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Statistical Inference for a Relative Risk Measure

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For monitoring systemic risk from regulators' point of view, this article proposes a relative risk measure, which is sensitive to the market comovement. The asymptotic normality of a nonparametric estimator and its smoothed version is established when the observations are independent. To effectively construct an interval without complicated asymptotic variance estimation, a jackknife empirical likelihood inference procedure based on the smoothed nonparametric estimation is provided with a Wilks type of result in case of independent observations. When data follow from AR-GARCH models, the relative risk measure with respect to the errors becomes useful and so we propose a corresponding nonparametric estimator. A simulation study and real-life data analysis show that the proposed relative risk measure is useful in monitoring systemic risk.

KEY WORDS: Copula; Expected shortfall; Jackknife empirical likelihood; Nonparametric estimation; Systemic risk.

1. INTRODUCTION

The recent financial crisis has highlighted the impact of systemic risk on the stability of the financial system. On the quantitative side, academics and policy makers have called for advanced statistical tools to measure systemic risk. Although a formal definition of systemic risk does not exist arguably, it is commonly agreed that systemic risk involves the co-movement of several key financial variables. Many measures have been proposed in the literature on banking industry; see the excellent review article (Bisias et al. 2012). Studies on systemic risk in the insurance/reinsurance industry have started to attract attention too. For example, Chen et al. (2014) used public data to investigate the Granger causality effect between banks and insurers by using some existing systemic risk measures, Chen et al. (2015) used network to analyze global insurers, and Berdin and Sottocornola (2015) examined the relationship between insurance activities and systemic risk. The existing literature provides a diversified and controversial picture of the systemic relevance of the insurance/reinsurance industry; see Billio et al. (2012), Chen et al. (2014), Weiss and Muhlnickel (2014), and Bierth, Irresberger, and Weiss (2015).

From regulators' point of view, having risk measures from each agency does not help understand/measure systemic risk at all. Instead, it would be more meaningful to have some relative risk measures reported by each agency with respect to a common benchmark, and hence, regulators could focus on further modeling, analyzing and monitoring those agencies with a larger relative risk. Therefore, an interesting question becomes (i) how to define a relative risk measure, which should be quite sensitive to the market co-movement for the purpose of studying systemic risk, and (ii) how to infer such a relative risk measure.

Let X and Y denote the random losses, respectively, on an individual portfolio and some benchmark variable, say, a financial market index with joint continuous distribution function $F(x, y)$. Consider the commonly employed expected shortfall risk measure, at level $\alpha \in (0, 1)$, defined as

$$\mathrm{ES}_\alpha(X) = E[X|F_1(X) > 1-\alpha] \quad \text{and}$$
$$\mathrm{ES}_\alpha(Y) = E[Y|F_2(Y) > 1-\alpha],$$

where F_1 and F_2 are the marginal distributions of X and Y given by $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$. A quick way to compare these two risk measures is to look at their ratio $\mathrm{ES}_\alpha(X)/\mathrm{ES}_\alpha(Y)$ (or difference). However this ratio or difference is invariant to the copula of X and Y, that is, it is irrelevant to the market comovement. To capture the extreme dependence between X and Y, recently, Agarwal, Ruenzi, and

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Weigert (2017) proposed to multiply the above ratio by the coefficient of (upper) tail dependence

$$\lambda = \lim_{t\downarrow 0} P(F_1(X) > 1 - t | F_2(Y) > 1 - t),$$

which was proposed by Ledford and Tawn (1996) and Ledford and Tawn (1997) and widely studied by many others in modeling extreme events. Since the coefficient of tail dependence is defined in a limiting way, nonparametric estimator for it has a slower rate of convergence than that for the ratio of expected shortfalls. This means the variability of nonparametrically estimating the ratio of expected shortfalls does not impact the asymptotic behavior of the nonparametric estimator for the defined relative risk in Agarwal, Ruenzi, and Weigert (2017). In other words, nonparametric estimator for the proposed relative risk measure in Agarwal, Ruenzi, and Weigert (2017) is not sensitive to the variability of individual risks, which is not good to serve as a systemic risk. Indeed in the empirical study, Agarwal, Ruenzi, and Weigert (2017) computed both tail sensitivity and risk measures at the same level. That is, Agarwal, Ruenzi, and Weigert (2017) defined the following *relative risk* measure

$$\rho_\alpha = \rho_\alpha(X, Y) = P(F_1(X) > 1 - \alpha | F_2(Y) > 1 - \alpha) \frac{ES_\alpha(X)}{ES_\alpha(Y)}.$$

Although Agarwal, Ruenzi, and Weigert (2017) mentioned a nonparametric estimator for the above relative risk, which is called tail risk by them, there is no any theoretical justification. In general when people talk about tail risk, it usually means the level $\alpha = \alpha(n)$ goes to zero as the sample size $n \to \infty$. Also, it is important to quantify the uncertainty of a risk measure in risk management. Hence, this article aims to derive asymptotic limit for a nonparametric estimator and its smoothed version of the above relative risk measure and to provide an effective way to construct an interval by considering either a fixed level or an intermediate level.

To implement the above relative risk measure ρ_α at a fixed level $\alpha \in (0, 1)$, this article first proposes a nonparametric estimator and its smoothing version and derives an asymptotic normality result based on independent observations. Since the asymptotic variance is quite complicated, we further investigate the possibility of employing an empirical likelihood method to construct a confidence interval since the empirical likelihood method has shown to be quite useful in interval estimation and hypothesis testing. We refer to Owen (2001) for an overview of the method. Quantifying uncertainty is important in risk management, and applications of empirical likelihood methods to risk measures have appeared in Baysal and Staum (2008), Peng et al. (2012), Peng, Wang, and Zheng (2015), and Wang and Peng (2016). In general, an empirical likelihood method is quite effective for linear functionals and requires linearization for a nonlinear functional by introducing some nuisance parameters. Since it is hard to linearize the proposed relative risk measure, we propose to employ the smoothed jackknife empirical likelihood method to construct a confidence interval for the proposed relative risk measure as motivated by the study for copulas and tail copulas in Peng and Qi (2010) and Peng, Qi, and Van Keilegom (2012). Note that smoothed jackknife empirical likelihood method is a generalization of the jackknife empirical likelihood method proposed by Jing, Yuan, and Zhou (2009) for dealing with nonlinear functionals, and smoothing is generally necessary for a nonsmoothing nonlinear functional.

When the level α is close to zero, which is a key interest of regulators, and the sample size n is not large enough, it is useful to model α as a function of n. This is generally classified as two situations: intermediate level (i.e., $\alpha = \alpha_n \to 0$ and $\alpha_n n \to \infty$ as $n \to \infty$) and extreme level (i.e., $\alpha = \alpha_n \to 0$ and $n\alpha \to c \in [0, \infty)$ as $n \to \infty$). Such a divergent level relates to the so-called tail risk in financial econometrics, which plays an important role in risk management; see, for example, Kelly and Jiang (2014). In general, an extreme level requires extrapolating outside the data range. Here, we focus on the intermediate level and extend the above study for a fixed level to this case too. Like quantile estimation, we show that nonparametric estimation for the proposed relative risk has a different asymptotic limit for a fixed level and an intermediate level. However, the proposed smoothed jackknife empirical likelihood method gives a unified interval for ρ_α regardless of the level being fixed or intermediate.

The above study is based on independent observations. When data follow from time series models, it becomes more meaningful to consider the relative risk measure of errors. Motivated by our real data applications, we consider an AR-GARCH model and propose a profile empirical likelihood method to construct an interval for the relative risk measure of errors without estimating the complicated asymptotic variance of a nonparametric estimator.

We organize this article as follows. Section 2 presents our nonparametric estimation procedure, jackknife empirical likelihood method, and asymptotic results based on independent data. When data follow from AR-GARCH models, a profile empirical likelihood method is proposed to construct an interval for the relative risk measure of errors in Section 3. A simulation study is carried out in Section 4, and a data analysis in finance is provided in Section 5 to demonstrate the usefulness of the proposed relative measure in monitoring systemic risk. All proofs are deferred to the online supplementary material.

