Statistical Inference for a Relative Risk Measure

Y. He, Y. Hou, L. Peng and J. Sheng

January 28, 2021

Published by Journal of Business and Economics Statistics (2009).

Presented by Liujun Chen.

Table of Contents

Systemic Risk

- A formal definition of systemic risk does not exist arguably.
- It is commonly agreed that systemic risk involves the co-movement of several key financial variables.
- Many measures have been proposed in the literature.

Relative Risk Measure

- From regulators' point of view, having risk measures from each agency does not help understand/measure systemic risk at all.
- It would be more meaningful to have some relative risk measures reported by each agency with respect to a common benchmark.

Therefore, an interesting question becomes:

(a) how to define a relative risk measure, which should be quite sensitive to the market co-movement for the purpose of studying systemic risk

(b) how to infer such a relative risk measure.

Mathematical Definition

- Let X denote the loss of an individual portfolio; Y denote the loss of some benchmark (a financial market index).
- Consider the commonly employed expected shortfall risk measure at level $\alpha \in (0,1)$, defined as

$$\mathrm{ES}_\alpha(X) = E[X|F_1(X) > 1-\alpha]$$
$$\mathrm{ES}_\alpha(Y) = E[Y|F_2(X) > 1-\alpha]$$

- A quick way to compare these two risk measures is to look at their ratio

$$\frac{\mathrm{ES}_\alpha(X)}{\mathrm{ES}_\alpha(Y)}$$

Relative Risk

- this ratio is invariant to the copula of X and Y , that is, it is irrelevant to the market comovement.
- To capture the extreme dependence between X and Y, Agarwal et al.(2017) proposed to multiply the above ratio by the coefficient of (upper) tail dependence

$$\lambda = \lim_{t \downarrow 0} P\left(F_1(x) > 1 - t | F_2(Y) > 1 - t\right).$$

- Since the coefficient of tail dependence is defined in a limiting way, nonparametric estimator for it has a slower rate of convergence than that for the ratio of expected shortfalls.

Relative Risk

- Agarwal et al.(2017) proposes to define the relative risk as

$$\rho_\alpha = \rho_\alpha(X, Y) = P\left(F_1(x) > 1-\alpha | F_2(Y) > 1-\alpha\right) \frac{\mathrm{ES}_\alpha(X)}{\mathrm{ES}_\alpha(Y)}$$

- This article aims to derive asymptotic limit for a nonparametric estimator and its smoothed version of the above relative risk measure and to provide an effective way to construct an interval by considering either a fixed level or an intermediate level.

Table of Contents

Notations

- Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. random vectors with joint d.f. $F(x, y)$ and marginals F_1, F_2.
- Order the X_i's as $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ and Y_i's as $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$.
- $\bar{F}_i = 1 - F_i$ and $Q_i = F_i^{\leftarrow}$.
- Use $\bar{F}_{n1}(x)$ and $\bar{F}_{n2}(y)$ denote the empirical survival function of X and Y respectively.
- Define survival copula function

$$C(u, v) = P\left(\bar{F}_1(X) < u, \bar{F}_2(Y) < v\right).$$

Non-parametric Estimation

We can rewrite

$$\rho_\alpha = \frac{1}{\alpha} C(\alpha, \alpha) \frac{\mathrm{ES}_\alpha(X)}{\mathrm{ES}_\alpha(Y)}.$$

Substituting the right-hand-side components by their empirical counterparts yields our nonparametric estimator

$$\rho_\alpha = \frac{1}{\alpha} \widetilde{C}(\alpha, \alpha) \frac{\widetilde{\mathrm{ES}}_\alpha(X)}{\widetilde{\mathrm{ES}}_\alpha(Y)}.$$

Smoothed Estimator

With some density function k, its distribution function K and the bandwidth $h = h(n) > 0$, a smoothed estimator of ρ_α is given by

$$\rho_\alpha = \frac{1}{\alpha}\widehat{C}(\alpha,\alpha)\frac{\widehat{\mathrm{ES}}_\alpha(X)}{\widehat{\mathrm{ES}}_\alpha(Y)},$$

where

$$\widehat{C}(\alpha,\alpha) = \frac{1}{n}\sum_{j=1}^{n} K\left(\frac{1-\bar{F}_{n1}(X_i)/a}{n}\right) K\left(\frac{1-\bar{F}_{n2}(Y_i)/a}{n}\right)$$

$$\widehat{\mathrm{ES}}_\alpha(X) = \frac{1}{n\alpha}\sum_{j=1}^{n}(X_i - X_{n-[n\alpha],n})K\left(\frac{1-\bar{F}_{n1}(X_i)/a}{n}\right) + X_{n-[n\alpha],n}$$

$$\widehat{\mathrm{ES}}_\alpha(Y) = \frac{1}{n\alpha}\sum_{j=1}^{n}(Y_i - Y_{n-[n\alpha],n})K\left(\frac{1-\bar{F}_{n2}(Y_i)/a}{n}\right) + Y_{n-[n\alpha],n}$$

Assumption 1 (Fixed level).

