FINM3003

Week 1

Forwards

S_t = Spot price at t
T = Delivery date
r = Risk free interest rate
F_t = Forward price

No Income

Price

$$F_t = S_t e^{r(T-t)}$$

Long value

$$V_t = (F_t - F_0) e^{-r(T-t)}$$

Short value

$$V_t = -(F_t - F_0) e^{-r(T-t)}$$

Income

Price

$$F_t = (S_t - D_t) e^{r(T-t)}$$

Long value

$$V_t = (F_t - F_0) e^{-r(T-t)}$$

Short value

$$V_t = -(F_t - F_0) e^{-r(T-t)}$$

Dividend yield (Continuous)

Price

$$F_t = S_t e^{(r-q)(T-t)}$$

Long value

$$V_t = (F_t - F_0) e^{-r(T-t)}$$

Short value

$$V_t = -(F_t - F_0) e^{-r(T-t)}$$

Futures

Value prior to closing

$$V_1 = f_1 - f_0$$

Basis

Basis = Spot price − Futures price

Week 2

Options

Euro = Exercise only at end
American = Exercise anytime (Never early)

K = Strike price
c/p = Euro
C/P = US

No Income

Euro

Parity

$$C_t - P_t = S_t - K e^{-r(T-t)}$$

Bounds

$$\max(S_t - Ke^{-r(T-t)}, 0) \leq C_t \leq S_t$$

$$\max(Ke^{-r(T-t)} - S_t, 0) \leq P_t \leq Ke^{-r(T-t)}$$

US

Parity

Lower bound $C_t - P_t \leq S_t - K e^{-r(T-t)}$

Upper bound $C_t - P_t \geq S_t - K$

Dividends

Euro

Parity

$$C_t - P_t = S_t - D_t - K e^{-r(T-t)}$$

Bounds

$$\max(S_t - D_t - Ke^{-r(T-t)}, 0) \leq C_t$$

$$\max(D_t + Ke^{-r(T-t)} - S_t, 0) \leq P_t$$

US

Parity

$$(S_t - D_t) - K \leq C_t - P_t \leq S_t - K e^{-r(T-t)}$$

- May be optimal to exercise early

Dividend yield

Euro

$$C_t - P_t = S_t e^{-q(T-t)} - K e^{-r(T-t)}$$

US

$$S_t e^{-q(T-t)} - K \leq C_t - P_t$$

Risk Neutral

$$p = \frac{e^{rT} - d}{u - d} \qquad 0 < p < 1$$

$$p = e^{rT} - d(1-p) \ ??$$

$$\Delta t = \frac{T}{n}$$

Definitions

360 trading days in a year

FINM3003 Week 3

Cox-Ross-Rubinstein Binomial Model

S_t = Risky asset
B_t = Risk-free asset
$\mathcal{F}_t$ = History of processes until t
Φ = Trading strategy / Dynamic portfolio
$A_S(t)$ = Stocks held at t
$A_B(t)$ = Bonds held at t

Value

$B_t = B_0 e^{rT}$

$S_t = S_0 \prod_{i=1}^{t} Z_i$

$V_\Phi(t) = A_S(t) S_t + A_B(t) B_t \quad \forall\ 1 \le t \le T$

Initial investment (Value after t prices observed, before portfolio changes)

$$V_\Phi(0) = A_S(0) S_0 + A_B(0) B_0 = A_S(1) S_0 + A_B(1) B_0$$

Change in market value $(t-1, t)$

$$\Delta V_\Phi = A_S(t) \times \Delta S_t + A_B(t) \times \Delta B_t$$

Gain Process

$$G_\Phi(t) = \sum_{i=1}^{t} (A_S(i) \Delta S_i + A_B(i) \Delta B_i)$$

Self-Financing iff

$$V_\Phi(t) = V_\Phi(0) + G_\Phi(t) \quad \forall\ 0 \le t \le T$$

$u = e^{\sigma\sqrt{\Delta t}}$ $\quad d = \frac{1}{u}$ $\quad A_B = \frac{e^{-r\Delta t}(uC_{ud} - dC_{u^2})}{u-d}$ $\quad A_S = \frac{C_{u^2} - C_{ud}}{S_{u^2} - S_{ud}}$

Replicating Strat

$M_0 S_0 u + B_0 e^{r\Delta t} = V_u$

Stock Process

Non-Dividend

$u = e^{\sigma\sqrt{\Delta t}}$

$d = \frac{1}{u}$

RNP

Binomial (n, p)

