Estimation of Extreme Quantile

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Introduction

- Mean vs Quantile
- Quantile vs Extreme Quantile
- Regression vs Quantile Regression
- Quantile Regression vs Extreme Quantile Regression

Introduction

Let $X_1, \ldots, X_{100}$ be i.i.d. random variables with coommon distribution F.

- Estimate 50% quantile.
- Estimate 80% quantile.
- Estimate 95% quantile.
- Estimate 99% quantile.

Introduction

- Now, we are ready to consider extreme quantile estimation.
- Let $x_p := U(1/p)$ be the quantile we want to estimate.
- We are particularly intertesd in the cases in which the mean number oberservations above x_p, np equals to a very small number.

$$U(p_n) \approx b(n/k) + a(n/k)\frac{\left(\frac{k}{np_n}\right)^{\gamma} - 1}{\gamma}$$

- Estimation of γ.(well discussed in Chapter 3)
- Estimation of $b(n/k)$.
- Estimation of $a(n/k)$.

Scale Estimation

Recall that

$$\frac{M_n^{(1)}}{a(n/k)/U(n/k)} \xrightarrow{P} \frac{1}{1-\gamma_-}.$$

Then we can estimate $a(n/k)$ by

$$\hat{\sigma}_M := X_{n-k,n} M_n^{(1)} (1 - \hat{\gamma}_-).$$

Scale Estimation

Assume $F \in D(G_\gamma)$ and $k \to \infty, k/n \to 0$ as $n \to \infty$. Then as $n \to \infty$,

$$\frac{\hat{\sigma}_M}{a(n/k)} \to 1.$$

Assume second order condition, then

$$\sqrt{k}\left(\frac{\hat{\sigma}_M}{a(n/k)} - 1\right) \xrightarrow{d} N(\lambda b_{\gamma,\rho}, var_\gamma)$$

Threshold Estimation

Indeed, we can always take $b_n = U(n/k)$.
From Theorem 2.4.1, we have

$$\frac{X_{n-k,n} - U(n/k)}{a(n/k)} \xrightarrow{d} N(0,1).$$

Extreme Quantile Estimation

Let k be an intermediate sequence. Suppose that for suitable estimators $\hat{\gamma}, \hat{a}(n/k), \hat{b}(n/k)$,

$$\sqrt{k}\left(\hat{\gamma}-\gamma, \frac{\hat{a}(n/k)}{a(n/k)}-1, \frac{\hat{b}(n/k)-X_{n-k,n}}{a(n/k)}\right) \xrightarrow{d} (\Gamma, \Lambda, B).$$

Define

$$\hat{x}_{p_n} := \hat{b}(n/k)+\hat{a}(n/k)\frac{\left(\frac{k}{np_n}\right)^{\hat{\gamma}}-1}{\hat{\gamma}}$$

Theorem 4.3.1

Suppose second order condition holds and

- $\rho < 0$ or $\rho = 0, \gamma < 0$.
- $\sqrt{k} A(n/k) \to \lambda$.
- $np_n = o(k)$ and $\log(np_n) = o(\sqrt{k})$.

Then as $n \to \infty$,

$$\sqrt{k}\frac{\hat{x}_{p_n} - x_{p_n}}{a(n/k)q_\lambda(d_n)} \xrightarrow{d} \Gamma + (\gamma_-)^2 B - \gamma_- \Lambda - \lambda \frac{\gamma_-}{\gamma_- + \rho}$$

with $d_n = k/(np_n)$ and

$$q_\lambda(t) := \int_1^t s^{\gamma - 1} \log s ds.$$

Tail Probability Estimation

Now, we consider the dual problem, estimate

$$p = 1 - F(x).$$

To estimate the probability, we can use

$$\hat{p}_n = \frac{k}{n} \left\{ \max \left(0, 1 + \hat{\gamma} \frac{x_n - \hat{b}(n/k)}{\hat{a}(n/k)} \right) \right\}^{-1/\hat{\gamma}}$$

Tail Probaility Estimation

Suppose that $\rho > -1/2$ and second order condition holds. Suppose

- $\rho < 0$ or $\rho = 0, \gamma < 0$.
- $k \to \infty, k/n \to 0$ and $\sqrt{k}A(n/k) \to \lambda$.
- $d_n \to \infty$ and $w_\gamma(d_n) = o(\sqrt{k})$ where

$$w_\gamma(d_n) = t^{-\gamma} \int_1^t s^{\gamma-1} \log s ds.$$

Then as $n \to \infty$,

$$\frac{\sqrt{k}}{w_\gamma(d_n)} \left(\frac{\hat{p}_n}{p_n} - 1 \right) \xrightarrow{d} \Gamma + (\gamma_-)^2 B - \gamma_- \Lambda - \lambda \frac{\gamma_-}{\gamma_- + \rho}$$

Endpoint Estimation

We can estimate x^* by

$$\hat{x}^* := \hat{b}(n/k) - \frac{\hat{a}(n/k)}{\hat{\gamma}}.$$