(1.a) For $j = 1, 2$, Q_j is Lipschitz continuous in a neighborhood of $1 - \alpha$ with $Q_j(1-\alpha) > 0$, and F_j is strictly increasing and differentiable in a neighborhood of $Q_j(1-\alpha)$. Moreover, for some $\delta > 0$, $\mathbb{E}(X_+^{2+\delta}) < \infty$ and $\mathbb{E}(Y_+^{2+\delta}) < \infty$

(1.b) The copula C has continuous first-order derivatives $C_1(x, \alpha) = \frac{\partial C(x,\alpha)}{\partial x}$ and $C_2(x, \alpha) = \frac{\partial C(\alpha,y)}{\partial y}$ in a neighborhood of, respectively, $x = \alpha$ and $y = \alpha$.

Theorem (fixed level)

For any $\alpha \in (0,1)$ satisfying $C(\alpha,\alpha) > 0$, Assumption 1 implies that

$$\sqrt{n\alpha}\left(\frac{\tilde{\rho}_\alpha}{\rho_\alpha} - 1\right) \xrightarrow{d} N(0, \sigma_\alpha^2),$$

with $\sigma_\alpha^2 = \mathrm{Var}(\Lambda_\alpha + \Theta_{\alpha,1} - \Theta_{\alpha,2})$, where

$$\begin{aligned}\Lambda_\alpha &= \frac{\sqrt{\alpha}}{C(\alpha,\alpha)}\{B_C(\alpha,\alpha) - C_1(\alpha,\alpha)B_C(\alpha,1)\\ &\quad - C_2(\alpha,\alpha)B_C(1,\alpha)\},\\ \Theta_{\alpha,1} &= -\frac{\frac{1}{\sqrt{\alpha}}\int_0^1 B_C(\alpha x,1)dQ_1(1-\alpha x)}{ES_\alpha(X)},\\ \Theta_{\alpha,2} &= -\frac{\frac{1}{\sqrt{\alpha}}\int_0^1 B_C(1,\alpha y)dQ_2(1-\alpha y)}{ES_\alpha(Y)}.\end{aligned}$$

Here, B_C is a C-Brownian bridge, that is, a zero-mean Gaussian process with covariance function

$$\begin{aligned}\mathbb{E}(B_C(u_1,v_1)B_C(u_2,v_2)) &= C(u_1 \wedge u_2, v_1 \wedge v_2) - C(u_1,v_1)\\ &\quad \times C(u_2,v_2), \quad (u_1,v_1),(u_2,v_2) \in [0,1]^2.\end{aligned}$$

fixed level

Furthermore, if k is a symmetric density with support $[-1,1]$ and bounded first derivative and the bandwidth $h = h(n) > 0$ satisfies

$$nh^2 \to \infty \quad \text{and} \quad nh^4 \to 0,$$

then we have that, as $n \to \infty$,

$$\sqrt{n\alpha}\left(\frac{\hat{\rho}_\alpha - \tilde{\rho}_\alpha}{\rho_\alpha}\right) \xrightarrow{P} 0.$$

Intermediate level

When α is close to zero (but not extremely), it is often useful to model α as an intermediate sequence of n in such a way that as $\alpha = \alpha_n \to 0$ and $n\alpha_n \to \infty$.

For the study of an intermediate level α, we need to control the tail behaviour of the distribution.

Assumption 2(intermediate level)

- For some $\gamma_j \in (0, 1/2), \beta_j \le 0$ and function A_j with a constant sign near infinity,
$$\lim_{t\to\infty} \frac{1}{A_j(1/\bar{F}_j(t))}\left(\frac{\bar{F}_j(tx)}{\bar{F}_j(t)} - x^{-1/\gamma_j}\right) = x^{-1/\gamma_j}\frac{x^{\beta_j/\gamma_j}-1}{\gamma_j\beta_j}.$$
- There exists a function $R : (0,\infty)^2 \to [0,\infty]$ such that
$$\lim_{t\to\infty} C(t^{-1}x, t^{-1}y) = R(x, y)$$
and it has continuous first-order derivatives on a neighborhood of $(1, 1)$.
- The function C has first-order derivatives on $(0, \delta)^2$ for some $\delta > 0$ and as $t \to \infty$,
$$\sup_{x,y\in(1-\delta,1+\delta)} \left|C_i(t^{-1}x, t^{-1}y) - R_i(x, y)\right| \to 0.$$