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Testing the independence of maxima: from bivariate vectors to spatial extreme fields

Asymptotic independence of extremes

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Abstract Characterizing the behaviour of multivariate or spatial extreme values is of fundamental interest to understand how extreme events tend to occur. In this paper we propose to test for the asymptotic independence of bivariate maxima vectors. Our test statistic is derived from a madogram, a notion classically used in geostatistics to capture spatial structures. The test can be applied to bivariate vectors, and a generalization to the spatial context is proposed. For bivariate vectors, a comparison to the test by Falk and Michel (Ann Inst Stat Math 58:261–290, 2006) is conducted through a simulation study. In the spatial case, special attention is paid to pairwise dependence. A multiple test procedure is designed to determine at which lag asymptotic independence takes place. This new procedure is based on the bootstrap distribution of the number of times the null hypothesis is rejected. It is then tested on maxima of three classical spatial models and finally applied to two climate datasets.

Keywords Bivariate extremes · Asymptotic independence · Max-stable random fields · Spatial processes

AMS 2000 Subject Classifications Primary—62G32 · 62H11 · 62G10; Secondary—60G70 · 62M30

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1 Introduction

For a wide class of environmental and climate studies, spatial extreme values are of fundamental interest since extreme events may have dramatic consequences. If standard geostatistic approaches perform well for statistical inference on the mean behaviour of spatial processes, it is well known that they are of limited interest when dealing with extreme realizations. As in the univariate case, the mean and the extremes of a spatial process are essentially of different nature and spatial dependencies related to mean values and to extreme ones can be very different. As a consequence, specific approaches and models have to be developed to infer on the extremal behaviour of spatial processes.

When addressing environmental or climatological issues involving spatial data, the recourse to a multivariate framework cannot be avoided. The multivariate extreme value theory (MEVT) offers various notions to capture the main characteristics of the underlying dependence structure (see for example Beirlant et al. 2004 and the references therein). A useful one is asymptotic independence. Roughly speaking, a random vector has its components asymptotically independent if an increasing number of independent copies tend to have their componentwise maxima independent. For instance, Sibuya (1960) showed that jointly normal variables which are not perfectly correlated are asymptotically independent. Asymptotic independence is a rather complicated notion which has little to do with the amount of independence of the component of the original vector. Resnick (2002) and Maulik and Resnick (2004) characterized how asymptotic independence is related to distribution refinements such as hidden regular variation and multivariate second order regular variation. In practice, asymptotic dependence is difficult to detect. In a series of papers, Ledford and Tawn (1996, 1997) proposed statistical methods for estimating the joint tail of a multivariate distribution. In particular, they introduced a quite general submodel for the joint survivor distribution where the nature of the tail dependence is characterized by a coefficient $\eta \in (0, 1]$, usually called *coefficient of tail dependence*. Asymptotic independence corresponds to $\eta \neq 1$ and the value of η quantifies the strength of the tail dependence in asymptotic independence (see Ledford and Tawn 1997 for details). In the last years, parametric and non parametric inference for η have been studied (Ledford and Tawn 1996, 1997; Peng 1999; Draisma et al. 2004). Note that Coles et al. (1999) introduced two functions $\chi(\cdot)$ and $\bar{\chi}(\cdot)$ whose graphs allow to distinguish between asymptotic independence and dependence (see e.g. Coles 2001). They also propose a model-free dependence measure $\bar{\chi}$, obtained as a limit of the function $\bar{\chi}(\cdot)$, which is equal to $2\eta - 1$ under the Ledford and Tawn model assumption.

In this paper, we are concerned with testing for pairwise independence of maxima from spatial data. Our objective is not to model extremal dependence of joint tails but rather to set up a statistical test for independence in a spatial context. Such a test should facilitate the modelling of the spatial data by a random field with appropriate extremal behaviour. At first, a new and simple test for the asymptotic independence of bivariate vectors of maxima is proposed. An extension to the spatial case is then derived, leading to a test for asymptotic independence on several classes of distance. It involves a standard geostatistical tool that is called a madogram (Matheron 1989).

The paper is organized as follows. Bivariate vectors of maxima are first considered. In Section 2, the notion of asymptotic independence for bivariate vectors is reviewed and several classical distribution-free statistical tests of asymptotic independence are presented. The test we propose is introduced in Section 3. Its performance is compared with that of Falk and Michel (2006) using simulations. The spatial framework is then considered. A test for asymptotic independence applicable to random fields of maxima is proposed in Section 4. Two cases are distinguished depending on whether a single or multiple realizations are available. Because rejecting the null hypothesis of asymptotic independence on several classes of distances requires a multi-test approach on (spatially) correlated data, a bootstrap procedure controlling the False Discovery Rate (FDR) is set up. This procedure is finally applied, firstly in Section 4 on simulated random fields with known asymptotic dependence behaviours, secondly in Section 5 on two real data sets (annual maximal temperatures of 29 French towns and annual maximal precipitations over a 30 year period in the French region of Burgundy). Conclusions are drawn and some perspectives for future work are discussed in Section 6.

2 Bivariate extreme distributions and asymptotic independence

Let $(X_1, Y_1), (X_2, Y_2) \ldots$ be independent copies of a bivariate random vector (X, Y) with distribution function K and marginals F_X and F_Y. The classical bivariate extreme theory is concerned with the limit behaviour of

$$(M_n(X), M_n(Y)) \equiv \left(\max_{i=1,\ldots,n} X_i, \max_{i=1,\ldots,n} Y_i \right)$$

as $n \to +\infty$. Because of this definition, the normalized marginals of $(M_n(X), M_n(Y))$ belong to the generalized extreme value (GEV) distribution family. The general form of a GEV distribution is $GEV_{\mu,\sigma,\xi}(x) = \exp(-[1 + \xi \frac{x-\mu}{\sigma}]^{-1/\xi})$ with $\mu \in \mathbb{R}, \sigma > 0, \xi \in \mathbb{R}, 1 + \xi(x - \mu)/\sigma > 0$ (Coles 2001).

From now onward, it is assumed without loss of generality that $F_X \equiv F_Y \equiv F$, where $F(\cdot)$ is the unit Fréchet distribution $F(z) = \exp(-1/z)$, $z > 0$. Hence, the limit distribution of $M_n(X)/n$ and $M_n(Y)/n$ is also unit Fréchet.

The following theorem (de Haan and Resnick 1977) characterizes the limit joint normalized distribution of $(M_n(X), M_n(Y))$.

Theorem 1 *If $P(M_n(X) \le nx, M_n(Y) \le ny) \underset{n\to\infty}{\longrightarrow} G(x, y)$, where G is a non-degenerate distribution function, then $G(\cdot, \cdot)$ takes the form $G(x, y) = \exp(-V(x, y))$ with*

$$V(x, y) = 2 \int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) dH(\omega)$$

and H is a distribution on $[0, 1]$ with mean $\frac{1}{2}$ (spectral measure).

