Modeling Maxima with autoregressive conditional Fréchet model

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Introduction

Maximum observations, as the representation of extreme behavior, are of particular interest.

- mutual fund managers: the potential maximum daily loss across all stocks in their managed portfolio
- high-frequency traders: the level of potential intra-day maximum loss

Introduction

Most applications of EVT focus on modeling extreme events in time series with a static approach under equilibrium distribution.

- The behavior of the underlying time series may change through time.
- Financial time series tends to exhibit structural changes and time-varying dynamics such as volatility clustering.

Introduction

Model time series of maxima $\{Q_t\}$, where $\{Q_t\}$ is a univariate time series of maxima based on a set of underlying financial time series $\{X_{it}\}_{i=1}^{p}$. There are mainly two types of Q_t.

- time series of cross-sectional maxima e.g. modeling the maximum daily loss across a group of stocks in a portfolio
- the time series of intra-period maxima e.g. intra-day maxima of high-frequency trading losses that occur on the same day.

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Dynamic Model

- A dynamic GEV framework: a conditional evolution scheme is designed for the parameters (μ_t, σ_t, ξ_t) of GEV, so that time dependency of $\{Q_t\}$ can be captured.
- Due to the heavy-tailedness of financial data, $\{Q_t\}$ can marginally can be well modeled by Fréchet distribution.
- $$Q_t = \mu_t + \sigma_t Y_t^{1/\alpha_t},$$

  where Y_t is a sequence of i.i.d. unit Fréchet random variables.

Should α_t be dynamic?

- Ad-hoc moving-window GEV analysis on the cross-sectional maxima of negative daily log-returns of S&P 100 index
- The observation period is from January 1, 2000 to December 31, 2014 with 3773 trading days.
- For each trading day t, we record the maximum daily loss across the 100 stocks and denote it by Q_t, Hence the time series $\{Q_t\}$ has 3773 observations
- For each $500 \leq t \leq 3273$, a GEV model is fitted using $\{Q_k\}_{k=t-499}^{t+500}$.

Moving Window Estimatio of Tail Index α

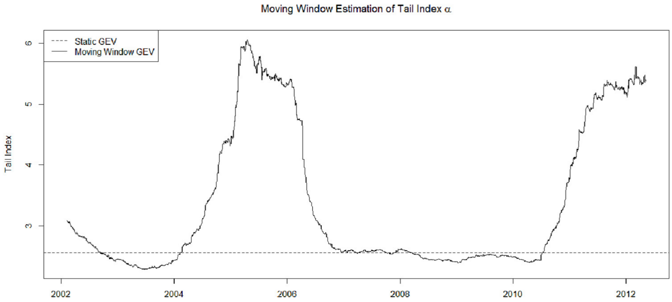

Fig. 1. Tail index $\hat{\alpha}_t$ estimated by moving window of size 1000 (solid curve) v.s. tail index $\hat{\alpha}$ estimated by the static GEV model based on total observations (dashed line).

Model Specification

Specifically, the autoregressive conditional Fréchet(AcF) model assumes the form

$$\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} + \eta_1(Q_{t-1})$$
$$\log \alpha_t = \gamma_0 + \gamma_1 \log \sigma_{t-1} + \eta_2(Q_{t-1}),$$

where $\beta_1, \gamma_1 \geq 0$. The log transform is used to ensure the positivity of the parameters.

Assume that η_1 is a continuous increasing function and η_2 is a continuous decreasing function of Q_{t-1}. Choose η_1, η_2 to be the simple exponential function $a_0 \exp(-a_1 x)$.

Model Specifiction

For the rest of the paper, we consider the following model:

$$Q_t = \mu + \sigma_t Y_t^{1/\alpha_t},$$

$$\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} - \beta_2 \exp(-\beta_3 Q_{t-1}), \tag{*}$$

$$\log \alpha_t = \gamma_0 + \gamma_1 \log \sigma_{t-1} + \gamma_2 \exp(-\gamma_3 Q_{t-1}),$$

where $\{Y_t\}$ is a sequence of i.i.d. unit Fréchet random variables, $0 \leq \beta_1 \neq \gamma_1 < 1$, $\beta_2, \beta_3, \gamma_2, \gamma_3 > 0$.

Stationarity and ergodicity

The model can be written as

$$\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} - \beta_2 \exp(-\beta_3(\mu + \sigma_{t-1} Y_{t-1}^{1/\alpha_{t-1}})),$$

$$\log \alpha_t = \gamma_0 + \gamma_1 \log \sigma_{t-1} + \gamma_2 \exp(-\gamma_3(\mu + \sigma_{t-1} Y_{t-1}^{1/\alpha_{t-1}})),$$

Hence, $\{\sigma_t, \alpha_t\}$ form a homogeneous Markov chain in $\mathbb{R}^2$.

Theorem (1)

For an AcF with $\beta_2, \beta_3, \gamma_2, \gamma_3 > 0$, $\beta_0, \gamma_0, \mu \in \mathbb{R}$ and $0 \leq \beta_1 \neq \gamma_1 < 1$, the latent process $\{\sigma_t, \alpha_t\}$ is stationary and geometrically ergodic.