$p = \frac{e^{r\Delta t} - d}{u - d}$

Dividend Yield: q

Approximation method for small Δt

Use same u & d

RNP

Binomial $(n, \tilde{p})$

$\tilde{p} = \frac{e^{(r-q)\Delta t} - d}{u - d}$

Dividend Fixed: D

Define

$S_0^* = S_0 - D_{t_D} e^{-r(t_D)}$

use p, u, d

Adjust S_0^* to S_0 for valuation

Self Replicating Portfolio: π

For long call

$M_0 = \frac{C_u - C_d}{S_0(u-d)}$

$B_0 = e^{-r\Delta t}\frac{(uC_d - dC_u)}{u-d}$

$V_\pi(0) = \sum \frac{M_0 S_0}{B_0} = e^{-r\Delta t}(p C_u + (1-p) C_d)$ ← Proof that π is self replicating $\therefore = V_c(0)$

Binomial to Normal Approximation

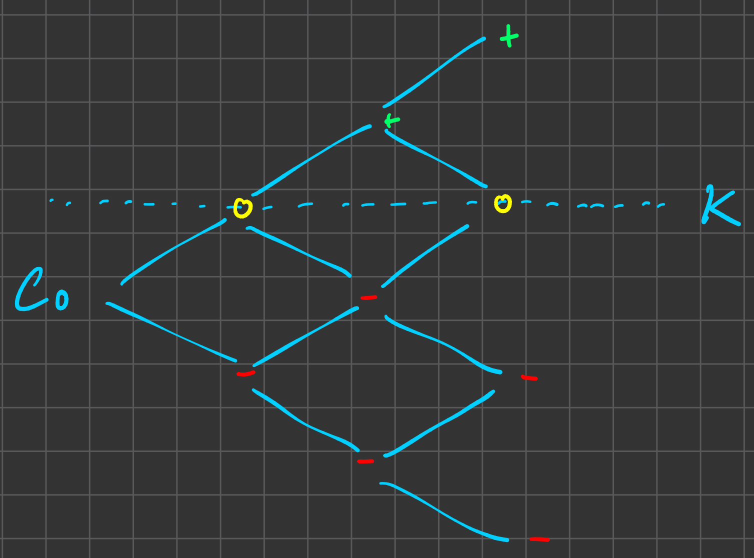
If

$$X_i = \begin{cases} u & \mathbb{P}(p) \\ d & \mathbb{P}(1-p) \end{cases}$$

$S_n = S_{n-1} X_n$

$\quad = S_0 \prod X_i$

$$\ln\left(\frac{S_n}{S_0}\right) = \sum^n X_i \sim (\ln u - \ln d)\left(Bin(n,p) + \frac{n \ln d}{\ln u - \ln d}\right.$$



If K, then only nodes above 2u need to be evaluated

For $n \to \infty$, binomial to normal approximation

FINM3003 — Week 4

Always $[t_0, t_1)$ because we don't know future

Brownian Motion

Standard Brownian Motion / Weiner Process

1) $Z_t \sim N(0,t) \quad \forall\ 0 \le t$
2) $Z_t - Z_s \perp\!\!\!\perp Z_u \quad \forall\ u \le s \le t$ — Independent increments
3) $Z_t - Z_s \overset{D}{=} Z_{t-s} \quad \forall\ 0 \le s \le t$

Other properties

- $E(Z_t) = 0 \quad Var(Z_t) = t$ ☆
- $Z_0 = 0 \quad Z_1 \sim N(0,1)$ ☆
- Z_t is continuous in t
- SBM not differentiable at any t

Increments of BM

$\Delta Z_t \overset{D}{=} \epsilon\sqrt{\Delta t}$

where

$\epsilon \sim N(0,1)$

Normal Distribution

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

☆ Normalise

$$Z = \frac{(y-\mu)}{\sigma}$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \rightarrow$$

Can substitute with $y = \frac{z^2}{2}$ to eval integral

$E(X^3) = 0$

$$\int_0^\infty z \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Let $y = \frac{z^2}{2} \quad dy = z\,dz \quad dz = \frac{dy}{z}$

$$\int_{0}^{\infty} z \times \frac{1}{\sqrt{2\pi}} e^{-y} \frac{dy}{z}$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y}\,dy$$

$$\frac{1}{\sqrt{2\pi}}\left[-e^{-y}\right]_0^\infty$$

Geometric Brownian Motion

SDE Process

☆ $dS_t = \mu S_t\,dt + \sigma S_t\,dZ_t$ — GBM with Drift $= \mu$, Volatility $= \sigma$

