Estimation of the marginal expected shortfall: the mean when a related variable is extreme

J.J. Cai, J.H.J Einmahl, L. de Haan and C. Zhou

Published on J.R. Statist. Soc. B (2015).

Table of Contents

1. Introduction
2. Main Results
3. Simulation
4. Application

Expected shortfall

We use capital letter to denote the loss of an asset.

Expected shortfall of an asset X at probability level p is defined as

$$E\left(X|X \geq Q_X(1-p)\right)$$

where

$$F_X(x) = P\left(X \leq x\right)$$

and Q_X denotes the inverse function of F_X.

Marginal Expected Shortfall (MES)

- A financial institute holds a portfolio $R = \sum_i y_i R_i$
- Expected shortfall at probability level p

$$E\left(R | R > Q_R(1-p)\right)$$

- Can be decomposed as

$$\sum_i y_i E\left(R_i | R > Q_R(1-p)\right)$$

- The sensitivity to the i-th asset is

$$E\left(R_i | R > Q_p(1-p)\right).$$

MES

- Marginal expectation shortfall is also often used to measure the contribution of a financial institute to a systemic crisis.
- It is defined as an institute' s expected equity loss when market falls below a certain threshold.
- "R_i": a particular finiancial institute, "R": the total market.

MES

- More generally: consider a random vector (X, Y)

- Marginal expected shortfall (MES) of X at level p is

$$E\left(X | Y > Q_Y(1-p)\right)$$

MES

- In this paper, we are interested in MES under exceptional stress conditions of the kind that have occurred very rarely or even not at all. (p is at an extremely low level that can be even lower than $1/n$)

- We assume that $p = p_n \to 0$, (intermediate level) $np_n \to \infty$, and (extreme level) $np_n = O(1)$ as $n \to \infty$.

- We want to estimate $E\left(X|Y > Q_Y(1-p)\right)$ for small p on the basis of i.i.d. observations

$$(X_1, Y_1), (X_2, Y_2,), \ldots, (X_n, Y_n)$$

Table of Contents

1 Introduction

2 Main Results

3 Simulation

4 Application

Notations

$$U_1(t) = Q_X\left(1 - \frac{1}{t}\right) = F_X^{\leftarrow}\left(1 - \frac{1}{t}\right)$$

$$U_2(t) = Q_Y\left(1 - \frac{1}{t}\right)$$

$$\theta_p = E\left(X | Y > U_2\left(\frac{1}{p}\right)\right)$$

For the time being we suppose that $X > 0$.

$$
\begin{aligned}
\theta_p &= E\left(X \middle| Y > U_2\left(\frac{1}{p}\right)\right) \\
&= \frac{\int_0^\infty P\left\{X > x, Y > U_2\left(\frac{1}{p}\right)\right\} dx}{P\left\{Y > U_2\left(\frac{1}{p}\right)\right\}} \\
&= \frac{1}{p}\int_0^\infty P\left\{X > x, Y > U_2\left(\frac{1}{p}\right)\right\} dx \\
&= \frac{1}{p} U_1\left(\frac{1}{p}\right)\int_0^\infty P\left\{X > xU_1\left(\frac{1}{p}\right), Y > U_2\left(\frac{1}{p}\right)\right\} dx.
\end{aligned}
$$

Thus,

$$
\frac{\theta_p}{U_1\left(\frac{1}{p}\right)} = \frac{1}{p}\int_0^\infty P\left\{X > xU_1\left(\frac{1}{p}\right), Y > U_2\left(\frac{1}{p}\right)\right\} dx.
$$

Condition (1)

First note (take $x = 1$ upstairs)

$$P\left\{X > U_1\left(\frac{1}{p}\right), Y > U_2\left(\frac{1}{p}\right)\right\}$$
$$= P\left\{1 - F_1(X) < p, 1 - F_2(Y) < p\right\}.$$

This is a copula.

Condition (1)

We impose conditions on the copula as $p \to 0$.

