

## Real Analysis

**Summary:** The fundamental concepts in calculus are limits, continuity, differentiation, and integration. In turn, all these ideas are built off  $\epsilon$  and  $\delta$  based reasoning. A good foundation in real analysis will cover up to the uniform convergence of functions.

- **Basic Object:** The Real numbers.
- **Basic Maps:** Continuous and Differentiable functions.
- **Basic Goal:** The Fundamental Theorem of Calculus.

**Def:** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a *limit*  $L$  at the point  $a$  if given  $\epsilon > 0$  there is a  $\delta > 0$  such that  $\forall x$  with  $0 < |x - a| < \delta$  we have  $|f(x) - L| < \epsilon$ . This is denoted by

$$\lim_{x \rightarrow a} f(x) = L$$

**Note:** The definition of the limit says that in order for  $f(x)$  to be close to  $L$ ,  $x$  must be close to  $a$ . We can take the required restriction for  $\epsilon$ ,  $|f(x) - L| < \epsilon$ , and find what our  $\delta$  must be starting from  $0 < |x - a| < \delta$ .

**Def:** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *continuous* at  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

**Note:** In order for the limit to exist, the right and left limits must be the same value.

**Def:** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $a$  if given any  $\epsilon > 0$ , there is some  $\delta > 0$  such that  $\forall x$  with  $0 < |x - a| < \delta$ , we have  $|f(x) - f(a)| < \epsilon$ .

**Def:** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $a$  if

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. This limit is called the *derivative* and is denoted by  $f'(a)$  or  $\frac{df}{dx}(a)$ .

**Note:** The slope of a tangent line to a function  $f$  at  $a$  is given by

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Note:** Functions with sharp points such as  $f(x) = |x|$  do not have derivatives at the sharp points.

**Def:** Let  $f(x)$  be a real-valued function defined on the closed interval  $[a, b]$ . For each positive integer  $n$ , let the lower sum of  $f(x)$  be

$$L(f, n) = \sum_{k=1}^n f(l_k) \Delta t$$

and the upper sum be

$$U(f, n) = \sum_{k=1}^n f(u_k) \Delta t$$

**Note:** the lower sum is the sum of the areas of the rectangles below our curve while the upper sum is the sum of the areas of the rectangles below our curve.

**Def:** A real-valued function  $f(x)$  with the domain  $[a, b]$  is said to be integrable if the following condition holds:

$$\lim_{n \rightarrow \infty} L(f, n) = \lim_{n \rightarrow \infty} U(f, n)$$

If these limits exist and are equal, we denote the quantity (both limits are the same real number) by

$$\int_a^b f(x) dx$$

and call it the *integral* of  $f(x)$ .

**Fundamental Theorem of Calculus** Let  $f(x)$  be a real-valued continuous function defined on the closed interval  $[a, b]$  and define

$$F(x) = \int_a^x f(t) dt$$

Then:

1. The function  $F(x)$  is differentiable and

$$\frac{dF(x)}{dx} = \frac{d \int_a^x f(t) dt}{dx} = f(x)$$

2. If  $G(x)$  is a real-valued differentiable function defined on the closed interval  $[a, b]$  whose derivative is

$$\frac{dG(x)}{dx} = f(x)$$

then

$$\int_a^b f(x) dx = G(b) - G(a)$$

**Note:** This theorem essentially states that the derivative and the integral are opposites of each other.

**Def:** Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of functions

$$f_1(x), f_2(x), f_3(x), \dots$$

defined on an interval  $[a, b] = \{x | a \leq x \leq b\}$ . This sequence, denoted  $\{f_n(x)\}$  will *converge pointwise* to a function  $f(x) : [a, b] \rightarrow \mathbb{R}$  if  $\forall \alpha \in [a, b]$ ,

$$\lim_{n \rightarrow \infty} f_n(\alpha) = f(\alpha)$$

**Note:** In  $\epsilon$  and  $\delta$  notation, we would say that  $\{f_n(x)\}$  converges pointwise to  $f(x)$  if  $\forall \alpha \in [a, b]$  and given any  $\epsilon > 0$ , there is a positive integer  $N$  such that  $\forall n \geq N$ , we have  $|f(\alpha) - f_n(\alpha)| < \epsilon$ .

