Estimation of Extreme Value Index

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Hill estimator applies for $\gamma > 0$. Recall that $F \in D(G_\gamma)$ for $\gamma > 0$ if and only if

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma}.$$

And Remark 1.2.3 shows that this is equivalent to

$$\lim_{t\to\infty} E(\log X - \log t | X > t) = \gamma.$$

Hence, we have

$$\frac{\int_t^\infty (\log u - \log t) dF(u)}{1-F(t)} \to \gamma.$$

Replace t by $X_{n-k,n}$ and F by F_n, we then get Hill estimator

$$\hat{\gamma}_H := \frac{1}{k}\sum_{i=0}^{k-1} \log X_{n-i,n} - \log X_{n-k,n}$$

.

Suppose $F \in D(G_\gamma)$ with $\gamma > 0$. Then as $n \to \infty, k = k(n) \to \infty, k/n \to 0$,

$$\hat{\gamma}_H \xrightarrow{P} \gamma.$$

Lemma 3.2.3

Let $E_i \overset{i.i.d.}{\sim} \exp(1)$. Let $E_{1,n} \leq E_{2,n} \leq \cdots \leq E_{n,n}$ be the order statistics. let f be such that $Varf(E) < \infty$. Then

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=0}^{k-1} f(E_{n-i,n} - E_{n-k,n}) - Ef(E)\right)$$

is independent of $E_{n-k,n}$ and asymptotically normal with mean zero and variance $Varf(E)$ as $n \to \infty, k \to \infty, k/n \to 0$.

Theorem 3.2.5

Suppose F satisfies the second order condition. Then

$$\sqrt{k}(\hat{\gamma}_H - \gamma) \xrightarrow{d} N(\lambda/(1-\rho), \gamma^2)$$

provided that $k \to \infty, k/n \to 0, n \to \infty$ and

$$\lim_{n\to\infty} \sqrt{k} A(\frac{n}{k}) = \lambda.$$

The Pickands Estimator

Pickands estimator applies for $\gamma \in \mathbb{R}$. The Pickands estimator is defined as

$$\hat{\gamma}_P := (\log 2)^{-1} \log \frac{X_{n-k,n} - X_{n-2k,n}}{X_{n-2k,n} - X_{n-4k,n}}$$

Assume $n \to \infty, k \to \infty, k/n \to 0$ and $F \in D(G_\gamma)$. Then

$$\hat{\gamma}_P \to \gamma.$$

Theorem 3.3.5

Assume the second order condition. Then as $n \to \infty, k \to \infty, k/n \to 0$,

$$\sqrt{k}(\hat{\gamma}_P - \gamma) \xrightarrow{d} N(\lambda b_{\gamma,\rho}, var_\gamma)$$

MLE

The MLE applies for $\gamma > -1/2$. Recall that

$$\lim_{t \to x^*} P\left(\frac{X-t}{f(t)} > x | X > t\right) = 1 - H_\gamma(x) := (1 + \gamma x)^{-1/\gamma}$$

MLE

Define $h_{\gamma,\sigma}(x) = \partial H_\gamma(x/\sigma)/\partial x$.
The approximate likelihood is

$$\prod_{i=1}^{k} h_{\gamma,\sigma}(z_i)$$

with $z_i = x_{n-i+1} - x_{n-k,n}$.
Suppose the second order condition, we can prove the asymptotically normal of MLE. See Theorem 3.4.2.

Lemma 3.5.1

Suppose $F \in D(G_\gamma)$. Define for $j = 1, 2$,

$$M_n^{(j)} := \frac{1}{k} \sum_{i=0}^{k-1} (\log X_{n-i,n} - \log X_{n-k,n})^j.$$

Then for $k = k(n) \to \infty, k/n \to 0, n \to \infty$,

$$\frac{M_n^{(j)}}{(a(n/k)/U(n/k))^j} \xrightarrow{P} \prod_{i=1}^{j} \frac{i}{1 - i\gamma_-}$$

Moment Estimator

By Lemma 3.5.1, we have

$$\frac{(M_n^{(1)})^2}{M_n^{(2)}} \xrightarrow{P} \frac{1-2\gamma_-}{2(1-\gamma_-)}$$

We then define the Moment estimator as

$$\hat{\gamma}_M := M_n^{(1)} + 1 - 1/2\left(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}}\right)^{-1}.$$

Similarly, under some second order condition, we can prove the asymptotical normality for the moment estimator.

- Probability Weigted Moment estimator(PWM): holds for $\gamma < 1$.
- Negative Hill Estimator: holds for $\gamma < -1/2$.

- MLE(Bücher and Segers(2018)), PMW(de Haan and Ferrira(2015))
- GPD vs GEV ? (Bücher and Zhou)