Chomsky Grammars - 9/10/2021

Definition

Grammars are more expressive languages than regular languages. They can generate all languages that regular expressions can, but regular expressions cannot generate all languages that grammars can.

Recall the definitions of:

- a vocabulary or alphabet V of symbols
- a word or sentence over V is a string of finite length of elements in V
- the empty string λ (sometimes ϵ) is the string containing no symbols
- the set of **all words** over V is denoted by V^*

Phrase-Structured Grammar

A phrase-structured grammar $G = (V, T, S, P)$ where

- V is a vocabulary
- T is a set of **terminals** (all possible symbols in all strings generated by the grammar)
- $S \in V$, the starting nonterminal
- $P = \{A \to a, \ldots\}$ a set of production rules (left-hand side **produces** right-hand side)
    - We can use '|' to denote all possible productions produced by the same nonterminal. Think of it as the word 'or'.
        - e.g. $\{S \to \lambda|1|2\}$ means S can produce λ or 1 or 2

Other definitions:

- N is the set of non-terminals, $V - T$ (all symbols $s \in V \text{ s.t. } s \notin T$)
    - Think of elements of N as intermediary variables in **derivation**.
- $V = N \cup T$

Definition of a Language Generated from a Phrase-Structured Grammar

For a phrase-structured grammar G, we say the language generated by G is $L(G)$, which is the set of all strings of terminals that are **derivable** from the starting state S. I.e. (and just pretend the single arrow down there is a double arrow like this one: $\Rightarrow$)

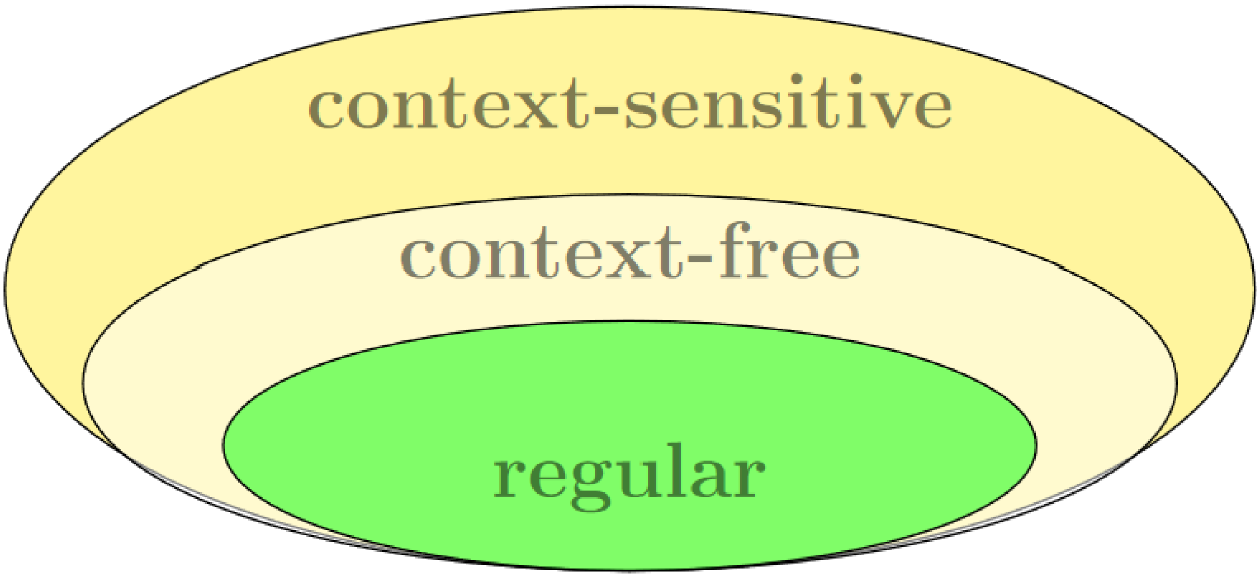
$$L(G) = \{w \in T^* | S \overset{*}{\to} w\}$$

- The starred arrow just means that any number of productions could have gone into producing the right-hand side, when starting from the left-hand side.
- The process of *derivation* is starting at the start symbol S and then using the production rules until there are only terminals in the resulting string.

Context-Free Grammars (Type 2 Grammars)

Definition

Context-free grammars are a subset of all phrase-structured grammars. They are all grammars such that the left-hand side of each production rule is a *single*, *nonterminal*. Grammars who have production rules with more than one non-terminal on the left-hand side are called **context-sensitive** (**Type 1** grammars) (e.g. $aAb \rightarrow \ \ldots$). See the following visualization:



Linking Grammars and Languages

For a given language L and context-free grammar G which generates that language:

- The vocabulary of L is the same as the set of terminals, T, of G
- The strings of L are the terminal strings of G

It's a bit confusing I know but stay with me. The key to understanding grammars are the productions.