Solution

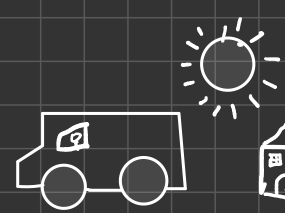
$$\frac{S_t}{S_0} = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

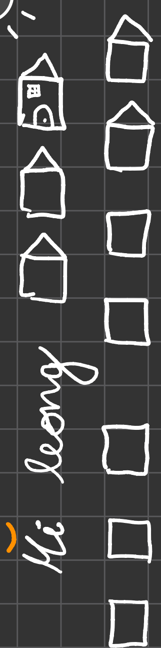
☆ Do increment calculations e.g. $S_6 - S_0$. Then to find S_6, subtract S_0

☆ $W_T - W_t = \sqrt{T-t}\ Z$

FINM3003 — Week 5

Itô





Itô Process Form → For some $a, b > 0$, a & b derived from R

☆ $dx_t = a(x_t, t)\,dt + b(x_t, t)\,dZ_t$

where Z_t is SBM

Itô's Lemma (Payoff)

$$dG = \left(a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2}\right)dt + b\frac{\partial G}{\partial x}dZ_t$$

Drift Term → For a single variable function G

$$a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2} = \left(0 + \mu + \frac{1}{2}\sigma^2\right)e^{\mu t + \sigma x}$$

Volatility Term

$$b\frac{\partial G}{\partial x} = \sigma e^{\mu t + \sigma x}$$

Log-Normal Distribution

If, $Z \sim N(a, b^2)$, $Y = e^Z$

$\ln Y \sim N(a,b)$

$Y \sim LN$

$Y > 0$

$E(Y) = e^{a + \frac{1}{2}b^2}$

$Var(Y) = e^{2a+b^2} \times (e^{b^2} - 1)$

Mi leong

Multi-dimensional Ito

For Bi-dimensional process,

$$dG = \left[\frac{dG}{di}a_i + \frac{dG}{dj}a_j + \frac{dG}{dt} + \frac{1}{2}\left(\frac{d^2G}{di^2}b_i^2 + \frac{d^2G}{dj^2}b_j^2 + 2\frac{d^2G}{di\,dj}b_i b_j \rho_{ij}\right]dt + \frac{dG}{di}b_i\,dZ_i + \frac{dG}{dj}b_j\,dZ_j$$

Where

$\rho_{ij} = Cor(dZ_i, dZ_j)$

$\frac{d^2G}{di\,dj}$ = Derive wrt i, then wrt j

FINM3003 — Week 5

Stochastic Differential Equation for Stock Price

$$d\left(\int_0^t e^{as}\, dZ_s\right) = e^{at} Z_t$$

☆ $r \to$ Risk Neutral
$\mu \to$ Real world

SDE for GBM

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

Solution

$$\frac{S_t}{S_0} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t}$$

or

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

$\nearrow$ BM with drift $\mu - \frac{\sigma^2}{2}$ & Volatility σ $\to$ If underlying is S, then it is GBM

P dist of rate of return

Stock Price Modeling

$$\ln(S_t) \sim N\left[\ln(S_0) + \left(\mu - \left(\tfrac{1}{2}\right)\sigma^2\right)t, \sigma^2 t\right]$$ Conditional on S_0

☆ $E(S_t) = E(S_T \mid S_0) = S_0 e^{\mu t}$ $\to$ Derived from SDE, not Solution.

$Var(S_t) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$

$$\ln\left(\frac{S_n}{S_0}\right) \approx N(n\mu, \sigma^2 n)$$

$\mu = p \ln\left(\frac{u}{d}\right) + \ln(d)$ $\to$ E(Return Cont. Com.)

$\sigma = \ln\left(\frac{u}{d}\right)\sqrt{p(1-p)}$ $\to$ Volatility

Volatility Parameter Estimation

If,

$$y_i = \ln\frac{S_i}{S_{i-1}} \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right)$$ $\to$ Under GBM model

$$\hat{\sigma}^2 = \frac{1}{\tau}\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$$

