Calculus for Vector-Valued Functions

Def: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *vector-valued.* Such functions have the form

$$f(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$$

Def: Let $a = (a_1, \dots, a_n)$ and let $b = (b_1, \dots, b_n)$ be two points in $\mathbb{R}^n$. Then the *distance* between a and b, denoted by $|a - b|$, is

$$|a - b| = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}$$

Def: The *length* of a is defined by

$$|a| = \sqrt{a_1^2 + \cdots + a_n^2}$$

Note: The length of a is the same as the distance between a and the origin.

Note: You may have noticed that there is a slight difference between vectors and points. A point is a location in space while a vector can be treated as a set of instructions to get from the origin to its associated point. More generally, remember that vectors in $\mathbb{R}^n$ are a particular case of the mathematical object called a vector. These vectors form a vector space over $\mathbb{R}$ while points do not. To illustrate this, consider the whether it makes any sense to add two points together or to scale a point. All these operations are meaningful in the context of vector spaces, but are meaningless when applied to locations.