Relational Algebra

CS236 - Discrete Structures
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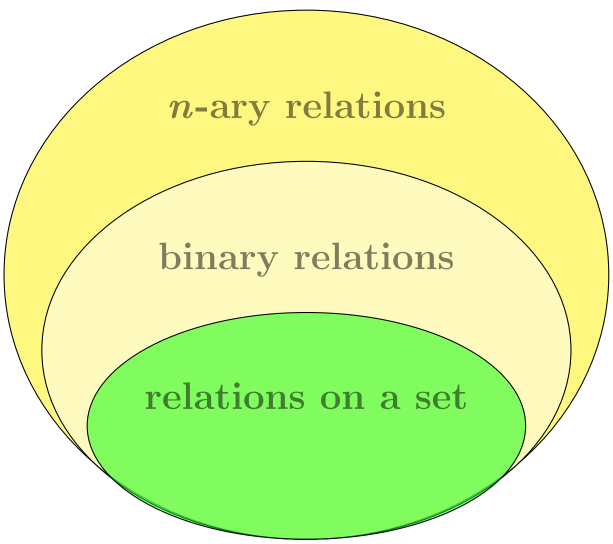
Relation Hierarchy

A *relation* is a set of tuples made up of elements from sets. Specifically, a relation is a subset of the Cartesian product of some number of sets. Consider the following sets, A_1, A_2, and A_3, and a relation R from these sets:

$A_1 = \{a, b, c, d\}$
$A_2 = \{2, 4, 6, 8, 10\}$
$A_3 = \{1, 3, 5\}$
$R = \{(a, 2, 1), (b, 8, 1), (c, 6, 5)\}$

R above is a ternary (3-ary) relation that is the subset of the Cartesian product of A_1, A_2, and A_3 ($R \subseteq A_1 \times A_2 \times A_3$). The following diagram shows the hierarchy of relations:



n-ary Relation: Definition 1, Section 9.2.2*

Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Binary Relation: Definition 1, Section 9.1.1*

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

Relations on a Set: Definition 2, Section 9.1.3*

A *relation on a set* A is a relation from A to A [a subset of $A \times A$].

Basic Relation Operations

We'll now discuss three relation operations: *union*, *intersection*, *difference*. Note that a relation is just a set of tuples, so these first three operations are just set operations. We will also discuss the *cross-product* of two relations (essentially Cartesian product for sets).

Relation Union Example:

Consider the following sets A and B, and relations R_1 and R_2. What is the result of the operation $R_1 \cup R_2$?

$A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4\}$
$R_1 = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$
$R_2 = \{(a, 1), (a, 2), (a, 3), (a, 4)\}$

The result is simply all unique tuples in both relations: $R_1 \cup R_2 = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 2), (c, 3), (d, 4)\}$. Note that we do *not* include $(a, 1)$ twice. Also, note that $R_1 \cup R_2 = R_2 \cup R_1$.

Relation Intersection Example:

Consider the following sets A and B, and relations R_1 and R_2. What is the result of the operation $R_1 \cap R_2$?

$A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4\}$
$R_1 = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$
$R_2 = \{(a, 1), (a, 2), (a, 3), (a, 4)\}$

The result is simply all tuples contained by both relations: $R_1 \cap R_2 = \{(a, 1)\}$. Note that $R_1 \cap R_2 = R_2 \cap R_1$.

Relation Difference Example:

Consider the following sets A and B, and relations R_1 and R_2. What is the result of the operation $R_1 - R_2$?

$A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4\}$
$R_1 = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$
$R_2 = \{(a, 1), (a, 2), (a, 3), (a, 4)\}$

The result is all tuples contained only in R_1 (we remove tuples from R_1 that are also in R_2): $R_1 - R_2 = \{(b,2),(c,3),(d,4)\}$. Note that $R_1 - R_2 \neq R_2 - R_1$ ($R_2 - R_1 = \{(a,2),(a,3),(a,4)\}$).

Relation Cross-product Example:

The cross-product of two relations is essentially the same as the Cartesian product of two sets. Consider the following sets A and B, and relations R_1 and R_2. What is the result of the operation $R_1 \times R_2$?

$A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4\}$
$R_1 = \{(a,1),(b,2)\}$
$R_2 = \{(c,3),(c,4)\}$

The result is as follows: $R_1 \times R_2 = \{(a,1,c,3),(a,1,c,4),(b,2,c,3),(b,2,c,4)\}$. Note that order matters. $R_2 \times R_1 = \{(c,3,a,1),(c,3,b,2),(c,4,a,1),(c,4,b,2)\}$. Notice that we don't have to specify the result as tuples of tuples – this is unnecessary for this course.

Relation on a Set: Matrix Representation

Relations on a set can be represented as a zero-one matrix that has a one in each cell that corresponds to each tuple of the relation (for details on matrices, see Section 2.6*). The matrix representation is often helpful for understanding and using relations. Let's consider the following relations on the set A:

$A = \{a, b, c, d\}$
$R_1 = \{(a,a),(b,b)\}$
$R_2 = \{(a,a),(a,c),(b,b),(d,a),(d,c)\}$

R_1 can be represented as the matrix M_{R_1}, and R_2 as M_{R_2}, as shown below:

$$M_{R_1} = \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \begin{array}{c} \begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M_{R_2} = \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \begin{array}{c} \begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Conclusion

Relational algebra is the foundation for relation data models, as we shall see. We will also study relations on a set in more detail. See the book* for further examples and details.

*All definitions are from *Discrete Mathematics and Its Applications*, by Kenneth H. Rosen.