$90 < n < 180$ samples
$\tau = 1$ day
} Usually

FINM3003

Week 5

Always $[t_0, t_1)$ because we don't know future

$dx = a\,dt + b\,dz$

Stock Price Process

Discrete Calculation

For a stock with

- No dividends
- Expected return $= \mu$ pacc
- Volatility $= \sigma$ pa

Process

$$\frac{dS}{S} = \mu\, dt + \sigma\, dZ$$

Discrete Approximation

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

where

$\epsilon \sim N(0,1)$

No variability

Model

$$\Delta S = \mu S \times \Delta t$$

Limit: As $\Delta t \to 0$

$$dS = \mu S \times dt$$

$$\frac{dS}{dt} = \mu S$$

Integrate for time 0 to T

$$S_T = S_0 e^{\mu T}$$

Log-Normal Property

Assume

$$dS_t = \mu S_t\, dt + \sigma S_t\, dZ_t$$

GBM

- Drift rate $= \mu S_t$
- Variance rate $= \sigma^2$

Use Ito to find the process followed by $\ln S$

Let

$G = \ln x$

Apply Ito

$$\frac{dG}{dx} = \frac{1}{x} \qquad \frac{d^2G}{dx^2} = -\frac{1}{x^2} \qquad \frac{dG}{dt} = 0$$

$$dG = \left[\mu S \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \times -\frac{1}{S^2}\right] dt + \sigma S \frac{1}{S}\, dZ_t$$

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ_t$$

Tells us:

① $\ln S$ $\left\}\begin{array}{l}\text{Drift rate} = \mu - \frac{\sigma^2}{2} \\ \text{Variance rate} = \sigma^2\end{array}\right.$ (Graded near peers)

② $\ln S \sim N\left[\ln S_0 + \left(\mu + \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$

Distribution of Returns

If

$$S_T = S_0 e^{rT}$$

$\therefore$

$$r = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$r \sim N\left[\mu - \frac{\sigma^2}{2},\ \frac{\sigma^2}{T}\right]$$

Solution / Process

$$d(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ_t$$

$$\int_0^T d(\ln S_u) = \left(\mu - \frac{\sigma^2}{2}\right)\int_0^T dt + \sigma \int_0^T dZ_t$$

$$\ln S_T - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma Z_T$$

$$S_T = S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\, Z}$$

$$\ln \frac{S_t}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma Z_t$$

Tells us

- $\ln S_t \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right)$
- $E(S_t \mid S_0) = S_0 e^{\mu t}$

Notation

$$dX_t = a_t\, dt + b_t\, dW_t$$

$$X_t = X_0 + \int_0^t a_u\, du + \int_0^t b_u\, dW_u$$

$$dZ_t = \lambda_t\, dt + dZ_t$$

$$Z_t = \int_0^t \lambda_s\, ds + Z_t$$

SDE

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t$$

$$X_t = X_0 + \int_0^t a(X_u, u)\, du + \int_0^t b(X_u, u)\, dW_u$$

FINM3003 Black-Sholes Market Model

Calculator: $\Phi(d)$
Menu → 5,5,2

Given

S follows GBM (μ, σ)

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t$$

$\hat{P}$

$$\ln S_T \sim N\left[\ln S_0 + \left(r + \frac{\sigma^2}{2}\right)T\ ,\ \sigma^2 T\right]$$

Euro Call

$$C_T = \max(S_T - K, 0)$$

$$C_0 = e^{-rT} \times \hat{E}[\max(S_T - K, 0)]$$

Finding the expectation

Let

$$\left.\begin{aligned} u &= \ln S_0 + \left(r + \frac{\sigma^2}{2}\right)T \\ v &= \sigma\sqrt{T} \end{aligned}\right\} \ln S_T \sim N(u, v^2)$$

CDF of $\ln S_T$

$$F_{\ln S_T}(x) = P(\ln S_T \le x)$$

$$= \int_{-\infty}^{x} \frac{1}{v\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-u}{v}\right)^2} dy$$

$$= \frac{1}{v\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{y-u}{v}\right)^2} dy$$

Let

$$\left.\begin{aligned} S &= e^x \\ x &= \ln(S) \end{aligned}\right\} P(\ln S_T \le x) = P(S_T \le e^x)$$

$$P(\ln S_T \le x) = P(S_T \le e^x)$$

$$= P(S_T \le S) = F_{S_T}(S)$$

$$= \frac{1}{v\sqrt{2\pi}} \int_{-\infty}^{\ln S} e^{-\frac{1}{2}\left(\frac{y-u}{v}\right)^2} dy$$

PDF of S_T

$$f_{S_T}(S) = \frac{d}{dS} F_{S_T}(S)$$

$$= \frac{1}{Sv\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\ln S - u}{v}\right)^2}$$