2. MAIN RESULTS FOR INDEPENDENT DATA

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent and identically distributed random vectors with distribution function $F(x, y)$ and marginals $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$. Order the X_i's as $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ and the Y_i's as $Y_{1:n} \le Y_{2:n} \le \cdots \le Y_{n:n}$. Define the survival functions $\bar{F}_i(\cdot) = 1 - F_i(\cdot)$ and quantile functions $Q_i(\cdot) = F_i^{\leftarrow}(\cdot)$ for $i = 1, 2$, where $F_i^{\leftarrow}$ denotes the (generalized) inverse function of F_i. The empirical survival functions are given by

$$\bar{F}_{n1}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i > x), \qquad \bar{F}_{n2}(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(Y_i > y), \qquad x, y \in \mathbb{R}.$$

We introduce the so-called *survival* copula function

$$C(u, v) = P(\bar{F}_1(X) < u, \bar{F}_2(Y) < v), \quad u, v \in [0, 1],$$

and we can rewrite

$$\rho_\alpha = \frac{1}{\alpha} C(\alpha, \alpha) \frac{ES_\alpha(X)}{ES_\alpha(Y)}.$$

Substituting the right-hand-side components by their empirical counterparts yields our nonparametric estimator

$$\widetilde{\rho}_\alpha = \widetilde{\rho}_\alpha(X, Y) = \frac{1}{\alpha}\widetilde{C}(\alpha, \alpha)\frac{\widetilde{ES}_\alpha(X)}{\widetilde{ES}_\alpha(Y)},$$

where, with $\lceil\cdot\rceil$ denoting the ceiling function,

$$\widetilde{C}(\alpha, \alpha) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\left[\bar{F}_{n1}(X_i) < \alpha, \bar{F}_{n2}(Y_i) < \alpha\right],$$

$$\widetilde{\mathrm{ES}}_\alpha(X) = \frac{1}{n\alpha}\sum_{i=1}^{n} X_i \mathbb{1}[X_i > X_{n-\lceil n\alpha\rceil:n}],$$

$$\widetilde{ES}_\alpha(Y) = \frac{1}{n\alpha}\sum_{i=1}^{n} Y_i \mathbb{1}[Y_i > Y_{n-\lceil n\alpha\rceil:n}].$$

Like smooth distribution (copula) estimation, we may consider a smooth version of the above nonparametric estimation. More specifically, with some density function k, its distribution function $K(x) = \int_{-\infty}^{x} k(s)ds$ and bandwidth $h = h(n) > 0$, a smoothed estimator of ρ_α is given by

$$\widehat{\rho}_\alpha = \frac{1}{\alpha}\widehat{C}(\alpha, \alpha)\frac{\widehat{\mathrm{ES}}_\alpha(X)}{\widehat{\mathrm{ES}}_\alpha(Y)},$$

where

$$\begin{cases} \widehat{C}(\alpha, \alpha) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{1-\bar{F}_{n1}(X_i)/\alpha}{h}\right) K\left(\frac{1-\bar{F}_{n2}(Y_i)/\alpha}{h}\right), \\ \widehat{\mathrm{ES}}_\alpha(X) = \frac{1}{n\alpha}\sum_{i=1}^{n}(X_i - X_{n-\lceil n\alpha\rceil:n})K\left(\frac{1-\bar{F}_{n1}(X_i)/\alpha}{h}\right) + X_{n-\lceil n\alpha\rceil:n}, \\ \widehat{\mathrm{ES}}_\alpha(Y) = \frac{1}{n\alpha}\sum_{i=1}^{n}(Y_i - Y_{n-\lceil n\alpha\rceil:n})K\left(\frac{1-\bar{F}_{n2}(Y_i)/\alpha}{h}\right) + Y_{n-\lceil n\alpha\rceil:n}. \end{cases}$$

To establish the asymptotic normality of $\widetilde{\rho}_\alpha$ and $\widehat{\rho}_\alpha$ for a fixed level $\alpha \in (0, 1)$, we will need the following regularity conditions.

Assumption 1 (Fixed level).

(1.a) For $j = 1, 2$, Q_j is Lipschitz continuous in a neighborhood of $1 - \alpha$ with $Q_j(1 - \alpha) > 0$, and F_j is strictly increasing and differentiable in a neighborhood of $Q_j(1 - \alpha)$. Moreover, for some $\delta > 0$, $\mathbb{E}(X_+^{2+\delta}) < \infty$ and $\mathbb{E}(Y_+^{2+\delta}) < \infty$, where $x_+ = \max\{x, 0\}$.

(1.b) C has continuous first-order derivatives $C_1(x, \alpha) = \frac{\partial C(x,\alpha)}{\partial x}$ and $C_2(\alpha, y) = \frac{\partial C(\alpha,y)}{\partial y}$ in a neighborhood of, respectively, $x = \alpha$ and of $y = \alpha$.

Assumption (1.a) contains standard conditions, which require underlying local continuity of the marginal distributions together with finite moments for the positive losses; see, for example, Chen (2008). Assumption (1.b) ensures the application of the standard empirical copula process result; see, for example, Section V in Ganssler and Stute (1987). Below is an asymptotic normality result, where '$\xrightarrow{d}$' denotes convergence in distribution and '$\xrightarrow{P}$' denotes convergence in probability.

Theorem 1 (Fixed level). For an $\alpha \in (0, 1)$ satisfying $C(\alpha, \alpha) > 0$, Assumption 1 implies that

$$\sqrt{n\alpha}\left(\frac{\widetilde{\rho}_\alpha}{\rho_\alpha} - 1\right) \xrightarrow{d} N\left(0, \sigma_\alpha^2\right),$$

as $n \to \infty$, with $\sigma_\alpha^2 = \mathrm{var}(\Lambda_\alpha + \Theta_{\alpha,1} - \Theta_{\alpha,2})$ and the zero-mean Gaussian random variables

$$\begin{aligned} \Lambda_\alpha &= \frac{\sqrt{\alpha}}{C(\alpha, \alpha)}\{B_C(\alpha, \alpha) - C_1(\alpha, \alpha)B_C(\alpha, 1) \\ &\quad - C_2(\alpha, \alpha)B_C(1, \alpha)\}, \\ \Theta_{\alpha,1} &= -\frac{\frac{1}{\sqrt{\alpha}}\int_0^1 B_C(\alpha x, 1)dQ_1(1 - \alpha x)}{ES_\alpha(X)}, \\ \Theta_{\alpha,2} &= -\frac{\frac{1}{\sqrt{\alpha}}\int_0^1 B_C(1, \alpha y)dQ_2(1 - \alpha y)}{ES_\alpha(Y)}. \end{aligned}$$

Here, B_C is a C-Brownian bridge, that is, a zero-mean Gaussian process with covariance function

$$\begin{aligned} \mathbb{E}(B_C(u_1, v_1)B_C(u_2, v_2)) &= C(u_1 \wedge u_2, v_1 \wedge v_2) - C(u_1, v_1) \\ &\quad \times C(u_2, v_2), \quad (u_1, v_1), (u_2, v_2) \in [0, 1]^2. \end{aligned}$$

Furthermore, if k is a symmetric density with support $[-1,1]$ and bounded first derivative and the bandwidth $h = h(n) > 0$ satisfies

$$nh^2 \to \infty \quad \text{and} \quad nh^4 \to 0,$$

then we have that, as $n \to \infty$,

$$\sqrt{n\alpha}\left(\frac{\widehat{\rho}_\alpha - \widetilde{\rho}_\alpha}{\rho_\alpha}\right) \xrightarrow{P} 0.$$

Theorem 1 states that, under weak regularity conditions, both the nonsmoothed estimator $\widetilde{\rho}_\alpha$ and smoothed estimator $\widehat{\rho}_\alpha$ are asymptotically normal with the same limiting distribution.

When α is close to zero (but not extremely), as discussed in Section 1, it is often useful to model α as an intermediate sequence of n in such a way that $\alpha = \alpha_n \to 0$ and $n\alpha_n \to \infty$, as $n \to \infty$. For the study of an intermediate level α, in the context of extreme value theory, one needs some conditions on the tail behavior of the underlying variables as follows.

Assumption 2 (Intermediate level).

(2.a) For some $\gamma_j \in (0, 1/2)$, $\beta_j \le 0$ and function A_j with a constant sign near infinity,

$$\lim_{t\to\infty}\frac{1}{A_j(1/\bar{F}_j(t))}\left(\frac{\bar{F}_j(tx)}{\bar{F}_j(t)} - x^{-1/\gamma_j}\right) = x^{-1/\gamma_j}\frac{x^{\beta_j/\gamma_j} - 1}{\gamma_j\beta_j}, \quad x > 0, \tag{2.1}$$

for all $j = 1, 2$.

(2.b) There exists a function $R : (0, \infty)^2 \to [0, \infty)$ such that

$$\lim_{t\to\infty} tC(t^{-1}x, t^{-1}y) = R(x, y), \quad (x, y) \in (0, \infty)^2, \tag{2.2}$$

and it has continuous first-order derivatives $R_1(x, y) = \frac{\partial R(x,y)}{\partial x}$ and $R_2(x, y) = \frac{\partial R(x,y)}{\partial y}$ on a neighborhood of $(1, 1)$.

