# Estimating Extreme Value Index by Subsampling for Massive Datasets with Heavy-Tailed Distributions

Yongxin Li<sup>1</sup>, Liujun Chen<sup>2</sup>, Deyuan Li<sup>2</sup> and Hansheng Wang<sup>3</sup>

<sup>1</sup> Center for Statistical Science, Peking University, Beijing, China; <sup>2</sup> School of Management, Fudan University, Shanghai, China; <sup>3</sup> Guanghua School of Management, Peking University, Beijing, China

### **Abstract**

Modern statistical analyses often encounter datasets with massive sizes and heavy-tailed distributions. For datasets with massive sizes, traditional estimation methods can hardly be used to estimate the extreme value index directly. To address the issue, we propose here a subsampling-based method. Specifically, multiple subsamples are drawn from the whole dataset by using the technique of simple random subsampling with replacement. Based on each subsample, an approximate maximum likelihood estimator can be computed. The resulting estimators are then averaged to form a more accurate one. Under appropriate regularity conditions, we show theoretically that the proposed estimator is consistent and asymptotically normal. With the help of the estimated extreme value index, a normal range can be established for a heavy-tailed random variable. Observations that fall outside the range should be treated as suspected records and can be practically regarded as outliers. Extensive simulation experiments are provided to demonstrate the promising performance of our method. A real data analysis is also presented for illustration purpose.

**KEY WORDS:** Extreme value index; Heavy-tailed distribution; Outlier detection; Massive dataset; Subsampling.

# 1. INTRODUCTION

Extreme value theory (EVT) is an elegant probability theory for describing the asymptotic behavior of sample extremes (e.g., maximum or minimum). It has profound applications in many research fields. For example, Gillia and Këllezib (2006) found that EVT is a useful tool for capturing the tail risk of several major financial market indices. Thomas et al. (2016) applied EVT to predict the extremes of annual influenza mortality. In the area of geoscience, EVT has been widely used to model flooding and draughts (Katz et al., 2002), wind speed (An and Pandey, 2005), temperature (Villarini et al., 2011) and others. For more applications, we refer to Coles (2001) and Beirlant et al. (2004) for an excellent summary.

In this paper, we focus on another emerging but important application for the EVT. That is outlier detection for massive datasets. Here massive datasets refer to the type of datasets, which are too large to be read into a typical computer's memory as a whole. For instance, the default size of RAM pre-installed in a standard MacBook Pro is up to 16 GB. Hence the datasets of sizes larger than (or even close to) 16 GB can hardly be loaded into the computer's memory as a whole. Thus, they can be regarded as massive for this particular computer. Massive datasets are becoming increasingly available due to the speedy advances of information technology. For datasets of these sizes, outliers (or abnormal observations) seem inevitable. The inevitability mainly lies in the fact that the sample sizes are too huge, and thus practically it is extremely difficult (or even impossible) to assure that there is no abnormal observation to be included in the datasets. As one can expect, those outliers would pose serious challenges to the subsequent statistical analysis. Accordingly, appropriate measures have to be undertaken to detect them and attenuate their influence.

In these regards, there exist at least two different cases. In the first case, the

outliers are truly abnormal observations (e.g., incorrect records), in the sense that they simply do not come from the same probability distribution as the rest majority of data points. Accordingly, those abnormal observations should be excluded from the subsequent analysis. In fact, they probably should be treated separately. Once these abnormal data points are excluded, the distribution of the rest data points can become much usual. As a result, the outliers in this case can be easily detected. The data distributions of this case are often light-tailed. A light-tailed distribution refers to the type of distributions with exponentially decaying tail probability. For this kind of data, a subjective judgement could be enough for outlier detection. Nevertheless, the second case is considerably more complicated, where the distribution of the normal data (i.e., zero outliers) is inherently heavy-tailed. In this work, we refer to a distribution as heavy-tailed, if the decaying rate of its tail probability is polynomial. In this case, the outliers might be mixed with normal observations. It is typically found that "outliers" are very difficult to be completely removed. New "outliers" keep on emerging after the old ones having been excluded. Consequently, naively eliminating "outliers" by subjective judgement could be problematic. Instead, a principled method with solid theoretical support should be a better choice.

To develop a principled statistical method in this regards, we start with a dataset consisting of independent and identically distributed observations, where the distribution function is heavy-tailed. We focus on heavy-tailed distribution in this work. As discussed before, the problem of outlier detection for this case is considerably more challenging than light-tailed one. Mathematically, we assume that the distribution function of normal data points has a polynomially decaying tail (Hall, 1982; Wang and Tsai, 2009; Sun et al., 2020). By the EVT, we know that the sample maximum in this case should follow a Fréchet type extreme value distribution asymptotically. The shape

of the distribution is mainly determined by an extreme value index  $\gamma$  (De Haan and Ferreira, 2006). Once the parameter  $\gamma$  (together with some other necessary parameters) is consistently estimated, a probabilistic upper bound for the normal data points can be constructed, at the desired level of rate. This becomes a natural benchmark to differentiate the truly abnormal observation from those normal but heavy-tailed ones.

To estimate the extreme value index  $\gamma$ , various methods have been proposed. According to the estimable range of  $\gamma$ , those methodologies can be grouped into four classes. The first class contains estimators which are applicable only if  $\gamma > 0$ . The most representative example is the Hill estimator (Hill, 1975; Haeusler and Teugels, 1985; De Haan and Ferreira, 2006) and its regression extension (Wang and Tsai, 2009). The second class contains the estimators that can be applied when  $\gamma > -1/2$ . Various maximum likelihood estimators belong to this category (Prescott and Walden, 1980, 1983; Hosking, 1985; Smith, 1985). The third class consists of estimators that are only suitable for  $\gamma < 1/2$ . This class includes the probability weighted moment estimator suggested by Hosking et al. (1985). While the last one contains estimators that can be used for all settings ( $\gamma \in \mathbb{R}$ ). The moment estimator of Dekkers and De Haan (1989) falls into this category.

All those pioneer estimation methods have been found useful in the traditional setup, where the dataset is not massive in size. As a result, the dataset can be read into memory as a whole without much difficulty. Nevertheless, the story changes in the context of massive datasets. In this case, the dataset is too huge in size to be comfortably hold in a standard memory as a whole. Consequently, the data have to be processed in a batch-wise manner. Accordingly, various divide-and-conquer (Lin and Xi, 2011; Chen and Xie, 2014; Jordan et al., 2019) and subsampling (Ma et al., 2015; Kleiner et al., 2014; Sengupta et al., 2016) methods become popular. Unfortunately,

these existing methods are not developed for extreme value index estimation and cannot be immediately used. How to develop methods of similar type for extreme value index estimation becomes a problem of great interest.

To fulfill this theoretical gap, we develop a subsample-based methodology in this work. Here the subsample is a subset of the entire dataset. It is obtained by simple random sampling with replacement. The subsample size should be much smaller than the whole sample size so that it can be read into memory as a whole. For an accurate estimation, subsamples should be randomly replicated many times. With the help of subsamples, the classical peaks over threshold (POT) method (Davison and Smith, 1990) is carefully studied. The main idea of the POT method is only to retain the observations whose values exceed a high threshold for estimation. Based on these observations, an approximate maximum likelihood estimator for the extreme value index can be computed for each subsample. Next, estimators obtained from different subsamples are averaged together so that a more powerful estimator can be constructed. We refer to the final estimator as the averaged maximum likelihood (AML) estimator. The consistency and the asymptotic normality of our new estimator are also established. Based on the AML estimator, we are able to estimate extreme quantiles for the underlying data distribution consistently. As a result, the quantile estimator with a pre-specified high quantile level can be served as the upper bound to detect outliers.

The rest of this paper is organized as follows. Section 2 introduces an averaged maximum likelihood method and investigates its asymptotic properties. Numerical studies are given in Section 3. Both simulation experiments and a real airline dataset analysis are included. Finally, the paper is concluded with a brief discussion in Section 4. All the proofs, together with additional lemmas, are deferred to the appendices.

# 2. THE METHODOLOGY

### 2.1. Averaged Maximum Likelihood Estimation

Let  $X_i$ 's  $\in \mathbb{R}$   $(i=1,2,\cdots,N)$  be independent and identically distributed copies of a random variable  $X \in \mathbb{R}$  with cumulative distribution function F. In this work, we focus on heavy-tailed distributions only. We follow the tradition (Hall, 1982; Wang and Tsai, 2009; Sun et al., 2020) and assume that the decaying rate of the distribution's tail probability is polynomial. Specifically, we assume that there exist two positive constants  $\beta$  and  $\gamma$  such that

$$\frac{1 - F(x)}{\beta x^{-1/\gamma}} \to 1 \tag{2.1}$$

as  $x \to \infty$ . This suggests that a principled probabilistic upper bound can be constructed as long as the parameter  $\gamma$  (or plus  $\beta$ ) can be consistently estimated.

To estimate the extreme value index  $\gamma$ , a classical estimation method is to model the conditional excess probability over a high threshold. This technique is called the peaks over threshold (POT) method (Davison and Smith, 1990). Specifically, for random variable X satisfying (2.1) and a given high threshold u, it can be shown that

$$\lim_{u \to \infty} P(X > tu | X > u) = \lim_{u \to \infty} \frac{1 - F(tu)}{1 - F(u)} = t^{-1/\gamma}, \tag{2.2}$$

for t > 1. In other words, the exceedances over the high threshold u can be approximated by a Generalized Pareto distribution with parameters  $\gamma$  (Pickands III, 1975; Coles, 2001).

Inspired by the classical POT method, we develop here a subsample-based counterpart. To this end, denote the index set of the full dataset as  $\mathbb{S} = \{1, \dots, N\}$ . We randomly draw K subsamples of size n with replacement from the full dataset. For the k-th subsample  $(1 \leq k \leq K)$ , the corresponding index set is denoted as  $S_k = \{m_1^{(k)}, m_2^{(k)}, \dots, m_n^{(k)}\} \subset \mathbb{S}$ . Conditioning on  $\mathbb{S}$ ,  $\{m_i^{(k)}: 1 \leq i \leq n, 1 \leq k \leq K\}$ 

are independent and identically distributed with  $P(m_i^{(k)} = j) = 1/N$  for any  $j \in \mathbb{S}$ . Then we can derive the following approximate log-likelihood function for the k-the subsample as

$$\mathcal{T}_k(\gamma) = \sum_{i \in \mathcal{S}_k} \left\{ -\log \gamma - \log u - \left(1 + 1/\gamma\right) \log \left(X_i/u\right) \right\} I(X_i > u).$$

By maximizing  $\mathcal{T}_k(\gamma)$  with respect to  $\gamma$ , we get an approximate maximum likelihood estimator as  $\hat{\gamma}_k = \arg\max_{\gamma} \mathcal{T}_k(\gamma) = (n_k^u)^{-1} \sum_{i \in \mathcal{S}_k} \log(X_i/u) I(X_i > u)$ . Here,  $n_k^u = \sum_{i \in \mathcal{S}_k} I(X_i > u)$  is the number of observations over a pre-defined threshold u for the k-th  $(1 \le k \le K)$  subsample. We then combine these subsample-based estimators together. This leads to the averaged final estimator

$$\hat{\gamma}_{\text{AML}} = K^{-1} \sum_{k=1}^{K} \hat{\gamma}_k = K^{-1} \sum_{k=1}^{K} (n_k^u)^{-1} \sum_{i \in \mathcal{S}_k} \log (X_i/u) I(X_i > u).$$

We refer to  $\hat{\gamma}_{\text{\tiny AML}}$  as the averaged maximum likelihood (AML) estimator.