Note that the marginal distributions of G are unit Fréchet. In the particular case $H = (\delta_0 + \delta_1)/2$ where δ_z denotes a Dirac measure at z, then $G(x, y) = F(x)F(y)$. We then say that X and Y are *asymptotically independent*. It has been established by Joe (1993) that (X, Y) are asymptotically independent if and only if

$$\chi \equiv \lim_{z\to\infty} \frac{P(X > z, Y > z)}{1 - F(z)} = \lim_{z\to\infty} P(Y > z \mid X > z) = 0.$$

Inference on χ is difficult because few observations are available as $z \to \infty$. Using the properties of V, it can be shown that χ can be written as $2 - \theta$ where $1 \leq \theta \leq 2$ is a coefficient satisfying $G(x, x) = \exp(-\theta/x) \equiv F(x)^{\theta}$. θ is called the *extremal coefficient* (see Schlather and Tawn 2003 for details). Asymptotic independence corresponds to the case $\theta = 2$.

3 Testing for bivariate asymptotic independence

3.1 Two usual tests

Several approaches to test for bivariate asymptotic independence have been proposed in the literature (see Ledford and Tawn 1997; Coles et al. 1999; Draisma et al. 2004; Ramos and Ledford 2005; Hüsler and Deyuan 2009, and the references therein). In the sequel, two of them which are distribution-free are discussed , namely a graphical test by de Haan and de Ronde (1998) and a statistical test by Falk and Michel (2006).

3.1.1 A graphical test

Suppose that G is non-degenerate, and consider the function

$$\ell(x, y) = \lim_{n\to\infty} nP\left(1 - F(X) < \frac{x}{n} \text{ or } 1 - F(Y) < \frac{y}{n}\right) \qquad x, y > 0.$$

Its level sets $\mathcal{Q}_c = \{(x, y) \mid \ell(x, y) = c\}$ satisfy a number of interesting properties established by de Haan and de Ronde (1998). In particular, the curve $\mathcal{Q}_c$ along with the axes $x = 0$ and $y = 0$ delimits a convex domain D_c such that $c \leq c' \Rightarrow D_c \subset D_{c'}$. Moreover the graph of this curve is closely related to the strength of the dependence between X and Y. $\mathcal{Q}_c$ is equal to $\{(x, y) \mid x + y = c\}$ if X and Y are asymptotically independent, and $\{(x, y) \mid \max(x, y) = c\}$ if X and Y are fully dependent. A graphical test based on $\mathcal{Q}_c$ curves with different c values can be used: asymptotic independence comes out as a straight line between $(0, c)$ and $(c, 0)$. Such an approach is only graphical but can be helpful in practical situations.

3.1.2 A statistical test

Originally set up for bivariate distributions with reverse exponential margins, the test by Falk and Michel (2006) can be adapted to unit Fréchet margins. More specifically, let $\varepsilon > 0$ and $t \in [0, 1]$. When ε tends to 0, the conditional distribution function

$$K_\varepsilon(t) \equiv P\{X^{-1} + Y^{-1} < \varepsilon t \mid X^{-1} + Y^{-1} < \varepsilon)$$

tends to t^2 if X and Y are asymptotically independent, and t otherwise. This result can be used to test for the asymptotic independence of X and Y using classical goodness-of-fit tests such as the Kolmogorov–Smirnov or the likelihood ratio as well as the chi-square test.

Note that Frick et al. (2007) recently proposed a generalization of Falk and Michel's work, based on a second order differential expansion of the spectral decomposition of G. They focused on the case of a null hypothesis of tail dependence against a composite alternative representing the various degrees of tail independence. Since these hypotheses differ notably from those we consider, we will restrict our comparison to the Falk and Michel approach.

3.2 A statistical test based on a madogram

In this section, the test we propose for bivariate vectors is presented. Simulations are then used to compare its performances with those of the two aforementioned tests.

3.2.1 A new statistical test for bivariate vectors

Suppose again that G is non-degenerate with spectral measure H, and consider the random variable

$$W = \frac{1}{2} \mid F(X) - F(Y) \mid .$$

Because $F(X)$ and $F(Y)$ are uniformly distributed on $[0, 1)$, their dependence relationships specify the distribution of W. If $X = Y$ almost surely, then the distribution of W is a Dirac distribution at 0. If X and Y are independent, then W admits the p.d.f. $f_W(z) = 4 - 8z$ on $[0, \frac{1}{2})$. Of course many intermediary situations are possible between perfect dependence and full independence. In other words, the distribution of W provides information about the asymptotic dependence between X and Y. Cooley et al. (2006) have shown that the mean of W is related to the extremal coefficient θ by the formula

$$\mathbb{E}(W) = \frac{1}{2}\frac{\theta - 1}{\theta + 1} \equiv \nu_W. \tag{1}$$

This formula and some variations (Bel et al. 2008), can be used for estimating θ. Regarding the variance of W, a proof is given in the Appendix that

$$\sigma_W^2 = \frac{1}{6} - \frac{1}{4}\left(\frac{\theta - 1}{\theta + 1}\right)^2 - \frac{1}{2}\int_0^1 \frac{dt}{\left[1 + A(t)\right]^2} \tag{2}$$

where A is the classical Pickands dependence function

$$A(t) = 2\int_0^1 \max\big(\omega(1-t), (1-\omega)t\big)\, dH(\omega).$$

In the case of asymptotic independence, $A(t) \equiv 1$ and $\sigma_W^2 = \frac{1}{72}$.

Formula 1 shows that ν_W is a strictly monotonic increasing function of θ with $\nu_W = \frac{1}{6}$ if and only if $\theta = 2$. Accordingly, the asymptotic independence of X and Y can be checked by testing the null hypothesis $H_0 : \nu_W = \frac{1}{6}$ against $H_1 : \nu_W < \frac{1}{6}$. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sequence of independent copies of (X, Y). Then a natural estimator for ν_W is

$$\widehat{\nu_W} = \frac{1}{2n}\sum_{i=1}^{n} \mid \hat{F}(X_i) - \hat{F}(Y_i) \mid$$

where $\hat{F}$ is the empirical distribution function as provided by the X_i's and the Y_i's.

It can be shown (Fermanian et al. 2004) that

$$\sqrt{n}\frac{\widehat{\nu_W} - \frac{1}{6}}{\widehat{\sigma_W}} \xrightarrow{\mathscr{D}} \mathscr{N}(0, 1) \tag{3}$$

as $n \to \infty$ under H_0, providing a straightforward test for the asymptotic independence of bivariate random vectors.

As the expectation ν_W is related to the well-known madogram used in spatial statistics, we call this test the madogram test.

