Proofs with Predicate Logic

CS236 - Discrete Structures
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Rules of Inference

Now that we have introduced predicate logic, we will discuss key rules of inference in predicate logic and how to use them in proofs.

Universal Instantiation

Let's look at universal instantiation first:

$\forall x\ P(x)$	x is universal
$\therefore\ P(c)$	c is an instance

Universal instantiation is stating that if $P(x)$ is true for all values of x, then for a specific instance of the domain, c, $P(c)$ must also be true.

Universal Generalization

Here is universal generalization:

$P(c)$	for an arbitrary c
$\therefore\ \forall x\ P(x)$	

This one looks odd at first, so let's see the description from the book, Section 1.6.5*:

[Universal generalization] states that $\forall x\ P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x\ P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true [see the book* for a further explanation].

Existential Instantiation

Here is existential instantiation:

$$\begin{array}{ll} \exists x\ P(x) & \\ \hline \therefore\ \ P(c) & \text{for some element } c \end{array}$$

Existential instantiation is stating that there must be some element (it is identified as c above) that makes the predicate true ($P(c)$ is true) if $\exists x\ P(x)$. We don't make a claim as to which element c is in the domain.

Existential Generalization

Here is existential generalization:

$$\begin{array}{ll} P(c) & \text{for some element } c \\ \hline \therefore\ \ \exists x\ P(x) & \end{array}$$

Existential generalization states that if for some element c, $P(c)$ is true then we can generalize that fact by stating there exists some x such that $P(x)$ is true.

Universal Modus Ponens

We can combine universal instantiation with modus ponens into a new rule of inference: universal modus ponens:

$$\begin{array}{ll} \forall x(P(x) \to Q(x)) & \\ P(c) & \text{where } c \text{ is a specific element of the domain} \\ \hline \therefore\ \ Q(c) & \end{array}$$

Let's prove this:

Proof. Prove universal modus ponens

$$\begin{array}{ll} 1.\ \forall x(P(x) \to Q(x)) & \text{starting premise} \\ 2.\ P(a) \to Q(a) & \text{universal instantiation on 1} \\ 2.\ \neg P(a) \lor Q(a) & \text{De Morgan's law on 2} \\ 3.\ P(a) & \text{starting premise} \\ \hline \therefore\ \ Q(a) & \text{resolution on 2 and 3} \end{array}$$

□

Universal Modus Tollens

We can combine universal instantiation with modus tollens into a new rule of inference: universal modus tollens:

$$\begin{array}{ll} \forall x(P(x) \to Q(x)) & \\ \neg Q(c) & \text{where } c \text{ is a specific element of the domain} \\ \hline \therefore \quad \neg P(c) & \end{array}$$

See if you can create the proof for universal modus tollens as we did with universal modus ponens.

Resolution with Predicates Example:

Let's see how we can use resolution with predicates. Given the following premises, what can we deduce (or prove)?

$$\begin{array}{ll} 1.\ \forall x(P(x) \to (Q(x) \wedge R(x))) & \text{premise} \\ 2.\ P(a), \quad \text{some element } a & \text{premise} \\ 3.\ P(a) \to (Q(a) \wedge R(a)) & \text{universal instantiation on 1} \\ 4.\ \neg P(a) \vee (Q(a) \wedge R(a)) & \text{conditional-disjunction equivalence} \\ 5.\ (\neg P(a) \vee Q(a)) \wedge (\neg P(a) \vee R(a)) & \text{distribution on 4} \\ 6.\ \neg P(a) \vee Q(a) & \text{simplification on 5} \\ 7.\ \neg P(a) \vee R(a) & \text{simplification on 5} \\ \hline \therefore \quad Q(a) & \text{resolution on 6 and 2} \\ \therefore \quad R(a) & \text{resolution on 7 and 2} \end{array}$$

Using Universal Generalization

To this point, we've seen proofs that only use universal instantiation to prove that some predicate for a specific value in the domain is true (as in our proof of $Q(a)$ in the previous section. Sometimes, we want to prove a quantified statement. In this case, we will use universal instantiation, but we want an arbitrary value of the domain. To do this, we will use the following annotation in our proof: *w.l.o.g.*, which means "without loss of generality." This means that even though we have to use a particular case, it does not affect the validity of the proof in general, because we chose an arbitrary value that is representative of all possible values. This allows us to use universal generalization later in the proof. Recall that universal generalization works for an *arbitrary* value of the domain. That is why we need to use universal instantiation without loss of generality, meaning we used universal instantiation to come up with an arbitrary value of the domain that is representative of all values in the domain.

You may be wondering what all the fuss is about. The issue is that we may only use the rules of inferences of propositional logic on propositions. Remember that propositions cannot contain quantifiers. Thus, we sometimes have to convert our premises from predicate logic to propositional logic – to make use of propositional logic rules of inference – and then back to predicate logic. The next section gives an example proof.

Using Universal Generalization Example

Prove $\forall x R(x)$ given the following premises (the domain is $\{1, 2, 3, 4, 5\}$):

1. $\forall y(P(y) \lor R(y))$	premise
2. $\neg\exists y P(y)$	premise
3. $\forall y \neg P(y)$	De Morgan's Law
4. $P(5) \lor R(5)$	universal instantiation on 1 w.l.o.g.
5. $\neg P(5)$	universal instantiation on 2 w.l.o.g.
6. $R(5)$	resolution on 4 and 5
$\therefore\ \forall x R(x)$	universal generalization on 6
	the above only works because of w.l.o.g. on 4 and 5

Note that we cannot use resolution on premises 3 and 1, because both premises are quantified statements, not propositional statements.

Variable Scope

Variables in predicate logic are either *bound*, by a quantifier, or *free* in their scope. By convention, any free variables will be replaced with universal quantification, with the justification being universal generalization. Thus, if we have a predicate $P(x)$, where x is a variable and not in the domain, then we can convert $P(x)$ to $\forall x P(x)$ by universal generalization.

Conclusion

We will use predicate logic proofs in the next few topics of study. It is highly suggested to study Sections 1.6.5-1.6.6 in the book* for more examples and details.

*All definitions are from *Discrete Mathematics and Its Applications*, by Kenneth H. Rosen.