Let

$x = \ln S$

$S = e^x$

$\frac{dS}{S} = dx$

$$\hat{E}(\max(S_T - K, 0)) = \int_K^{\infty} (S - K) \times f_{S_T}(S)\, dS$$

$$= \int_K^{\infty} (S-K) \times \frac{1}{Sv\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\ln S - u}{v}\right)^2} dS$$

$$= \int_{\ln K}^{\infty} (e^x - K) \times \frac{1}{\cancel{e^x}\, v\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2}\, \cancel{e^x}\, dx$$

$$= \frac{1}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} (e^x - K) \times e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2} dx$$

$$= \frac{1}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} e^x e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2} dx - \frac{K}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2} dx$$

where

$$e^x e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2} = e^{-\frac{(x-u)^2}{2v^2} + w}$$

$$-\frac{(x-u)^2}{2v^2} + w = \frac{-(x-(u+v^2))^2 + (u+v^2)^2 - u^2}{2v^2}$$

$$= \frac{1}{v\sqrt{2\pi}}\, e^{\frac{(u+v^2)^2 - u^2}{2v^2}} \times \int_{\ln K}^{\infty} e^{\frac{-(x-(u+v^2))^2}{2v^2}} dx - \frac{K}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2} dx$$

Integration via Substitution

Let

$$y = -\frac{x - (u+v^2)}{v}$$

$$z = -\frac{x-u}{v}$$

FINM3003

Black-Sholes Market Model

Calculator: $\Phi(d)$
Menu → 5, 5, 2

$$\hat{E}(\max(S_T-K,0)) = \frac{1}{v\sqrt{2\pi}} e^{\frac{(u+v^2)^2-u^2}{2v^2}} \times \int_{\ln K}^{\infty} e^{-\frac{(x-(u+v^2))^2}{2v^2}}\,dx - \frac{K}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{x-u}{v}\right)^2}\,dx$$

Integration via Substitution

Let

$$y = -\frac{x-(u+v^2)}{v}$$

$$z = -\frac{x-u}{v}$$

$$= \frac{e^{\frac{(u+v^2)^2-u}{2v^2}}}{\sqrt{2\pi}} \times \int_{-\infty}^{-\frac{\ln K-(u+v)^2}{v}} e^{-\frac{1}{2}y^2}\,dy - \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\ln K-u}{v}} e^{-\frac{1}{2}z^2}$$

$$= e^{\frac{(u+v^2)^2-u}{2v^2}} N\left(\frac{u+v^2-\ln K}{v}\right) - K N\left(\frac{u-\ln K}{v}\right)$$

where

$u = \ln S_0 + (r - \frac{\sigma^2}{2})T$

$v = \sigma\sqrt{T}$

$u+v^2 = \ln S_0 + (r+\frac{1}{2}\sigma^2)T$

$\frac{(u+v^2)^2-u^2}{2v^2} = \ln S_0 + rT$

$$C_0 = e^{-rT} \times \hat{E}(\max(S_T-K,0))$$

$$= S_0 N(d_1) - K e^{-rT} N(d_2)$$

Black-Sholes Market Model

Calculator: $\Phi(d)$
Menu → 5, 5, 2

With:

Bond process: $B_t = B_0 e^{rt}$

Stock process: $S_t \sim GBM(\mu, \sigma)$

Value of Portfolio

$$V(t) = A_S(t) S_t + A_B(t) B_t$$

Gain Process

$$dG_t = A_S(t)\, dS_t + A_B(t)\, dB_t$$

$$G_t = \int_0^t A_S(u)\, dS_u + A_B(u)\, dB_u$$

Self Financing

$$V(t) = V(0) + G(t)$$

Arbitrage

$$V(0) = 0 \quad \text{or} \quad \mathbb{P}(V(T) \geq 0) = 1 \quad \text{or} \quad \mathbb{P}(V(T) > 0) = 0$$

Stock Price Process & Dynamics

$$R_t = \frac{S_t}{S_{t-1}}$$

$\mu \approx \frac{\delta f}{\delta t}$?

$\mu = E(R_t)$

σ = Annual Volatility

1 Stock price process

$$S_t = e^{\mu t + \sigma Z_t}$$

2 Apply Itô

$x_t = Z_t$ is Itô process with $a = 0, b = 1$

$$G(S,t) = e^{\mu t + \sigma x}$$

where

$$\frac{\delta G}{\delta x} = \sigma e^{\mu t + \sigma x}$$

$$\frac{\delta^2 G}{\delta x^2} = \sigma^2 e^{\mu t + \sigma x}$$

$$\frac{\delta G}{\delta t} = \mu e^{\mu t + \sigma x}$$

$$dG = \left(\mu + \frac{1}{2}\sigma^2\right) e^{\mu t + \sigma x} + \sigma e^{\mu t + \sigma x}\, dZ_t$$