3.2.2 Simulation study

Samples of independent bivariate vectors of maxima are simulated and the null hypothesis *asymptotic independence* is tested against *non asymptotic independence*. Three bivariate max-stable distributions $G(\cdot, \cdot)$ and an empirical distribution of bivariate Gaussian maxima are considered. The four distributions are:

(D1) Logistic distribution (Tawn 1988)
Defined as

$$G(x, y) = \exp\left\{-\left(x^{-1/\alpha} + y^{-1/\alpha}\right)^{\alpha}\right\} \qquad x, y > 0,$$

this bivariate distribution depends on one parameter $0 < \alpha \leq 1$. When $\alpha = 1$, we have $G(x, y) = \exp\{-(x^{-1} + y^{-1})\}$, which is the independence case. When $\alpha \to 0$, then we get $G(x, y) \to \exp\{-\max(x^{-1}, y^{-1})\}$, which is the perfect dependence case.

(D2) Asymmetric logistic distribution (Tawn 1988)
This generalization of the logistic distribution allows asymmetry and nonexchangeability. Its general form is given by

$$G(x, y) = \exp\left\{-\frac{1-\psi_1}{x} - \frac{1-\psi_2}{y} - \left(\left(\frac{\psi_1}{x}\right)^{1/\alpha} + \left(\frac{\psi_2}{y}\right)^{1/\alpha}\right)^{\alpha}\right\}$$
$$x,\ y > 0$$

It depends on three parameters, namely $0 < \alpha \leq 1$ and $0 \leq \psi_1, \psi_2 \leq 1$. Independence corresponds to $\alpha = 1$ or $\psi_1\psi_2 = 0$, whereas perfect dependence is obtained when $\psi_1\psi_2 = 1$ and $\alpha \to 0$.

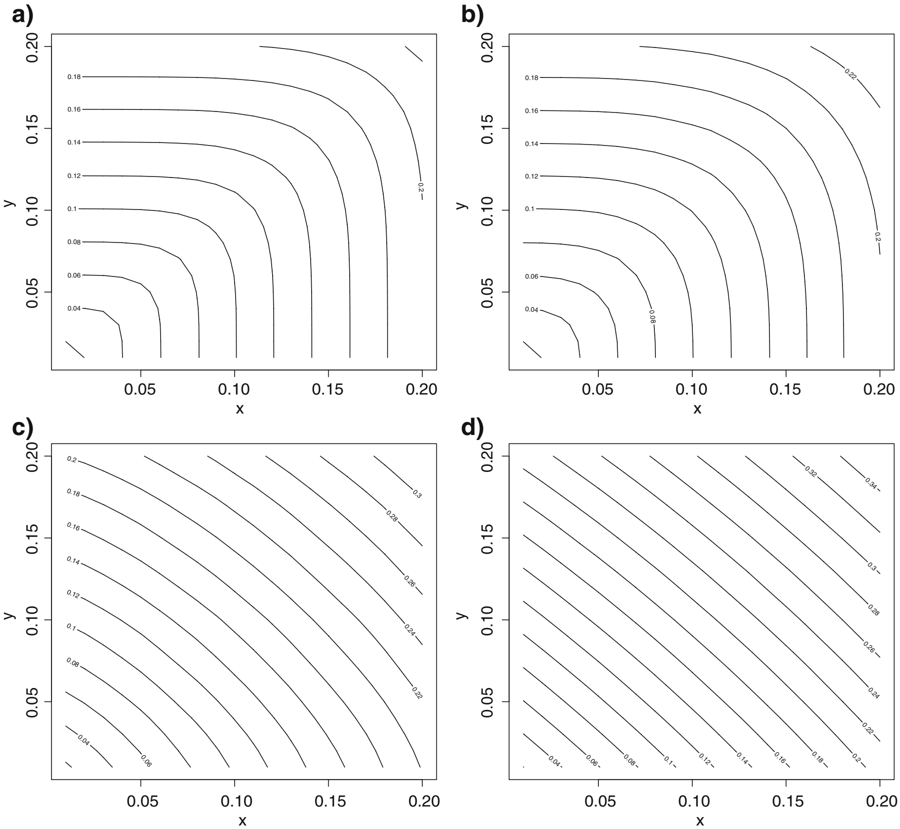

Fig. 1 de Haan graphical test for models **a**) logistic ($\alpha = 0.2$), **b**) asymmetric logistic ($\psi_1 = \psi_2 = 0.4$), **c**) Hüsler–Reiss ($\alpha = 0.9$), **d**) Gaussian ($\rho = 0.5$).

(D3) Hüsler–Reiss distribution (1989)
This bivariate distribution is defined as

$$G(x, y) = \exp\left\{-\frac{1}{x}\Phi\left(\frac{1}{\alpha}+\frac{\alpha}{2}\log\frac{y}{x}\right)-\frac{1}{y}\Phi\left(\frac{1}{\alpha}+\frac{\alpha}{2}\log\frac{x}{y}\right)\right\}$$
$$x, y > 0,$$

where $\Phi(\cdot)$ is the standard normal distribution function. It depends on one parameter $\alpha > 0$. Independence and perfect dependence are limit cases respectively obtained when $\alpha \to 0$ and $\alpha \to \infty$.

(D4) Empirical distribution of bivariate Gaussian maxima.
A sample of realizations is generated by taking 500 times the componentwise maxima over 10,000 realizations of a bivariate Gaussian vector with a fixed correlation parameter ρ.

For each distribution, 500 samples of size 500 are simulated. Distributions D1, D2 and D3 have been simulated using the R-package *evd* (Stephenson 2002) with specified parameter values. Regarding distribution D4, the *mnormt* package has been used.

Figure 1 shows how the graphical test performs on those 500×500 samples. It can be checked that the level lines produced by D4 are almost straight. Curvatures of the level lines are slightly more pronounced for D3, and definitely more pronounced for D1 and D2. In other words panels (a) and (b) lead to a clear unambiguous conclusion regarding asymptotic dependence whereas panel (c) could lead to the same conclusion and panel (d) to the conclusion of asymptotic independence but with some ambiguity.