(2.c) The function C has first-order derivatives $C_1(x, y) = \frac{\partial C(x,y)}{\partial x}$ and $C_2(x, y) = \frac{\partial C(x,y)}{\partial y}$ on $(0, \delta)^2$ for some $\delta > 0$, and, as $t \to \infty$,

$$\sup_{x,y\in(1-\delta,1+\delta)}\left|C_1(t^{-1}x, t^{-1}y) - R_1(x, y)\right| \to 0,$$

$$\sup_{x,y\in(1-\delta,1+\delta)}\left|C_2(t^{-1}x, t^{-1}y) - R_2(x, y)\right| \to 0.$$

Assumption (2.a) is a standard second-order condition in univariate extreme value theory; see, for example, Section 2.3 in Haan and Ferreira (2006). The condition $\gamma_1, \gamma_2 < \frac{1}{2}$ implies that there exists some $\delta_1 > 0$ such that $EX_+^{2+\delta_1} < \infty$ and $EY_+^{2+\delta_1} < \infty$. Assumption (2.b) can be viewed as a tail analog of Assumption (1.b) and it is commonly assumed when applying the theory of tail copula process; see, for example, Cai et al. (2015), Einmahl, de Haan, and Li (2006), and Theorem 7.2.2 in Haan and Ferreira (2006). The R-function defined therein fully characterizes the so-called *stable tail dependence function* l in such a way that

$$l(x, y) = x + y - R(x, y), \quad x, y \geq 0;$$

see, for example, Drees and Huang (1998) and Section 8.2 in Beirlant et al. (2006).

Theorem 2 (Intermediate level). Let $\alpha = \alpha_n$ be an intermediate sequence, that is, $\alpha_n \to 0$ and $n\alpha_n \to \infty$, as $n \to \infty$. Given $R(1, 1) > 0$, Assumption 2 implies that, as $n \to \infty$,

$$\sqrt{n\alpha}\left(\frac{\widetilde{\rho}_\alpha}{\rho_\alpha} - 1\right) \xrightarrow{d} N\left(0, \sigma_0^2\right),$$

with $\sigma_0^2 = \mathrm{var}(\Lambda_0 + \Theta_{0,1} - \Theta_{0,2})$ and the zero-mean Gaussian random variables

$$\Lambda_0 = \frac{W_R(1,1) - R_1(1,1)W_R(1,\infty) - R_2(1,1)W_R(\infty,1)}{R(1,1)},$$

$$\Theta_{0,1} = (\gamma_1 - 1)\int_0^1 W_R(x, \infty)dx^{-\gamma_1},$$

$$\Theta_{0,2} = (\gamma_2 - 1)\int_0^1 W_R(\infty, y)dy^{-\gamma_2}.$$

Here, W_R is a R-Brownian motion, that is, a zero-mean Gaussian process with covariance function

$$\mathbb{E}(W_R(u_1, v_1)W_R(u_2, v_2)) = R(u_1 \wedge u_2, v_1 \wedge v_2) \quad \text{for} \quad (u_1, v_1), (u_2, v_2) \in (0, \infty]^2 \setminus \{\infty, \infty\}.$$

Furthermore, if k is a symmetric density with support $[-1,1]$ and bounded first derivative and the bandwidth $h = h(n) > 0$ satisfies

$$n\alpha h^2 \to \infty, \quad n\alpha h^4 \to 0, \quad \text{and}$$
$$n\alpha h^2 A_i^2(1/\alpha) = O(1) \quad \text{for} \quad i = 1, 2,$$

then we have that, as $n \to \infty$,

$$\sqrt{n\alpha}\left(\frac{\widehat{\rho}_\alpha - \widetilde{\rho}_\alpha}{\rho_\alpha}\right) \xrightarrow{P} 0.$$

Theorem 2 is a tail analog of Theorem 1, despite slightly stronger conditions are imposed to eliminate the asymptotic bias due to the extreme-value approximations.

Based on these two asymptotic normality results, one can construct a confidence interval of ρ_α based on either $\widetilde{\rho}_\alpha$ or $\widehat{\rho}_\alpha$. Estimating the asymptotic variance of $\widetilde{\rho}_\alpha$ or $\widehat{\rho}_\alpha$ requires some (empirical) approximation of the copula function C or the function R, say, $\widehat{C}$ and $\widehat{R}$, respectively. A usual approach requires simulating the Gaussian process $B_{\widehat{C}}$ or $W_{\widehat{R}}$ with some empirical approximations of the limiting covariance functions. It is also necessary to estimate the first-order partial derivatives of the (tail) copula function and even, when α_n is an intermediate sequence, the tail indices of the marginal distributions. This approach is often quite computationally intensive, and its finite-sample performance can be quite poor by aggregating all the estimation uncertainties discussed above.

Instead, we investigate the possibility of employing the empirical likelihood method. Although this method proposed by Owen (1988) and Owen (1990) has proved to be quite effective in interval estimation and hypothesis testing, it has a serious problem in handling a nonlinear statistic. For example, it can lead to computational difficulties by solving a number (dependent on n) of simultaneous equations. Recently, Jing, Yuan, and Zhou (2009) proposed a so-called jackknife empirical likelihood method for dealing with nonlinear statistics such as U-statistics. Like inference for receiver operating characteristic (ROC) curves, copulas, and tail copulas in Gong, Peng, and Qi (2010), Peng, Qi, and Van Keilegom (2012), and Peng and Qi (2010), a smoothed version is needed for the proposed relative risk measure.

Hence, we shall establish our jackknife empirical likelihood inference method for ρ_α based on the smoothed nonparametric estimation. To apply the smoothed jackknife empirical likelihood method, we first need to construct a jackknife pseudo sample of ρ_α given by

$$\widehat{V}_{\rho,i} = n\widehat{\rho}_\alpha - (n-1)\widehat{\rho}_{\alpha,i}, \quad i = 1, \ldots, n,$$

where

$$\widehat{\rho}_{\alpha,i} = \frac{1}{\alpha}\widehat{C}_i(\alpha, \alpha)\frac{\widehat{ES}_{\alpha,i}(X)}{\widehat{ES}_{\alpha,i}(Y)}$$

with

$$\begin{cases} \widehat{C}_i(\alpha, \alpha) = \frac{1}{n-1}\sum_{j \neq i} K\left(\frac{1 - \bar{F}_{n1,i}(X_j)/\alpha}{h}\right) K\left(\frac{1 - \bar{F}_{n2,i}(Y_j)/\alpha}{h}\right), \\ \widehat{\mathrm{ES}}_{\alpha,i}(X) = \frac{1}{(n-1)\alpha}\sum_{j \neq i}(X_j - X_{n-\lceil n\alpha \rceil:n})K\left(\frac{1 - \bar{F}_{n1,i}(X_j)/\alpha}{h}\right) \\ \qquad + X_{n-\lceil n\alpha \rceil:n}, \\ \widehat{\mathrm{ES}}_{\alpha,i}(Y) = \frac{1}{(n-1)\alpha}\sum_{j \neq i}(Y_j - Y_{n-\lceil n\alpha \rceil:n})K\left(\frac{1 - \bar{F}_{n2,i}(X_j)/\alpha}{h}\right) \\ \qquad + Y_{n-\lceil n\alpha \rceil:n}, \end{cases}$$

and

$$\bar{F}_{n1,i}(x) = \frac{1}{n-1}\sum_{j \neq i} \mathbb{1}[X_j > x],$$

$$\bar{F}_{n2,i}(y) = \frac{1}{n-1}\sum_{j \neq i} \mathbb{1}[Y_j > y], \quad x, y \in \mathbb{R}.$$

Based on this pseudo sample, the jackknife empirical likelihood ratio function for $\theta = \rho_\alpha$ can be defined by

$$\widehat{\mathcal{R}}(\theta) = \sup\left\{\Pi_{i=1}^n np_i : p_1 > 0, \ldots, p_n > 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i\widehat{V}_{\rho,i} = \theta\right\}.$$

Applying the Lagrange multiplier method yields

$$p_i = \frac{1}{n}\frac{1}{1 + \lambda(\widehat{V}_{\rho,i} - \theta)}, \tag{2.3}$$

where $\lambda = \lambda(\theta)$ solves the equation

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{V}_{\rho,i}-\theta}{1+\lambda(\widehat{V}_{\rho,i}-\theta)}=0. \tag{2.4}$$

It follows that the log empirical likelihood ratio is

$$-2\log\widehat{\mathcal{R}}(\theta)=2\sum_{i=1}^{n}\log\left\{1+\lambda(\widehat{V}_{\rho,i}-\theta)\right\}.$$

To unify our jackknife empirical likelihood procedure for fixed and intermediate level α, we need one more assumption.

Assumption 3. For some $\tau > \max\{\gamma_1, \gamma_2\}$ such that

$$\lim_{t\downarrow 0}\sup_{0<x,y\le 1}\frac{\left|t^{-1}C(tx,ty)-R(x,y)\right|}{(x\wedge y)^{\tau}}=0, \tag{2.5}$$

where $x \wedge y := \min\{x, y\}$.