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MLE

- denote all the parameters in the model by $\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \mu)$

- denote $\Theta_s = \{\theta | \beta_0, \gamma_0, \mu \in \mathbb{R}, 0 \leq \beta_1, \gamma_1 < 1, \beta_2, \beta_3, \gamma_2, \gamma_3 > 0\}$

- Assume that all allowable parameters are in Θ_s

- Denote the true parameter by $\beta_0 = (\beta_0^0, \beta_1^0, \beta_2^0, \beta_3^0, \gamma_0^0, \gamma_1^0, \gamma_2^0, \gamma_3^0, \mu_0)$

MLE

The conditional p.d.f. of Q_t given $(\mu_t, \sigma_t, \alpha_t)$ is

$$f_t(\theta) = f(Q_t|\sigma_t, \alpha_t) = \alpha_t \sigma_t^{\sigma_t}(Q_t - \mu)^{-(\alpha_t+1)} \exp\left\{-\sigma_t^{\alpha_t}(Q_t - \mu)^{-\alpha_t}\right\}.$$

Hence, by conditional independence, the log-likelihood function with observations $\{Q_t\}_{t=1}^n$ is

$$\begin{aligned} L_n(\theta) &= \frac{1}{n}\sum_{t=1}^{n} \log f_t(\theta) \\ &= \frac{1}{n}\sum_{i=1}^{n} \log \alpha_t + \sigma_t \log \sigma_t - (\alpha_t + 1)\log(Q_t - \mu) - \sigma_t^{\alpha_t}(Q_t - \mu)^{-\alpha_t} \end{aligned}$$

where $\{\sigma_t, \alpha_t\}$ can be obtained recursively through (*) with an initial value (σ_1, α_1).

Initial Value

- Notice here the true value of (σ_1, α_1), denoted as (σ_1^0, α_1^0), is an unknown since the state variables $\{\sigma_t, \alpha_t\}$ is a hidden process.
- Fortunately, with $0 \leq \beta_1, \gamma_1 < 1$, the influence of (σ_1, α_1) on future (σ_t, α_t) decays exponentially as t increases
- Hence its impact on parameter estimation will be minimum with a sufficiently large sample size.
- It will be shown that the consistency and asymptotic normality does not depend on the initial value.

Asymptotic Properties

Theorem (Consistency)

Assume the parameter space Θ is a compact set of Θ_s. Suppose the observations $\{Q_t\}_{t=1}^{n}$ are generated by a stationary and ergodic AcF with true parameter θ_0 and θ_0 is in the interior of Θ, then there exists a sequence θ_n of local maximizer of $L_n(\theta)$ such that $\theta_n \xrightarrow{p} \theta_0$ and $||\theta_n - \theta_0|| \leq \tau_n$ where $\tau_n = O_p(n^{-r}), 0 < r < 1/2$. Hence $\hat{\theta}_n$ is consistent.

Asymptotic Properties

Proposition (Asymptotic Uniqueness)

Denote $V_n = \{\theta \in \Theta | \mu \leq cQ_{n,1} + (1-c)\mu_0\}$ *where* $Q_{n,1} = \min_{1 \leq t \leq n} Q_t$*, under the conditions in the Theorem for consistency, for any fixed* $0 < c < 1$*, there exists a sequence of* $\hat{\theta}_n = \arg\max_{\theta \in V_n} \tilde{L}(\theta)$ *such that,* $\hat{\theta}_n \xrightarrow{P} \theta_0$*,* $||\theta_n - \theta_0|| \leq \tau_n$ *where* $\tau_n = O_p(n^{-r}), 0 < r < 1/2$*, and* $P(\hat{\theta}_n$ *is the unique global maximizer of* $\tilde{L}_n(\theta)$ *over* $V_n) \to 1$*.*

Asymptotic Properties

Theorem (Asymptotic Normality)

Under the conditions in Theorem for consistency, we have that $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, M_0^{-1})$*, where* $\hat{\theta}_n$ *is that in Theorem of consistency and* M_0 *is the Fisher Information matrix evaluated at* θ_0*. Further, the sample variance of plug-in estimated score function* $\left\{ \frac{\partial}{\partial\theta} \log f_t(\hat{\theta}_n) \right\}_{t=1}^{n}$ *is a consistent estimator of* M_0*.*

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Convergence of maxima in factor model

- we conduct numerical experiments to investigate the finite sample behavior of Q_t
- Specifically, we study the convergence of the marginal distribution of Q_t to its Fréchet limit under a one-time period factor model.
- To simplify notation, we drop the time index t in this section.

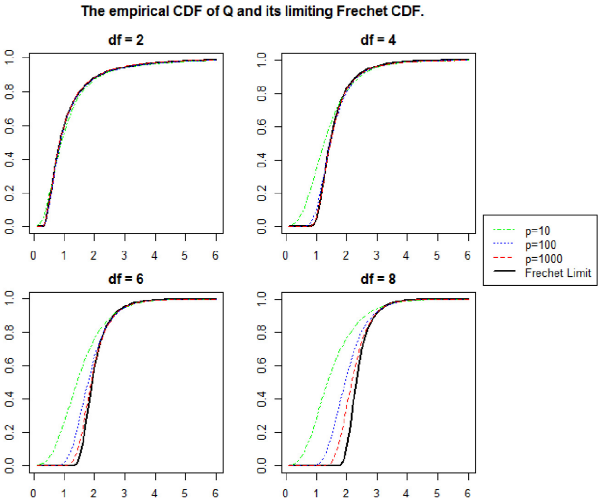
Convergence of maxima in factor model

We simulate data from the following one-factor linear model,

$$X_i = \beta_i Z + \sigma_i \varepsilon_i, \quad i = 1, \ldots, p,$$

- $Z \sim N(0,1)$ is the latent model,
- β_i are i.i.d. random coefficients generated from a uniform distribution $U(-2,2)$
- ε_i are i.i.d. t-distributions with degree of freedom v
- σ_i are i.i.d. random variables generated from a mixture of uniform distribution $\frac{1}{2}U(0.5,1.5) + \frac{1}{2}U(0.75,1.25)$

Finite sample empirical distribution of the maxima Q and its corresponding Fréchet limit



AcF estimation for conditional VaR of maxima