Chapter 6

de Haan and Ferreira(2006)

EVT: biviraite case

Suppose $(X_1.Y_1),(X_2,Y_2),\ldots$ be i.i.d. random vectors with distribution function F. Suppose that there exist sequences of constants $a_n, c_n > 0, b_n, d_n \in \mathbb{R}$ a distribution function G with non -degenerate marginals such that for all continuity points (x,y) of G,

$$\lim_{n\to\infty} P\left(\frac{\max(X_1,X_2,\ldots,X_n)-b_n}{a_n} \le x, \frac{\max(Y_1,Y_2,\ldots,Y_n)-d_n}{c_n} \le y\right) = G(x,y). \tag{6.1.1}$$

Any limit distribution function G in (6.1.1) with non-degenerate marginals is called a multivariate extreme value distribution.

EVT: biviraite case

Since (6.1.1) implies convergence of the one-dimensional two marginal distribution, we have

$$\lim_{n\to\infty} P\left(\frac{\max\left(X_1, X_2, \ldots, X_n\right) - b_n}{a_n} \le x\right) = G(x, \infty),$$

and

$$\lim_{n\to\infty} P\left(\frac{\max\left(Y_1, Y_2, \ldots, Y_n\right) - d_n}{c_n} \le y\right) = G(\infty, y).$$

EVT: bivirate case

Let F_1, F_2 denote the marginal distribution of F.

Define $U_i(t) := F_i^{\leftarrow}(1 - 1/t), i = 1, 2$. Then there exist constants a_n, b_n, c_n, d_n such that

$$\begin{aligned} \lim_{t\to\infty} \frac{U_1(nx) - b_n}{a_n} &= \frac{x_1^{\gamma} - 1}{\gamma_1}, \\ \lim_{t\to\infty} \frac{U_2(nx) - d_n}{c_n} &= \frac{x_2^{\gamma} - 1}{\gamma_2}. \end{aligned} \tag{1}$$

EVT: bivirate case

Now, we return to (6.1.1), which can be written as

$$\lim_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n) = G(x, y). \tag{6.1.8}$$

If $x_n \to u, y_n \to v$, then

$$\lim_{n\to\infty} F^n(a_n x_n + b_n, c_n y_n + d_n) = G(u, v). \tag{6.1.9}$$

Apply (6.1.9) with

$$x_n = \frac{U_1(nx) - b_n}{a_n}, y_n = \frac{U_2(ny) - d_n}{c_n}$$

then

$$\lim_{n\to\infty} F^n(U_1(nx), U_2(ny)) = G\left(\frac{x^\gamma - 1}{\gamma}, \frac{y^\gamma - 1}{\gamma}\right) := G_0(x, y)$$

- The marginal distribution of function G_0 is $\exp(-x^{-1})$ and $\exp(-y^{-1})$.
- The marginal distribution does not depend on γ and other parameters.
- Now, we can only consider the dependence structure.

Corollary 6.1.3

For any (x, y) for which $0 < G_0(x, y) < 1$,

$$\lim_{n\to\infty} n\left\{1 - F(U_1(nx), U_2(ny))\right\} = -\log G_0(x, y) \tag{6.1.11}$$

This also holds by replacing n by t, where t runs through the real numbers.

Exponent Measure

There are set functions v, v_1, v_2 defined for all Borel sets $A \subset \mathbb{R}_+^2$ with

$$\inf_{(x,y)\in A} \max(x,y) > 0$$

such that

1.

$$v_n \left\{(s,t) \in \mathbb{R}_+^2 : s > x \ or \ t > y\right\} = n\left(1 - F(U_1(nx), U_2(ny))\right),$$
$$v \left\{(s,t) \in \mathbb{R}_+^2 : s > x \ or \ t > y\right\} = -\log G_0(x,y)$$

2. for all $a > 0$ the set functions $v, v_1, v_2, \ldots$ are finite measures on $\mathbb{R}_+^2$ $[0,a]^2$

3.for each Borel set $A \subset \mathbb{R}_+^2$ with $\inf_{(x,y)\in A} \max(x,y) > 0$ and $v(\partial A) = 0$,

$$\lim_{n\to\infty} v_n(A) = v(A).$$

The measure v is sometimes called the exponent measure of the extreme value distribution G_0.