This is very similar to the condition (a) in Cai et al. (2015) but we allow an arbitrary rate of convergence here. We can show that (2.5) is satisfied with $\tau = 1$ if

$$\lim_{t\downarrow 0}\sup_{x\ge 1}\left|t^{-1}C(tx,t)-R(x,1)\right|=0 \quad \text{and}$$

$$\lim_{t\downarrow 0}\sup_{y\ge 1}\left|t^{-1}C(t,ty)-R(1,y)\right|=0,$$

which is weaker than the usual second-order condition (7.2.8) in Haan and Ferreira (2006).

Below is a Wilks type result for our JEL approach.

Theorem 3. Either if the conditions of Theorem 1 hold, or if the conditions of Theorem 2 in conjunction with Assumption 3 hold, then $-2\log\widehat{\mathcal{R}}(\rho_\alpha)$ converges in distribution to a chi-squared limit with one degree of freedom as $n \to \infty$.

Based on Theorem 3, an asymptotic confidence interval with level ψ for ρ_α is given by

$$I_\psi=\left\{\theta\in\mathbb{R}:-2\log\widehat{\mathcal{R}}(\theta)\le\chi^2_{1,\psi}\right\}$$

where $\chi^2_{1,\psi}$ is the ψ-th quantile of the chi-squared distribution with one degree of freedom. This interval has the asymptotically correct coverage probability regardless of the level α being fixed or intermediate. In other words, for certain sample size n and small level α, both asymptotic embeddings lead to the same approximation. This interval can be efficiently determined using a standard search algorithm; for more details we refer to Section 2.9 in Owen (2001).

3. RELATIVE RISK MEASURE FOR AR-GARCH MODELS

When observations follow from a strictly stationary sequence, it is a bit straightforward to derive the asymptotic limit of the proposed nonparametric estimator for the relative risk measure, while the construction of an interval generally relies on a blockwise bootstrap method or a blockwise empirical likelihood, which involves the difficult choice of a tuning parameter. Motivated by the considered real data analysis in Section 5, we consider the following AR-GARCH/IGARCH models

$$\begin{cases} X_t=\mu_x+\sum_{i=1}^{P_x}a_{x,i}X_{t-i}+\varepsilon_{x,t},\\ \varepsilon_{x,t}=h_{x,t}^{1/2}\eta_{x,t},\quad h_{x,t}=w_x+\sum_{i=1}^{q_x}\alpha_{x,i}\varepsilon^2_{x,t-i}+\sum_{j=1}^{p_x}\beta_{x,j}h_{x,t-j}\end{cases} \tag{3.1}$$

and

$$\begin{cases} Y_t=\mu_y+\sum_{i=1}^{P_y}a_{y,i}Y_{t-i}+\varepsilon_{y,t},\\ \varepsilon_{y,t}=h_{y,t}^{1/2}\eta_{y,t},\quad h_{y,t}=w_y+\sum_{i=1}^{q_y}\alpha_{y,i}\varepsilon^2_{y,t-i}+\sum_{j=1}^{p_y}\beta_{y,j}h_{y,t-j},\end{cases} \tag{3.2}$$

where the $(\eta_{x,t}, \eta_{y,t})$'s are independent and identically distributed random vectors with means zero and variances one. In this case, we are concerned with both point and interval estimation of the relative risk measure of $\eta_{x,t}$ and $\eta_{y,t}$, which involves inference for unknown parameters in (3.1) and (3.2).

An obvious estimator for the unknown parameters in (3.1) and (3.2) is the so-called quasi maximum likelihood estimator (QMLE), and its asymptotic normality is available in Francq and Zakoïan (2004), which requires finite fourth moments of $\varepsilon_{x,t}, \eta_{x,t}, \varepsilon_{y,t}, \eta_{y,t}$. However, in practice it is quite often that $\sum_{i=1}^{q_x}\alpha_{x,i}+\sum_{j=1}^{p_x}\beta_{x,j}$ is less than one, but quite close to one, and so assuming $E\varepsilon^4_{x,t} < \infty$ may be problematic. Here we propose to employ the self-weighted QMLE in Ling (2007) which requires weaker moment conditions when we use the weights

$$\delta_{x,t}=\left\{\max\left(1,\frac{1}{C_x}\sum_{k=1}^{t-1}\frac{|X_{t-k}|I(|X_{t-k}|>C_x)}{k^9}\right)\right\}^{-4}$$

and

$$\delta_{y,t}=\left\{\max\left(1,\frac{1}{C_y}\sum_{k=1}^{t-1}\frac{|Y_{t-k}|I(|Y_{t-k}|>C_y)}{k^9}\right)\right\}^{-4},$$

for some constants $C_x, C_y > 0$. These two constants are chosen as the 90% sample quantiles of $\{X_t\}_{t=1}^n$ and $\{Y_t\}_{t=1}^n$, respectively, as suggested by Zhu and Ling (2011). In the simulation study, we use 95% sample quantiles of both samples, which performs better. Based on the self-weighted QMLE in Ling (2007) with the above weight functions, we employ the corresponding score equations to formulate an empirical likelihood method to construct a confidence interval for the relative risk measure of $\eta_{x,t}$ and $\eta_{y,t}$ as follows.

Given the observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ and the initial value $(\bar{X}_0, \bar{Y}_0) = \{(X_t, Y_t) : t \le 0\}$ generated by the above model, we can write the parametric model as

$$\varepsilon_{x,t}(\phi_x)=X_t-\mu_x-\sum_{i=1}^{P_x}a_{x,i}X_{t-i},$$

$$\varepsilon_{y,t}(\phi_y)=Y_t-\mu_y-\sum_{i=1}^{P_y}a_{y,i}Y_{t-i},$$

$$\eta_t(\psi_x)=\varepsilon_t(\phi_x)/\sqrt{h_t(\psi_x)},\quad \eta_t(\psi_y)=\varepsilon_t(\phi_y)/\sqrt{h_t(\psi_y)},$$

$$h_{x,t}(\psi_x)=w_x+\sum_{i=1}^{q_x}\alpha_{x,i}\varepsilon^2_{x,t-i}(\phi_x)+\sum_{i=1}^{p_x}\beta_{x,i}h_{x,t-i}(\psi_x),$$

$$h_{y,t}(\psi_y)=w_y+\sum_{i=1}^{q_y}\alpha_{y,i}\varepsilon^2_{y,t-i}(\phi_y)+\sum_{i=1}^{p_y}\beta_{y,i}h_{y,t-i}(\psi_y),$$

where $\phi_x = (\mu_x, a_{x,1}, \ldots, a_{x,P_x})$, $\phi_y = (\mu_y, a_{y,1}, \ldots, a_{y,P_y})$, $\phi_{hx} = (w_x, \alpha_{x,1}, \ldots \alpha_{x,q_x}, \beta_{x,1}, \ldots, \beta_{x,p_x})$, $\phi_{hy} = (w_y, \alpha_{y,1}, \ldots \alpha_{y,q_y}, \beta_{y,1}, \ldots, \beta_{y,p_y})$, $\psi_x = (\phi_x, \phi_{hx})$, $\psi_y = (\phi_y, \phi_{hy})$.

Then, the self-weighted QMLEs in Ling (2007) for ψ_x and ψ_y solve the score equations

$$\sum_{t=1}^n \delta_{x,t} \frac{\partial l_{x,t}(\psi_x)}{\partial \psi_x} := \sum_{t=1}^n \delta_{x,t} \frac{\partial}{\partial \psi_x} \left\{\eta_{x,t}^2(\psi_x) + \log h_{x,t}(\psi_x)\right\} = 0,$$

$$\sum_{t=1}^n \delta_{y,t} \frac{\partial l_{y,t}(\psi_y)}{\partial \psi_y} := \sum_{t=1}^n \delta_{y,t} \frac{\partial}{\partial \psi_y} \left\{\eta_{y,t}^2(\psi_y) + \log h_{y,t}(\psi_y)\right\} = 0,$$

respectively. Put the true parameters $\theta_1^0 = F_{\eta_{x,1}}^{\leftarrow}(1-\alpha)$, $\theta_2^0 = F_{\eta_{y,1}}^{\leftarrow}(1-\alpha)$, $\theta_3^0 = E\{\eta_{x,1}\mathbf{1}(\eta_{x,1} > \theta_1^0)\}$, $\theta_4^0 = E\{\eta_{y,1}\mathbf{1}(\eta_{y,1} > \theta_2^0)\}$, $\theta_5^0 = \rho_\alpha(\eta_{x,1}, \eta_{y,1})$. Then $F_{\eta_{x,1}}(\theta_1^0) = 1-\alpha$, $F_{\eta_{y,1}}(\theta_2^0) = 1-\alpha$, $\theta_3^0 = -E\{\eta_{x,1}\mathbf{1}(\eta_{x,1} \le \theta_1^0)$, $\theta_4^0 = -E\{\eta_{y,1}\mathbf{1}(\eta_{y,1} \le \theta_2^0)$, $\theta_5^0\alpha\theta_4^0/\theta_3^0 = P(\eta_{x,1} > \theta_1^0, \eta_{y,1} > \theta_2^0) = 2\alpha - 1 + P(\eta_{x,1} \le \theta_1^0, \eta_{y,1} \le \theta_2^0)$, which motivate the following definitions:

