Graphical Models for Extremes

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Introduction

Extreme Value theory in multivariate case. Assume $\boldsymbol{X} \in \mathbb{R}^d$.

- Max-stable distributions arise as limits of normalized maxima of independent copies of $\boldsymbol{X}$.
- Multivariate Pareto distributions describe the random vector $\boldsymbol{X}$ conditioned on the event that at least one component exceeds a high threshold.

Sparse Multivariate Models

- Sparse multivariate models require the notion of conditional independence.
- The problem is how to define the conditional independence for tail dependence.
- If (Z_1, Z_2, Z_3) is a max-stable random vector with positive continuous density, then the conditional independence of $Z_1 \perp\!\!\!\perp Z_3 \mid Z_2$ already implies the independence $Z_1 \perp\!\!\!\perp Z_3$.
- Meaningful conditional independence structures can thus only be obtained for max-stable distributions with discrete spectral measure.

Graphical Models

We now consider the Multivariate Pareto distribution $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$.

For an undirected graph $\mathcal{G} = (V, E)$ with nodes $V = \{1, 2, \ldots, d\}$ and edge set E, we say that $\boldsymbol{Y}$ is an extremal graphical model if it satisfies the pairwise Markov property

$$\boldsymbol{Y}_i \perp_{\mathrm{e}} \boldsymbol{Y}_j \mid \boldsymbol{Y}_{\setminus\{i,j\}}, \quad (i,j) \notin E,$$

where we use $\perp_{\mathrm{e}}$ to stress that it is designed for extremes. (And we will define this later.)

The main advantage of conditional independence and graphical models is that they imply a simple probabilistic structure and possibly sparse patterns in multivariate random vectors.

Multivariate extreme value theory

- Let $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{id}), i = 1, \ldots, n$ be independent copies of $\boldsymbol{X}$ and denote the componentwise maximum by $\mathrm{M}_n = (M_{1n}, \ldots, M_{dn}) = (\max_{i=1}^n X_{i1}, \ldots, \max_{i=1}^n X_{id})$.
- Assume there are sequences of normalizing constants $b_{jn} \in \mathbb{R}, a_{jn} > 0$, such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_{jn} - b_{jn}}{a_{jn}} \leqslant x \right) = G_j(x) = \exp\left\{ - \left(1 + \xi_j x\right)_+^{-1/\xi_j} \right\}, \quad x \in \mathbb{R}.$$

Multivariate extreme value theory

Assume

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1,\dots,n} X_{i1} \leqslant nz_1, \dots, \max_{i=1,\dots,n} X_{id} \leqslant nz_d \right) = \mathbb{P}(\boldsymbol{Z} \leqslant \boldsymbol{z})$$

In this case, $\boldsymbol{Z}$ is max-stable with standard Fréchet marginals $\mathbb{P}(Z_j \le z) = \exp(-1/z)$ and we may write

$$\mathbb{P}(\boldsymbol{z} \leqslant \boldsymbol{z}) = \exp\{-\Lambda(\boldsymbol{z})\}, \quad \boldsymbol{z} \in \mathcal{E}$$

where the exponent measure Λ is a Radon measure on the cone $\mathcal{E} = [0,\infty)^d \backslash \{0\}$ and $\Lambda(\boldsymbol{z})$ is shorthand for $\Lambda(\mathcal{E}\backslash[0,\boldsymbol{z}])$.

Multivariate extreme value theory

If Λ is absolutely continuous with respect to Lebesgue measure on $\mathcal{E}$, its Radon–Nikodym derivative, denoted by λ has the following properties

- homogeneity of order $-(d+1)$, i.e. $\lambda(t\boldsymbol{y}) = t^{-(d+1)}\lambda(\boldsymbol{y})$.
- normalized marginals, i.e. for any $i = 1, \ldots, d$,

$$\int_{\mathrm{y} \in \mathcal{E}: y_i > 1} \lambda(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = 1$$

Multivariate extreme value theory

Another perspective on multivariate extremes is through threshold exceedances

$$\lim_{u\to\infty} u\{1 - \mathbb{P}(\boldsymbol{X} \leqslant u\boldsymbol{z})\} = \Lambda(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathcal{E}$$

Consequently, the multivariate distribution of the threshold exceedances of $\boldsymbol{X}$ satisfies

$$\mathbb{P}(\mathrm{Y} \leqslant \boldsymbol{z}) = \lim_{u\to\infty} \mathbb{P}\left(\frac{\boldsymbol{X}}{u} \leqslant \boldsymbol{z} \mid \|\boldsymbol{X}\|_\infty > u\right) = \frac{\Lambda(\boldsymbol{z} \wedge 1) - \Lambda(\boldsymbol{z})}{\Lambda(1)}.$$

The distribution of the limiting random vector $\boldsymbol{Y}$ is called amultivariate Pareto distribution.

It is defined through the exponent measure Λ of the max-stable distribution $\boldsymbol{Z}$, with support on the L-shaped space $\mathcal{L} = \{\mathrm{x} \in \mathcal{E} : \|\boldsymbol{x}\|_\infty > 1\}$.