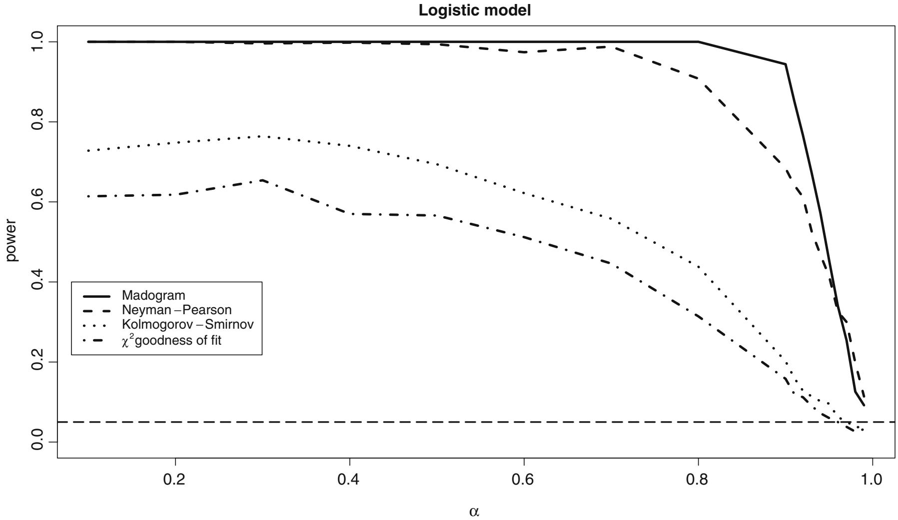

Fig. 2 Power function for the logistic model. The *dashed horizontal line* indicates significant threshold 0.05. The parameter α is related to the strength of the dependence.

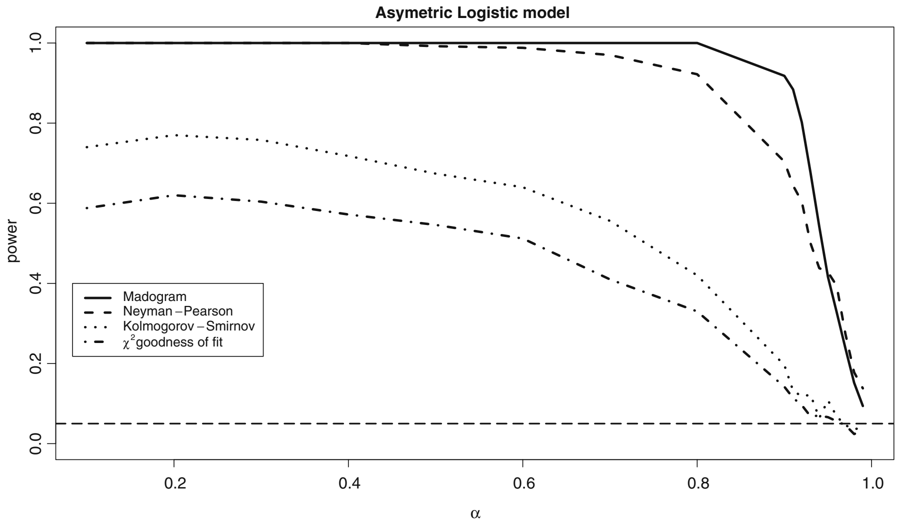

Fig. 3 Power function for the asymmetric logistic model. The *dashed horizontal line* indicates significant threshold 0.05. The parameter α is related to the strength of the dependence and asymmetry parameters are fixed to 0.4.

Asymptotic independence is now tested using four different statistics, namely the madogram statistic and three other statistics derived from the Falk and Michel (2006) test procedure (Neyman–Pearson, Kolmogorov–Smirnov and Chi-square statistics). This procedure requires a constant ε that has been chosen as the 90% quantile of the distribution of $X + Y$.

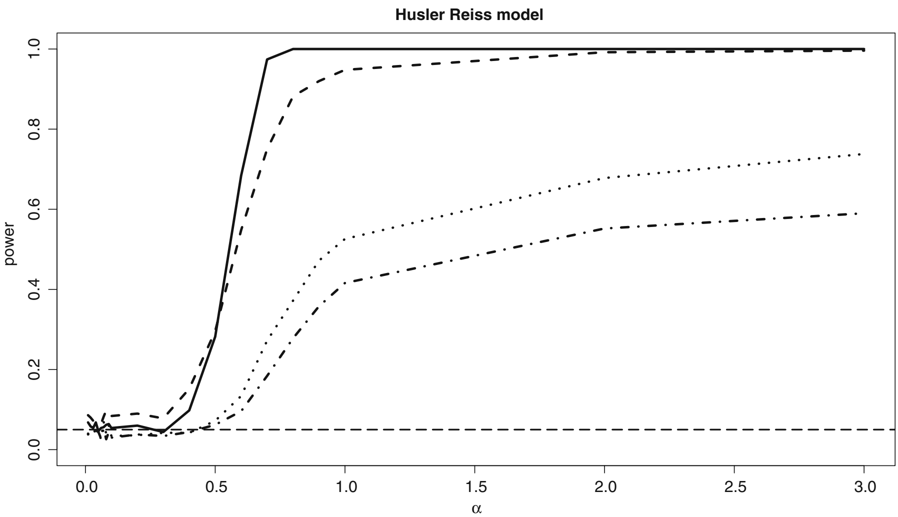

Fig. 4 Power function for the Hüsler–Reiss model. The *dashed horizontal line* indicates significant threshold 0.05. The parameter α is related to the strength of the dependence.

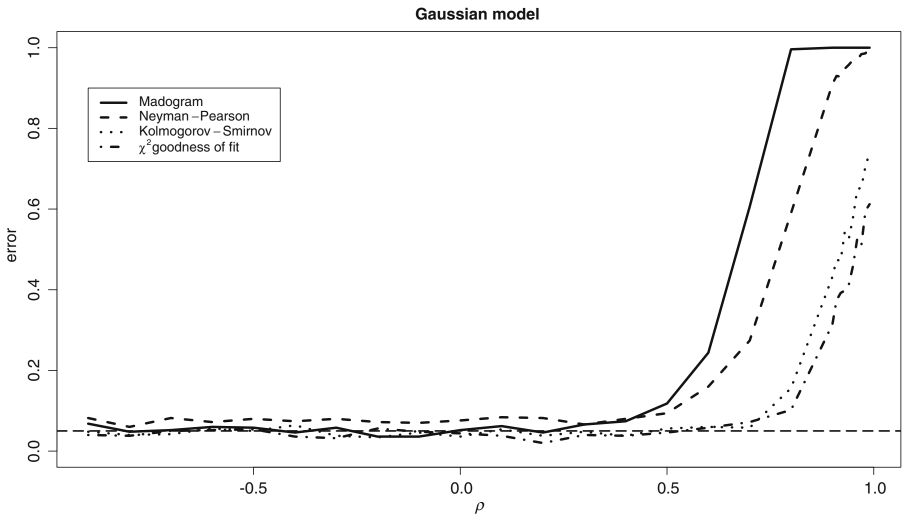

Fig. 5 Type I error function for the maxima of Gaussian model. The *dashed horizontal line* indicates significant threshold 0.05. Parameter ρ is the correlation.

The power function, that is the probability of rejecting the asymptotic independence hypothesis for max-stable bivariate vectors, as a function of the parameter distribution, is calculated on the three max-stable models. For D1 and D2, α varies from 0.1 to 0.9 by steps of 0.1, and from 0.90 to 0.99 by steps of 0.01. The two parameters ψ_1 and ψ_2 of the asymmetric model are set to 0.4. For D3, α varies from 0.01 to 0.09, from 0.1 to 1.0 and from 1 to 3 by respective steps of 0.01, 0.1 and 1.