$$\begin{aligned}
\boldsymbol{Z}_t(\theta_5, \boldsymbol{\nu}) = \Bigg(&\delta_{x,t}\frac{\partial l_{x,t}(\psi_x)}{\partial \psi_x}, K\left(\frac{\theta_1 - \eta_{x,t}(\psi_x)}{h}\right) \\
&-1+\alpha, -\eta_{x,t}(\psi_x)K\left(\frac{\theta_1 - \eta_{x,t}(\psi_x)}{h}\right) - \theta_3, \\
&\delta_{y,t}\frac{\partial l_{y,t}(\psi_y)}{\partial \psi_y}, K\left(\frac{\theta_2 - \eta_{y,t}(\psi_y)}{h}\right) \\
&-1+\alpha, -\eta_{y,t}(\psi_y)K\left(\frac{\theta_2 - \eta_{y,t}(\psi_y)}{h}\right) - \theta_4, \\
&2\alpha - 1 + K\left(\frac{\theta_1 - \eta_{x,t}(\psi_x)}{h}\right)K \\
&\times \left(\frac{\theta_2 - \eta_{y,t}(\psi_x)}{h}\right) - \alpha\theta_5\theta_4/\theta_3\Bigg) \qquad (3.3)
\end{aligned}$$

where $\boldsymbol{\nu} := (\psi_x, \psi_y, \theta_1, \ldots, \theta_4)$.

Then the empirical likelihood function for θ_5 and $\boldsymbol{\nu}$ is defined as

$$L(\theta_5, \boldsymbol{\nu}) = \sup\left\{\prod_{t=1}^n (nr_t) : r_1 \ge 0, \ldots, r_n \ge 0, \sum_{t=1}^n r_t = 1, \sum_{t=1}^n r_t \boldsymbol{Z}_t(\theta_5, \boldsymbol{\nu}) = 0\right\}.$$

Since we are only interested in $\theta_5^0 = \rho_\alpha(\eta_{x,1}, \eta_{y,1})$, we consider the profile empirical likelihood function

$$L^P(\theta_5) = \max_{\boldsymbol{\nu}} L(\theta_5, \boldsymbol{\nu}).$$

To show that Wilks theorem holds for the above proposed profile empirical likelihood method, we assume the following regularity conditions:

A1. Let Θ_x and Θ_y denote the parameter spaces for ψ_x and ψ_y, separately. Assume the true values of ψ_x and ψ_y are an interior of Θ_x and Θ_y, respectively;

A2. $1 - \sum_{i=1}^{P_x} a_{x,i}z^i \ne 0$ for all $|z| \le 1$ and $\psi_x \in \Theta_x$, and $1 - \sum_{i=1}^{P_y} a_{y,i}z^i \ne 0$ for all $|z| \le 1$ and $\psi_y \in \Theta_y$;

A3. There is no common root for equations $\sum_{i=1}^{q_x} \alpha_{x,i}z^i = 0$ and $1 - \sum_{j=1}^{p_x} \beta_{x,j}z^j = 0$, $\sum_{i=1}^{q_x} \alpha_{x,i} \ne 0$, $\alpha_{x,q_x} + \beta_{x,p_x} \ne 0$ and $\sum_{j=1}^{p_x} \beta_{x,j} < 1$ for each $\psi_x \in \Theta_x$. Similarly there is no common root for equations $\sum_{i=1}^{q_y} \alpha_{y,i}z^i = 0$ and $1 - \sum_{j=1}^{p_y} \beta_{y,j}z^j = 0$, $\sum_{i=1}^{q_y} \alpha_{y,i} \ne 0$, $\alpha_{y,q_y} + \beta_{y,p_y} \ne 0$ and $\sum_{j=1}^{p_y} \beta_{y,j} < 1$; for each $\psi_y \in \Theta_y$.

A4. $\{(\eta_{x,t}, \eta_{y,t})^T\}_{t=1}^n$ is a sequence of independent and identically distributed random vectors satisfying $E\eta_{x,t} = 0$, $E\eta_{x,t}^2 = 1$, $E|\eta_{x,t}|^{4+\delta_0} < \infty$, $E\eta_{y,t} = 0$, $E\eta_{y,t}^2 = 1$ and $E|\eta_{y,t}|^{4+\delta_0} < \infty$ for some $\delta_0 > 0$. $\eta_{x,t}$ and $\eta_{y,t}$ are not perfectly correlated.

A5. $\sum_{i=1}^{q_x} \alpha_{x,i} + \sum_{i=1}^{p_x} \beta_{x,i} \le 1$ and $\sum_{i=1}^{q_y} \alpha_{y,i} + \sum_{i=1}^{p_y} \beta_{y,i} \le 1$. Moreover, $\eta_{x,t}$ or/and $\eta_{y,t}$ have positive density on $\mathbb{R}$ when $\sum_{i=1}^{q_x} \alpha_{x,i} + \sum_{i=1}^{p_x} \beta_{x,i} = 1$ or/and $\sum_{i=1}^{q_y} \alpha_{y,i} + \sum_{i=1}^{p_y} \beta_{y,i} = 1$.

A6. Assumption 1 in Theorem 1 holds for $\eta_{x,t}$ and $\eta_{y,t}$, and other conditions in Theorem 1 are true too. Furthermore, $nh^{2+\delta_h} \to \infty$ for some $\delta_h > 0$.

Assumptions A1–A3 are standard conditions for the stationarity and identifiability of the AR-GARCH/IGARCH model. Assumption A4 implies finite fourth moments of $\eta_{x,t}$ and $\eta_{y,t}$, which are necessary for the asymptotic normality of self-weighted QMLE. Assumption A5 implies that $E(|\varepsilon_{x,t}|^{2\iota} + |\varepsilon_{y,t}|^{2\iota}) < \infty$ for all $\iota \in (0, 1)$. See, for example, Sections 2 and 3 in Ling (2007) for more discussions.

Note that we focus on the case of a fixed level α. When $\alpha \to 0$, the first step in estimating unknown parameters using self-weighted QMLE or local QMLE from Ling (2007) in models (3.1) and (3.2) does not play a role asymptotically in estimating the relative risk measure of $\eta_{x,t}$ and $\eta_{y,t}$ due to their fast rate of convergence under suitable conditions. In other words, Theorem 3 is still applicable to models (3.1) and (3.2) in case of an intermediate level under Assumptions A1–A5 and the conditions in Theorem 2 on $\eta_{x,t}$ and $\eta_{y,t}$.

Theorem 4. Assume models (3.1) and (3.2) hold for conditions A1–A6. Then, as $n \to \infty$, $-2\log L^P(\rho_\alpha(\eta_{x,1}, \eta_{y,1}))$ converges in distribution to a chi-squared limit with one degree of freedom.

As before, an empirical likelihood confidence interval for the relative risk measure $\rho_\alpha(\eta_{x,1}, \eta_{y,1})$ based on models (3.1) and (3.2) can be obtained via the above theorem.

Remark 1. Similarly under models (3.1) and (3.2) we can develop a profile empirical likelihood interval for the conditional relative risk measure of (X_{n+1}, Y_{n+1}) given $(X_1, Y_1), \ldots, (X_n, Y_n)$ and $h_{x,n}, \ldots, h_{x,n-p_x}, h_{y,n}, \ldots, h_{y,n-p_y}$ by noting that $h_{x,n+1}$ and $h_{y,n+1}$ can be expressed as functions of conditional variables and unknown parameters in the above AR-GARCH models.

4. SIMULATION STUDY

4.1 Independent Data

In this subsection, a simulation study is carried out to evaluate the finite-sample behavior of the proposed jackknife empirical likelihood method for our proposed relative risk measure ρ_α. The survival copula in our simulation study is a so-called t-copula with multiple parameters of degrees of freedom which is a generalization of the grouped t-copula; see Luo and Shevchenko (2010) for details. The distribution of a two-dimensional t-copula with multiple parameters of degrees of freedom is

$$C^{\Sigma}_{\nu_1,\nu_2}(u_1,u_2)=\int_0^1 \Phi_\Sigma(z_1(u_1,s),z_2(u_2,s))ds, \quad u_1,u_2\in[0,1],$$

- Φ_Σ is the distribution function of a bivariate normal random vector with zero means, unit variances, and positive correlation ρ;
- $z_i(u_i,s)=t^{-1}_{\nu_i}(u_i)/\omega_i(s)$, $\omega_i(s)=\sqrt{\nu_i/\chi^{-1}_{\nu_i}(s)}$, $i=1,2$;
- t_{ν_i} and $t^{-1}_{\nu_i}$ denote the distribution function and quantile of a student-t random variable with ν_i degrees of freedom respectively, $i=1,2$;
- χ_{ν_i} and $\chi^{-1}_{\nu_i}$ denote the distribution function and quantile of a chi-squared random variable with ν_i degrees of freedom respectively, $i=1,2$.