Suppose there exists a positive function $R(x, y)$ such that for all

$$0 \leq x, y \leq \infty, x \vee y > 0, x \wedge y < \infty$$

$$\lim_{p \to 0} \frac{1}{p} P \left\{ X > U_1\left(\frac{1}{xp}\right), Y > U_2\left(\frac{1}{yp}\right) \right\} = R(x, y).$$

i.e.,

$$\lim_{p \to 0} \frac{1}{p} P \left\{ 1 - F_1(X) < px, 1 - F_2(Y) < py \right\} = R(x, y).$$

This condition indicates and specifies dependence in the tail. (usual condition in extreme value theory)

Condition (2)

Compare: in the definition of θ_p we have

$$P\left\{X > xU_1\left(\frac{1}{p}\right), Y > U_2\left(\frac{1}{p}\right)\right\}$$

and in the condition we have (for $y = 1$)

$$P\left\{X > U_1\left(\frac{1}{xp}\right), Y > U_2\left(\frac{1}{p}\right)\right\}.$$

Condition (2)

In order to connect the two we impose a second condition, on the U : for $x > 0$

$$\lim_{t \to \infty} \frac{U_1(tx)}{U_1(t)} = x^{\gamma_1}, \quad x > 0.$$

where $\gamma_1 > 0$.

Proposition 1

Under these conditions, we get the first result:

$$\lim_{p\to 0} \frac{\theta_p}{U_1\left(\frac{1}{p}\right)} = \lim_{p\to 0} \frac{E\left(X|Y > U_2\left(\frac{1}{p}\right)\right)}{U_1\left(\frac{1}{p}\right)} = \int_0^\infty R(x^{-1/\gamma_1}, 1)dx.$$

Hence, θ_p goes to infinity as $p \to 0$ as the same rate as $U_1\left(\frac{1}{p}\right)$, the value at risk for X.

Estimation for θ_p.

Now we go to statistics and look at how to estimate θ_p.
Wo do that in stages:

- First, we estimate $\theta_{k/n}$ where $k = k(n) \to \infty, k/n \to 0$ as $n \to \infty$. We can estimate $\theta_{k/n}$ non-parametrically.
- The second stage will be the extrapolation from $\theta_{k/n}$ to θ_p with $np_n = O(1)$.

Estimation for $\theta_{k/n}$.

Recall $\theta_{k/n} = E\left(X | Y > U_2\left(\frac{n}{k}\right)\right)$

- Replace quantile $U_2(n/k)$ by corresponding sample quantile $Y_{n-k,n}$ (k-th order statistics from above)
- Replace the expectation by the sample mean.

The obvious estimation of $\theta_{k/n}$ is then

$$\hat{\theta}_{k/n} := \frac{\frac{1}{n}\sum_{i=1}^{n} X_i 1_{\{Y_i > Y_{n-k,n}\}}}{P\left(Y > U_2\left(\frac{n}{k}\right)\right)} = \frac{1}{k}\sum_{i=1}^{n} X_i 1_{\{Y_i > Y_{n-k,n}\}}$$

Under some strengthening of our conditions (relating to R and to the sequences $k(n)$),

$$\sqrt{k}\left(\frac{\hat{\theta}_{k/n}}{\theta_{k/n}}-1\right)\xrightarrow{d}\Theta,$$

a normal random variable that we describe now.
Background of limit result is the assumption

$$\lim_{p\to 0}\frac{1}{p}P\left(1-F_1(X)<px, 1-F_2(Y)<py\right)=R(x,y).$$

Now define $V := 1-F_1(X)$
$W := 1-F_2(Y)$.
V and W have a uniform distribution, their joint distribution is a copula.

Tail Copula

- Now consider the i.i.d. r.v.'s

$$(V_i, W_i) = (1 - F_1(X_i), 1 - F_2(Y_i)) \quad (i \leq n)$$

- Empirical distribution function: $\frac{1}{n}\sum_{i=1}^{n} 1_{\{V_i \leq x, W_i \leq y\}}$.
- We consider the left tail of (V_i, W_i) i.e., the right tail for (X_i, Y_i).
- We define the tail version

$$T_n(x, y) := \frac{1}{k}\sum_{i=1}^{n} 1_{\{V_i \leq \frac{kx}{n}, W_i \leq \frac{ky}{n}\}}.$$