### 2.2. Theoretical properties

To investigate the asymptotic behavior of  $\hat{\gamma}_k$  and the averaged estimator  $\hat{\gamma}_{\text{AML}}$ , the following conditions are needed.

- (C1) (TAIL PROBABILITY) Define  $\alpha_u = P(X > u)$ . As  $N \to \infty$ , assume that  $u \to \infty$  and  $\alpha_u = \beta u^{-1/\gamma} \left\{ 1 + Cu^{-\delta} + o(u^{-\delta}) \right\}$  for  $\beta > 0, \gamma > 0, \delta > 0$  and  $C \in \mathbb{R}$ .
- (C2) (Subsample Ratio) As  $N \to \infty$ , assume that  $n \to \infty$  and n = o(N).
- (C3) (DIVERGENCE RATE) As  $N \to \infty$ , we assume that the threshold u satisfies  $u \to \infty$ ,  $un^{-\gamma} \to 0$  and  $un^{-\gamma/(2\delta\gamma+1)} \to \infty$ .

(C4) (Number of Subsamples) As  $N \to \infty$ , we assume that  $K \to \infty$  and  $K = o(\min\{n^{-1}u^{1/\gamma+2\delta}, nu^{-1/\gamma}\})$ .

Condition (C1) quantifies the decaying speed of the tail probability. A similar condition has been popularly assumed in extreme value literature (Hall, 1982; Smith, 1987; Resnick, 2007; Wang and Tsai, 2009). It is equivalent to replace  $\beta$  and C in Condition (C1) by two smooth functions  $\beta(u)$  and C(u) respectively, as long as  $\beta(u) \to 0$  $\beta$  and  $C(u) \to C$  as  $u \to \infty$ . Condition (C2) is widely used in subsampling literature (Kleiner et al., 2014; Wang et al., 2018; Wang and Ma, 2020). It enforces that the subsample size n should diverge to infinity at an appropriate speed. Specifically, nshould not be too small. Otherwise, the asymptotic theory cannot be developed. On the other hand, n should not be too large. Otherwise the subsample is too large to be loaded into memory. Condition (C3) imposes a restriction on the divergence speed of the threshold level u. On the one side, it should not be too small. Otherwise the tail probability cannot be well approximated by the polynomial function as given in Condition (C1). On the other side, it cannot be too large either. Otherwise the exceedance size of each subsample  $n_k^u$  is too small to support a consistent estimator. Condition (C4) requires that the number of subsamples K should diverge to infinity as  $N \to \infty$ . Obviously, Larger K leads to smaller variability of  $\hat{\gamma}_{\text{AML}}$ . Since the local estimators  $\hat{\gamma}_k$ s are identically distributed,  $\hat{\gamma}_{\text{AML}}$  shares the same amount of bias as  $\hat{\gamma}_k$ . Accordingly, the bias of  $\hat{\gamma}_{AML}$  cannot be reduced by increasing K. As a consequence, if K is too large, the bias suffered by  $\hat{\gamma}_{\text{\tiny AML}}$  becomes nonnegligible, as compared with its standard deviation. Consequently, the number of subsamples K cannot be too large either. Otherwise the benefit introduced by larger K in terms of variability reduction can be completely offset by its bias.

**Theorem 1.** Let  $n_*^u = \sum_{k=1}^K n_k^u$  be the size of the total exceedances. Assume Conditions

(C1)-(C4) hold, then we have (1)  $\sqrt{n_k^u}(\hat{\gamma}_k - \gamma) \xrightarrow{d} N(0, \gamma^2)$  for  $1 \leq k \leq K$ ; (2)  $\sqrt{n_*^u}(\hat{\gamma}_{AML} - \gamma) \xrightarrow{d} N(0, \gamma^2)$ .

According to Theorem 1, we know that both the local estimator  $\hat{\gamma}_k$  and the averaged estimator  $\hat{\gamma}_{AML}$  are consistent and asymptotically normal. The rate of convergence is  $\sqrt{n_k^u}$  for  $\hat{\gamma}_k$  and  $\sqrt{n_*^u}$  for  $\hat{\gamma}_{AML}$ , respectively. Given a confidence level  $1-\alpha$ , an asymptotically valid confidence interval for  $\gamma$  can be constructed as  $\hat{\gamma}_{\text{\tiny AML}} \pm \Phi^{-1}(1-1)$  $\alpha/2)\sqrt{\hat{\gamma}_{\text{AML}}^2/(n_*^u)}$ , where  $\Phi$  is the cumulative distribution function of the standard normal distribution. In the meanwhile, assume a high threshold u > 0, (2.2) indicates that  $P(X > x) \approx \alpha_u (x/u)^{-1/\gamma}$  for any x > u. Therefore, for a quantile level  $1 - \tau$  with sufficiently small positive  $\tau$ , we have  $P(X > q_{1-\tau}^{(a)}) \approx \tau$  with  $q_{1-\tau}^{(a)} = u \left(\alpha_u/\tau\right)^{\gamma}$ . Denote the exact  $(1-\tau)$ -th quantile of distribution F as  $q_{1-\tau}$ . That is  $P(X>q_{1-\tau})=\tau$ . We should expect  $q_{1-\tau}^{(a)} \approx q_{1-\tau}$ . A natural estimator for  $q_{1-\tau}^{(a)}$  is given by  $\hat{q}_{1-\tau}^{(a)} = u \left(\hat{\alpha}_u/\tau\right)^{\hat{\gamma}_{\text{AML}}}$ with  $\hat{\alpha}_u = (nK)^{-1}n_*^u$ , provided  $\tau < \alpha_u$ . This leads to a useful method to approximate the high level quantile for X, if  $\gamma$  can be consistently estimated. In this way, it is natural to use  $\hat{q}_{1-\tau}^{(a)}$  as the probabilistic upper bound for normal observations. Observations which exceed the upper bound should be considered as suspected records and might be practically treated as outliers. As a result, outlier detection can be automatically conducted with guaranteed statistical validity. Let  $\hat{\tau} = P(X > \hat{q}_{1-\tau}^{(a)})$ , The following theorem describes the asymptotic behavior of  $\hat{q}_{1-\tau}^{(a)}$  and  $\hat{\tau}$ .

**Theorem 2.** Assume  $0 < \tau < \alpha_u$ ,  $\log(\alpha_u/\tau) = o(\sqrt{nK\alpha_u})$ , and Conditions (C1)–(C4) hold, then we have (i)  $\hat{q}_{1-\tau}^{(a)}/q_{1-\tau} \xrightarrow{p} 1$ ; and (ii)  $\hat{\tau}/\tau \xrightarrow{p} 1$ .

Theorem 2 indicates that we can estimate a very high-level quantile  $q_{1-\tau}$  consistently. The resulting tail probability  $\hat{\tau}$  is also ratio-consistent for the intended tail probability. It is remarkable that this can be done even if  $\tau$  is too small to have sufficient number of extreme observations (i.e.  $X_i$ s such that  $X_i > q_{1-\tau}$ ). To fix the idea,

take an example for illustration. Consider a very small level  $\tau = 10^{-5}$ . To estimate the corresponding quantile  $q_{1-\tau}$ , very large sample size (e.g.  $N \gg 10^5$ ) is needed if a traditional method is used. However, under the help of  $\hat{\gamma}_{\text{AML}}$ , we can estimate  $q_{1-\tau}$  consistently with K = 10 and  $n = 10^3$ . Accordingly, the tail probability  $\tau$  can also be estimated consistently.

### 2.3. The Outlier Effect

Recall that an important intended application of this work is to detect outliers. As mentioned before, for a dataset with massive size, abnormal observations seem inevitable. The proportion of outliers could be extremely small as compared with the whole data size. However, its total amount could be large and can still cause serious trouble for computational stability and statistical validity. We thus need to have a principled method to detect them automatically. To this end, we need to address one critical issue. That is how the proposed AML estimator  $\hat{\gamma}_{\text{AML}}$  would be effected by outliers.

To address this issue, we assume that there exist a total of d outliers in the whole sample. As mentioned before, the outliers considered here refer to abnormal observations, which are too extremal to be explained by the tail probability distribution of the majority data. Theoretically, we allow  $d \to \infty$  as  $N \to \infty$ . However, the percentage d/N should be very small. It also reflects the reality in practice. Obviously, the AML estimator will not be affected by outliers if there are no outliers be sampled within total K subsamples. In this case, we shall have  $\mathcal{D} \cap \{\bigcup_{k=1}^K \mathcal{S}_k\} = \emptyset$ , where  $\mathcal{D} = \{j : X_j \text{ is a outlier}, j \in \mathbb{S}\}$ . We next study how likely this would happen. The following theorem quantifies the probability.

**Theorem 3.** As  $N \to \infty$ , if dnK = o(N), then the probability that there exists no

outliers in all K subsamples converges to one, i.e.,

$$P\left(\mathcal{D} \cap \left\{ \bigcup_{k=1}^{K} \mathcal{S}_k \right\} = \emptyset \right) \to 1. \tag{2.3}$$

The proof of Theorem 3 is straightforward and thus is omitted here. By Conditions (C2)-(C4), we have  $n = O(N^a)$  and  $K = O(n^{b\delta/(\gamma^{-1}+\delta)})$  for some  $a, b \in (0,1)$ . Accordingly,  $nK = O(N^{a+ab\delta/(\gamma^{-1}+\delta)})$ . As a consequence, as long as  $d = O(N^{1-a-ab})$ , the probability of zero-outlier impact shall approach one asymptotically.

## 3. NUMERICAL STUDIES

### 3.1. Simulation Models

To demonstrate the finite sample performance of the proposed estimators, we conduct a number of simulation studies in this section. To this end, we need to generate  $X_i$ s from a distribution whose tail probability satisfies Condition (C1). Specifically, we consider the following examples.