We draw 1000 random samples of size $n=500$ and 1000 from a bivariate distribution with a t-copula with two parameters of degrees of freedom $\nu=(\nu_1,\nu_2)\in\{(3,3),(3,5),(5,3),(5,5)\}$ and two marginal t distributions with degrees of freedom ν_1 and ν_2, respectively. We consider two cut-off levels $\alpha=0.05, 0.1$ and two confidence levels $\psi=0.9, 0.95$. In all cases, we set $\rho=0.2$.

The empirical coverage probability of the jackknife empirical likelihood-based confidence interval is compared to that of the bootstrap confidence interval. The bootstrap confidence interval is obtained by using 1000 bootstrap samples of size n from each sample $X_1,\dots,X_n$. Specifically, for each bootstrap sample, we calculate the empirical estimate of ρ_α, which results in 1000 bootstrapped empirical estimates of ρ_α, denoted as $\widetilde{\rho}^{*1}_\alpha,\dots,\widetilde{\rho}^{*1000}_\alpha$, and therefore 1000 bootstrap differences $\delta^{*i}=\widetilde{\rho}^{*i}_\alpha-\rho_\alpha$, $i=1,\dots,1000$. Ordering these bootstrap differences by $\delta^{*[1]}\le\cdots\le\delta^{*[1000]}$, the bootstrap confidence interval at level ψ is then calculated as

$$I^*_\psi=\left[\widetilde{\rho}_\alpha-\delta^{*[n2]},\ \widetilde{\rho}_\alpha-\delta^{*[n1]}\right],$$

where n_1 and n_2 denote the integer part of $500(1-\psi)$ and $500(1+\psi)$, respectively. Motivated by the optimal bandwidth choice in smoothing distribution function estimation, we choose $h=d(n\alpha)^{-1/3}$ for various $d=0.5, 1, 1.5, 2, 2.5, 3$.

We report the empirical coverage probabilities in Tables 1 and 2, which show that the proposed jackknife empirical likelihood method performs better than the bootstrap method in terms of coverage accuracy, and the results are quite stable with respect to the different choices of bandwidth h especially with $d=1, 1.5, 2$. For $\alpha=0.1$, it clearly shows that a larger sample size improves the accuracy.

4.2 AR-GARCH Models

In addition to the independent cases, we also implement the simulation studies for models (3.1) and (3.2) by using similar settings to the real data in Section 5. In real data, we multiply observations (loss) by 100 for proper scaling, and then fit the models (3.1) and (3.2) which turn out AR(1)+GARCH(1,1) fits well for both margins by checking the autocorrelation functions of estimated residuals. Note that we do not use AIC or BIC to choose the best fitting and simply prefer AR-GARCH models with a smaller number of parameters due to the heavy computation in the profile empirical likelihood method. We also find that the innovations in models (3.1) and (3.2) are fitted well by marginal t distributions with degrees of freedom 5.5, 8.9 and a t-copula in real analysis. Therefore, we simulate data from the following AR(1)-GARCH(1,1) models

$$X_t=\mu_x+a_xX_{t-1}+\varepsilon_{x,t},\quad \varepsilon_{x,t}=h^{1/2}_{x,t}\eta_{x,t},$$
$$h_{x,t}=\omega_x+\alpha_x\varepsilon^2_{x,t-1}+\beta_xh_{x,t-1},$$

Table 1. Empirical coverage probabilities for the jackknife empirical likelihood-based confidence interval $I_\psi(h)$ and the bootstrap confidence interval I^*_ψ of ρ_α with cutoff level $\alpha=0.05$, sample size $n=500$, 1000 and confidence levels $\psi=0.90, 0.95$. Bandwidths are chosen as $h_1=0.5(n\alpha)^{-\frac{1}{3}}$, $h_2=(n\alpha)^{-\frac{1}{3}}$, $h_3=1.5(n\alpha)^{-\frac{1}{3}}$, $h_4=2(n\alpha)^{-\frac{1}{3}}$, $h_5=2.5(n\alpha)^{-\frac{1}{3}}$, $h_6=3(n\alpha)^{-\frac{1}{3}}$

	$n=500$				$n=1000$			
(ν_1,ν_2)	(3, 3)	(3, 5)	(5, 3)	(5, 5)	(3, 3)	(3, 5)	(5, 3)	(5, 5)
$I^*_{0.95}$	0.880	0.873	0.882	0.860	0.920	0.922	0.914	0.919
$I_{0.95}(h_1)$	0.938	0.954	0.939	0.945	0.934	0.934	0.937	0.943
$I_{0.95}(h_2)$	0.947	0.959	0.953	0.960	0.947	0.945	0.951	0.939
$I_{0.95}(h_3)$	0.945	0.954	0.956	0.958	0.945	0.940	0.955	0.946
$I_{0.95}(h_4)$	0.940	0.950	0.952	0.955	0.932	0.932	0.935	0.939
$I_{0.95}(h_5)$	0.941	0.944	0.949	0.955	0.927	0.934	0.933	0.935
$I_{0.95}(h_6)$	0.925	0.938	0.947	0.950	0.923	0.924	0.927	0.932
$I^*_{0.90}$	0.834	0.824	0.838	0.831	0.868	0.870	0.874	0.867
$I_{0.90}(h_1)$	0.890	0.903	0.886	0.889	0.883	0.897	0.892	0.890
$I_{0.90}(h_2)$	0.900	0.914	0.911	0.915	0.899	0.892	0.890	0.900
$I_{0.90}(h_3)$	0.896	0.903	0.903	0.910	0.902	0.893	0.899	0.898
$I_{0.90}(h_4)$	0.885	0.899	0.905	0.900	0.884	0.881	0.885	0.889
$I_{0.90}(h_5)$	0.880	0.892	0.901	0.903	0.879	0.876	0.887	0.890
$I_{0.90}(h_6)$	0.865	0.882	0.885	0.896	0.875	0.873	0.876	0.889

Table 2. Empirical coverage probabilities for the jackknife empirical likelihood-based confidence interval $I_\psi(h)$ and the bootstrap confidence interval I^*_ψ of ρ_α with cutoff levels $\alpha=0.1$, sample size $n=500$, 1000 and confidence levels $\psi=0.90, 0.95$. Bandwidths are chosen as $h_1=0.5(n\alpha)^{-\frac{1}{3}}$, $h_2=(n\alpha)^{-\frac{1}{3}}$, $h_3=1.5(n\alpha)^{-\frac{1}{3}}$, $h_4=2(n\alpha)^{-\frac{1}{3}}$, $h_5=2.5(n\alpha)^{-\frac{1}{3}}$, $h_6=3(n\alpha)^{-\frac{1}{3}}$

	$n=500$				$n=1000$			
(ν_1,ν_2)	(3, 3)	(3, 5)	(5, 3)	(5, 5)	(3, 3)	(3, 5)	(5, 3)	(5, 5)
$I^*_{0.95}$	0.920	0.922	0.914	0.919	0.935	0.939	0.937	0.935
$I_{0.95}(h_1)$	0.930	0.940	0.938	0.934	0.950	0.943	0.948	0.953
$I_{0.95}(h_2)$	0.938	0.946	0.947	0.950	0.948	0.946	0.954	0.959
$I_{0.95}(h_3)$	0.942	0.949	0.944	0.947	0.946	0.947	0.950	0.956
$I_{0.95}(h_4)$	0.932	0.932	0.935	0.939	0.951	0.952	0.943	0.946
$I_{0.95}(h_5)$	0.927	0.934	0.933	0.935	0.948	0.947	0.937	0.946
$I_{0.95}(h_6)$	0.923	0.924	0.927	0.932	0.943	0.947	0.939	0.940
$I^*_{0.90}$	0.868	0.870	0.874	0.867	0.874	0.887	0.874	0.890
$I_{0.90}(h_1)$	0.880	0.885	0.877	0.882	0.911	0.890	0.906	0.905
$I_{0.90}(h_2)$	0.884	0.891	0.884	0.891	0.909	0.906	0.901	0.904
$I_{0.90}(h_3)$	0.879	0.894	0.888	0.891	0.910	0.903	0.897	0.907
$I_{0.90}(h_4)$	0.884	0.881	0.885	0.889	0.897	0.903	0.893	0.896
$I_{0.90}(h_5)$	0.879	0.876	0.887	0.890	0.887	0.899	0.891	0.899
$I_{0.90}(h_6)$	0.875	0.873	0.876	0.889	0.880	0.892	0.886	0.893