**Note:** It is important to note that the pointwise limit of a sequence of continuous functions need not be continuous. What this means is that a bunch of continuous functions can actually converge to a discontinuous function.

**Def:** A sequence of functions  $f_n : [a, b] \rightarrow \mathbb{R}$  will converge *uniformly* to a function  $f : [a, b] \rightarrow \mathbb{R}$  if given any  $\epsilon > 0$ ,  $\exists N > 0$  such that  $\forall n \geq N$ , we have

$$|f(x) - f_n(x)| < \epsilon$$

for all points  $x$ .

**Note:** Uniform convergence is stronger than pointwise convergence since the sequence will have to uniformly converge for  $N$  for all points  $x$ . A common way to phrase uniform convergence is to say that the sequence converges independently of  $x$ .

**Thm.** Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of continuous functions converging uniformly to a function  $f(x)$ . Then  $f(x)$  will be continuous.

**Def:** Let  $f_1(x), f_2(x), \dots$  be a sequence of functions. The series of functions

$$f_1(x) + f_2(x) + \dots = \sum_{k=1}^{\infty} f_k(x)$$

converges uniformly to a function  $f(x)$  if the sequence of partial sums  $f_1(x), f_1(x) + f_2(x), \dots$  converges uniformly to  $f(x)$ .

**Note:** What this theorem allows us to do is to look at the sum up to a certain number and figure out if the series converges when  $n \rightarrow \infty$ .

**Thm.** If each function  $f_k(x)$  is continuous and if  $\sum_{k=1}^{\infty} f_k(x)$  converges uniformly to  $f(x)$ , then  $f(x)$  must be continuous.

**Note:** This last result allows us to express any continuous function in terms of a series of continuous functions. You will be very thankful for this result when you are working with Holomorphic functions in Complex Analysis and even when solving differential equations.

**Def:** A series converges *absolutely* if

$$\sum_{k=1}^{\infty} |f_k(x)|$$

converges uniformly.

**The Weierstrass M-Test:** Let  $\sum_{k=1}^{\infty} f_k(x)$  be series of functions, with each function  $f_k(x)$  defined on a subset  $A$  of the real numbers. Suppose  $\sum_{k=1}^{\infty} M_k$  is a series of numbers such that:

1.  $0 \leq |f_k(x)| \leq M_k$  for all  $x \in A$ .
2. The series  $\sum_{k=1}^{\infty} M_k$  converges.

Then  $\sum_{k=1}^{\infty} f_k(x)$  converges uniformly and absolutely.

**Note:** The Weierstrass M-Test is a very useful tool for determining uniform convergence using basic calculus tools for testing a series convergence. The basic idea is that if the absolute value of each term in the series can be bounded by a number and if the series of these numbers converges, then the original series will converge uniformly.

### Exercises

1. Let  $f(x)$  and  $g(x)$  be differentiable functions. Using the definition of the derivative, show

1.  $(f + g)' = f' + g'$
2.  $(fg)' = f'g + fg'$
3. Assume that  $f(x) = c$ , where  $c$  is a constant. Show that  $f'(x) = 0$ .

1.1:

$$\begin{aligned} (f+g)' &= \lim_{h \rightarrow 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = f' + g' \end{aligned}$$

■

1.2:

$$\begin{aligned} (fg)' &= \lim_{h \rightarrow 0} \frac{(fg)(x+h) - fg(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x) - f(x+h)g(x) + f(x+h)g(x)}{h} \\ &= \lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h} \end{aligned}$$

$$= fg' + gf'$$

■ 1.3:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

■

2. Let  $f(x)$  and  $g(x)$  be integrable functions.

1. Using the definition of integration, show that the sum  $f(x) + g(x)$  is an integrable function.
2. Using the Fundamental Theorem of Calculus and problem 1.1, show that the sum  $f(x) + g(x)$  is an integrable function.