EXAMPLE 1. (STUDENT'S t-DISTRIBUTION) The first example considered here is the student's t-distribution. It has been popularly used in extreme value modeling literature (Blattberg and Gonedes, 1974; Stoyanov et al., 2011). The distribution is heavy-tailed on both sides (right and left). Specifically, let X follow a student's t-distribution with v degrees of freedom, then the right tail probability becomes (Beirlant et al., 2004)

$$P(X > u) = \int_{u}^{\infty} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{x^{2}}{v}\right)^{-\frac{v+1}{2}} dx$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)v^{(v-1)/2}}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)} u^{-v} \left\{1 - \frac{v^{2}(v+1)}{2(v+2)} u^{-2} + o(u^{-2})\right\}.$$

Consequently, the corresponding extreme value index is given by  $\gamma = 1/v$ . The degree of freedom parameter v controls the tail behavior of the student's t-distribution. In this simulation study, we consider student's t-distributions with v = 1 and v = 2, denoted as t(1) and t(2).

EXAMPLE 2. (PARETO DISTRIBUTION) Pareto distribution is another widely adopted heavy-tailed distribution. It is found to be particularly useful in income distribution research (Wold and Whittle, 1957; Nirei and Aoki, 2016). The right tail probability of Pareto( $x_m, \alpha$ ) where  $\alpha > 0$  is given by  $P(X > u) = (u/x_m)^{-\alpha}$  with  $u \geq x_m$ . Accordingly, the extreme value index is given by  $\gamma = 1/\alpha$ . The tail heaviness is controlled by the parameter  $\alpha$ . In this simulation study, we consider cases Pareto(2, 1) and Pareto(2, 2).

EXAMPLE 3. (FRÉCHET DISTRIBUTION) We study here the Fréchet distribution. The tail probability of Fréchet( $\alpha$ ) where  $\alpha > 0$  is given by  $P(X > u) = 1 - \exp(-u^{-\alpha}) = u^{-\alpha} \{1 - 1/2u^{-\alpha} + o(u^{-\alpha})\}$ . As a consequence, the extreme value index is given by  $\gamma = 1/\alpha$ . Thus, the tail behavior of a Fréchet distribution is determined by  $\alpha$ . In this simulation study, we consider Fréchet(1) and Fréchet(2) cases.

The above examples specify the generating distribution for X. Once the generating distribution and the whole sample size N are given, we can simulate the full dataset. Here, we consider four different N values  $(N=10^5, 5\times 10^5, 10^6, 5\times 10^6)$ . In view of Condition (C2) requiring that n=O(N), we set  $n=\lfloor N^{1/2}\rfloor$ . Here,  $\lfloor x\rfloor$  stands for the largest integer no larger than x. Condition (C3) requires that  $u=n^{\gamma/(1+h\gamma)}$  with  $h\in(0,2\delta)$ . We fix  $h=0.8\delta$ . It implies that  $P(X>u)=O(n^{-1/(1+h\gamma)})=O(n^{-1/(1+0.8\delta\gamma)})$ . Therefore, the threshold value is set to be the  $(1-n^{-1/(1+0.8\delta\gamma)})$ -th quantile of the given generating distribution. Moreover, Condition (C4) indicates that the divergence speed of K is at most  $O(n^{\delta/(\gamma^{-1}+\delta)})$ . Hence, we let  $K=\lfloor n^{C_K\delta/(\gamma^{-1}+\delta)}\rfloor$ 

with coefficients  $C_K = 0.3, 0.5, 0.7$ . The parameter  $\delta$  of Examples 1–3 is set to be 2, 5 and  $\alpha$  respectively. Thereafter, the averaged estimator  $\hat{\gamma}_{\text{AML}}$  can be computed.

### 3.2. Simulation Results

To obtain a reliable evaluation, we randomly replicate each simulation experiment for a total of R=1,000 times. Let  $\hat{\gamma}_{\text{AML}}^{(r)}$  be the averaged estimator obtained in the r-th  $(1 \leq r \leq R)$  replication. We then define the root mean squared error (RMSE) as

$$RMSE(\hat{\gamma}_{AML}) = \left\{ R^{-1} \sum_{r=1}^{R} \left( \hat{\gamma}_{AML}^{(r)} - \gamma \right)^{2} \right\}^{1/2}.$$

Additionally, an asymptotic  $1-\alpha$  confidence interval for  $\gamma$  can be constructed as  $\mathrm{CI}^{(r)} = \left(\hat{\gamma}_{\mathrm{AML}}^{(r)} - \Phi^{-1}(1-\alpha/2)\hat{\gamma}_{\mathrm{AML}}^{(r)}\sqrt{n_{*(r)}^u},\hat{\gamma}_{\mathrm{AML}}^{(r)} + \Phi^{-1}(1-\alpha/2)\hat{\gamma}_{\mathrm{AML}}^{(r)}/\sqrt{n_{*(r)}^u}\right)$ . Then the empirical coverage probability is given by  $\mathrm{ECP} = R^{-1}\sum_{r=1}^R I(\gamma \in \mathrm{CI}^{(r)})$ . Here we fix  $\alpha = 0.05$ . We can further obtain a  $(1-\tau)$ -th quantile estimator for X in the r-th replication as  $\hat{q}_{1-\tau}^{(a,r)}$ . We can compute the tail probability  $\hat{\tau}^{(r)} = P(X > \hat{q}_{1-\tau}^{(a,r)})$  according to its theoretical formula. This leads to relative accuracy (RA) for tail probability as

RA = 
$$\left\{ R^{-1} \sum_{r=1}^{R} \left( \hat{\tau}^{(r)} / \tau - 1 \right)^{2} \right\}^{1/2}$$
.

In this simulation study, we set  $\tau = 10^{-3}$ .

The simulation results are summarized in Tables 1–3. First, we find that the RMSE of  $\hat{\gamma}_{\text{AML}}$  decreases towards 0 as the whole exceedances size  $n_*^u$  increases for all the examples. In particular, the standard deviation decreases towards 0 as  $n_*^u$  goes to infinity. However, with a fixed whole sample size N, the bias of  $\hat{\gamma}_{\text{AML}}$  has little changes as the number of subsamples K increases. This is expected because the bias is mainly controlled by  $n_*^u$  and has nothing to do with K. Second, the empirical coverage prob-

abilities (ECP) are all around 95%. This confirms the asymptotical normality of  $\hat{\gamma}_{\text{AML}}$ . Moreover, the relative accuracy of the tail probability estimator (the last column) approaches 0 as  $n_*^u$  increases towards infinity. This corroborates the theoretical findings given in Theorem 2. Additionally, we find that larger exceedance sizes (i.e., lower  $1-\tau$  and larger K) are typically required by distributions with heavier tails, for comparable performances.

### 3.3. The Competing Estimators

For comparison purpose, we also evaluate here a number of competing estimators. They are, respectively, the probability weighted moment estimator (Hosking et al., 1985) and the moment estimator (Dekkers and De Haan, 1989). As mentioned before, the maximum likelihood estimator in this work is closely related to the Hill estimator (Hill, 1975). Thus, the classical Hill estimator is not included for comparison. Similar to the AML estimator, the moment estimator and the probability weighted moment estimator of  $\gamma$  can be computed on each subsample  $\mathcal{S}_k$ . This leads to a total of Kestimators. They are then averaged to form the final ones. The resulting estimators are referred to as an averaged moment (AMO) estimator and an averaged probability weighted moment (APWM) estimator respectively. Note that the classical probability weighted moment estimator is consistent only for  $\gamma < 1$  and is asymptotically normal only for  $\gamma < 1/2$ . For a fair comparison, we only consider here generating distributions with  $\gamma < 1/2$ . Specifically, those distributions are t(3), t(5), Pareto(2,3), Pareto(2,5), Fréchet(3) and Fréchet(5), respectively. The whole sample size N varies form  $10^6$  to  $10^7$ . The subsample size n is fixed at  $n = \lfloor N^{1/2} \rfloor$ . The number of subsamples is given by K=10. The threshold value u is set to be the  $(1-n^{-3/5})$ -th quantile of X. We compare the RMSE values of  $\hat{\gamma}$  for three different estimation methods (AML, AMO) and APWM). The simulation results are shown in Figure 1. From Figure 1, we can

observe that the AML estimator always performs best in all cases.

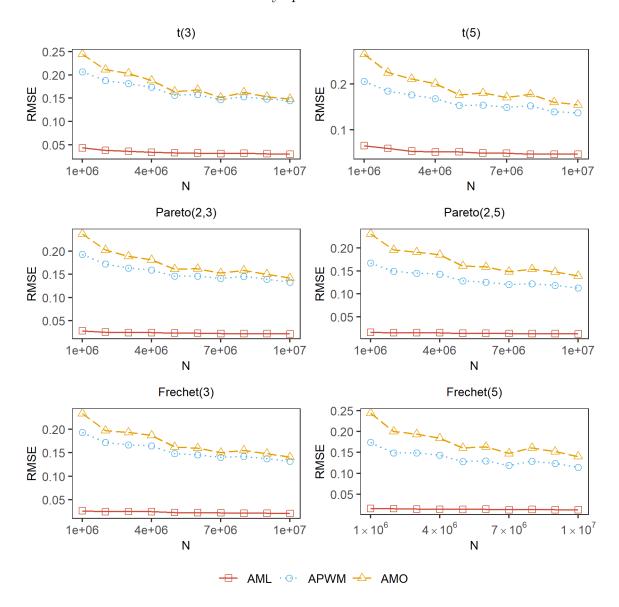


Figure 1: Root mean squared error (RMSE) of various estimators with N ranging from  $10^6$  to  $10^7$ ,  $n = \lfloor N^{1/2} \rfloor$  and K = 10. The RMSE is computed based on 1,000 replications. Different plots correspond to different distributions.

However, it is remarkable that the optimal threshold value for different estimators could be different. Thus, comparing the performance of different estimators under one pre-specified threshold value might be unconvincing. To address the issue, we fix  $N=5\times 10^6$  and  $n=\lfloor N^{1/2}\rfloor=2,236$ . We compare the best finite sample performances

of the three estimators. That is their minimal RMSE values across a wide range of threshold values. In this study, a total of 50 threshold values are considered. Their corresponding tail probabilities are given by 0.5+i/100 for  $i=0,1\cdots,49$ . We present the detailed results in Figure 2. As one can see, the minimal RMSE values of the AML estimators remain the smallest.

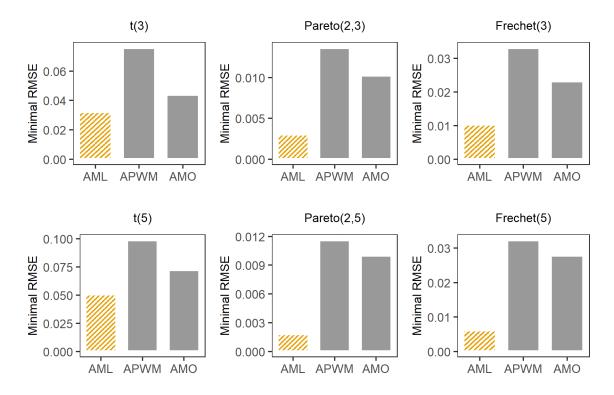


Figure 2: The minimal RMSE values of three estimators under different distributions. In each plot, different bars correspond to different estimators. The bar heights are equal to the minimal RMSE values, which is computed across 50 different threshold values. The bar with diagonal-stripes pattern represents the estimator with the global minimal RMSE value. In this study,  $N = 5 \times 10^6$ , n = 2,236 and K = 10. The RMSE is computed based on 1,000 replications.