Homogeneity of v

For any Borel set $A \subset \mathbb{R}_+^2$,with $\inf_{(x,y)\in A} \max(x,y) > 0$ and $v(\partial A) = 0$,

$$v(aA) = a^{-1} v(A)$$

Proof.

It is easy to verify this property by Corollary 6.1.3. $\square$

The Spectral Measure

The homogeneity property of the exponent measure v suggests a coordinate transformation in order to capitalize on that.
Examples are

$$\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ d(x,y) = \arctan\frac{y}{x} \end{cases}$$

$$\begin{cases} r(x,y) = x+y \\ d(x,y) = \frac{x}{x+y} \end{cases}$$

$$\begin{cases} r(x,y) = x \vee y \\ d(x,y) = \arctan\frac{x}{y} \end{cases}$$

The Spectral Measure

Let us start with the first transformation. Define for constants $r > 0$ and $\theta \in [0, \pi/2]$ the set

$$B_{r,\theta} = \left\{ (x, y) \in \mathbb{R}_+^{2*} : \sqrt{x^2 + y^2} > r \textit{ and } \arctan \frac{y}{x} \leq \theta \right\}$$

Clearly $B_{r,\theta} = rB_{1,\theta}$ and hence

$$v(B_{r,\theta}) := r^{-1} v(B_{1,\theta}).$$

Set for $0 \leq \theta \leq \pi/2$,

$$\Psi(\theta) := v(B_{1,\theta}).$$

The Spectral Measure

Write $s = r\cos\theta, t = r\sin\theta$. Take $x, y > 0$,

$$\begin{aligned}
-\log G_0(x,y) &= v\left\{(s,t) : s > x \ or \ t > y\right\} \\
&= v\left\{(r,\theta) : r\cos\theta > x \ or \ r\sin\theta > y\right\} \\
&= v\left\{(r,\theta) : r > \frac{x}{\cos\theta} \wedge \frac{y}{\sin\theta}\right\} \\
&= \int_{x/\cos\theta < y/\sin\theta} \int_{r > x/\cos\theta} \frac{\partial^2 v(B(r,\theta))}{\partial r \partial\theta} dr d\theta + \\
&\quad + \int_{x/\cos\theta > y/\sin\theta} \int_{r > y/\sin\theta} \frac{\partial^2 v(B(r,\theta))}{\partial r \partial\theta} dr d\theta \\
&= \int_{x/\cos\theta < y/\sin\theta} \int_{r > x/\cos\theta} \frac{dr}{r^2} \frac{\partial v(B_{1,\theta})}{\partial\theta} d\theta \\
&\quad + \int_{x/\cos\theta > y/\sin\theta} \int_{r > y/\sin\theta} \frac{dr}{r^2} \frac{\partial v(B_{1,\theta})}{\partial\theta} d\theta
\end{aligned}$$

Theorem 6.1.14

There exist a finite measure on $[0, \pi]$ such that for $x, y > 0$,

$$G_0(x, y) = \exp\left(- \int_0^{\pi/2} \left(\frac{\cos\theta}{x} \vee \frac{\sin\theta}{y} \right) \Psi(d\theta) \right)$$

with the side functions

$$\int_0^{\pi/2} \cos\theta \Psi(d\theta) = \int_0^{\pi/2} \sin\theta \Psi(d\theta) = 1.$$

Copula

We could also describe the dependence using copulas.