Table 3. Empirical coverage probabilities for the jackknife empirical likelihood-based confidence interval $I_\psi(h)$ of ρ_α with cutoff levels $\alpha = 0.05$, sample size $n = 2000$ and 4000, and confidence level $\psi = 0.95$. Bandwidths are chosen as $h_1 = 0.5(n\alpha)^{-\frac{1}{3}}$, $h_2 = (n\alpha)^{-\frac{1}{3}}$, $h_3 = 1.5(n\alpha)^{-\frac{1}{3}}$. The parameters in AR(1)-GARCH(1,1) are $\mu_x = -0.030$, $\mu_y = -0.053$, $\omega_x = -0.029$, $\omega_y = 0.012$, $\alpha_x = 0.072$, $\alpha_y = 0.074$

	$I_{0.95}\&n = 2000$			$I_{0.95}\&n = 4000$		
$(\beta_x, \beta_y, \gamma)$	h_1	h_2	h_3	h_1	h_2	h_3
(0.924, 0.917, 0.72)	0.941	0.932	0.934	0.936	0.940	0.928
(0.824, 0.817, 0.72)	0.933	0.931	0.923	0.941	0.933	0.930
(0.924, 0.917, 0)	0.969	0.968	0.966	0.958	0.951	0.951
(0.824, 0.817, 0)	0.967	0.971	0.967	0.955	0.943	0.943

$$
\begin{aligned}
Y_t &= \mu_y + a_y Y_{t-1} + \varepsilon_{y,t}, \quad \varepsilon_{y,t} = h_{y,t}^{1/2}\eta_{y,t}, \\
h_{y,t} &= \omega_y + \alpha_y \varepsilon_{y,t-1}^2 + \beta_y h_{y,t-1},
\end{aligned}
$$

where the iid innovations $\{(\eta_{x,t}, \eta_{y,t})\}_{t=1}^n$ have marginal t-distribution with degrees of freedom 5.5, 8.9 scaled to unit variance and a t-copula with correlation γ and common degree of freedom 7. Details of the parameter setting from real data can be found in Section 5.

The results of eight simulation cases are summarized in Table 3, which shows the proposed method works well for the considered time series models, and accuracy improves as the sample size becomes larger.

5. REAL-LIFE DATA ANALYSIS

In this section, we study our relative risk measure in a real-life dataset, which contains daily stock losses on three U.S. Banks, Bank of America (BOA), JP Morgan (JPM), Wells Fargo (WFG), and Standard & Poor's 500 index (SNP) (benchmark) between February 1st, 2002 and March 31st, 2011 from the Center for Research in Security Prices (CRSP), and (weekly) levels of the Adjusted National Financial Conditional Index (ANFCI) between September 1st, 2006 and March 30, 2011. As usual, our collected stock returns exhibit so-called *volatility clustering* behavior widely documented in the empirical finance literature: the univariate squared stock returns are moderately auto-correlated. Hence, we shall work on a filtered version of the univariate losses. For loss data of each bank combined with S&P500 index, we calibrate an autoregressive moving average model ARMA(p, q) with generalized autoregressive conditional heteroskedasticity (proposed in Bollerslev (1986)) GARCH(P, Q) errors. Due to the heavy computation in the proposed profile empirical likelihood method, we prefer ARMA-GARCH models with a smaller number of parameters as long as the estimated residuals are uncorrelated. Hence, we use the following AR(1)-GARCH(1,1) models:

$$
\begin{aligned}
X_t &= \mu_x + a_x X_{t-1} + \varepsilon_{x,t}, \quad \varepsilon_{x,t} = h_{x,t}^{1/2}\eta_{x,t}, \\
h_{x,t} &= \omega_x + \alpha_x \varepsilon_{x,t-1}^2 + \beta_x h_{x,t-1}, \\
Y_t &= \mu_y + a_y Y_{t-1} + \varepsilon_{y,t}, \quad \varepsilon_{y,t} = h_{y,t}^{1/2}\eta_{y,t}, \\
h_{y,t} &= \omega_y + \alpha_y \varepsilon_{y,t-1}^2 + \beta_y h_{y,t-1},
\end{aligned}
$$

where X_t is the stock loss on an individual institution and Y_t is the benchmark (loss on S&P 500 in our study).

After estimating the parameters in the above AR-GARCH models by the self-weighted QMLE in Ling (2007), we plot the autocorrelation functions of the estimated residuals in Figure 1, which shows the assumption of independent $(\eta_{x,t}, \eta_{y,t})$'s is reasonable. That is, the proposed profile empirical likelihood method is applicable.

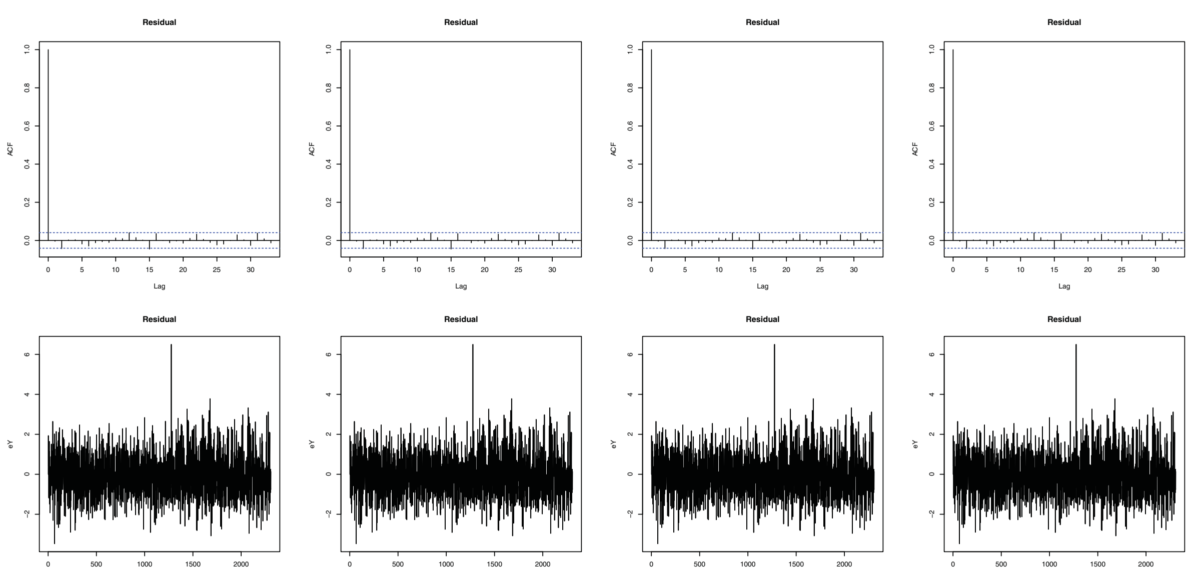

Figure 1. The columns from left to right are for Bank of America, JP Morgan, Wells Fargo, and S&P500. The first row are ACF plots from the AR(1)+GARCH(1,1) models of three banks and S&P500, and the second row are residual plots.

Table 4. Estimated coefficients of AR(1)-GARCH(1,1) models for three banks and S&P500 index

	df(Marginal)	γ	df(tcopula)	μ	a	ω	α	β
BOA	5.5	0.72	7	– 0.021	– 0.029	0.020	0.067	0.929
JPM	6.3	0.77	7	– 0.064	– 0.037	0.014	0.076	0.923
WFG	6.5	0.72	7	– 0.055	– 0.104	0.011	0.099	0.900
SNP	8.9	N/A	N/A	– 0.052	– 0.083	0.012	0.074	0.917

Since we use the same setting to generate data to evaluate the finite sample performance of the proposed method in the simulation study, we need a parametric model for $(\eta_{x,t}, \eta_{y,t})^T$. Here, we fit innovations $\{(\eta_{x,t}, \eta_{y,t})\}_{t=1}^n$ by marginal t-distributions with degrees of freedom df_x, df_y scaled to unit variance and a t-copula with correlation γ and common degree of freedom df. The fitted results are summarized in Table 4. Note that $\eta_{x,t}$ is related to either BOA or JPM or WFG, and $\eta_{y,t}$ is related to SNP.