## 3.4. Outlier Detection

In this subsection, we focus on two issues. The first one is how the existence of outliers might affect the performance of the AML estimators. The second one is how the proposed outlier detection method can be used to capture outliers automatically. We start with the first issue. The whole sample data are simulated in the same way as before with  $N=5\times 10^6$ . Then we set  $n=\lfloor N^{1/2}\rfloor=2,236$  and K=10. In order to introduce outliers, we independently generate a binary random variable  $Z_i$  for each  $X_i$  with  $P(Z_i=1)=C_o(Kn\log\log N)^{-1}$ . As a result, the technical condition proposed in Section 3.3 can be satisfied. The coefficient  $C_o$  controls the number of outliers. In this simulation study,  $C_0$  varies from 0 to 1. If  $C_o=0$ , there exists no outliers; If  $C_o=1$ , that leads to  $O(N^{7/20}/\log\log N)$  outliers. If  $Z_i=1$ , then the value  $X_i$  is replaced by an outlying value  $10q_{1-\tau}$ . Here,  $q_{1-\tau}$  is the  $(1-\tau)$ -th quantile of X with  $\tau=N^{-1}$ . To estimate  $\gamma$ , the threshold value u is fix to be the  $(1-n^{-3/5})$ -th quantile of X. We replicate the experiment R times and this leads to a total of R values of  $\hat{\gamma}_{\rm AML}$ . The boxplots of these  $\hat{\gamma}_{\rm AML}$  values are presented in Figure 3. From Figure 3, we find that the finite sample performances of  $\hat{\gamma}_{\rm AML}$  are almost identical across different  $C_o$  values. They exhibit comparable medians and variabilities.

We next investigate the second problem. That is how to detect outliers automatically. We fix  $C_0 = 1$  in this study. The whole sample size N varies from  $10^6$  to  $10^7$ . We then compute the  $(1 - \tilde{\tau})$ -th quantile for X with the help of AML estimator. Here,  $\tilde{\tau}$  is set to be  $N^{-1}$ . Next, we screen out all the suspected observations as  $\tilde{\mathcal{D}} = \{i : X_i > \hat{q}_{1-\tilde{\tau}}\}$ . Define the detection rate as  $\pi = |\tilde{\mathcal{D}} \cap \mathcal{D}|/|\tilde{\mathcal{D}} \cup \mathcal{D}|$ , where |A| denotes the size of an arbitrary set A. Thus,  $\pi = 1$  implies a perfect detection. In that case, all the outliers are correctly captured, without misidentifying any normal observations. We average the detection rates over R simulations and plot it in Figure 4. From Figure 4, we can see that all the detection rates are very high (larger than 0.95). Additionally, the detection rate approaches 1 as N increases.

### 3.5. The Airline Data Analysis

To demonstrate the practical usefulness of the proposed method, we present here

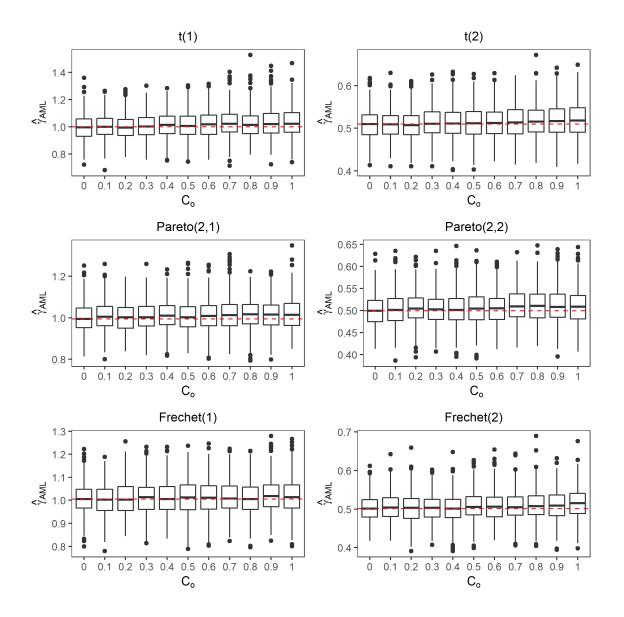


Figure 3: Boxplots of R=1,000 replications of  $\hat{\gamma}_{\text{AML}}$  values under different outlier factor  $C_o$ s. The value of  $C_o$  varies from 0 to 1.  $C_o=0$  corresponds to zero-outlier scenario and  $C_o=1$  corresponds to scenario with the largest number of outliers. For comparison convenience, the horizontal dashed line in each plot represents the median of  $\hat{\gamma}_{\text{AML}}$  values under  $C_o=0$ .

a real dataset analysis. The dataset is referred to as the airline data. It can be freely downloaded from the website http://stat-computing.org/dataexpo/2009/. Each record contains detailed information for one particular commercial flight within the USA, from 1987 and 2008. Specifically, a total of 29 variables are included in each record. Among these variables, 13 of them are continuous variables. However, most of

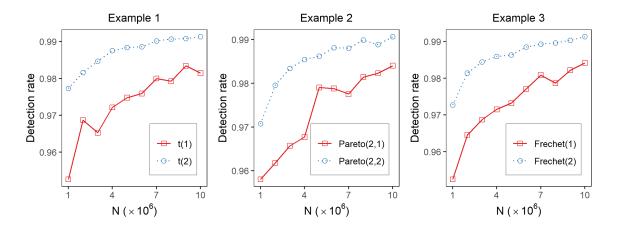


Figure 4: The averaged detection rate versus the whole sample size N. Different plots correspond to different distribution examples.

these continuous variables suffer from severe missing problems. Only 5 of them have missing rates less than 10%. They are, respectively, the ActualElapsedTime (actual elapsed time), CRSElapsedTime (scheduled elapsed time), ArrDelay (arrival delay), DepDelay (departure delay) and Distance variables. Therefore, we will only focus on these 5 variables for illustration purpose. More details about these variables can be found in the downloading website.

Our aim in this study is to construct normal ranges for these 5 continuous variables. Accordingly, those values that fall outside the normal ranges will be regarded as suspected records and might be practically treated as outliers. Note that the original dataset contains about 120 million records. It takes up about 12 GB on a hard drive. As a consequence, it is too large to be read into a usual personal computer's memory as a whole. We have to rely on subsampling technique for data analysis. To this end, we first randomly draw a subsample with size n = 10,000 for a quick review. Based on the subsample, we are able to obtain a number of useful descriptive statistics for each variable. The detailed results are reported in Table 4. From Table 4, we can see that all the variables have kurtosises larger than 3. While the kurtosises of the ArrDelay and DepDelay variables are larger than 9. This indicates that all the five variables

under study have heavier tails than a normal distribution. Moreover, two of them are even more severely heavy-tailed compared with t(5). The outlier detection for these variables becomes imperative and challenging.

To address the issue, we use the proposed subsampling-based method to estimate the normal value upper and lower bound for each variable. In this study, we fix n = 10,000 and K = 10. For each subsample, we set the threshold value u as the  $(1-n^{-3/5})$ -th empirical quantile. This leads to the corresponding  $\hat{\gamma}_k$ . By combining  $\hat{\gamma}_{k}$ s  $(1 \leq k \leq K)$  together, we can obtain the final estimator  $\hat{\gamma}_{AML}$ . Next, we reset uas the average of K empirical quantile values instead. Then we are able to compute the value of the upper bound estimator  $\hat{q}_{1-\tau}^{(a)}$  according to its formula. The quantile level  $\tau$  referring to the upper bound is fixed to be  $0.5 \times 10^{-3}$  (the lower bound is equivalent to the upper bound for the opposite values). In other words, for each variable, the corresponding normal range should cover 99.9% of the normal records. Before the formal data analysis, the original data are appropriately shifted so that its tail distribution can be better approximated by the tail probability which specified in Condition (C1). The detailed results are reported in Table 4. From Table 4, we observe that, for nearly all the variables under study, approximately 1\% of suspected outliers are detected. It is remarkable that this is a percentage significantly larger than the nominal level 0.1%. This suggests that at least some of those suspected outliers are genuine ones. Their extreme behaviors can hardly be explained by the inherent tailheaviness of the data distribution. Therefore, practically it might be better to treat them separately as outliers.

## 4. CONCLUDING REMARKS

In this paper, we develop a subsampling-based method to estimate the extreme value index for heavy-tailed data. The method is developed particularly for massive datasets with memory constraints. By repeatedly subsampling from the whole data, we can estimate the extreme value index consistently. With the help of the extreme value index estimator, the normal range can be constructed. Observations that fall outside the normal range can be regarded as outliers.

To conclude this paper, we would like to discuss a number of interesting problems for further research. First, it would be worthy to extend our method to multivariate outlier detection. Since the large-scale data in reality are often high-dimensional. The tail dependence between different variables might be nonnegligible. How to take this information into consideration deserves further investigation. Second, recall that in this paper,  $X_i$ s are assumed to be independent and identically distributed random variables with extreme value index  $\gamma$ . Let  $\xi$  be an arbitrary but fixed constant. Theoretically speaking, both variables  $X_i$  and  $X_i + \xi$  should have the same extreme value index. Accordingly, the interested extreme value index can be estimated by either the original sample  $\{X_i\}$  or the shifted one  $\{X_i + \xi\}$ . Both samples should lead to consistent estimators for  $\gamma$ . However, our numerical experience suggests that their finite sample performances could be very different. An appropriately shifted sample can yield an estimator substantially more efficient than that of the original one. As a consequence, how to select the optimal shifting parameter should be another interesting topic for future study. Third, how to choose the upper bound value to capture anomalies is also of great importance. In this work, we use the estimated  $(1-\tau)$ -th quantile with  $au=10^{-3}$  as the cutoff value. This choice is subjective. It is well worthy of study to devise a data-driven method to determine the value.

# APPENDIX A. PROOF OF THEOREMS

In this appendix, we provide the full proof of Theorem 1 and Theorem 2. To finish the proof, we need several supporting lemmas. Those lemmas with detailed proofs are given in Appendix B.