Tail Copula

Now, $T_n(x, y)$ is close to its mean which is

$$\frac{n}{k} P \left\{ 1 - F_1(X) \leq \frac{kx}{n}, 1 - F_2(Y) \leq \frac{ky}{n} \right\}$$

and this is close to $R(x, y)$ (as $n \to \infty$.)
"Hence" $T_n(x, y) \stackrel{P}{\to} R(x, y)$ and

$$\sqrt{k} \left(T_n(X, Y) - R(x, y) \right)$$

converges in distribution to a mean zero Gaussian process W_R.
This stochastic process $W_R(x, y)$ has independent increments that is,

$$EW_R(x_1, y_1) W_R(x_2, y_2) = R(x_1 \vee x_2, y_1 \vee y_2).$$

Convergence for $\hat{\theta}_{k/n}$

$$
\begin{aligned}
\int_0^\infty T_n(x,1)\,dx^{-\gamma_1} &= \frac{1}{k}\sum_{i=1}^{n}\int_0^\infty 1_{\left\{X_i > U_1\left(\frac{n}{kx}\right),\, Y_i > U_2\left(\frac{n}{k}\right)\right\}}\,dx^{-\gamma_1} \\
&\overset{U_1 \in R.V.}{\approx} \frac{1}{k}\sum_{i=1}^{n}\int_0^\infty 1_{\left\{X_i > x^{-\gamma_1} U_1\left(\frac{n}{k}\right),\, Y_i > U_2\left(\frac{n}{k}\right)\right\}}\,dx^{-\gamma_1} \\
&= \frac{1}{k}\sum_{i=1}^{n}\int_0^\infty 1_{\left\{X_i > xU_1\left(\frac{n}{k}\right),\, Y_i > U_2\left(\frac{n}{k}\right)\right\}}\,dx \\
&= \frac{1}{k}\sum_{i=1}^{n}\int_0^\infty 1_{\left\{X_i > xU_1\left(\frac{n}{k}\right)\right\}} 1_{\left\{Y_i > U_2\left(\frac{n}{k}\right)\right\}}\,dx
\end{aligned}
$$

Cont.

$$
\begin{aligned}
&= \frac{1}{k} \sum_{i=1}^{n} 1_{\left\{Y_i > U_2\left(\frac{n}{k}\right)\right\}} \int_0^{X_i / U_1(n/k)} dx \\
&= \frac{1}{k} \sum_{i=1}^{n} \frac{X_i}{U_1\left(\frac{n}{k}\right)} 1_{\left\{Y_i > U_2\left(\frac{n}{k}\right)\right\}} \\
&\approx \frac{1}{k} \sum_{i=1}^{n} \frac{X_i}{U_1\left(\frac{n}{k}\right)} 1_{\left\{Y_i > Y_{n-k,n}\right\}} = \frac{\hat{\theta}_{k/n}}{U_1(n/k)}.
\end{aligned}
$$

Cont.

Hence,

$$\frac{\sqrt{k}}{U_1(n/k)}\left(\hat{\theta}_{k/n}-\theta_{k/n}\right)\approx\sqrt{k}\int_0^{\infty}\left\{T_n(x,1)-R(x,1)\right\}dx^{-1/\gamma}$$

and we get that

$$\sqrt{k}\left\{\frac{\hat{\theta}_{k/n}}{\theta_{k/n}}-1\right\}$$
$$\xrightarrow{d}(\gamma_1-1)W_R(\infty,1)+\left(\int_0^{\infty}R(s,1)ds^{-\gamma_1}\right)^{-1}\int_0^{\infty}W_R(s,1)ds^{-\gamma_1}$$

Extrapolation

- We need to extrapolate from $\theta_{k/n}$ to θ_p.
- Consider our first (non-statistical) result again:

$$\lim_{p\to 0} \frac{E\left(X|Y > U_2(1/p)\right)}{U_1(1/p)} = \int_0^\infty R(x^{-1/\gamma_1}, 1)dx$$

- In particular this holds for $p = k/n$, i.e,

$$\lim_{n\to\infty} \frac{E\left(X|Y > U_2(n/k)\right)}{U_1(n/k)} = \int_0^\infty R(x^{-1/\gamma_1}, 1)dx$$

- Thus, we have that, for suffiently large n,

$$\frac{\theta_p}{U_1(1/p)} \approx \frac{\theta_{k/n}}{U_1(n/k)}.$$

Extrapolation

We have that,

$$\begin{aligned}\theta_p &= E\left(X|Y > U_2(1/p)\right) \\ &\sim \frac{U_1(1/p)}{U_1(n/k)} E\left(X|Y > U_2(n/k)\right) \\ &= \frac{U_1(1/p)}{U_1(n/k)} \theta_{k/n}\end{aligned}$$

This leads to an estimate for θ_p

$$\hat{\theta}_p = \frac{\widehat{U_1(1/p)}}{\widehat{U_1(n/k)}} \hat{\theta}_{k/n}$$

Here, $\hat{\theta}_{k/n}$ is the estimator we discussed before and $\widehat{U_1(n/k)} = X_{n-k,n}$.