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right) S_t\, dt + \sigma S_t\, dZ_t$$

$$\frac{dS_t}{S_t} = \left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma\, dZ_t$$

Euro Value

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$P_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Dividends

$$C = S_0 e^{-qT} N(\tilde{d}_1) - Ke^{-rT} N(\tilde{d}_2)$$

$$P = Ke^{-rT} N(-\tilde{d}_2) - S_0 e^{-qT} N(-\tilde{d}_1)$$

$$C + Ke^{-rT} = P + S_0 e^{-qT}$$

CRR Model

$$u = e^{\sigma\sqrt{\Delta T}}$$

$$d = \frac{1}{u}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

☆ **Risk-neutral Probability Distribution: $\hat{P}$**

$$\ln S_t \sim N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right)$$

Equivalent Martingale Measure

$$\hat{E}(e^{-rt} S_t \mid F_u) = e^{-ru} S_u \quad \forall u \leq t$$

$$\hat{E}(S_t \mid F_0) = S_0 e^{-rt}$$

$N(d_1) \approx \mathbb{P}$(Expire ITM)

$N(d_2) \approx \mathbb{P}$(Expire OTM)

☆ **Risk-less Asset**

$$dg = r \times g \times dt$$

1 Assume $S_t \sim LN$

↳ Corresponding stochastic process is GBM → S_t follows GBM with expected return μ & volatility σ

$$N(d_2) = \mathbb{P}[S_T > K] = \mathbb{P}(\text{Call exercise}) \quad \neq \Delta_C$$

$$\approx \mathbb{1}_{[S_T > K]}$$

$$N(-d_2) = \mathbb{P}(S_T < K) = \mathbb{P}(\text{Put exercise})$$

σ = Volatility

σ^2 = Variance Rate of GBM process

$$d_1 = d_2 + \sigma\sqrt{T}$$

$$-d_1 = -d_2 - \sigma\sqrt{T}$$

$$-d_1 - \sigma\sqrt{T} = -d_2 - 2\sigma\sqrt{T}$$

$$d_1 = \frac{\ln\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$\tilde{d}_1 = \frac{\ln\frac{S_0}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T}$$

FINM3003 Week 6

Implied Volatility

From market options, there is a unique $\hat{\sigma}$ such that,

$c(\hat{\sigma})$ = Observed value of call

Options on Currencies

Interest Rate Parity

$$\frac{F_t}{S_t} = \frac{e^{r_d(T-t)}}{e^{r_f(T-t)}}$$

F_t = Futures price
S_t = Spot exchange rate
Value of 1 unit of foreign currency, in AUD
r_d = Domestic risk-free rate
r_f = Foreign risk-free rate

$$F_t = S_t\, e^{(r_d - r_f)\times(T-t)}$$

Options

$$C_0 = S_0 e^{-r_f T} N(\tilde{d}_1) - K e^{-r_d T} N(\tilde{d}_2)$$

$$P_0 = K e^{-r_d T} N(-\tilde{d}_2) - S_0 e^{-r_f T} N(-\tilde{d}_1)$$

$$C + K e^{-r_d \Delta T} = P + S_0 e^{-r_f \Delta T}$$

$$c^{\$} \times S_0 \times K = p^{€}$$

$$\tilde{d}_1 = \frac{\ln\frac{S_0}{K} + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T}$$

$$\Delta c = e^{-r_f T} N(d_1)$$

Delta Hedging

$$\Delta_F = \frac{\partial F}{\partial S} = e^{(r-q)(T-t)}$$

→ $[\Delta_F]$ no. of futures has the same sensitivity to stock price movements, as 1 stock

Options on Futures (Euro)

$$C_T = e^{(r-q)(T_f - T)} \max\left[f_T - \tilde{K}, 0\right]$$

$$f_t = e^{(r-q)T_f} S_t$$

$$\tilde{K} = e^{-(r-q)(T_f - T)} K$$

☆ $$C_0 = e^{-rT} \times \left[f_0 N(\tilde{d}_1) - K N(\tilde{d}_2)\right]$$

☆ $$P_0 = e^{-rT} \times \left[K N(-\tilde{d}_2) - f_0 N(-\tilde{d}_1)\right]$$

$$\tilde{d}_1 = \frac{\ln\frac{f_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T}$$

