Classical Stokes Theorems

Def: A *vector field* on $\mathbb{R}^n$ is a vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^m$. If $x_1, \ldots, x_n$ are coordinates for $\mathbb{R}^n$, then the vector field F will be described by m real-valued functions $f_k : \mathbb{R}^n \to \mathbb{R}$ as follows

$$F(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix}$$

Def: A vector field is *continuous* if each real-valued function f_k is continuous.

Def: A vector field is *differentiable* if each real-valued function f_k is differentiable.

Note: A vector field is essentially a function that you plug a vector in and get a vector out. There is no connection to the abstract algebra idea of a field. (Stupid physicists, always ruining perfectly good math)

Def: A *differentiable manifold* M of dimension k in $\mathbb{R}^n$ is a set of points in $\mathbb{R}^n$ such that for any point $p \in M$, there is a small open neighborhood U of p, a vector-valued differentiable function $F : \mathbb{R}^k \to \mathbb{R}^n$ and an open set V in $\mathbb{R}^k$ with

1. $F(V) = U \cap M$
2. The Jacobian of F has rank k at every point in V

The function F is called the *parametrization* of the manifold.

Note: The reason why the Jacobian needs to have rank equal to the dimension of the manifold is so the Jacobian will have a $k \times k$ minor.

Examples: A circle is a 1-dimensional manifold in 2-space. A cone is a 2-dimensional manifold in 3-space.

Def: The *closure* of M, denoted $\bar{M}$, is the set of all points x in $\mathbb{R}^n$ such that there is a sequence of points $\{x_n\}$ in the manifold M with

$$\lim_{n\to\infty} x_n = x$$

The *boundary* of M, denoted ∂M, is:

$$\partial M = \bar{M} - M$$

Note: The boundary of an n-dimensional manifold is usually either empty or is a $(n-1)$-dimensional manifold.