# A.1. Proof of Theorem 1

### 1. Proof of Conclusion (i).

Write  $\sqrt{n_k^u}(\hat{\gamma}_k - \gamma) = (n_k^u)^{-1/2} \sum_{i \in \mathcal{S}_k} \{ \log(X_i/u) - \gamma \} I(X_i > u) = Q_{1k}Q_{2k}$ , where  $Q_{1k} = (\alpha_u n/n_k^u)^{1/2}$  and  $Q_{2k} = (\alpha_u n)^{-1/2} \sum_{i \in \mathcal{S}_k} \{ \log(X_i/u) - \gamma \} I(X_i > u)$ . To prove the result in Theorem 1, it suffices to show that  $Q_{1k} \stackrel{p}{\to} 1$  and  $Q_{2k} \stackrel{d}{\to} N(0, \gamma^2)$ . By Lemma 1 in Appendix B, we know that  $\alpha_u n/n_k^u \stackrel{p}{\to} 1$ . This implies that  $Q_{1k} \stackrel{p}{\to} 1$ . Moreover,  $Q_{2k}$  is a normalized sum of independent and identically distributed random variables conditioned on  $\mathbb{S}$ . Denote  $Y_i = \alpha_u^{-1/2} \{ \log (X_i/u) - \gamma \} I(X_i > u)$ . Thus, by the Lindeberg-Feller theorem, the asymptotical normality of  $Q_{2k}$  holds if we are able to show that (i)  $E(Q_{2k}) = o(1)$ , (ii)  $\text{var}(Q_{2k}) = \gamma^2 + o(1)$  and (iii)  $\sum_{i \in \mathcal{S}_k} E(n^{-1}|Y_i|^2 1\{n^{-1/2}|Y_i| > \epsilon\}) = o(1)$  for any  $\epsilon > 0$ . The proof details are given below.

STEP 1. We study  $E(Q_{2k})$  first. Note that  $E(Q_{2k}) = E\{E(Q_{2k}|\mathbb{S})\}$ . Given full data  $\mathbb{S}$ , the conditional expectation of  $Q_{2k}$  equals

$$E(Q_{2k}|\mathbb{S}) = n^{-1/2} E\left[\sum_{i \in \mathcal{S}_k} \alpha_u^{-1/2} \left\{ \log\left(X_i/u\right) - \gamma \right\} I(X_i > u) \middle| \mathbb{S} \right]$$
$$= n^{1/2} N^{-1} \sum_{i \in \mathbb{S}} Y_i, \tag{A.1}$$

Hence,  $E(Q_{2k}) = n^{1/2}E(Y)$  where  $Y = \alpha_u^{-1/2} \{ \log (X/u) - \gamma \} I(X > u)$ . Next we exam the expectation of Y.

By the definition of Y,  $E(Y) = \alpha_u^{-1/2} \left[ E\left\{ \log \left( X/u \right) I(X > u) \right\} - \gamma P(X > u) \right]$ . Note that

$$E\Big\{\log\big(X/u\big)I\big(X>u\big)\Big\} = \int_0^\infty P\Big\{\log\big(X/u\big) > t\Big\}dt = \int_0^\infty P\big(X>ue^t\big)dt$$

$$= \beta u^{-1/\gamma} \int_0^\infty e^{-t/\gamma}dt + \beta C u^{-1/\gamma-\delta} \int_0^\infty e^{-(1/\gamma+\delta)t}dt + o(1)$$

$$= \beta u^{-1/\gamma} \gamma + \frac{\beta C}{1/\gamma + \delta} u^{-1/\gamma-\delta} + o(1). \tag{A.2}$$

The third equality in (A.2) follows from Condition (C1). Therefore, we have  $E(Y) = \alpha_u^{-1/2} \{ (1/\gamma + \delta)^{-1} - \gamma \} \beta C u^{-1/\gamma - \delta} + o(1)$ . Thus,

$$E(Q_{2k}) = n^{1/2}E(Y) = \left(\frac{1}{1/\gamma + \delta} - \gamma\right)\beta C\left(nu^{-1/\gamma - 2\delta}\right)^{1/2} \left(\frac{u^{-1/\gamma}}{\alpha_u}\right)^{1/2} + o(1).$$

By Condition (C3), we have  $nu^{-1/\gamma-2\delta} \to 0$ . And by Condition (C1), we have  $\alpha_u^{-1}u^{-1/\gamma} = O(1)$ . This implies that  $E(Q_{2k}) = o(1)$ .

STEP 2. We next study the variance of  $Q_{2k}$ . Here,  $\operatorname{var}(Q_{2k}) = P_1 + P_2$ , where  $P_1 = \operatorname{var}\{E(Q_{2k}|\mathbb{S})\}$ ,  $P_2 = E\{\operatorname{var}(Q_{2k}|\mathbb{S})\}$ . For  $P_1$ , note that  $E(Q_{2k}|\mathbb{S}) = n^{1/2}N^{-1}\sum_{i\in\mathbb{S}}Y_i$  by (A.1). Hence,  $P_1 = nN^{-1}\operatorname{var}(Y)$ . For  $P_2$ , notice that the conditional variance of  $Q_{2k}$  equals

$$\operatorname{var}(Q_{2k}|\mathbb{S}) = n^{-1}\operatorname{var}\left[\alpha_u^{-1/2} \sum_{i \in \mathcal{S}_k} \left\{\log\left(X_i/u\right) - \gamma\right\} I(X_i > u) \Big| \mathbb{S}\right]$$
$$= N^{-1} \sum_{i \in \mathbb{S}} \left(Y_i - \overline{Y}\right)^2.$$

where  $\overline{Y} = N^{-1} \sum_{i \in \mathbb{S}} Y_i$ . Thus,  $P_2 = E\{ \operatorname{var}(Q_{2k}|\mathbb{S}) \} = (N-1)\operatorname{var}(Y)/N$ . Then we have  $\operatorname{var}(Q_{2k}) = P_1 + P_2 = (n+N-1)\operatorname{var}(Y)/N$ . Since by Condition (C2),  $(n+N-1)/N \to 1$  as  $n \to \infty$  and  $N \to \infty$ . It suffices to show that  $\operatorname{var}(Y) \to \gamma^2$ .

Next we exam the variance of Y.

According to the result of Step 1, we have  $E(Y) = n^{-1/2}E(Q_{2k}) = o(1)$ . Consequently, we obtain that

$$var(Y) = E(Y^{2}) + o(1) = \alpha_{u}^{-1} E\left[\left\{\log(X/u) - \gamma\right\}^{2} I(X > u)\right] + o(1)$$

$$= \alpha_{u}^{-1} \left\{\int_{0}^{\infty} P\left\{\log^{2}(X/u) > t\right\} dt - 2\gamma \int_{0}^{\infty} P\left\{\log\left(X/u\right) > t\right\} dt + \gamma^{2} P(X > u)\right\} + o(1). \tag{A.3}$$

Similar to (A.2), we can obtain that  $\int_0^\infty P\{\log(X/u) > t\} dt = \beta u^{-1/\gamma} \gamma + (1/\gamma + \delta)^{-1} \beta C u^{-1/\gamma - \delta} + o(1)$ . Moreover, we can also prove that  $\int_0^\infty P\{\log^2(X/u) > t\} dt = 2\beta u^{-1/\gamma} \gamma^2 + 2(1/\gamma + \delta)^{-2} \beta C u^{-1/\gamma - \delta} + o(1)$ . Substituting the two equations into (A.3), we obtain that

$$var(Y) = \alpha_u^{-1} \left\{ 2\beta u^{-1/\gamma} \gamma^2 - 2\beta u^{-1/\gamma} \gamma^2 + o\left(u^{-1/\gamma}\right) \right\} + \gamma^2 + o(1)$$
$$= \left(\alpha_u u^{1/\gamma}\right)^{-1} o(1) + \gamma^2 + o(1). \tag{A.4}$$

By Condition (C1),  $\alpha_u u^{1/\gamma} = \beta + o(1)$ , so we then have  $(\alpha_u u^{1/\gamma})^{-1} o(1) \to 0$ . Thus,  $var(Y) \to \gamma^2$  as  $u \to \infty$ .

STEP 3. We finally check the Lindeberg condition. According to the proof in step 2, we can show that for any  $i \in \mathcal{S}_k$ ,  $E(|Y_i|^2)$  converges to  $\gamma^2$ . Note that for any  $\epsilon > 0$ ,  $\sum_{i \in \mathcal{S}_k} E(n^{-1}|Y_i|^2 1\{n^{-1/2}|Y_i| > \epsilon\}) = E(|Y_i|^2 1\{|Y_i| > n^{1/2}\epsilon\}).$  Then by the dominated convergence theorem, we have  $E(|Y_i|^2 1\{|Y_i| > n^{1/2}\epsilon\}) \to 0$  as  $n \to \infty$ . This completes the proof.

# 2. Proof of Conclusion (ii).

Note that  $\sqrt{n_*^u}(\hat{\gamma}_{AML} - \gamma) = (n_*^u/\alpha_u nK)^{1/2} \sqrt{\alpha_u nK}(\hat{\gamma}_{AML} - \gamma)$ . Similar to the poof of Lemma 1 in Appendix B, it can be shown that  $E(n_*^u/\alpha_u nK) = 1 + o(1)$  and  $var(n_*^u/\alpha_u nK) = o(1)$ . Therefore,  $(n_*^u/\alpha_u nK)^{1/2} \xrightarrow{p} 1$  and it suffices to show that  $\sqrt{\alpha_u nK}(\hat{\gamma}_{AML} - \gamma) \xrightarrow{d} N(0, \gamma^2)$ .

According to to the poof of conclusion 1, we have  $\sqrt{\alpha_u n}(\hat{\gamma}_k - \gamma) = (\alpha_u n/n_k^u)(\alpha_u n)^{-1/2}$  $\sum_{i \in \mathcal{S}_k} \{ \log(X_i/u) - \gamma \} I(X_i > u) = Q_{1k}^2 Q_{2k}, \text{ where } Q_{1k} = (\alpha_u n/n_k^u)^{1/2} \text{ and } Q_{2k} = (\alpha_u n)^{-1/2} \sum_{i \in \mathcal{S}_k} \{ \log(X_i/u) - \gamma \} I(X_i > u), \text{ following the same definition in the proof of Theorem 1. Hence, } \sqrt{\alpha_u n}(\hat{\gamma}_k - \gamma) = Q_{1k}^2 Q_{2k} = Q_{2k} + (Q_{1k}^2 - 1)Q_{2k}. \text{ Therefore, for the averaged estimator } \hat{\gamma}_{\text{AML}}, \text{ we have}$ 

$$\sqrt{\alpha_u n K} (\hat{\gamma}_{AML} - \gamma) = K^{-1/2} \sum_{k=1}^K \sqrt{\alpha_u n} (\hat{\gamma}_k - \gamma) = K^{-1/2} \sum_{k=1}^K \left\{ Q_{2k} + (Q_{1k}^2 - 1) Q_{2k} \right\}$$

$$= K^{-1/2} \sum_{k=1}^K Q_{2k} + K^{-1/2} \sum_{k=1}^K (Q_{1k}^2 - 1) Q_{2k}. \tag{A.5}$$

For the first term on the right hand side of (A.5), it can be proven that under Conditions (C1)-(C4),  $E(K^{-1/2}\sum_{k=1}^K Q_{2k}) = (nK)^{1/2}E(Y) = o(1)$  and  $\operatorname{var}(K^{-1/2}\sum_{k=1}^K Q_{2k}) = nN^{-1}\operatorname{var}(Y) + (NK-1)\operatorname{var}(Y)/NK = \gamma^2 + o(1)$ , where Y is defined in Appendix A.1 The rest of proof is similar to that of  $Q_{2k} \xrightarrow{d} N(0, \gamma^2)$  in Appendix A.1 and finally we can conclude that  $K^{-1/2}\sum_{k=1}^K Q_{2k} \xrightarrow{d} N(0, \gamma^2)$ . For the second term, Lemma 4 in Appendix B indicates that  $K^{-1/2}\sum_{k=1}^K Q_{2k} \xrightarrow{d} N(0, \gamma^2)$ . Combining the results together yields that  $\sqrt{\alpha_u nK}(\hat{\gamma}_{\text{AML}} - \gamma) \xrightarrow{d} N(0, \gamma^2)$ , which completes the proof.