1.1:

The upper sum of  $f(x) + g(x)$  can be split as follows:

$$\begin{aligned} \lim_{n \rightarrow \infty} U(f(x) + g(x), n) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (f + g)(l_k) \Delta t = \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n f(l_k) + \sum_{k=1}^n g(l_k) \right] \\ &= \lim_{n \rightarrow \infty} U(f, n) + \lim_{n \rightarrow \infty} U(g, n) \end{aligned}$$

Similarly,

$$\lim_{n \rightarrow \infty} L(f(x) + g(x), n) = \lim_{n \rightarrow \infty} L(f, n) + \lim_{n \rightarrow \infty} L(g, n)$$

Because  $f(x)$  and  $g(x)$  are integrable functions, we have

$$\lim_{n \rightarrow \infty} U(f, n) = \lim_{n \rightarrow \infty} L(f, n)$$

and

$$\lim_{n \rightarrow \infty} U(g, n) = \lim_{n \rightarrow \infty} L(g, n)$$

Therefore,

$$\lim_{n \rightarrow \infty} U(f, n) + \lim_{n \rightarrow \infty} U(g, n) = \lim_{n \rightarrow \infty} L(f, n) + \lim_{n \rightarrow \infty} L(g, n)$$

and we have

$$\lim_{n \rightarrow \infty} U(f(x) + g(x), n) = \lim_{n \rightarrow \infty} L(f(x) + g(x), n)$$

as desired. ■

2.2:

Since  $f(x)$  and  $g(x)$  are integrable, they have anti-derivatives  $F(x)$  and  $G(x)$  such that  $F'(x) = f(x)$  and  $G'(x) = g(x)$ . By the product rule,  $(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x)$ . Therefore,  $f(x) + g(x)$  is integrable and has the form  $F(x) + G(x)$ . ■

**3.** Then goal of this problem is to calculate  $\int_0^1 x dx$  three ways. The first two methods are not supposed to be challenging.

1. Look at the graph of the function  $y = x$ . Note what type of geometric object this is, and then get the area under the curve.
2. Find a function  $f(x)$  such that  $f'(x) = x$  and then use the Fundamental Theorem of Calculus to find  $\int_0^1 x dx$ .
3. First show by induction that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Then use the definition of the integral to find  $\int_0^1 x dx$ .

*3.1:*

The area under  $y = x$  will be the right triangle with vertices at  $(0,0)$ ,  $(1,0)$ , and  $(1,1)$ . Therefore, the area is  $A = \frac{1}{2}(1)(1) = \frac{1}{2}$ .

*3.2:*

$\int x dx = \frac{1}{2}x^2 + C$ . Therefore,  $\int_0^1 x dx = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$ .

*3.3:*

If  $n = 1$ ,  $\sum_{i=1}^1 i = 1$  and  $\frac{1(1+1)}{2} = 1$ . Now suppose for  $n = k$  that

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

Now with  $n = k + 1$

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

By the definition of the integral,

$$\int_0^1 x dx = \lim_{n \rightarrow \infty} L(x, n) = \sum_{k=1}^{\infty} l_k \Delta t$$

(don't really know where to go from here...)

**4.** Let  $f(x)$  be differentiable. Show that  $f(x)$  must be continuous.

*Proof:*

If  $f(x)$  is differentiable, then

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists for all  $x$ . By the quotient law for limits, we know that the numerator exists, and that

$$\lim_{x \rightarrow a} f(x) - f(a) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} f(a) = f(a) - f(a) = 0$$

Therefore,

$$\lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} f(a) = 0$$

and

$$\lim_{x \rightarrow a} f(x) = f(a)$$

■

**5.** On the interval  $[0, 1]$ , define

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is not rational} \end{cases}$$

Show that  $f(x)$  is not integrable.

*Proof:*