Chapter 6

de Haan and Ferreira(2006)

EVT: biviraite case

Suppose $(X_1.Y_1), (X_2, Y_2), \ldots$ be i.i.d. random vectors with distribution function F. Suppose that there exist sequences of constants $a_n, c_n > 0, b_n, d_n \in \mathbb{R}$ a distribution function G with non -degenerate marginals such that for all continuity points (x, y) of G,

$$\lim_{n\to\infty} P\left(\frac{\max(X_1, X_2, \ldots, X_n) - b_n}{a_n} \le x, \frac{\max(X_1, X_2, \ldots, X_n) - d_n}{c_n} \le y\right) = G(x, y). \tag{6.1.1}$$

Any limit distribution function G in (6.1.1) with non-degenerate marginals is called a multivariate extreme value distribution.

EVT: bivirate case

Let F_1, F_2 denote the marginal distribution of F. Define $U_i(t) := F_i^{\leftarrow}(1 - 1/t), i = 1, 2$. Then

$$\begin{aligned} \lim_{t\to\infty} \frac{U_1(nx) - b_n}{a_n} &= \frac{x_1^{\gamma} - 1}{\gamma_1}, \\ \lim_{t\to\infty} \frac{U_2(nx) - d_n}{c_n} &= \frac{x_2^{\gamma} - 1}{\gamma_2}, \end{aligned} \tag{1}$$

EVT: bivirate case

Now, we return to (6.1.1), which can be written as

$$\lim_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n) = G(x, y). \tag{6.1.8}$$

If $x_n \to u, y_n \to v$, then

$$\lim_{n\to\infty} F^n(a_n x_n + b_n, c_n y_n + d_n) = G(u, v). \tag{6.1.9}$$

Apply (6.1.9) with

$$x_n = \frac{U_1(nx) - b_n}{a_n}, y_n = \frac{U_2(ny) - d_n}{c_n}$$

then

$$\lim_{n\to\infty} F^n(U_1(nx), U_2(ny)) = G\left(\frac{x^\gamma - 1}{\gamma}, \frac{y^\gamma - 1}{\gamma}\right) := G_0(, xy)$$

Corollary 6.1.3

For any (x, y) for which $0 < G_0(x, y) < 1$,

$$\lim_{n\to\infty} n\left\{1 - F(U_1(nx), U_2(ny))\right\} = -\log G_0(x, y) \tag{6.1.11}$$

This also holds by replacing n by t, where t runs through the real numbers.

Exponent Measure

There are set functions v, v_1, v_2 defined for all Borel sets $A \subset \mathbb{R}_+^2$ with

$$\inf_{(x,y)\in A} \max(x,y) > 0$$

such that
1.

$$v_n\left\{(s,t) \in \mathbb{R}_+^2 : s > x \textit{ or } t > y\right\} = n\left(1 - F(U_1(nx), U_2(ny))\right),$$
$$v\left\{(s,t) \in \mathbb{R}_+^2 : s > x \textit{ or } t > y\right\} = -\log G_0(x,y)$$

2. for all $a > 0$ the set functions $v, v_1, v_2, \ldots$ are finite measures on $\mathbb{R}_+^2$ $[0,a]^2$
3.for each Borel set $A \subset \mathbb{R}_+^2$ with $\inf_{(x,y)\in A} \max(x,y) > 0$ and $v(\partial A) = 0$,

$$\lim_{n\to\infty} v_n(A) = v(A).$$

The measure v is sometimes called the exponent measure of the extreme value distribution G_0.

Homogeneity of v

For any Borel set $A \subset \mathbb{R}_+^2$,with $\inf_{(x,y)\in A} \max(x,y) > 0$ and $v(\partial A) = 0$,

$$v(aA) = a^{-1} v(A)$$

The Spectral Measure

The homogeneity property of the exponent measure v suggests a coordinate transformation in order to capitalize on that.
Examples are

$$\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ d(x,y) = \arctan \frac{y}{x} \end{cases}$$

$$\begin{cases} r(x,y) = x+y \\ d(x,y) = \frac{x}{x+y} \end{cases}$$

$$\begin{cases} r(x,y) = x \vee y \\ d(x,y) = \arctan \frac{x}{y} \end{cases}$$

The Spectral Measure

Let us start with the first transformation. Define for constants $r > 0$ and $\theta \in [0, \pi/2]$ the set

$$B_{r,\theta} = \left\{ (x, y) \in \mathbb{R}_+^{2*} : \sqrt{x^2 + y^2} > r \textit{ and } \arctan \frac{y}{x} \leq \theta \right\}$$

Clearly $B_{r,\theta} = rB_{1,\theta}$ and hence

$$v(B_{r,\theta}) := r^{-1} v(B_{1,\theta}).$$

Set for $0 \leq \theta \leq \pi/2$,

$$\Psi(\theta) := v(B_{1,\theta}).$$

Theorem 6.1.4

There exist a finite measure on $[0,\pi]$ such that for $x, y > 0$,

$$G_0(x,y) = \exp\left(-\int_0^{\pi/2}\left(\frac{\cos\theta}{x} \vee \frac{\sin\theta}{y}\right)\Psi(d\theta)\right)$$

with the side functions

$$\int_0^{\pi/2} \cos\theta\Psi(d\theta) = \int_0^{\pi/2} \sin\theta\Psi(d\theta) = 1.$$

L

Define

$$L(x, y) = v\left\{(s,t) \in \mathbb{R}^2_+ : s > 1/x \ or \ t > 1/y\right\}.$$

Properties of the function L,

- $L(ax.ay) = aL(x, y)$
- $L(x, 0) = L(0, x) = x$
- $x \vee y \leq L(x + y) \leq x + y$
- If X, Y are independent, then $L(x, y) = x + y$. If X, Y are completely positive dependent, then $L(x, y) = x \vee y$.
- L is continuous.
- L is convex.

Q_1, R, χ

Define the set Q_1 by

$$Q_1 := \left\{(x, y) \in \mathbb{R}_+^2 : -\log G_0(1/x, 1/y) \leq 1\right\}$$

The function R is defined as

$$R(x, y) = x + y - L(x, y)$$

The function χ is defined as

$$\chi(t) = -R(t, 1)$$

Theorem 6.2.1

The followings are equivalent.

1.

$$\lim_{t\to\infty} \frac{1 - F(U_1(tx), U_2(ty))}{1 - F(U_1(t), U_2(t))} = S(x, y)$$

with $S(x, y) = \log G((x^{\gamma_1} - 1)/\gamma, (y^{\gamma_2} - 1)/\gamma)/\log G(0, 0)$.

2. For all $r > 1$ and all $\theta \in [0, \pi/2]$ that are continuity point of Ψ,

$$P\left(V^2 + W^2 > t^2 r^2 \; and \frac{W}{V} \leq \tan\theta | V^2 + W^2 > t^2\right) \to r^{-1} \frac{\Psi(\theta)}{\Psi(\pi/2)}$$

Asymptotic Independence

Let $(X_1, \ldots, X_d)$ be a random vector with distribution function F. If

$$\frac{P(X_i > U_i(t), X_j > U_j(t))}{P(X_i > U_i(t))} = 0$$

for all $1 \leq i < j \leq d$, then

$$\lim_{n\to\infty} F^n(a_n^{(1)}x_1 + b_n^{(1)}, \cdots, a_n^{(d)}x_1 + b_n^{(d)}) = \exp\left(-\sum_{i=1}^{d}(1+\gamma_i x_i)^{-1/\gamma_i}\right).$$