# 1. Proof of Conclusion (i).

The conclusion (i) follows if we can prove (a)  $q_{1-\tau}^{(a)}/q_{1-\tau} \stackrel{p}{\to} 1$  and (b)  $\hat{q}_{1-\tau}^{(a)}/q_{1-\tau}^{(a)} \stackrel{p}{\to} 1$ . For the first part, according to the definition of  $q_{1-\tau}$ , we have  $\tau = \beta q_{1-\tau}^{-1/\gamma} \{1 + \beta q_{1-\tau}^{-1/\gamma}\}$ 

 $Cq_{1-\tau}^{-\delta} + o(q_{1-\tau}^{-\delta})$ . Moreover, since  $q_{1-\tau}^{(a)} = u\left(\alpha_u/\tau\right)^{\gamma}$ , we can also obtain that  $\tau = (q_{1-\tau}^{(a)})^{-1/\gamma}u^{1/\gamma}\alpha_u$ . Combining the two equations together gives  $\beta q_{1-\tau}^{-1/\gamma}\{1 + Cq_{1-\tau}^{-\delta} + o(q_{1-\tau}^{-\delta})\} = (q_{1-\tau}^{(a)})^{-1/\gamma}u^{1/\gamma}\beta u^{-1/\gamma}\{1 + Cu^{-\delta} + o(u^{-\delta})\}$ . Rearrange it yields that

$$(q_{1-\tau}^{(a)}/q_{1-\tau})^{-1/\gamma} = \frac{1 + Cq_{1-\tau}^{-\delta} + o(q_{1-\tau}^{-1/\delta})}{1 + Cu^{-\delta} + o(u^{-\delta})}.$$
 (A.6)

Condition (C3) assumes that  $u \to \infty$  and thus  $\alpha_u \to 0$ . Hence,  $\tau \to 0$  and then  $q_{1-\tau} \to \infty$ . As a consequence, the right-hand side of (A.6) converges to 1, implying that  $q_{1-\tau}^{(a)}/q_{1-\tau} \xrightarrow{p} 1$ .

As for the second part, note that  $\hat{q}_{1-\tau}^{(a)} = u\left(\hat{\alpha}_u/\tau\right)^{\hat{\gamma}_{\text{AML}}}$  and  $q_{1-\tau}^{(a)} = u\left(\alpha_u/\tau\right)^{\gamma}$ , we have

$$\log(\hat{q}_{1-\tau}^{(a)}/q_{1-\tau}^{(a)}) = \hat{\gamma}_{\text{AML}} \log(\hat{\alpha}_u/\tau) - \gamma \log(\alpha_u/\tau)$$

$$= (\hat{\gamma}_{\text{AML}} - \gamma)(\log \alpha_u - \log \tau) + \hat{\gamma}_{\text{AML}}(\log \hat{\alpha}_u - \log \alpha_u)$$
(A.7)

The first term in (A.7) can be rewritten as  $(\hat{\gamma}_{AML} - \gamma)(\log \alpha_u - \log \tau) = \sqrt{n_*^u}(\hat{\gamma}_{AML} - \gamma)(n_*^u)^{-1/2}\log(\alpha_u/\tau)$ . Here,  $\sqrt{n_*^u}(\hat{\gamma}_{AML} - \gamma) = O_p(1)$  by Theorem 1. According to the proof of Lemma 1, we are also able to show that  $n_*^u = nK\alpha_u\{1 + o_p(1)\}$ . Since we assume that  $\log(\alpha_u/\tau) = o(\sqrt{nK\alpha_u})$ , we have  $(n_*^u)^{-1/2}\log(\alpha_u/\tau) = o_p(1)$ . Therefore, the first term equals to  $o_p(1)$ .

While for the second term, following an approach similar to that in the proof of Lemma 1, it can be verified that  $\hat{\alpha}_u/\alpha_u \stackrel{p}{\to} 1$  and thus  $\log \hat{\alpha}_u - \log \alpha_u \stackrel{p}{\to} 0$ . Since Theorem 1 implies that  $\hat{\gamma}_{\text{AML}}$  is a consistent estimator of  $\gamma$ , we can conclude that  $\hat{\gamma}_{\text{AML}}(\log \hat{\alpha}_u - \log \alpha_u) = o_p(1)$ . Combining these two terms together, we have  $\log(\hat{q}_{1-\tau}^{(a)}/q_{1-\tau}^{(a)}) \stackrel{p}{\to} 0$ . This means that  $\hat{q}_{1-\tau}^{(a)}/q_{1-\tau}^{(a)} \stackrel{p}{\to} 1$ . The proof is completed.

# 2. Proof of Conclusion (ii).

Note that  $\hat{\tau} = P(X_i > \hat{q}_{1-\tau}^{(a)}) = \beta(\hat{q}_{1-\tau}^{(a)})^{-1/\gamma} [1 + C(\hat{q}_{1-\tau}^{(a)})^{-\delta} + o\{(\hat{q}_{1-\tau}^{(a)})^{-\delta}\}]$  and  $\tau = \beta q_{1-\tau}^{-1/\gamma} \{1 + Cq_{1-\tau}^{-\delta} + o(q_{1-\tau}^{-\delta})\}$ . We have

$$\hat{\tau}/\tau = (\hat{q}_{1-\tau}^{(a)}/q_{1-\tau})^{-1/\gamma} \frac{1 + C(\hat{q}_{1-\tau}^{(a)})^{-\delta} + o\{(\hat{q}_{1-\tau}^{(a)})^{-\delta}\}}{1 + Cq_{1-\tau}^{-\delta} + o(q_{1-\tau}^{-\delta})}$$

Since we have shown that  $\hat{q}_{1-\tau}^{(a)}/q_{1-\tau} \xrightarrow{p} 1$  in the first conclusion,  $\hat{\tau}/\tau \xrightarrow{p} 1$  can be proven.

# APPENDIX B. LEMMAS

We present here several useful lemmas for proofs in Appendix A. To prove the first conclusion in Theorem 1, the following lemma is needed. Similar lemma was also developed in Wang and Tsai (2009). The difference is that the lemma given in Wang and Tsai (2009) was developed for original data while ours is developed for subsampled data. The details are given below.

**Lemma 1.** Under Conditions (C1) and (C2), we have  $n_k^u/n = \beta u^{-1/\gamma} \{1 + o_p(1)\}$ .

Proof. The conclusion follows, if we can show that  $A = \beta^{-1}u^{1/\gamma}n_k^u/n = 1 + o_p(1)$ . To this end, it suffices to show that E(A) = 1 + o(1) and var(A) = o(1). For the expectation of A, let  $\alpha_u^* = P(X_i > u)$  where  $i \in \mathcal{S}_k$  for any  $1 \le k \le K$ . Then we have  $E(A) = \beta^{-1}u^{1/\gamma}\alpha_u^*$ . Note that

$$\alpha_u^* = P(X_i > u) = E\{I(X_i > u)\} = E[E\{I(X_i > u)|S\}]$$
$$= E\{N^{-1}\sum_{i \in S} I(X_i > u)\} = P(X > u) = \alpha_u.$$

Moreover, Condition (C1) implies that  $\alpha_u = \beta^{-1}u^{1/\gamma}\{1 + o(1)\}$ . Therefore,  $E(A) = \beta^{-1}u^{1/\gamma}\alpha_u = 1 + o(1)$ . For the variance of A, write  $U_i = I(X_i > u)$ , then  $var(A) = I(X_i > u)$ 

 $(\beta^{-1}u^{1/\gamma})^2 \operatorname{var}(n_k^u/n) = n^{-2}(\beta^{-1}u^{1/\gamma})^2 \operatorname{var}\left(\sum_{i \in \mathcal{S}_k} U_i\right)$ . Recall that

$$\operatorname{var}\left(\sum_{i\in\mathcal{S}_k} U_i\right) = E\left\{\operatorname{var}\left(\sum_{i\in\mathcal{S}_k} U_i\middle|\mathbb{S}\right)\right\} + \operatorname{var}\left\{E\left(\sum_{i\in\mathcal{S}_k} U_i\middle|\mathbb{S}\right)\right\}$$
$$= E\left\{nN^{-1}\sum_{i\in\mathbb{S}} \left(U_i - \overline{U}\right)^2\right\} + \operatorname{var}\left(nN^{-1}\sum_{i\in\mathbb{S}} U_i\right)$$
$$= n(n+N-1)N^{-1}\operatorname{var}(U).$$

where  $\overline{U} = N^{-1} \sum_{i \in \mathbb{S}} U_i$ , U = I(X > u),. Moreover,  $var(U) = P(X > u)\{1 - P(X > u)\} \le P(X > u)$ . Therefore, combining with Condition (C1), we have

$$\operatorname{var}(A) \le (\beta^{-1}u^{1/\gamma})^2 (n+N-1)(nN)^{-1} P(X > u)$$
$$= \beta^{-1} (un^{-\gamma})^{1/\gamma} (n+N-1)N^{-1} \{1 + o(1)\}.$$

By Condition (C2) and (C3), we know that n = o(N) and  $u = o(n^{\gamma})$ . Hence,  $var(A) \rightarrow 0$ . This completes the proof.

To prove the second conclusion in Theorem 1, we need the following three lemmas. We first develop an Bernstein-type inequality for subsampled data in Lemma 2. This lemma can be viewed as the extension of the classical Bernstein inequality (Bennett, 1962) but for subsampled data.

