Set Theory

CS236 - Discrete Structures
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Set Theory

We used sets earlier in our study of languages. Recall we had an input alphabet, I, that we denoted as $I = \{a, b, c\}$, where the braces denoted which elements were a part of the set which we labeled the input alphabet. We use the braces to denote a set.

Set: Definition 1, Section 2.1.1*

A *set* is an unordered collection of objects, called *elements* or *members* of a set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Set Notation

As seen in the definition above, we often denote a set with a capital letter. For sets with finite elements, we put all elements contained by the set between braces $\{\ \}$. For example, the set A containing the elements 1, 2, and 3 is defined as: $A = \{1, 2, 3\}$.

Empty Set

The following is called the empty set: $\{\}$. The empty set is important enough to be denoted by the following symbol: $\emptyset$. We can combine the two, $\{\emptyset\}$, to denote that there is a set that only contains the empty set. Note that this shows that an element of a set can be a set itself. Here is a set that contains three other sets: $\{\{1\}, \{2\}, \{3\}\}$. The empty set is *not* an element of every set.

Set Builder Notation

Set builder notation allows us to describe what elements are in a set without being explicit: $A = \{x | x \text{ satisfies some property}\}$. This may be read as '$A$ is the set that contains element x such that x satisfies some property.'.

For example, $A = \{x \mid x \text{ is odd} \wedge 0 < x < 10\}$ is equivalent to $A = \{1, 3, 5, 7, 9\}$. Another example: $A = \{x \mid 0 < x < 4\}$ is $A = \{1, 2, 3\}$. Often we'd like to restrict x to some domain. $B = \{x \in Z^+ \mid x \text{ is even} \wedge x < 12\}$ where Z^+ is the set of all positive integers (the book* gives other useful domains such a Z, the set of all integers). Thus, $B = \{2, 4, 6, 8, 10\}$. Consider another example: $C = \{x \in B \mid x \text{ is a power of two}\}$. Then, $C = \{2, 4, 8\}$.

Set Equivalence: Definition 2, Section 2.1.1*

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \iff x \in B)$. We write $A = B$ if A and B are equal sets.

Subset: Definition 3, Section 2.1.3*

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Subsets

Note that $A \subseteq B$ is not interchangeable with $B \subseteq A$. Each has a separate meaning. If both $A \subseteq B$ and $B \subseteq A$ are true then A and B contain all the exact same elements. Thus we have another definition for set equality: $(A \subseteq B \wedge B \subseteq A) \iff A = B$.

Subset Example:

Consider the following sets A and B is $A \subseteq B$ true or false? Is $B \subseteq A$ true or false?:

$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A \subseteq B$ is true because every element of A is also contained in B. $B \subseteq A$ is false, with a counter example of element 2. This element is not a member of the set A, thus $B \nsubseteq A$.

Subset Theorem: Theorem 1, Section 2.1.3*

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proper Subset

To emphasis that A is a subset of B, but that $A \neq B$, we say that A is a proper subset of B, denoted as $A \subset B$ (note the similarity to the difference between $<$ and $\leq$).

Set Size: Definition 4 & 5, Section 2.1.4*

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$. A set is said to be *infinite* if it is not finite.

Set Size Example:

$|S|$ denotes the cardinality, or size, of a set. Determine the cardinality of each of the following sets:

1) $S = \{a, b, c, d, e\}$
2) $S = \{1, 2, 3, 5, 8, 13\}$

Count the distinct elements in each set to determine the cardinality. The size of each set is as follows: 1) $|S| = 5$, 2) $|S| = 6$.

Power Set: Definition 6, Section 2.1.5*

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Power Set Example:

Alternate notation for the power set of the set S is 2^S (this notation is often used show the cardinality of the power set of S; note $|\mathcal{P}(S)| = |2^S| = 2^{|S|}$).

What is $\mathcal{P}(A)$? What is 2^B?