On the D4 model, known to be asymptotically independent, the type I error, that is the probability of wrongly rejecting the asymptotic independence hypothesis is calculated as a function of the correlation ρ. ρ varies from -0.9 to 0.9 by steps of 0.1 and from 0.90 to 0.99 by steps of 0.01.

Results for the power functions are presented in Figs. 2, 3 and 4. Despite its simplicity, the madogram based test appears often the most powerful. The Falk and Michel procedure using the Neyman–Pearson goodness of fit test is slightly less powerful whereas the Chi-square and Kolmogorov–Smirnov tests definitely lack power. Regarding the Gaussian case, Fig. 5 shows that when ρ is growing the type I error increases up to the value 1 when the correlation becomes close to 1. This is more pronounced for the madogram test and the Neyman–Pearson version of the Falk and Michel test: it is the usual price to pay to get more powerful tests.

4 The spatial framework

4.1 A test for spatial asymptotic independence

Let $Z(\cdot)$ be a stationary, isotropic random field with unit Fréchet marginal distribution. In geostatistics, it is usual to characterize spatial dependence using the semi-variogram $\gamma(h) = \frac{1}{2}E(Z(s+h) - Z(s))^2$. However second order moments may not be adapted for extremes and in the sequel the madogram $\nu_Z(h) = \frac{1}{2}E \mid Z(s+h) - Z(s) \mid$ is considered instead. In order to determine whether asymptotic independence takes place at lag h, the previous results on bivariate random vectors (Section 3) can be applied to all pairs $(Z(s_k), Z(s_\ell))$ of samples such that $\|s_k - s_\ell\| = h$ or more generally $h - \varepsilon \leq \|s_k - s_\ell\| \leq h + \varepsilon$ for some prespecified $\varepsilon > 0$. Two different situations are considered depending on whether a single realization or more than one independent realizations of the random field are available at each (sampled) site because the procedure for testing asymptotic independence is not the same.

4.1.1 Case of multiple realizations

Let C be one class of distances. In order to test asymptotic independence (AI) on C, we are faced with a problem of multiple hypotheses testing that needs to be carefully handled. Controlling a global type I error, say α, under multiple hypotheses testing is known to be difficult, and the false discovery rate (FDR) approach (Benjamini and Hochberg 1995) appears as one of the most relevant to cope with it.

FDR is the expected proportion of incorrectly rejected null hypotheses among all the rejected hypotheses. Starting from the p-values associated to all pairs of sites of class C, the FDR approach makes it possible to determine which p-values are significant while controlling the fixed global type I error. In order to decide whether the AI hypothesis is admissible or not, we need to assess how many pairs would be wrongly rejected under the null hypothesis of global AI. A decision rule could be, for instance, that there are less than a fixed proportion α of significant p-values. In fact, this procedure would be justified if the p-values distribution was uniform under the null hypothesis. In our case, the test statistics are spatially dependent, so that the p-value distribution is unknown under the null hypothesis. A way to cope with this problem is to resort to a bootstrap procedure. The test statistics that we consider are approximatively normally distributed and positively correlated. Adapting the demonstration of Benjamini and Yekutieli (2001) for multivariate normal test statistics, it can be shown that this family of test statistics satisfies the PRDS property.[1] Then the standard Benjamini–Hochberg procedure for determining significant p-values controls the FDR as in an independent framework. Let R be the (data based) number of individual hypothesis rejected using this approach. The following procedure aims at simulating the distribution of R under the null hypothesis of AI while preserving the

[1] PRDS stands for *Positive Regression Dependency on Subset*. Let $\leq$ be a partial order on $\mathbb{R}^m$. A subset D of $\mathbb{R}^m$ is called an upper set if $x \in D$ and $y \geq x$ implies $y \in D$. Then the random vector $\mathbf{X} = (X_1, \ldots, X_m)$ is said to satisfy the PRDS property on $I_0 \subset \{1, \ldots, m\}$ if for each upper set D and for each $i \in I_0$, $P(\mathbf{X} \in D \mid X_i = x)$ is a non-decreasing function in x.

spatial dependence structure. In this procedure, S_ℓ stands for the set of all sites whose distance to a given site s_ℓ belongs to class C. Its cardinality is denoted by m_ℓ.

1. *Let* $\mathbf{Z}(s_\ell) = \left(Z^i(s_\ell), i = 1, \ldots, n\right)$ *be the vector of realizations at* s_ℓ *and denote by* $\mathbf{Z}^*(s_\ell) = \left(Z^{i*}(s_\ell), i = 1, \ldots, n\right)$ *the resampling vector at* s_ℓ *obtained from a standard bootstrap on* $\mathbf{Z}(s_\ell)$*;*
2. *For each site* $s_j \in S_\ell$ *compute* $\widehat{\nu^*}_{F(Z)}(s_\ell, s_j) = \dfrac{1}{2n}\displaystyle\sum_{i=1}^{n}\left|\widehat{F}(Z^{i*}(s_\ell)) - \widehat{F}(Z^i(s_j))\right|$*;*
3. *Compute the vector of the associated p-values* $(p_1^*, \ldots, p_{m_\ell}^*)$*;*
4. *Assign successively the role of* s_ℓ *to each sampled site and repeat (1) to (3). For each particular site* s_ℓ*, pairs of sites which have been already taken into account along the procedure are removed;*
5. *Consider the complete sample of p-values obtained at the end of (4) and use the FDR approach to count the number of rejected null hypotheses* R^* *among all the performed tests;*
6. *Repeat* (1) *to* (5) *a large number of times and derive the empirical distribution of* R^*.

Resampling values of one of the terms in the pair ensures the asymptotic independence, while not resampling the other term will let the p-values being spatially dependent. Hence this procedure gives an empirical distribution of the number R^* of rejected hypotheses under the assumption of global asymptotic independence in a spatial dependent framework. It then suffices to compute empirically $P(R^* \geq R)$ to conclude about the validity of this assumption.

4.1.2 Case of a single realization

When independent copies of the maxima field are not available, the stationarity property of the field is to be used in order to calculate the estimators. Marginal law is given by $\widehat{F}(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}_{\{Z(s_i)\leq z\}}$ and the madogram is estimated in each distance class C_k by

$$h \in C_k \qquad \widehat{\nu_F}(h) = \frac{1}{2n_k}\sum_{d(s_{\ell_1}, s_{\ell_2})\in C_k} \mid \hat{F}(Z(s_{\ell_1}) - \hat{F}(Z(s_{\ell_2})) \mid .$$

The asymptotic independence on each class can then be tested by the standard Benjamini–Hochberg procedure, but clearly the relevance of (3) will be affected because of spatial dependencies.