Recognizing Context-Free Languages

Finite-state machines are not quite enough to recognize languages. Instead we'll use something called **pushdown automata**. They are essentially finite-state automata but they can push and pop information to the stack at each transition, which is key in recognizing context-free languages.

Examples

Taking a symbol a^n to mean a concatenated to itself n times, consider the language:

$$L = \{0^n 1^n | n \geq 0\}$$

We can't do this with regular expressions because we have no concept of exponents. But with a stack we can push a 0 each time we read a zero, and pop a zero each time we read a 1, and see if we come up with an empty stack at the end to see if the criteria are met (that is, there are the same number of 0s and 1s). Let's define a context-free grammar to generate L:

$$G_L = (V, T, S, P) \text{ where}$$

$V = \{S, \lambda, 0, 1\}$
$T = \{\lambda, 0, 1\}$
$S = \{S\}$
$P = \{S \to 0S1, S \to \lambda\}$

So to attempt to **derive** all possible **terminal strings**, we can do the following:

string before	production	string after	terminal string?
S	$S \to \lambda$	λ	✓
S	$S \to 0S1$	$0S1$	
$0S1$	$S \to 0S1$	$00S11$	
$\ldots$	$\ldots$	$\ldots$	
$0 \ldots 0S1 \ldots 1$	$S \to \lambda$	$0 \ldots 01 \ldots 1$	✓

And thus we see that G_L produces the language

$$L = L(G_L) : \{\lambda, 01, 0011, 000111, \ldots\}$$

This one is mad weird

Consider the following grammar for a simple arithmetic language:
$G = (V, T, S, P)$ where

$N = \{E, D\}$
$T = \{0, 1, 2, \ldots, 9, +, -, *, /, (,)\}$
$S = E$
$P = \{E \rightarrow D|(E)|E + E|E - E|E * E|E/E$
$\quad D \rightarrow 0|1|2|\ldots|9\}$

Recall that $V = N \cup T$ (all elements in both sets). Note that we can use the '|' to separate all the productions produced by the same nonterminal. Also, note that the start symbol can be any nonterminal. Let's derive the terminal string $4 + 3$:

E
$E + E$
$E + D$
$E + 3$
$D + 3$
$4 + 3$

$E \overset{*}{\Rightarrow} 4 + 3.$

Backus-Naur (or Normal) Form

BNF is a standardization for the identification of terminals and nonterminals in a grammar. It was created to support the development of programming lanugages.

- Nonterminals are wrapped in angle brackets e.g. `<id>`
- Terminals are not in angle brackets e.g. `id`
- Production rules use `::=` instead of \rightarrow

e.g.

```
<definition> ::= <type> <id> = <idOrint>;
<type>       ::=   int | byte
<idOrint>    ::=  <id> | <int>
<id>         ::= a | b | c | ... | x | y | z
<int>        ::= 0 | 1 | 2 | ... | 9
```

Here, terminals

$$T = \{\texttt{int}, \texttt{byte}, \text{'}\texttt{=}\text{'}, \text{'}\texttt{;}\text{'}, \text{all letters of the alphabet, digits from '0' to '9'}\}$$

The non-terminals are

$$N = \{\texttt{<definition>}, \texttt{<type>}, \texttt{<idOrint>}, \texttt{<id>}, \texttt{<int>}\}$$

And S is usually just the first nonterminal, but will be specified otherwise.

BNF example

Derive `int b = 3` from the grammar specified above

```
<definition>
<type> <id> = <idOrint>; # <definition> ::= <type> <id> = <idOrint>;
int <id> = <idOrint>; # <type> ::= int | byte
int b = <idOrint>; # <id> ::= a | b | c | ... | x | y | z
int b = <int> # <idOrint> ::= <id> | <int>
int b = 3; # <int> ::= 0 | 1 | 2 | ... | 9
```

Derive `int a = b;`

```
<definition>
<type> <id> = <idOrint>;
int <id> = <idOrint>;
int a = <idOrint>;
int a = <id>;
int a = b;
```

Derive `byte z = 1;`

```
<definition>
<type> <id> = <idOrint>;
byte <id> = <idOrint>;
byte z = <idOrint>;
byte z = <int>;
byte z = 1;
```