**Lemma 2.** Let  $\mathcal{F}_N = \{X_1, \dots, X_N\}$  be independent and identically distributed bounded random variables such that  $EX_i = \mu$  and  $|X_i| \leq M$ . Subsamples  $\{X_1^*, \dots, X_n^*\}$  are drawn from  $\mathcal{F}_N$  independently with replacement. Then for any t > 0, we have

$$P\bigg(\Big|n^{-1}\sum_{i=1}^n X_i^* - \mu\Big| > t\bigg) \le 4\exp\bigg(-\frac{nt^2}{12\phi + 4Mt/3}\bigg) + 2\exp\bigg(-\frac{N\phi^2}{8\psi^2 + 4M^2\phi/3}\bigg),$$

where  $\phi = E(X_i^2)$  and  $\psi = var(X_i^2)$ .

*Proof.* Let  $\sigma^2 = \text{var}(X_i)$ . By Bernstein inequality (Bennett, 1962) we know that, for any t > 0,  $P(\sum_{i=1}^{N} X_i - N\mu > t) \le \exp\{-t^2/(2N\sigma^2 + 2Mt/3)\}$ . This is equivalent to

$$P(|\bar{X} - \mu| > t) \le 2 \exp\left(-\frac{Nt^2}{2\sigma^2 + 2Mt/3}\right).$$
 (B.1)

Similarly, note that given full dataset  $\mathcal{F}_N$ ,  $X_i^*$ s are conditionally independent with  $E(X_i^*|\mathcal{F}_N) = N^{-1} \sum_{i=1}^N X_i = \overline{X}$  and  $var(X_i^*|\mathcal{F}_N) = N^{-1} \sum_{i=1}^N (X_i - \overline{X})^2 \le N^{-1} \sum_{i=1}^N X_i^2$ , we can also obtain that

$$P\left(\left|n^{-1}\sum_{i=1}^{n}X_{i}^{*} - \bar{X}\right| > t \middle| \mathcal{F}_{N}\right) \le 2\exp\left(-\frac{nt^{2}}{2N^{-1}\sum_{i=1}^{N}X_{i}^{2} + 2Mt/3}\right).$$
(B.2)

Taking the expectation from both sides in (B.2) lead to

$$P(\left|n^{-1}\sum_{i=1}^{n}X_{i}^{*} - \bar{X}\right| > t) = E\left\{P(\left|n^{-1}\sum_{i=1}^{n}X_{i}^{*} - \bar{X}\right| > t \middle| \mathcal{F}_{N})\right\}$$

$$= E\left\{P(\left|n^{-1}\sum_{i=1}^{n}X_{i}^{*} - \bar{X}\right| > t \middle| \mathcal{F}_{N})I(\left|N^{-1}\sum_{i=1}^{N}X_{i}^{2} - \phi\right| < \phi/2) + P(\left|n^{-1}\sum_{i=1}^{n}X_{i}^{*} - \bar{X}\right| > t \middle| \mathcal{F}_{N})I(\left|N^{-1}\sum_{i=1}^{N}X_{i}^{2} - \phi\right| > \phi/2)\right\}$$

$$\leq 2\exp\left(-\frac{nt^{2}}{3\phi + 2Mt/3}\right) + P(\left|N^{-1}\sum_{i=1}^{N}X_{i}^{2} - \phi\right| > \phi/2). \tag{B.3}$$

For the second term on the right side of (B.3), notice that  $E(N^{-1}\sum_{i=1}^{N}X_i^2)=\phi$  and  $X_i^2 \leq M^2$ , applying Bernstein inequality (B.1) again and replacing t by  $\phi/2$  gives

$$P(\left|N^{-1}\sum_{i=1}^{N}X_{i}^{2}-\phi\right|>\phi/2)\leq 2\exp\left(-\frac{N\phi^{2}}{8\psi^{2}+4M^{2}\phi/3}\right).$$
 (B.4)

Combining (B.1)-(B.4) and  $n \leq N$  and  $2\sigma^2 < 3\phi$  yield that  $P(|n^{-1}\sum_{i=1}^n X_i^* - \mu| > 2t) \leq 4\exp\{-nt^2/(3\phi + 2Mt/3)\} + 2\exp\{-N\phi^2/(8\psi^2 + 4M^2\phi/3)\}$ . This completes

the proof.  $\Box$ 

**Lemma 3.** For  $0 < t \le 1$ , we have

$$P\left(\max_{1 \le k \le K} \left| \alpha_u n / n_k^u - 1 \right| > t \right) \le 8K \exp\left(-\alpha_u n t^2 / 51\right) + 4K \exp\left(-3\alpha_u N / 28\right).$$

*Proof.* For  $0 < t \le 1$ , we have

$$P\left(\max_{k} |\alpha_{u}n/n_{k}^{u} - 1| > t\right) \leq KP\left(|\alpha_{u}n/n_{k}^{u} - 1| > t\right)$$

$$\leq KP\left\{|n_{k}^{u}/(\alpha_{u}n) - 1| > tn_{k}^{u}/(\alpha_{u}n), |n_{k}^{u}/(\alpha_{u}n) - 1| \leq t/2\right\}$$

$$+ KP\left\{|n_{k}^{u}/(\alpha_{u}n) - 1| > t/2\right\}$$

$$\leq KP\left\{|n_{k}^{u}/(\alpha_{u}n) - 1| > t(1 - t/2)\right\} + KP\left\{|n_{k}^{u}/(\alpha_{u}n) - 1| > t/2\right\}$$

$$\leq 2KP\left\{|n_{k}^{u}/(\alpha_{u}n) - 1| > t/2\right\}$$

Notice that  $n_k^u/(\alpha_u n) = n^{-1} \sum_{i \in \mathcal{S}_k} H_i$ , where  $H_i = \alpha_u^{-1} I(X_i > u)$  bounded by  $M = \alpha_u^{-1}$ . According to the proof of lemma 1,  $E(H_i) = 1$  and  $\text{var}(H_i) = \alpha_u^{-1}(1 - \alpha_u) \leq \alpha_u^{-1}$ . Moreover,  $H_i^2 = \alpha_u^{-1} H_i$  implies that  $\phi = E(H_i^2) = \alpha_u^{-1}$  and  $\psi^2 = \text{var}(H_i^2) \leq \alpha_u^{-3}$ . Combining the results and Lemma 2 lead to

$$P(|n_k^u/(\alpha_u n) - 1| > t/2) \le 4 \exp\left(-\frac{nt^2}{48\phi + 8Mt/3}\right) + 2 \exp\left(-\frac{N\phi^2}{8\psi^2 + 4M^2\phi/3}\right)$$
  
 
$$\le 4 \exp\left(-\alpha_u nt^2/51\right) + 2 \exp\left(-3\alpha_u N/28\right).$$

Here, the second inequality follows that  $48\phi + 8Mt/3 \le \alpha_u^{-1}(48 + 8t/3) \le \alpha_u^{-1}(48 + 8t/3) \le 51\alpha_u^{-1}$  and  $8\psi^2 + 4M^2\phi/3 = 8\psi^2 + 4\alpha_u^{-3}/3 \le 8\alpha_u^{-3} + 4\alpha_u^{-3}/3 = 28\alpha_u^{-3}/3$ . This completes the proof.

**Lemma 4.** Under Conditions (C1)-(C4),  $K^{-1/2}\sum_{k=1}^K (Q_{1k}^2-1)Q_{2k} \xrightarrow{p} 0$  as  $N \to \infty$ .

*Proof.* This lemma conclusion follows if we can show that, for any  $\epsilon > 0$ , we have  $P\{|K^{-1/2}\sum_{k=1}^{K}(Q_{1k}^2-1)Q_{2k}| > \epsilon\} \to 0$  as  $N \to \infty$ . Note that

$$P\Big\{ \left| K^{-1/2} \sum_{k=1}^{K} (Q_{1k}^2 - 1) Q_{2k} \right| > \epsilon \Big\} \le P\Big( K^{-1/2} \sum_{k=1}^{K} \left| Q_{1k}^2 - 1 \right| \left| Q_{2k} \right| > \epsilon \Big)$$

$$\le P\Big( \max_{1 \le k \le K} \left| Q_{1k}^2 - 1 \right| K^{-1/2} \sum_{k=1}^{K} \left| Q_{2k} \right| > \epsilon \Big), \tag{B.5}$$

Therefore, the conclusion follows if we are able to show that  $\max_{1 \le k \le K} |Q_{1k}^2 - 1| K^{1/2} = o_p(1)$  and  $K^{-1} \sum_{k=1}^K |Q_{2k}| = O_p(1)$ . For the first one, since  $Q_{1k}^2 = \alpha_u n/n_k^u$ , then by Lemma 3 we have for any  $\varepsilon > 0$  such that  $\varepsilon/\sqrt{K} \le 1$ ,

$$P\left(\max_{1 \le k \le K} \left| Q_{1k}^2 - 1 \right| > \varepsilon / \sqrt{K} \right) \le 8K \exp\left(-\alpha_u n \varepsilon^2 / 51K\right) + 4K \exp\left(-3\alpha_u N / 28\right).$$

If  $\varepsilon/\sqrt{K} > 1$ , we can still obtain the upper bound by

$$P\left(\max_{1 \le k \le K} |Q_{1k}^2 - 1| > \varepsilon/\sqrt{K}\right) \le P\left(\max_{1 \le k \le K} |Q_{1k}^2 - 1| > 1\right)$$
$$\le 8K \exp\left(-\alpha_u n/51\right) + 4K \exp\left(-3\alpha_u N/28\right).$$

According to Conditions (C1)-(C4),  $\alpha_u n$  and  $K \to \infty$  as  $N \to \infty$ . Condition (C4) indicates that  $K = o(\alpha_u n)$ . Therefore, for any  $\varepsilon > 0$ ,  $P\{\max_{1 \le k \le K} |Q_{1k}^2 - 1|K^{1/2} > \varepsilon\} \to 0$  as  $N \to \infty$  and thus  $\max_{1 \le k \le K} |Q_{1k}^2 - 1|K^{1/2} = o_p(1)$ .

For the second one, note that in Appendix A.1, it has been shown that  $E(Q_{2k}) = o(1)$  and  $var(Q_{2k}) = \gamma^2 + o(1)$ . Therefore,  $E(K^{-1} \sum_{k=1}^K |Q_{2k}|) = E(|Q_{2k}|) \le E(Q_{2k}^2)^{1/2} = O(1)$ , implying that  $K^{-1} \sum_{k=1}^K |Q_{2k}| = O_p(1)$ . This completes the proof.

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Table 1: Simulation results for Example 1 with 1,000 replications. The numerical performance are evaluated for different coefficients  $C_K$  (i.e., different numbers of subsamples K) and the whole sample sizes N. The corresponding subsample sizes n and quantile levels  $1-\tau$  (induce the threshold value) are shown in the table. Apart from three measurements (RMSE, ECP and RA) defined in Section 3.2, the average of the total exceedance size  $(n_*^u)$ , bias (Bias) and standard deviation (SD) of  $\hat{\gamma}_{\text{AML}}$  are also reported. The target ECP is 95%.