4.2 Simulation study

Three random field models are considered, namely the Gaussian model, the storm model and the Gaussian extremal model. These models have been considered in Bel et al. (2008), with the goal of estimating the extremal function $\theta(h)$. Note that the last two models are max-stable but the first one is not. For each model let

$Z_1, Z_2, \ldots, Z_n$ be n independent copies of a stationary random field and $M_n = \frac{1}{n}\max(Z_1, Z_2, \ldots, Z_n)$ for n large enough. We consider N independent copies of the maxima random field M_n.

Gaussian model Here the Z_i's are two-dimensional Gaussian random fields with autocorrelation function ρ. The bivariate distribution of $(M_n(s), M_n(s+h))$ is asymptotically independent. Accordingly $\theta(h) = 2$ as soon as $h \neq 0$. Simulations are drawn on $S = 200$ points with $n = 10{,}000$ and $N = 100$. For each site $s \in \mathbb{R}^2$, the bivariate distribution of $(M_n(s), M_n(s+h))$ is asymptotically independent. Accordingly $\theta(h) = 2$ as soon as $h \neq 0$. 100 random fields of maxima over $n = 10{,}000$ Gaussian random fields have been simulated. The correlation function of each Gaussian field is $\rho(h) = exp(-h/20)$. Two hundred random values on a square of side 100 have been simulated this way.

Storm model Introduced by Smith (1990), this model is defined by $Z(s) = \sup_{j=1,2,\ldots} \zeta_j g(x_j - s)$, $s \in \mathbb{R}^2$, where $(\zeta_j, x_j)_j$ are the points of a Poisson process on $(0,\infty)\times\mathbb{R}^2$ with intensity measure $d\Lambda(\zeta, x) = \zeta^{-2} d\zeta dx$. The process Z is max-stable with unit Fréchet margins. Here the deterministic function g is chosen as a bivariate Gaussian pdf with covariance matrix C.

The random vector $(M_n(s), M_n(s+h))$ is Hüsler–Reiss distributed with extremal coefficient function $\theta(h) = 2\Phi(\frac{1}{2}\sqrt{h'C^{-1}h})$, where Φ denotes the standard Gaussian distribution function. As a consequence, asymptotic independence is expected as soon as h has its modulus large enough. Simulations are drawn on $S = 100$ points with $n = 30$ and $N = 100$.

Extremal Gaussian model Let Y be a stationary standard Gaussian random field with correlation function ρ, and let Π be a Poisson point process on $(0,\infty)$ with intensity measure $d\Lambda(\zeta) = \sqrt{2\pi}\zeta^{-2}d\zeta$. The extremal Gaussian process proposed by Schlather (2002) is defined as $Z(s) = \max_{\zeta\in\Pi} \zeta Y_\zeta(s)$, $s \in \mathbb{R}^2$. Then the random vector $(M_n(s), M_n(s+h))$ is asymptotically dependent with extremal function $\theta(h) = 1 + \sqrt{1-\rho(h)}/\sqrt{2}$. It should be pointed out that $\lim_{|h|\to\infty}\theta(h) = 1 + 1/\sqrt{2} < 2$, even if $\rho(h)$ tends to 0 as the modulus $|h|$ of h tends to ∞. Therefore asymptotic independence does not occur at any lag. Simulations are drawn on $S = 100$ points with $n = 30$ and $N = 100$.

4.2.1 Case of multiple realizations

Results are presented in Figs. 6, 7, and 8. For the maxima of Gaussian random fields the observed value R of the number of rejected hypotheses disagrees with the bootstrap distribution of R^* at short lags ($h < 8$). Accordingly, the global asymptotic independence is rejected for $h < 8$ and accepted for larger lags. Lags between 0 and 8 correspond to correlation between 1 and 0.75 and as in the bivariate case we observe that the test does not perform very well for the Gaussian case when the correlation is high. For the Gaussian storm process, the null hypothesis is rejected up to when

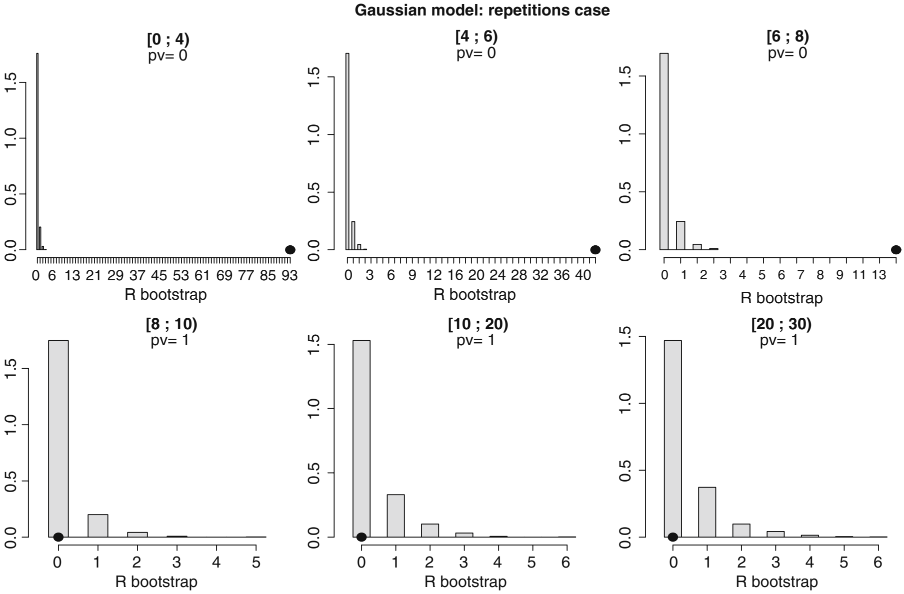

Fig. 6 Gaussian model, distribution of R^*, number of rejected hypotheses under the null hypothesis for some classes of distances. *Bullet*: observed R, pv=$P(R^* \geq R)$.