For american options, use binomial tree

Replicating a Call with Futures

$$m_{f_0} = \frac{C_u - C_d}{S_0 e^{r\Delta T}(u-d)} = \frac{\Delta \text{ Call price}}{\Delta \text{ Futures price}}$$

$$B_0 = e^{-r\Delta T}\left(\hat{p} C_u + (1-\hat{p}) C_d\right)$$

$$\hat{p} = \frac{e^{r\Delta T} - d}{u - d}$$

If $S_u \uparrow S_{uu}$

$$m_{f_1} = \frac{C_{uu} - C_{ud}}{S_u e^{r\Delta T}(u-d)}$$

$$B_1 = e^{-r\Delta T}\left(\hat{p} C_{uu} + (1-\hat{p}) C_{ud}\right)$$

FINM3003 Week 7

Greeks & Hedging

dS = Continuous time
ΔS = Discrete time

Delta

$$\Delta_{call} = \frac{\partial c}{\partial S} = N(d_1) \qquad = N(d_1)e^{-qT}$$

$$\Delta_{put} = N(d_1) - 1 \quad \text{or} \quad -N(-d_1)$$

Hedging

$$\Delta C = N(d_1)\Delta S$$

Replicating Portfolio

If hedge against n short calls at time t,

Sold $= C_t \times n$

Buy $= \Delta_{\pi t}$ Stocks

Borrow $= \$(C_t \times n - \Delta_{\pi t} S_t)$ & invest

$$V_{t=0} = \begin{cases} + C_t \times n \\ - \Delta_{\pi t} \times S_0 \\ + (C_t \times n - \Delta_{\pi t} S_t) \end{cases}$$

Gamma

$$\Gamma_{call} = \Gamma_{put} = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 c}{\partial S^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \times \frac{e^{-\frac{1}{2}d_1^2}}{S\sqrt{T}} > 0$$

Hedging

For portfolio π,

$$\Gamma_\pi = \frac{\partial \pi^2}{\partial S^2}$$

Add w_T of options, for gamma neutral, choose:

$$w_T = -\frac{\Gamma_\pi}{\Gamma_T}$$

Buy/Sell

$\frac{|\Gamma|}{\Gamma_T}\Delta$ of the stock

Theta

$$\Theta = \frac{\partial c(t)}{\partial t} < 0$$

$$\Theta_{call} = \frac{-S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$\Theta_{put} = \frac{-S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Vega

$$V_{call} = V_{put} = \frac{\partial c(\sigma)}{\partial \sigma}$$

$$= S\sqrt{T}\,N'(d_1)$$

Rho

$$\rho_{call} = \frac{\partial c}{\partial r} = TKe^{-rT}N(d_2) > 0$$

$$\rho_{put} = -TKe^{-rT}N(-d_2)$$

Call price vs Strike

$$\frac{\partial c}{\partial K} = -e^{-rT}N(d_2)$$

$$\frac{\partial c}{\partial K} = \frac{e^{rT}}{\sqrt{2\pi}} \times \frac{e^{-\frac{1}{2}d_2^2}}{K\sigma\sqrt{T}} > 0$$

$$\rightarrow S - Ke^{-rT} < c \leq S$$

BSM Partial Differentiation Equation

1) Find EOV
2) Apply Ito's lemma
3) Sub into BSM PDE ?

Assume

① $$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dZ_t$$

If $G(S,t)$ is a derivative on stock S,

② $$dG = \left[\mu S\frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}\right]dt + \sigma S\frac{\partial G}{\partial S}dZ_t \quad \text{Ito's lemma}$$

Consider portfolio π:

- Long $\Delta = \frac{\partial G}{\partial S}$ stock
- Short 1 derivative

$$\rightarrow V_\pi = \frac{\partial G}{\partial S} \times S + (-G)$$

Δ of portfolio

$$\Delta\pi = \frac{\partial G}{\partial S} \times \Delta S + (-\Delta G)$$

Discretise & Substitute

① $$\Delta S = \mu S\Delta t + \sigma S\Delta Z$$

② $$\Delta G = \left[\mu S\frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}\right]\Delta t + \sigma S\frac{\partial G}{\partial S}\Delta Z_t$$

↳ ③ $$\Delta\pi = \left[-\frac{\partial G}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}\right]\Delta t$$

No μ or Z_t term → Risk free

④ $$\Delta\pi = r\pi\Delta t$$

Sub 3 & 4

BSM PDE: $$\frac{\partial G}{\partial t} + rS\frac{\partial G}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2} = rG$$