- If F is the distribution function of the random vector (X, Y), the copula C associated with F is a distribution function that satisfies $F(x, y) = C(F_1(x), F_2(y))$.
- It contains complete information about the joint distribution of F apart from the marginal distribution.

$$\begin{aligned} F(F_1^{-1}(x), F_2^{-1}(y)) &= P(X \leq F_1^x, Y \leq F_2^{-1}(y)) \\ &= P(F_1(X) \leq x, F_2(y) \leq y) \\ &:= C(x, y) \end{aligned}$$

Then, $F(x, y) = C(F_1(x), F_2(y))$.

L function

Define for $0 < x, y < 1$,

$$C(x, y) := G_0(-1/\log x, -1/\log y).$$

Then, C is a copula and the homogeneity of the exponent measure implies that: for $0 < x, y < 1$, $a > 0$,

$$C(x^a, y^a) = C^a(x, y).$$

This relation is not very tractable for analysis, we instead consider the L function defined by

$$\begin{aligned} L(x, y) &:= -\log G_0(1/x, 1/y) \\ &= v\left\{(s, t) \in \mathbb{R}_+^2 : s > 1/x \textit{ or } t > 1/y\right\}. \end{aligned}$$

Properties of L function

1. $L(ax.ay) = aL(x, y)$, for all $a, x, y > 0$
2. $L(x, 0) = L(0, x) = x$, for all $x > 0$
3. $x \vee y \leq L(x + y) \leq x + y$
4. If X, Y are independent, then $L(x, y) = x + y$. If X, Y are completely positive dependent, then $L(x, y) = x \vee y$.
5. L is continuous.
6. $L(x, y)$ is a convex function.

Other Measures of dependence

Define the set Q_c by

$$Q_c := \left\{(x, y) \in \mathbb{R}_+^2 : -\log G_0(1/x, 1/y) \leq c\right\}$$

The function R is defined as

$$R(x, y) = x + y - L(x, y)$$

The function χ is defined as

$$\chi(t) = -R(t, 1)$$

The function A is defined as

$$A(t) := L(1 - t, t)$$

Domains of Attractions

- We have now discussed the multivariate extreme value distribution G.
- Now, we are going to discuss which F belongs to the max domain of attractions of multivariate extreme value distribution.

Theorem 6.2.1

If F belongs to the maximum domain of attraction, the followings are equivalent.

a.

$$\lim_{t\to\infty} \frac{1-F(U_1(tx),U_2(ty))}{1-F(U_1(t),U_2(t))} = S(x,y) \tag{6.2.1}$$

with $S(x,y) = \log G((x^{\gamma_1}-1)/\gamma,(y^{\gamma_2}-1)/\gamma)/\log G(0,0)$.

b. For all $r>1$ and all $\theta\in[0,\pi/2]$ that are continuity point of Ψ,

$$P\left(V^2+W^2>t^2r^2 \; and \frac{W}{V} \le \tan\theta | V^2+W^2>t^2\right) \to r^{-1}\frac{\Psi(\theta)}{\Psi(\pi/2)}, \tag{6.2.1}$$

where $V=1/(1-F_1(X)), W=1/(1-F_2(Y))$.

Conversely, if the continuous marginal distribution function F_i are in the domain of attraction of univariate extreme value distribution and any limit relation (6.1.1) or (6.1.2) holds, then F is in the domain of attraction of G.

Asymptotic Independence

Let $(X_1, \ldots, X_d)$ be a random vector with distribution function F. Let the marginal distribution F_i satisfies the domain of attraction condition. If

$$\frac{P(X_i > U_i(t), X_j > U_j(t))}{P(X_i > U_i(t))} \to 0$$

for all $1 \leq i < j \leq d$, then

$$\lim_{n\to\infty} F^n(a_n^{(1)}x_1 + b_n^{(1)}, \cdots, a_n^{(d)}x_1 + b_n^{(d)}) = \exp\left(-\sum_{i=1}^{d}(1+\gamma_i x_i)^{-1/\gamma_i}\right).$$