$C_K$	N	n	$1-\tau$	K	$n_*^u$	Bias	SD	RMSE	ECP	RA	
$\circ_K$	$(\times 10^{5})$	16	(%)	11	16*	$(\times 10^{-2})$			(%)	(%)	
Case 1: $t(1)$ distribution											
0.3	1.0	316	89.1	3	103.6	2.76	9.57	9.96	96.6	47.1	
	5.0	707	92.0	3	169.3	1.34	7.86	7.97	95.6	36.2	
	10.0	1,000	93.0	3	209.6	1.03	6.87	6.95	95.7	30.6	
	50.0	2,236	94.9	4	456.9	1.19	4.52	4.60	96.5	18.1	
0.5	1.0	316	89.1	6	206.4	2.47	7.28	7.69	94.9	36.4	
	5.0	707	92.0	8	452.3	1.21	4.62	4.77	96.1	21.0	
	10.0	1,000	93.0	9	630.4	1.11	4.01	4.16	94.5	17.6	
	50.0	2,236	94.9	13	1,482.7	0.43	2.57	2.50	95.0	10.4	
0.7	1.0	316	89.1	14	482.2	2.78	4.61	5.38	93.1	23.5	
	5.0	707	92.0	21	1,188.3	1.53	3.00	3.36	92.0	13.6	
	10.0	1,000	93.0	25	1,751.9	1.10	2.34	2.58	94.0	10.5	
	50.0	2,236	94.9	36	4,105.0	0.54	1.56	1.65	94.6	6.4	
Case 2: $t(2)$ distribution											
0.3	1.0	316	95.9	2	25.9	3.02	10.86	11.27	94.5	104.0	
	5.0	707	97.4	2	36.8	1.74	8.50	8.68	95.5	71.3	
	10.0	1,000	97.8	2	44.4	1.41	8.04	8.16	94.7	59.3	
	50.0	2,236	98.6	3	93.8	1.14	5.22	5.35	95.4	30.0	
0.5	1.0	316	95.9	4	52.2	3.34	7.52	8.23	95.6	61.1	
	5.0	707	97.4	5	91.8	2.25	5.58	6.02	95.0	36.7	
	10.0	1,000	97.8	5	110.2	1.66	4.89	5.16	95.4	32.9	
	50.0	2,236	98.6	6	187.8	1.21	3.85	4.03	94.2	21.4	
0.7	1.0	316	95.9	7	89.9	2.98	5.62	6.37	94.8	45.2	
	5.0	707	97.4	9	165.7	1.98	4.01	4.47	95.4	27.0	
	10.0	1,000	97.8	11	241.6	1.77	3.31	3.75	94.0	20.3	
	50.0	2,236	98.6	14	439.0	1.15	2.45	2.71	93.3	13.8	

Table 2: Simulation results for Example 2 with 1,000 replications. In this simulation, we set  $\delta=5$ . The numerical performance are evaluated for different coefficients  $C_K$  (i.e., different numbers of subsamples K) and the whole sample sizes N. The corresponding subsample sizes n and quantile levels  $1-\tau$  (induce the threshold value) are shown in the table. Apart from three measurements (RMSE, ECP and RA) defined in Section 3.2, the average of the total exceedance size  $(n_*^u)$ , bias (Bias) and standard deviation (SD) of  $\hat{\gamma}_{\text{AML}}$  are also reported. The target ECP is 95%.

$C_K$	N	n	$1-\tau$	K	$n_*^u$	Bias	SD	RMSE	ECP	RA
$C_K$	$(\times 10^{5})$	11	(%)	$I\lambda$	11.*		(×10 <sup>-</sup>	·2)	(%)	(%)
		(	CASE 1:	Par	RETO(2,1)					
0.3	1.0	316	68.4	4	399.3	0.01	5.02	5.02	94.6	32.9
	5.0	707	73.1	5	950.8	0.08	3.33	3.34	94.4	19.3
	10.0	1,000	74.9	5	1,255.1	0.01	2.76	2.76	95.7	15.7
	50.0	2,236	78.6	6	2,873.0	0.01	1.82	1.82	95.9	10.1
0.5	1.0	316	68.4	11	1,098.0	0.13	3.04	3.04	95.5	20.1
	5.0	707	73.1	15	2,853.4	0.07	1.88	1.88	94.7	11.3
	10.0	1,000	74.9	17	4,266.7	-0.06	1.56	1.56	94.7	9.1
	50.0	2,236	78.6	24	11,484.4	-0.00	0.94	0.94	94.6	5.2
0.7	1.0	316	68.4	28	2,796.5	-0.02	2.04	2.04	92.2	15.1
	5.0	707	73.1	45	8,558.0	0.05	1.14	1.14	93.7	7.7
	10.0	1,000	74.9	56	14,053.9	0.04	0.90	0.90	92.9	5.8
	50.0	2,236	78.6	89	42,581.1	-0.03	0.52	0.52	92.2	3.1
Case 2: Pareto $(2,2)$ distribution										
0.3	1.0	316	85.3	3	138.9	-0.12	4.30	4.30	93.2	52.5
	5.0	707	88.8	4	316.3	-0.04	2.89	2.89	94.2	30.4
	10.0	1,000	90.0	4	399.7	0.23	2.58	2.59	94.7	24.7
	50.0	2,236	92.4	5	849.3	-0.02	1.69	1.69	95.5	15.7
0.5	1.0	316	85.3	7	324.8	-0.11	2.83	2.83	94.1	30.9
	5.0	707	88.8	10	790.0	-0.04	1.69	1.69	95.6	17.4
	10.0	1,000	90.0	11	1,099.9	-0.02	1.54	1.54	95.0	14.9
	50.0	2,236	92.4	15	2,550.0	-0.00	1.01	1.01	94.9	8.9
0.7	1.0	316	85.3	17	786.7	-0.06	1.88	1.88	93.9	20.6
0.1	5.0	707	88.8	26	2,056.1	-0.02	1.13	1.13	94.7	11.3
	10.0	1,000	90.0	31	3,098.7	-0.02	0.92	0.92	94.2	9.0
	50.0	2,236	92.4	47	7,986.2	0.00	0.52	0.56	95.5	5.0
	55.0	_,	02.1		1 .,000.2	0.00	0.00	0.00		

Table 3: Simulation results for Example 3 with 1,000 replications. The numerical performance are evaluated for different coefficients  $C_K$  (i.e., different numbers of subsamples K) and the whole sample sizes N. The corresponding subsample sizes n and quantile levels  $1-\tau$  (induce the threshold value) are shown in the table. Apart from three measurements (RMSE, ECP and RA) defined in Section 3.2, the average of the total exceedance size  $(n_*^u)$ , bias (Bias) and standard deviation (SD) of  $\hat{\gamma}_{\text{AML}}$  are also reported. The target ECP is 95%.

$C_K$	N	n	$1-\tau$	K	$n_*^u$	Bias	SD	RMSE	ECP	RA
$\circ_K$	$(\times 10^{5})$	11	(%)	$I\Lambda$	114*		(×10 <sup>-2</sup>	2)	(%)	(%)
Case 1: Fréchet(1) distribution										
0.3	1.0	316	95.9	2	26.0	0.25	21.20	21.20	91.4	124.0
	5.0	707	97.4	2	36.8	0.24	17.03	17.03	93.4	75.4
	10.0	1,000	97.8	2	44.1	0.88	15.19	15.21	93.9	56.2
	50.0	2,236	98.6	3	93.8	0.09	10.84	10.84	93.1	32.2
0.5	1.0	316	95.9	4	51.5	1.76	14.79	14.90	93.9	67.3
	5.0	707	97.4	5	92.4	1.17	10.71	10.78	95.1	37.0
	10.0	1,000	97.8	5	109.8	0.45	9.95	9.96	94.1	34.5
	50.0	2,236	98.6	6	188.3	0.33	7.38	7.39	95.1	21.1
0.7	1.0	316	95.9	7	90.5	1.39	11.18	11.27	94.1	47.6
	5.0	707	97.4	9	165.2	0.86	8.16	8.20	95.2	28.7
	10.0	1,000	97.8	11	242.3	0.60	6.75	6.77	94.2	22.2
	50.0	2,236	98.6	14	438.3	0.33	4.74	4.75	95.0	13.7
Case 2: Fréchet(2) distribution										
0.3	1.0	316	95.9	2	26.0	0.50	42.40	42.41	91.4	124.0
	5.0	707	97.4	2	36.8	0.48	34.06	34.06	93.4	75.4
	10.0	1,000	97.8	2	44.1	1.75	30.38	30.43	93.9	56.2
	50.0	2,236	98.6	3	93.8	0.18	21.67	21.67	93.1	32.2
0.5	1.0	316	95.9	4	51.5	3.53	29.58	29.79	93.9	67.3
	5.0	707	97.4	5	92.4	2.35	21.42	21.55	95.1	37.0
	10.0	1,000	97.8	5	109.8	0.90	19.91	19.93	94.1	34.5
	50.0	2,236	98.6	6	188.3	0.65	14.77	14.78	95.1	21.1
0.7	1.0	316	95.9	7	90.5	2.79	22.37	22.54	94.1	47.6
0.7	5.0	707	95.9 97.4	9	165.2	2.79 1.71	22.37 16.31	16.40	94.1 $95.2$	$\frac{47.0}{28.7}$
	10.0	1,000	97.4 97.8	9 11	242.3	1.71	10.51 $13.50$	10.40 $13.55$	95.2 $94.2$	28.1 22.2
	50.0	2,236	97.8 98.6	$\frac{11}{14}$	438.3	0.66	9.47	9.50	94.2 $95.0$	13.7
	0.00	2,230	98.0	14	400.0	0.00	9.41	9.00	90.0	13.7

Table 4: Descriptive statistics for continuous variables we focused, based on 10,000 records. These statistics include mean, median (Med), minimum (Min), maximum (Max) and kurtosis (Kurt). The normal value regions (for 99.9% normal records) and the correlated suspected outlier probability (Sop) for each variable, are reported in the last three columns.

						Lower	Upper	Sop
Variable	Mean	Med	Min	Max	Kurt	Bound	Bound	(%)
ActualElapsedTime	121.4	102.0	16.0	656.0	5.6	29.9	365.2	0.9
CRSElapsedTime	122.5	103.0	18.0	660.0	5.8	31.7	366.0	1.0
ArrDelay	7.4	0.0	-49.0	1149.0	184.1	-31.6	175.7	0.9
DepDelay	8.5	0.0	-64.0	1428.0	255.6	-11.3	167.9	0.8
Distance	716.8	564.7	11.0	4962.0	6.6	72.3	25768	1.4