$A = \{a, b, c\}$
$B = \{1, 3, 5, 7\}$

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Again, note that $|\mathcal{P}(A)| = |2^A| = 2^{|A|} = 2^3 = 8$
$2^B = \{\emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, \{3, 5, 7\}, \{1, 3, 5, 7\}\}$
$|2^B| = 16$

Set Operations

We'll now discuss six important set operations: *union*, *intersection*, *difference*, *complement*, and the *Cartesian product*. In order to understand the Cartesian product we must introduce the concept of a *tuple*.

Set Union (see Definition 1, Section 2.2.1*)

We define the union of two sets A and B as follows: $A \cup B = \{x \mid x \in A \vee x \in B\}$. Thus, $A \cup B$ is a new set that contains all distinct elements contained in A or B (we don't include duplicates).

Set Intersection (see Definition 2, Section 2.2.1*)

We define the intersection of two sets A and B as follows: $A \cap B = \{x \mid x \in A \wedge x \in B\}$. Thus, $A \cap B$ is a new set that contains all distinct elements contained in both A and B.

Disjoint: Definition 3, Section 2.2.1*

Two sets are called *disjoint* if their intersection is the empty set.

Set Difference (see Definition 4, Section 2.2.1*)

We define the set difference of A and B as follows: $A - B = \{x \mid x \in A \wedge x \notin B\}$. Thus, the new set contains elements that are only contained by A and not by B.

Set Complement (see Definition 5, Section 2.2.1*)

In order to define set complement, we define first the set U which is called the universal set: it contains all possible elements of the domain. The complement of a set S is denoted as $\overline{S}$ and is defined as the difference of U and S as follows: $\overline{S} = U - S$.

Tuple: Definition 7, Section 2.1.6*

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Cartesian Product: Definition 8, Section 2.1.6*

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs [2-tuple] (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Cartesian Product: Definition 9, Section 2.1.6*

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i) \text{ for } i = 1, 2, \ldots, n\}.$$

Cartesian Product with the Empty Set

Let's consider the following: $\emptyset \times A$. What does it mean to perform the Cartesian product on the empty set and another set. Recall that the Cartesian product creates tuples for all the possible combinations of the elements of both sets. The empty set has no elements, thus there are no combinations. That means the result is also the empty set. The Cartesian product on the empty set and another set is similar to multiplication of zero and another number. Also note we can think about this in terms of cardinality. The cardinality of $A \times B$ is equal to the cardinality of A multiplied by B. Thus $|\emptyset \times A| = 0 \times |A| = 0$.

Cartesian Product in Code

Here is another way to think about the Cartesian product:

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AxB = {(i,j) | i in A AND j in B}
  for i in A
    for j in B
      create(i,j) then put in AxB
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Cartesian Product Example:

Consider the following sets A and B. What is $A \times B$?

$A = \{a, b, c\}$
$B = \{x, y\}$

$A \times B = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}$. Is $A \times B$ equal to $B \times A$? Let's check. $B \times A = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$. Because tuples are ordered (thus order does matter) the answer is *no*, $A \times B \neq B \times A$.

What is $A \times A$ and $B \times B \times B$?

$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$
$B \times B \times B = \{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$

Note that $|A \times A| = |A| \cdot |A| = 3 \cdot 3 = 9$ and that $|B \times B \times B| = 2 \cdot 2 \cdot 2 = 8$.

Bit-string Representation

We can use bit-strings to help us represent the elements in the universe of a domain. A set can then be represented as a bit-string, where a 1 in a specific digit denotes that the set contains a specific element. Let's consider a domain where the universe is $U = \{a, b, c, d, e\}$. There are five elements, so we need 5 bits to represent every possible combination of elements. We'll order the elements as a, b, c, d, e, meaning that the most significant digit (in our 5-bit number) will be a 1 when a is a member of a set and 0 otherwise. The universe is represented as having all one's because it contains all elements: bit-string for $U = 11111$. Now consider the set $A = \{a\}$. The bit-string representation for A is 10000. Let $X = \{a, c, e\}$. Then the bit-string representation for X is 10101. See Section 2.2.4* for more details.

Conclusion

We will use set theory as a foundation for future topics. It is important that you understand the notation, definitions, and operations of a set. See the book* for further examples and details.

*All definitions are from *Discrete Mathematics and Its Applications*, by Kenneth H. Rosen.