Tautologies and Logical Equivalences

CS236 - Discrete Structures
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Propositional Logic: Review

Review the different propositional operators by studying the table below:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
F	T	T	F	T	T	F
T	F	F	F	T	F	F
F	F	T	F	F	T	T

Compound Propositions

Some compound propositions have special properties, such as always being true or always being false, or neither. We categorize these properties, as they are useful for reasoning and proving mathematical and logical concepts.

Tautology: Definition 1, Section 1.3*

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.

Contradiction: Definition 1, Section 1.3*

A compound proposition that is always false is called a *contradiction*.

Contingency: Definition 1, Section 1.3*

A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Compound Propositions: Example

Consider the following compound propositions. Is each a tautology, contradiction, or contingency?

1. $p \vee T$
2. $p \vee F$
3. $p \wedge T$
4. $p \wedge F$
5. $p \vee \neg p$
6. $p \wedge \neg p$

The following are tautologies: 1 and 5. The following are contradictions: 4 and 6. The remaining (2 and 3) are contingencies.

Logical Equivalences: Definition 2, Section 1.3*

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Tautology Example:

Create the truth table for the following compound proposition: $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

Solution:

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$
T	T	F	F	T
F	T	T	T	T
T	F	T	T	T
F	F	T	T	T

Note that each cell in column $\neg(p \wedge q)$ and $\neg p \vee \neg q$ match exactly, and that the final column $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ only contains the value T. The bi-implication is always true, regardless of the truth values of p and q, thus we have a tautology, as well as, a logical equivalence. This logical equivalence is extremely important in logic; it is one of many logical equivalences know as De Morgan's laws.

Logical Equivalence Example:

Here are some other useful logical equivalences, also known as laws (for a more complete list of laws see Tables 6, 7, and 8 in Section 1.3.2*):

1. Identity: $p \wedge T \equiv p$; $p \vee F \equiv p$
2. Domination: $p \vee T \equiv T$; $p \wedge F \equiv F$
3. Distributive: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Conditional-Disjunctive Equality: $p \to q \equiv \neg p \vee q$
5. Negation: $p \vee \neg p \equiv T$; $p \wedge \neg p \equiv F$
6. De Morgan's Laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
   $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Let's show the truth table for Conditional-Disjunctive Equality shown above:

p	q	$p \to q$	$\neg p \vee q$	$p \to q \leftrightarrow \neg p \vee q$
T	T	T	T	T
F	T	T	T	T
T	F	F	F	T
F	F	T	T	T

Conjunctive Normal Form

Often it is desirable to express compound propositions in a specific form. Conjunctive Normal Form (CNF) is a form in which the propositional variables and expressions are grouped by disjunction and separated by conjunction. Negation must only be used as a literal, i.e. it can only precede an atomic proposition – no compound propositions can be negated in CNF.

For example, here is a compound proposition in CNF: $(p \vee q) \wedge (\neg p \vee r) \wedge (q \vee s)$

Conditional-Disjunctive Equality, the Distributive laws, and De Morgan's laws are helpful in converting compound propositions into CNF. Consider the following:

$(p \to q) \wedge r \wedge (r \to p)$

We convert the implication to disjunction (using Conditional-Disjunctive Equality) which gives us:

$(\neg p \vee q) \wedge r \wedge (\neg r \vee p)$

which is now in CNF.

Conclusion

We will use logical equivalences extensively throughout our study of propositional logic and all topics built thereon. See the book* for further examples and details, as well as many important logical equivalences (often referred to as laws).

*All definitions are from *Discrete Mathematics and Its Applications*, by Kenneth H. Rosen.