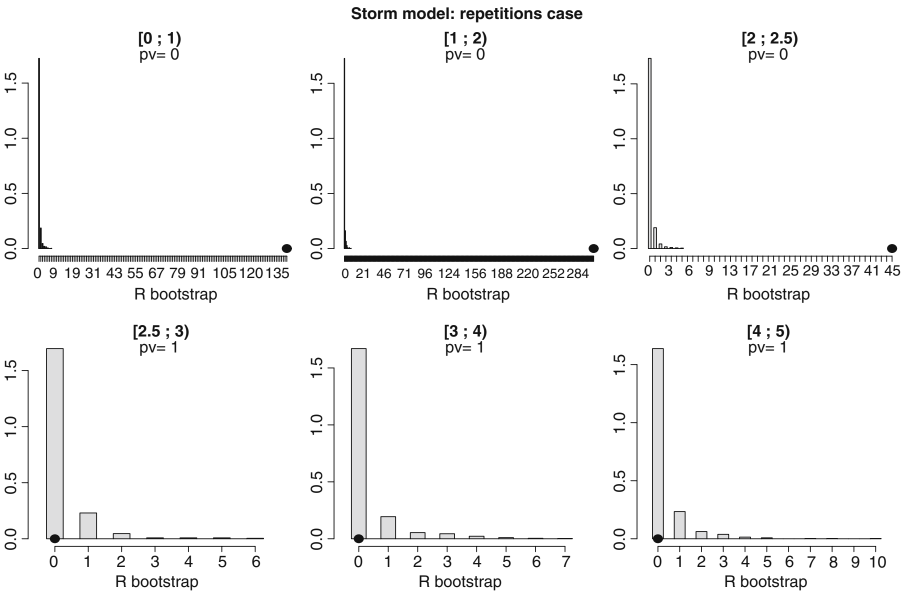

Fig. 7 Storm model, distribution of R^*, number of rejected hypotheses under the null hypothesis for some classes of distances. *Bullet*: observed R, pv=$P(R^* \geq R)$.

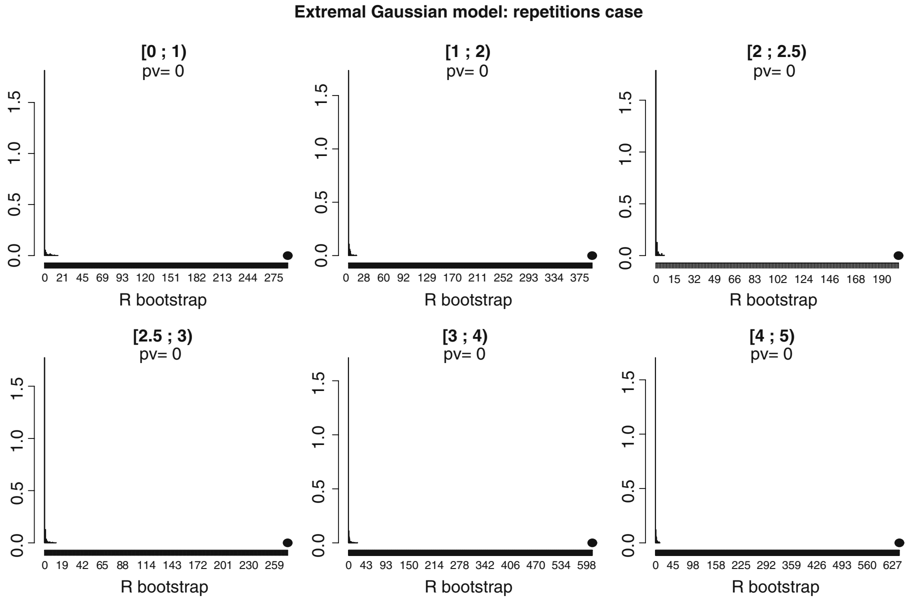

Fig. 8 Extremal Gaussian model, distribution of R^*, number of rejected hypotheses under the null hypothesis for some classes of distances. *Bullet*: observed R, pv=$P(R^* \geq R)$.

$|h| = 2.5$ and accepted beyond that distance. It can be mentioned that $\theta(2.5) = 1.79$ with the correlation matrix $C = Id$ considered for this exercise. For the extremal Gaussian model with correlation function $\rho(h) = \exp(-|h|)$, the null hypothesis is always rejected, whatever the class of distance considered, which was the expected conclusion as this random field does not display asymptotic independence at any lag.

4.2.2 Case of a single realization

For each random field of maxima, a p-value based on the madogram is computed at each distance class. Then standard FDR procedure is applied to determine which p-values are significant. The same operation is repeated 100 times. Tables 1, 2, and 3 give the number of significant and non significant p-values obtained on those realizations.

Table 1 Gaussian model, number of significant and non significant p-values for several classes of distances in the spatial case

Classes	[0 ; 4)	[4 ; 6)	[6 ; 8)	[8 ; 10)	[10 ; 20)	[20 ; 30)
Significant	99	58	42	11	8	3
Non significant	1	42	58	89	92	97

Table 2 Storm model, number of significant and non significant p-values for several classes of distances in the spatial case

Classes	[0 ; 1)	[1 ; 2)	[2 ; 2.5)	[2.5 ; 3)	[3 ; 4)	[4 ; 5)
Significant	100	84	33	17	12	8
Non significant	0	16	67	83	88	92

For the Gaussian random field and the first class of distances ($h < 0.05$), the p-values are given as significant for almost all copies. The correlation at small distances is too high for asymptotic independence to be detected. For the storm model, the number of non significant p-values becomes larger than that of significant p-values at distance $|h| = 2.0$, which is a bit smaller than what was obtained in the case of multiple realizations. For the Gaussian extremal model, the number of non significant p-values becomes larger than that of significant p-values at a distance $h = 3$, which should be never the case for this random field.

5 Applications: temperatures and precipitations

Extreme events in climatology such as very high temperature or large amount of precipitation have dramatic consequences. In order to prevent these consequences, it is important to predict to what extent those extreme values can arise. To do this, a model has to be designed that specifies the arrangement of the large values. Regarding the model choice, guidance is brought by the spatial test for asymptotic independence. It tells us which kind of model (max-stable or not) is more likely. In this section the spatial asymptotic independence of two climatological data sets are examined. These two data sets have already been considered in Bel et al. (2008) in an exercise to estimate the extremal coefficient function. The present paper aims at testing for asymptotic independence using the madogram test.

5.1 Temperature data

The data consist of daily maxima temperature recorded at 29 locations for more than 50 years, some of them up to 100 years. The closest sites are 40 km apart.

Table 3 Extremal Gaussian model, number of significant and non significant p-values for several classes of distances in the spatial case

Classes	[0 ; 1)	[1 ; 2)	[2 ; 2.5)	[2.5 ;3)	[3 ; 4)	[4 ; 5)
Significant	100	92	69	49	44	26
Non significant	0	8	31	51	56	74

Let $\text{Temp}_{ij}(s)$ denote the random temperature at location s and at day i of year j. It has been assumed that the random field $\text{Temp}(\cdot)$ can be written as

$$Temp_{i,j}(s) = \mu_i(s) + \varepsilon_{ij}(s)$$

where μ is the spatiotemporal trend. It is estimated by a two-stage procedure. First we average over years for each site and each day of the year

$$\tilde{\mu}_i(s) = \frac{1}{J_s}\sum_{j=1}^{J_s} Temp_{i,j}(s)$$

where J_s is the number of years available for station s. Then for each site we smooth the obtained profile by a moving average on time

$$\hat{\mu}_i(s) = \sum_{k=i-\ell}^{i+\ell} \omega_k \tilde{\mu}_k(s)$$

where $\ell = 8$ and $\omega_k = 0.05$ for $k \neq 0$, $\omega_0 = 0.2$ are set empirically.