Estimation for $\widehat{U_1(1/p)}$

- It remains to define and to study $\widehat{U_1(1/p)}$ with $np_n = O(1)$.
- Now, $U_1(1/p)$ is a one-dimensional object.
- Recall the condition $U \in RV$, i.e.,

$$\lim_{t\to\infty} \frac{U_1(tx)}{U_1(t)} = x^{\gamma_1}.$$

- Hence, for large t, $U_1(tx) \approx x^{\gamma_1} U_1(t)$
- Use this relation with $t := n/k, tx = 1/p$, we get

$$U_1(1/p) \approx U_1(n/k) \left(\frac{k}{np_n} \right)^{\gamma_1}.$$

- This suggests the estimator for $\widehat{U_1(1/p)}$:

$$\widehat{U_1(1/p)} = X_{n-k,n} \left(\frac{k}{np_n} \right)^{\hat{\gamma}_1}.$$

Estimation for γ_1

- Since $\gamma_1 > 0$, we use the well-known Hill estimator:

$$\hat{\gamma}_1 = \frac{1}{k_1} \sum_{i=1}^{k_1 - 1} \log X_{n-i,n} - \log X_{n-k_1,n}$$

- k_1 may differ from k but satisfies similar conditions.
- Property of Hill's estimator:

$$\sqrt{k_1}\,(\hat{\gamma}_1 - \gamma_1) \xrightarrow{d} \Gamma$$

Property of $\widehat{U_1(1/p)}$

- Property of $X_{n-k,n}$

$$\sqrt{k}\left(\frac{X_{n-k,n}}{U_1(n/k)}-1\right)\xrightarrow{d} N_0$$

- Combine the two relations:

$$\begin{aligned}\frac{\widehat{U_1(1/p)}}{U_1(1/p)} &= \frac{X_{n-k,n}}{U_1(n/k)}\frac{U_1(n/k)}{U_1(1/p_n)}\left(\frac{k}{np_n}\right)^{\hat{\gamma}_1}\\ &\approx \frac{X_{n-k,n}}{U_1(n/k)}\left(\frac{np_n}{k}\right)^{\gamma_1}\left(\frac{k}{np_n}\right)^{\hat{\gamma}_1}\\ &= \frac{X_{n-k,n}}{U_1(n/k_1)}\left(\frac{k}{np_n}\right)^{\hat{\gamma}_1-\gamma_1}\end{aligned}$$

Cont.

$$\approx \left(1 + \frac{N_0}{\sqrt{k}}\right) \exp\left\{ \sqrt{k_1}\,(\hat{\gamma}_1 - \gamma_1)\,\frac{\log \frac{k}{np_n}}{\sqrt{k_1}} \right\}.$$

- Now, assume that

$$\frac{\log \frac{k}{np_n}}{\sqrt{k_1}} \to 0,$$

  (this means that p can not be too small.)

- Then (expansion of function "exp")

$$\frac{\widehat{U_1(1/p)}}{U_1(1/p)} \approx \left(1 + \frac{N_0}{\sqrt{k}}\right) \left\{ 1 + \sqrt{k_1}\,(\hat{\gamma}_1 - \gamma_1)\,\frac{\log \frac{k}{np_n}}{\sqrt{k_1}} \right\}$$

Cont.

Hence,

$$\frac{\sqrt{k_1}}{\log \frac{k}{np_n}} \left(\frac{\widehat{U_1(1/p)}}{U_1(1/p)} - 1 \right) \xrightarrow{d} \Gamma$$

Theorem 1

Technical Conditions:

(a) There exists $\beta > \gamma_1$ and $\tau < 0$ such that as $t \to \infty$,

$$\sup_{\substack{0<x<\infty \\ 1/2 \leqslant y \leqslant 2}} \frac{|tP\{1 - F_1(X) < x/t, 1 - F_2(Y) < y/t\} - R(x,y)|}{x^\beta \wedge 1} = O\left(t^\tau\right)$$

(b) There exist $\rho_1 < 0$ and $A_1 \in RV(\rho_1)$, such that

$$\sup_{x>1} \left| x^{-\gamma_1} \frac{U_1(tx)}{U_1(t)} - 1 \right| = O\left\{A_1(t)\right\}.$$

(c) As $n \to \infty$, $\sqrt{k_1} A(n/k_1) \to 0$.

