Propositional Logic

CS236 - Discrete Structures
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Propositional Logic

We will now begin our study of logic. We will start with propositional logic. First, let's define the following terms: proposition, truth value, propositional logic, and compound propositions.

Proposition: Section 1.1*

A *proposition* is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Proposition Example:

Are the following propositions?

1. The earth is round.
2. Are you sure the earth is round?
3. The sky is blue.
4. Look up.
5. Why does my head hurt?
6. My head hurts.

The sentences 1, 3, and 6 are propositions. Sentences 2, 4 and 5 are not.

Propositional Variable: Section 1.1*

A *propositional variable* is a letter that represents a proposition (traditionally, we start with $p, q, r, s, ...$).

Propositional Variable Example:

Let $p =$ "The earth is round."
Let $q =$ "The sky is blue."

We often call propositional variables atomic propositions. Thus, compound propositions are made up of atomic propositions and logic operators.

Truth Value: Section 1.1*

The *truth value* of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

Propositional Variable Example:

Let $p =$ "The earth is round."
The truth value of p is T (yes, the earth really is round).
Let $q =$ "The sky is blue."
The truth value of q is T (let's not get too technical; people see the color blue when they look at the sky).

Propositional Logic: Section 1.1*

Propositional logic (also called propositional calculus) is the area of logic that deals with propositions.

Compound Propositions: Section 1.1*

New propositions formed from existing propositions using logical operators are called *compound propositions*.

Logic Operators

We will introduce five, core operators in propositional logic. These are negations, conjunction, disjunction, implication, and bi-implication, denoted as $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, respectively.

Negation: Definition 1, Section 1.1*

Let p be a proposition. The *negation* of p, denoted by $\neg p$, is the statement "It is not the case that p."

Truth Table: Negation

We often represent propositions in a format that shows all of their possible truth values. We call this a *truth table*. The following is the truth table for negation:

p	$\neg p$
T	F
F	T

This shows that if the truth value of p is true (T), then the truth value of $\neg p$ is false (F).

Negation Example:

Let $p =$ "The earth is round."
$\neg p =$ The earth is *not* round.

Conjunction: Definition 2, Section 1.1*

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth Table: Conjunction

The following is the truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction Example:

Let $p =$ "The earth is round."
Let $q =$ "The sky is blue."
$p \wedge q =$ "The earth is round, and the sky is blue."

Disjunction: Definition 3, Section 1.1*

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Truth Table: Disjunction

The following is the truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction Example:

Let $p =$ "The earth is round."
Let $q =$ "The sky is blue."
$p \vee q =$ "The earth is round, or the sky is blue (or both)."

Implication: Definition 5, Section 1.1*

Let p and q be propositions. The *conditional statement* (or *implication*) $p \rightarrow q$, is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and is true otherwise. In the conditional statement $p \rightarrow q$, p is called the *premise*, and q is called the *consequence*.

Truth Table: Implication

The following is the truth table for the conditional statement, or implication:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication Example:

Let $p =$ "I hit my head."
Let $q =$ "My head hurts."
$p \rightarrow q =$ "I hit my head *implies that* my head hurts."
Notice that it is fine if my head hurts even if I haven't hit it (maybe I have a migraine) – that doesn't invalidate the proposition. But, if I hit my head and it doesn't hurt, then the above proposition is a false claim.

Bi-implication: Definition 6, Section 1.1*

Let p and q be propositions. The *biconditional statement* (or *bi-implication*) $p \leftrightarrow q$, is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

Truth Table: Bi-implication

The following is the truth table for the biconditional statement, or bi-implication:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Bi-implication Example:

Let $p =$ "I hit my head."
Let $q =$ "My head hurts."
$p \leftrightarrow q =$ "I hit my head *if and only if* my head hurts."
Notice that it is now unacceptable for my head to hurt if I haven't hit my head (a migraine would invalidate the proposition).

Precedence of Logical Operators

The following table shows the precedence of the five logical operators introduced above:

Precedence	**Operator**
1	$\neg$
2	$\wedge$
3	$\vee$
4	$\rightarrow$
5	$\leftrightarrow$

Just like how multiplication,$*$, has higher precedence than addition, $+$, conjunction, $\wedge$ has higher precedence than disjunction, $\vee$. With numbers, negation, $-$, also has highest precedence (just as negation, $\neg$, does in propositional logic).

Propositional Logic Examples:

Let p, q, and r be propositional variables where $p = T$, $q = F$, and $r = T$. Determine the truth values of the following:

1. $\neg p \vee q$
2. $p \vee q \vee r$
3. $q \rightarrow (p \wedge r)$
4. $p \leftrightarrow r$
5. $\neg[q \vee (p \wedge r)]$

For 1, plug in the values for p and q: $\neg T \vee F$. The negation binds tighter than disjunction, so we have $F \vee F$, which results in F. For 2, we have $T \vee F \vee T$. We can evaluate from left to right to get $T \vee T$ (since $T \vee F = T$), which results in T. For 3, we have $F \rightarrow (T \wedge T)$, which simplifies to $F \rightarrow T$, which evaluates to T (look at implication again if this is confusing). For 4, $T \leftrightarrow T$ is T (remember for bi-implication the values must be the same on both sides of the operator for the proposition to evaluate to true). For 5, we have $\neg[F \vee (T \wedge T)]$, which simplifies to $\neg[F \vee T]$, which is $\neg[T]$, which evaluates to F. Note that the parentheses around $p \wedge r$ are redundant (since $\wedge$ has higher precedence that $\vee$), but the brackets around $q \vee (p \wedge r)$ are essential.

Conclusion

Propositional logic is the basis for reasoning in the field of Computer Science (as well as Philosophy). See the book* for further examples and details.

*All definitions are from *Discrete Mathematics and Its Applications*, by Kenneth H. Rosen.