of F are scalars.

Note: Let E be an extension field over a field F. Then E is vector space over F with the usual addition in E and scalar multiplication is the usual multiplication $\alpha a \in E$ where $\alpha \in E$ and $a \in F$.

Thm. If V is a vector space over F, then $0\alpha = 0, a0 = 0$ and $(-a)\alpha = a(-\alpha) = -(a\alpha)$ for all $a \in F$ and $\alpha \in V$.

Def: Let V be a vector space over F. The vectors in the subset $S = \{\alpha_i | i \in I\}$ of V *span* V if for every $\beta \in V$, we have

$$\beta = a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_n\alpha_{i_n}$$

for some $a_j \in F$ and $\alpha_{i_j} \in S, j = 1, \ldots, n$. A vector $\sum_{j=1}^{n} a_j\alpha_{i_j}$ is a *linear combination* of the α_{i_j}.

Example: Let F be a field and E an extension field of F. Let $\alpha \in E$ be algebraic over F. Then $F(\alpha)$ is a vector space over F and is spanned by vectors of of the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$, where $n = deg(\alpha, F)$.

Def: A vector space V over a field F is *finite dimensional* if there is a finite subset of V whose vectors span V.

Example: If $F \leq E$ and $\alpha \in E$ is algebraic over F, $F(\alpha)$ is a finite dimensional vector space over F. This happens because the degree of the irreducible polynomial of α is finite and therefore the spanning vector will have that number of components.

Def: The vectors in the subset $S = \{\alpha_i | i \in I\}$ of a vector space V over a field F are *linearly independent* over F if, for any distinct vectors $\alpha_{i_j} \in S$, coefficients $a_j \in F$ and $n \in \mathbb{Z}^+$, we have $\sum_{j=1}^{n} a_j\alpha_{i_j} = 0$ in V only if $a_j = 0$ for $j = 1, \ldots, n$. If the vectors are not linearly independent, they are *linearly dependent*.

Note: Another way to think of linear independence is that there is only way to represent the 0 vector. Also, if the vectors are linearly dependent, one will be a linear combination of the remaining vectors.

Example: Back to the $F(\alpha)$ case in previous examples, the vectors look like $v = a_0 + a_1\alpha, \ldots, a_{n-1}\alpha^{n-1}$. Setting $v = 0$,

$$0 = a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} = 0 + 0\alpha + \cdots + 0\alpha^{n-1}$$

Therefore, the α's are linearly independent.

Def: If V is a vector space over a field F, the vectors in a subset $B = \{\beta_i | i \in I\}$ of V form a *basis* for V over F if they span V and are linearly independent.

Lemma: Let V be a vector space over a field F, and let $\alpha \in V$. If α is a linear combination of vectors β_i in V for $i = 1, \ldots, m$ and each β_i is a linear combination of vectors γ_j in V for $j = 1, \ldots, n$, then α is a linear combination of the γ_i.

Thm. In a finite dimensional vector space, every finite set of vectors spanning the space contains a subset that is a basis.

Note: This theorem means that we can always generate a finite dimensional vector space from a finite number of basis vectors. (whoops, got ahead of the text)

Corollary: A finite dimensional vector space has a finite basis.

Thm. Let $S = \{\alpha_1, \ldots, \alpha_r\}$ be a finite set of linearly independent vectors of a finite dimensional vector space V over a field F. Then S can be enlarged to a basis for V over F. Furthermore, if $B = \{\beta_1, \ldots, \beta_n\}$ is any basis for V over F, then $r \leq n$.

Corollary: Any bases of a finite dimensional vector space V over F have the same number of elements.

Def: If V is a finite dimensional vector space over a field F, the number of elements in a basis is the *dimension* of V over F.

Example: Let E be an extension field of F and let $\alpha \in E$ be algebraic over F. If $deg(\alpha, F) = n$, then the dimension of $F(\alpha)$ is n.

Thm. Let E be an extension field of F, and let $\alpha \in E$ be algebraic over F. If $deg(\alpha, F) = n$, then $F(\alpha)$ is an n-dimensional vector space over F with basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$. Furthermore, every element $\beta \in F(\alpha)$ is algebraic over F, and $deg(\beta, F) \leq deg(\alpha, F)$.

Selected Exercises:

23. Prove that every finite dimensional vector space V of dimension n over a field F is isomorphic to the vector space F^n.

24a. Let $\{\beta_i | i \in I\}$ be a basis for V over F, show that a linear transformation $\phi : V \to V'$ is completely determined by the vectors $\phi(\beta_i) \in V'$.

26. Let V be a vector space over a field F, and let S be a subspace of V. Define the *quotient space* V/S, and show that it is a vector space over F.