(d) As $k \to \infty$, $k = O(n^\alpha)$ for some $\alpha < \min\left\{-2\tau/(-2\tau + 1), 2\gamma_1\rho_1/(2\gamma_1\rho_1 + \rho_1 - 1)\right\}$

Theorem 1

- Suppose the conditions (a)-(d) hold.
- Suppose $\gamma \in (0, 1/2)$.
- Suppose $X > 0$.
- Assume $d_n = \frac{k}{np_n} \geq 1$ and $\log d_n / \sqrt{k_1} \to 0$.

Denote $r = \lim_{n\to\infty} \frac{\sqrt{k}\log d_n}{\sqrt{k_1}} \in [0, \infty]$. Then, as $n \to \infty$,

$$\min\left(\sqrt{k}, \frac{\sqrt{k_1}}{\log d_n}\right)\left(\frac{\hat{\theta}_p}{\theta_p} - 1\right) \xrightarrow{d} \begin{cases} \Theta + r\Gamma, & \text{if } r \leq 1, \\ \frac{1}{r}\Theta + \Gamma, & \text{if } r > 1. \end{cases}$$

X real

So far we assumed $X > 0$.
For general $X \in \mathbb{R}$ we need some extra conditions

- Thinner left tail: $E\left|\min(X,0)\right|^{1/\gamma_1} < \infty$
- A further bound on $p = p_n$.

We estimate θ_p with

$$\left(\frac{k}{np_n}\right)^{\hat{\gamma}_1} \frac{1}{k} \sum_{i=1}^{n} X_i I(X_i > 0, Y_i > Y_{n-k,n})$$

Table of Contents

1. Introduction
2. Main Results
3. Simulation
4. Application

Simulation Distributions

- Generate Data from three bivariate distributions.
- Let (Z_1, Z_2) denotes a standard Cauchy distribution on $\mathbb{R}^2$ with density $(1/2\pi)(1+x^2+y^2)^{-3/2}$.

Simulation Distributions

- transformed Cauchy distribution on $(0,\infty)^2$ defined as

$$(X,Y) = (|Z_1|^{2/5}, |Z_2|).$$

It follows that $\gamma_1 = 2/5$ and $R(x,y) = x + y - \sqrt{(x^2+y^2)}, x, y \geq 0$.

- a Student t_3 -distribution on $(0,\infty)^2$ with density

$$f(x,y) = \frac{2}{\pi}\left(1+\frac{x^2+y^2}{3}\right)^{-5/2}, \quad x,y > 0.$$

We have $\gamma_1 = 1/3, R(x,y) =$
$x + y - (x^{4/3} + 1/2x^{2/3}y^{2/3} + y^{4/3})/\sqrt{(x^{2/3}+y^{2/3})}$.

- a transformed Cauchy distribution on the whole $\mathbb{R}^2$ defined as

$$(X,Y) = \Big(Z_1^{2/5} I\left(Z_1 \geqslant 0\right) + Z_1^{1/5} I\left(Z_1 < 0\right),$$
$$Z_2 I\left(Z_1 \geqslant 0\right) + Z_2^{1/3} I\left(Z_1 < 0\right)\Big).$$

We have $\gamma_1 = 2/5, R(x,y) = x/2 + y - \sqrt{(x^2/4+y)}$.

Other Estimators

Besides the estimator we propose, we construct two other estimators.

1. For $np \geq 1$, an empirical counterpart of θ_p, given by

$$\hat{\theta}_{\mathrm{emp}} = \frac{1}{\lfloor np \rfloor} \sum_{i=1}^{n} X_i I\left(Y_i > Y_{n-\lfloor np \rfloor, n}\right).$$

Other Estimators

2. Recall the first result,

$$\lim_{p\to 0}\frac{\theta_p}{U_1\left(\frac{1}{p}\right)}=\lim_{p\to 0}\frac{E\left(X|Y>U_2\left(\frac{1}{p}\right)\right)}{U_1\left(\frac{1}{p}\right)}=\int_0^\infty R(x^{-1/\gamma_1},1)dx.$$