ε is a residual random field. We assume it as stationary in space and in time, with small range dependence in time so the annual maxima at the same site can be

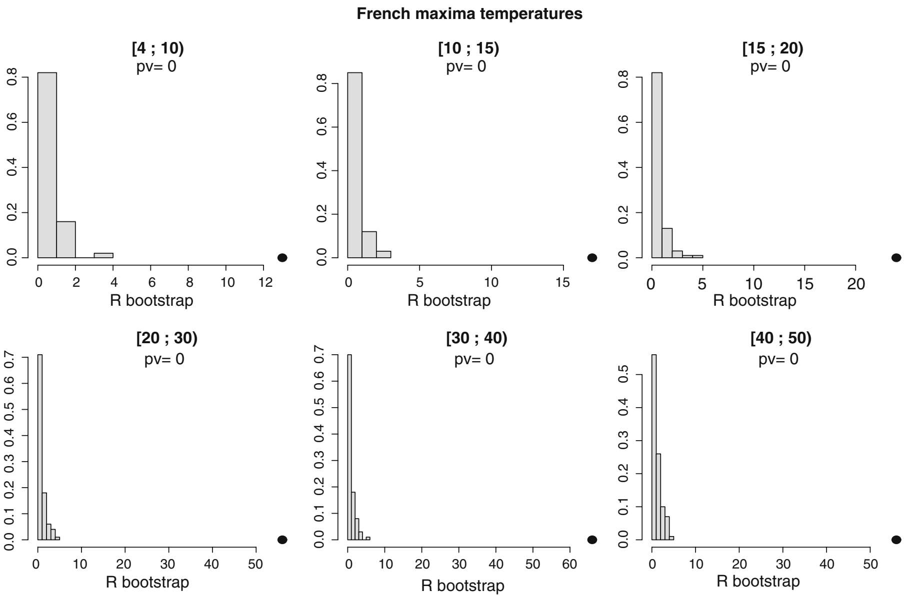

Fig. 9 Maxima French temperatures, distribution of R^*, number of rejected hypotheses under the null hypothesis for some classes of distances. *Bullet*: observed R, pv=$P(R^* \geq R)$.

Table 4 Burgundy maxima precipitations, *p*-values function of distance in testing asymptotic independence. Significant *p*-values are in italics

Distance (km)	0–15	15–25	25–30	30–35	35–40	40–45	45–50	50–55	55–60	60–65	65–70	70–75	75–80	80–85	85–90
p-value	*0.0000*	*0.0182*	*0.0002*	*0.0001*	*0.0028*	*0.0055*	0.2345	0.2096	0.5897	0.6299	0.6565	0.9943	0.9956	0.1932	0.4821

considered as independent. The jth maxima random field under study is then the annual maxima random field $M_j(\cdot)$

$$M_j(s) = \max_{i=1,365} (\widehat{\varepsilon}_{ij}(s)).$$

Figure 9 shows the distribution of R^* produced by the bootstrap procedure under the null hypothesis, together with the value of R observed on the data. The number of rejected hypotheses is always very high whatever the class of distance (actually almost all the p-values are significant even at large distances). We must conclude that the field of temperature exhibits a very strong asymptotic dependence at all distances. This could be explained by the fact that very high temperatures tend to simultaneously occur everywhere over the country during heatwaves. This suggests that the standard storm process is not well adapted to this data set because it exhibits aymptotic independence at finite distance. A model such as the extremal Gaussian process would certainly be more appropriate.

5.2 Precipitation data

The second example deals with the precipitations in the French region of Burgundy. The basic data consist of maximum daily precipitations recorded at 146 locations during 30 years. The closest and the farthest sites are respectively 3 and 200 km apart. The data were preprocessed by Meteo France research laboratory in order to make them compatible with a stationary and isotropic random field. As a result, only the maxima of the resultant field over the whole period were available, which precluded an analysis based on several realizations.

The classes of distances have been designed to contain between 500 and 1,000 pairs.

A p-value is computed for each class of distances based on the estimated madogram. Then the FDR procedure is performed to determine which p-values are significant, whereby one can derive the distance up to which the process shows asymptotic dependence. Table 4 shows that this distance is about $h = 50$ km. It is commonly accepted that large precipitations are local phenomena, which is consistent with the result obtained.

These data could be modeled by a storm model fitting the extremal coefficient such as to be close to 2 for $h = 50$, but we cannot eliminate the Gaussian model with a strong correlation for distances up to 50 km.

6 Conclusion

Prediction of spatial extreme events appears as a major challenge for various research communities such as environmental or climate ones. A possible way to predict spatial extreme events is to consider scenarios of extreme realizations using a simulation approach. Such an approach requires a stochastic model, and any available information must contribute to specify it. Here a procedure has been proposed to jointly discriminate between asymptotic independence and asymptotic dependence at various lags starting from realizations of maxima. This procedure has been tested on a number of stochastic models (Gaussian model, storm model, extremal Gaussian model). It generally provides correct, reliable and fast results. However a slow rate of convergence has been noted in the case of the Gaussian model for highly correlated pairs of variables. The source of this problem is under current investigation.

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Appendix

Let (X, Y) be a bivariate random vector with distribution $G(\cdot, \cdot)$ and margins Fréchet $F(\cdot)$. Denote by $A(\cdot)$ the related Pickands dependence function. Then,

$$G(x, y) = \exp\left(\left(-\frac{1}{x} - \frac{1}{y}\right) A\left(\frac{x}{x+y}\right)\right)$$

Let $U = F(X) = e^{-\frac{1}{X}}$ and $V = F(Y) = e^{-\frac{1}{Y}}$.

$$\begin{aligned}
E(UV) &= E\left(\int_0^1 \int_0^1 \mathbf{1}_{u<U} \mathbf{1}_{v<V} du dv\right) \\
&= \int_0^1 \int_0^1 P\left(X < \frac{-1}{\log u}, Y < \frac{-1}{\log v}\right) du dv \\
&= \int_0^1 \int_0^1 \exp\left((\log u + \log v) A\left(\frac{\log v}{\log u + \log v}\right)\right) du dv
\end{aligned}$$

The change of variables $u = e^z$, $v = e^{\frac{zt}{1-t}}$ gives

$$\begin{aligned}
E(UV) &= -\int_0^1 \int_{-\infty}^0 \exp\left(\frac{z}{(1-t)} A(t)\right) e^{\frac{z}{1-t}} \frac{z}{(1-t)^2} dz dt \\
&= \int_0^1 \frac{1}{(1 + A(t))^2} dt
\end{aligned}$$

leading to Eq. 2.

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