Given $(X_1, Y_1), \ldots, (X_t, Y_t)$, the relative risk measure of (X_{t+1}, Y_{t+1}) cannot be written as the relative risk measure of $(\eta_{x,t+1}, \eta_{y,t+1})$ multiplied by a constant, but the relative risk

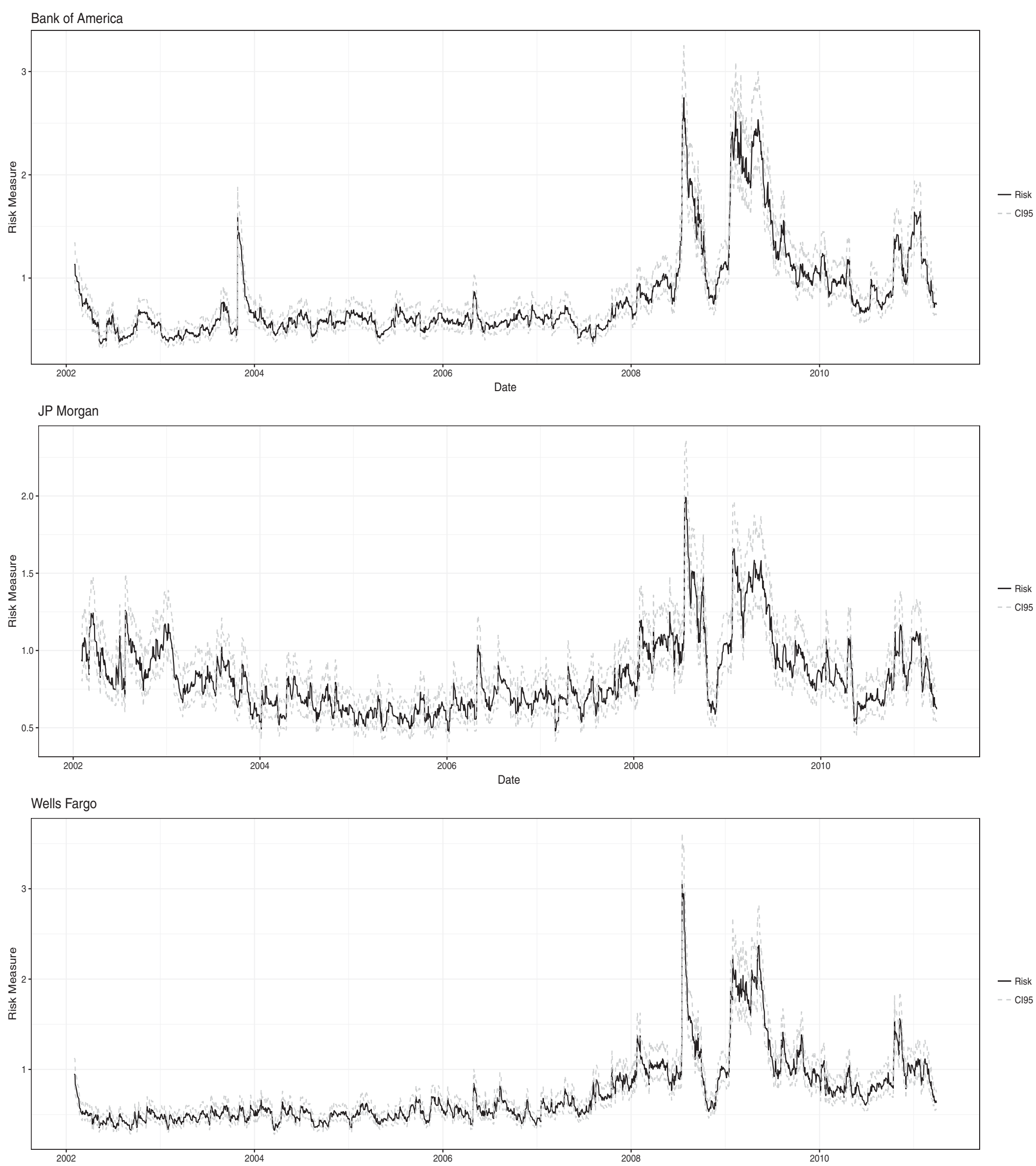

Figure 2. Time series from February 1st, 2002 to March 31st, 2011 and 95% intervals of the relative tail risk measure in (5.1) with cut-off level $\alpha = 0.05$ for each of the daily stock losses on J. P. Morgan, Bank of America and Wells Fargo against the daily loss on Standard & Poor's 500 index. $h = (n\alpha)^{-1/3}$.

measure of $(\varepsilon_{x,t+1}, \varepsilon_{y,t+1})$ can be written as the relative risk measure of $(\eta_{x,t+1}, \eta_{y,t+1})$ multiplied by the constant $h_{x,t}^{1/2}/h_{y,t}^{1/2}$. With a focus on volatility, we estimate the relative risk measure of $(\varepsilon_{x,t+1}, \varepsilon_{y,t+1})$ given (X_s, Y_s) for $s \le t$, that is,

$$\rho_{\alpha,t+1|t} = \sqrt{\frac{h_{x,t+1}}{h_{y,t+1}}}\rho_\alpha(\eta_{x,1}, \eta_{y,1}), \tag{5.1}$$

where $\rho_\alpha(\eta_{x,1}, \eta_{y,1})$ is the (unconditional) relative risk measure of $(\eta_{x,1}, \eta_{y,1})$.

We plot the estimated $\rho_{\alpha,t+1|t}$ for each of the three banks against S&P500 index in Figure 2 from February 1st, 2002, to March 31st, 2011. The intervals, at 95% level, are constructed by first the profile empirical likelihood method on the estimated innovations and then a simple multiplication of the estimated $\sqrt{\frac{h_{x,t+1}}{h_{y,t+1}}}$ at time $t+1$. Note that we ignore the uncertainty of estimating $h_{x,t+1}$ and $h_{y,t+1}$ in the above intervals. One can easily spot some features of the three time series in Figure 2. First, during the pre-crisis period (2002–2007), all three banks have relative risk measures below the market level, that is, unit level, for most of the time where Wells Fargo has the lowest variation (in the sense of shortest interval) and JP Morgan has the highest one. Second, during the crisis period (2008–2010), all three banks share a similar pattern of the relative risk measures, which grow suddenly to a peak value and then varies between high risk values. However, there is no obvious similar pattern during the noncrisis for the three banks. This suggests that the three banks may encounter (substantial) systemic risk during the crisis period. We observe that the average relative risk level of Bank of American is (more than 20%) higher than that of Wells Fargo and JP Morgan during April 2008 to March 2009, which is consistent with the ranking using sample marginal expected shortfalls in the same period; see Table 1 in Acharya et al. (2017).

Next, we document some empirical evidence of the predictive power of bank-specific relative risk on the distribution of future systemic shocks during the crisis period. We measure systemic shocks by the innovations to an autoregression to ANFCI. Using a rolling window size of 1154 days (half of our sample size), we construct the daily prediction of relative risks in a recursive manner using (5.1) where $\rho_\alpha(\eta_{x,1}, \eta_{y,1})$ is estimated by maximizing the profile empirical likelihood $L^P(\theta_5)$ defined in Section 3 and the AR(1)-GARCH(1,1) parameters are estimated by the self-weighted QMLE proposed in Ling (2007). Since the ANFCI is only reported weekly (every Friday), our predicted relative risks are converted into a weakly basis by taking the median of the daily estimates in the same week. Given some forecast horizon $L > 0$, forecast accuracy can be evaluated via an out-of-sample R^2 based on the mean squared loss function:

$$R^2 = 1 - \frac{\sum_t(\text{ANFCI}_t - \widehat{\text{ANFCI}}_t)^2}{\sum_t(\text{ANFCI}_t - \overline{\text{ANFCI}}_t)^2} \tag{5.2}$$

where $\widehat{\text{ANFCI}}_t$ is the fitted value from a predictive regression estimated through period $t-1$ using both the L-weeks lagged ANFCI and L-weeks lagged individual relative risk, and $\overline{\text{ANFCI}}_t$ is that from the corresponding autoregression using L-weeks lagged ANFCI only. The models including more lagged variables of ANFCI produce qualitatively the same results in our analysis.

We forecast ANFCI starting from December 2007 in $L = 8, 9, 10$ weeks (about two months) forecast horizon and the out-of-sample R^2 are reported in Table 5. The predictive power of the three banks seems to be quite different, although their estimated relative risk are highly correlated over time. We observe positive out-of-sample R^2's for JP Morgan and Wells Fargo, which means the predictive regression including their individual relative risk has lower average mean squared prediction error than that of the corresponding autoregressive model. The estimated relative risk of BOA, in contrast, has shown almost no predictive power (beyond the autoregression). In summary, the proposed relative risk measure and its intervals are useful in monitoring systemic risk.

Table 5. Out-of-sample R^2 (in percentage) for the predictive regressions with bank-specific relative risks in forecast horizon $L = 8, 9, 10$ weeks

R^2(%)	BOA	JPM	WFG
$L = 8$	− 2.14	1.70	2.31
$L = 9$	− 0.62	2.98	5.19
$L = 10$	1.07	2.52	4.80

SUPPLEMENTARY MATERIALS

Supplementary material available online includes proofs of Theorems 1–4 as well as the detailed proofs of all the lemmas in this article.

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