. Estimate

$$\hat{R}(x,y)=\frac{1}{k}\sum_{i=1}^{n}I\left(X_i>X_{n-\lfloor kx\rfloor,n},Y_i>Y_{n-\lfloor ky\rfloor,n}\right),\quad x,y\geqslant 0.$$

and $\hat{U}_1(1/p)=d_n^{\hat{\gamma}_1}X_{n-k,n}$, we define an alternative EVT estimator as

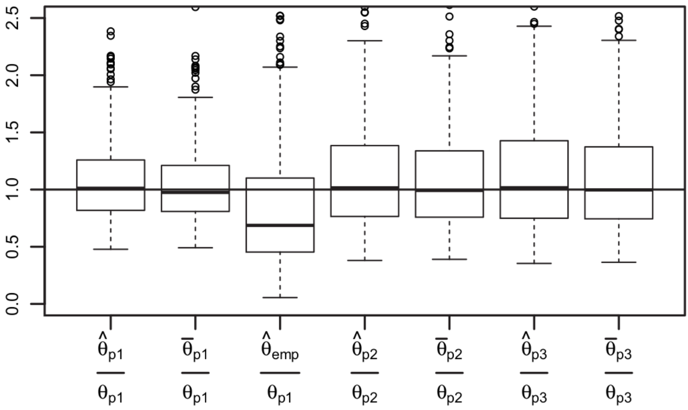
$$\begin{aligned}\bar{\theta}_p&=-\hat{U}_1\left(\frac{1}{p}\right)\int_0^\infty\hat{R}(x,1)\mathrm{d}x^{-\hat{\gamma}_1}\\&=d_n^{\hat{\gamma}_1}X_{n-k,n}\frac{1}{k}\sum_{i=1}^{n}I\left(Y_i>Y_{n-k,n}\right)\left\{\frac{n-\operatorname{rank}\left(X_i\right)+1}{k}\right\}^{-\hat{\gamma}_1}.\end{aligned}$$

Simulation Setting

- 500 samples from each distribution with sample sizes $n = 500$ and $n = 2000$.
- On the basis of each sample, we estimate θ_p for $p = 1/500, 1/5000$ or $1/10000$.

Boxplot of the estimates

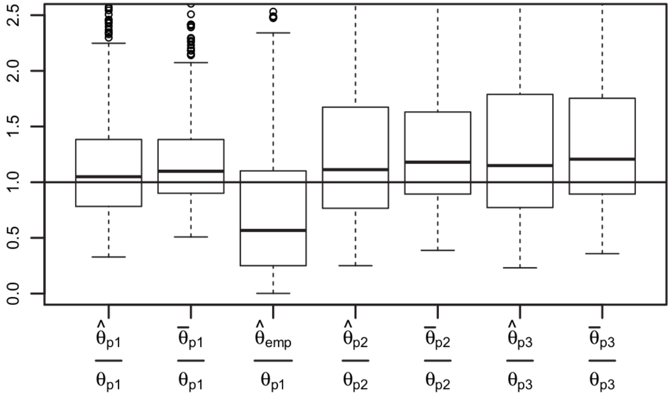
- Transformed Cauchy distribution 1.
- $n = 500, k = 75, k_1 = 75, p_1 = 1/500, p_2 = 1/5000, p_3 = 1/10000$



(a)

Boxplots of the estimates

- Student t_3 distribution
- $n = 500, k = 75, k_1 = 75, p_1 = 1/500, p_2 = 1/5000, p_3 = 1/10000$



(c)

Simulation Distributions

We also investigate the performance of our estimator when our assumptions are partially violated.

- The transformed Cauchy distribution 3 is defined as

$$(X, Y) = \left(|Z_1|^{0.7}, |Z_2|\right).$$

  $\gamma_1 = 0.7 > 1/2$.

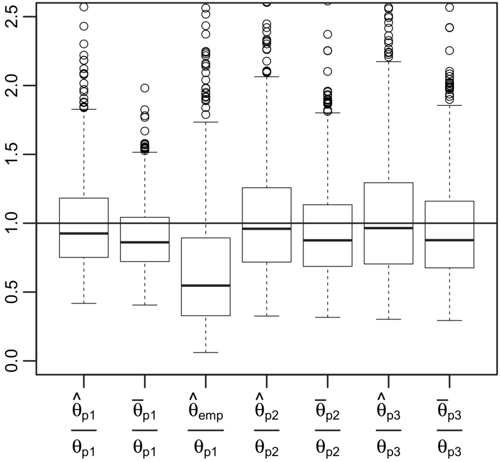
- The second distribution is an asymptotically independent distribution defined as

$$(X, Y) = (V_1 + W_1, V_2 + W_2),$$

  where (V_1, V_2) follows the Student t_3-distribution and W_1 and W_2 are Pareto distributed with density $(25/2)(1+5x)^{-7/2}, x > 0$. Moreover, (V_1, V_2), W_1 and W_2 are independent. This does not satisfy condition (a).

Boxplots of the estimates

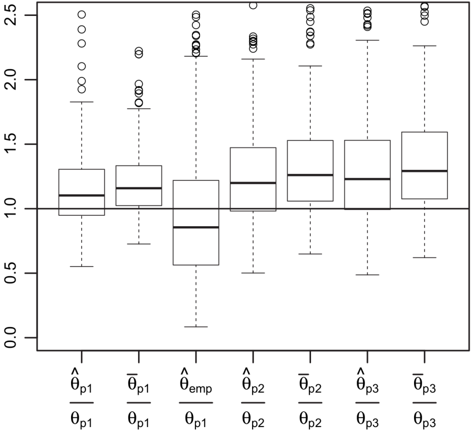
- transformed Cauchy distribution 3.



(a)

Boxplots of the estimates

- Asymptotically independent distribution.



(b)

Table of Contents

1 Introduction

2 Main Results

3 Simulation

4 Application

Datasets

- We apply our estimation method to estimate the MES for financial institutions.
- We consider three large investment banks in the USA, namely Goldman Sachs, Morgan Stanley and T. Rowe Price.
- Then, X refers to each of these banks, Y refers to the market index (value weighted index aggregating three markets: NYS, AE, NASDAQ).
- $n = 2513$(daily loss) and $n = 522$(weekly loss)
- $p = 1/n$, which corresponds to a once-per-decade systemic event.

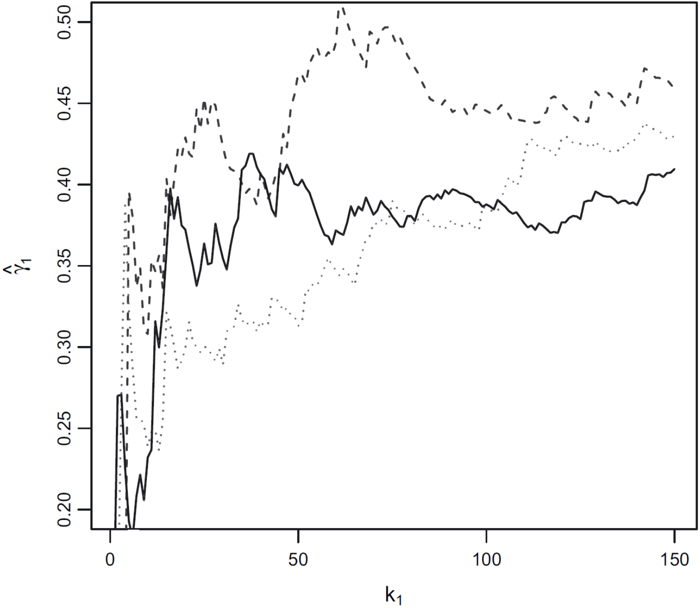


Fig. 4. Hill estimates based on daily loss returns of three investment banks: ———, Goldman Sachs; – – –, Morgan Stanley; · · · · · ·, T. Rowe Price

Table 1. MES of the three investment banks†

Bank	*Daily loss*		*Weekly loss*	
	$\hat{\gamma}_1$	$\hat{\theta}_p$	$\hat{\gamma}_1$	$\hat{\theta}_p$
Goldman Sachs	0.388	0.308	0.417	0.346
Morgan Stanley	0.465	0.608	0.483	0.654
T. Rowe Price	0.378	0.316	0.347	0.339

†The second and third columns report the results based on *daily* loss returns ($n = 2513$ and $p = 1/n$). The estimates $\hat{\gamma}_1$ are computed by taking the average for $k_1 \in [70, 100]$. The estimates of the MES are based on these values of $\hat{\gamma}_1$. We report the average of the MES estimates $\hat{\theta}_p$ for $k \in [70, 100]$. The last two columns report the results based on *weekly* loss returns from the same sample period ($n = 522$ and $p = 1/n$), where both k_1 and k are from the interval [20, 30].