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2\Gamma = rG$$

FINM3003 Week 7

$$n(d_1) = \frac{\delta}{\delta d_1} N(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$$

$$n(d_2) = \frac{\delta}{\delta d_2} N(d_2) = \frac{1}{\sqrt{2\pi}} \underbrace{e^{-\frac{1}{2}d_2^2}}$$

$$e^{-(d_1-\sigma\sqrt{T})^2 \times \frac{1}{2}}$$

$$e^{-\frac{1}{2}d_1^2} \times \underbrace{e^{d_1\sigma\sqrt{T}} \times e^{-\frac{1}{2}\sigma^2 T}}$$

$$e^{\ln\frac{S}{K} + (r+\frac{1}{2}\sigma^2)T} \times e^{-\frac{1}{2}\sigma^2 T}$$

$$\frac{S}{K} e^{rT}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \times \frac{S}{K} e^{rT}$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\ln(S) - \ln(K)$$

$$\frac{1}{\sigma\sqrt{T}} \times \frac{1}{S}$$

Delta

Non-dividend

$$C = S \times N(d_1) - Ke^{-rT}N(d_2)$$

$$\Delta = \frac{\delta C}{\delta S} = \frac{\delta}{\delta S} S \times N(d_1) + S \times \frac{\delta}{\delta S} N(d_1) - Ke^{-rT}\frac{\delta}{\delta S} N(d_2) \sim PR$$

$$= N(d_1) + \underbrace{S\frac{\delta}{\delta S} d_1 \times \frac{\delta}{\delta S} N(d_1) - Ke^{-rT} \frac{\delta}{\delta S} d_2 \frac{\delta}{\delta S} N(d_2)} \sim CR$$

$$\cancel{S}\frac{1}{\cancel{\sqrt{2\pi}}} \cancel{e^{-\frac{1}{2}d_1^2}} \frac{1}{\cancel{S\sigma\sqrt{T}}} - \cancel{Ke^{-rT}} \frac{1}{\cancel{\sqrt{2\pi}}} \cancel{e^{-\frac{1}{2}d_1^2}} \cancel{\frac{S}{K}} \cancel{e^{rT}} \frac{1}{\cancel{S\sigma\sqrt{T}}}$$

$$= N(d_1)$$

Dividend

$$\Delta = e^{-qT} N(d_1)$$

FINM3003 Week 8

Exotics

Pricing
① Write payoff
② Split into sub derivatives

Gap

$$g_T = (S_T - G)\,\mathbb{I}_{[S_T > K]}$$
$$= (S_T - K)\,\mathbb{I}_{[S_T > K]} + (K - G)\,\mathbb{I}_{[S_T > K]}$$
$$= \text{Call} + \text{CoN}$$

Paylater
Euro Call − CON

Forward start

As you like

Lookback

Call Payoff at T
Long = $\max(S_T - K, 0)$
Short = $\min(K - S_T, 0) = -\max(S_T - K, 0)$

Put
Long = $\max(K - S_T, 0) = -\min(S_T - K)$
Short = $\min(S_T - K, 0) = -\max(K - S_T, 0)$

$$\text{CoN}_0 = Q e^{-rT} N(d_2)$$

$$\text{AoN}_0 = S e^{-rT} N(d_1)$$

$$\tilde{d}_1 = \frac{\ln\left(\frac{S_T}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

FINM3003 Week 9

Martingales & Risk Measures

Market Price of Risk λ

If asset 1 & 2 are dependant on the same source of risk,

$$\frac{\mu_1 - r}{\sigma_1} = \lambda = \frac{\mu_2 - r}{\sigma_2}$$

r = Instantaneous risk free rate

Risk Variables

For a security f, dependent variables $\theta_1, \dots, \theta_n$

SDE

$$\frac{df}{f} = \mu\, dt + \sum_{i=1}^{n} \sigma_i\, dZ_i$$

σ_i = Risk attributed to θ_i

EMM

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Change of Numeraire

~ Price of f in units of g

When

MPR = Volatility of g

the ratio $\frac{f}{g}$ is martingale $\forall$ securities f

E.g. if $\sigma_g = \lambda$

$$\lambda = \sigma_g = \frac{\mu_f - r}{\sigma_f} \rightarrow \mu_f = r + \sigma_g \sigma_f$$

→ To get $\frac{f}{g}$, mus us $\ln f - \ln g$, then $e^\wedge(\ln \frac{f}{g})$

① $$\frac{df}{f} = (r + \sigma_g \sigma_f)\, dt + \sigma_f\, dZ_t$$

② $$\frac{dg}{g} = (r + \sigma_g^2)\, dt + \sigma_g\, dZ_t$$

By Ito,

$$d(\ln f) = \left(r + \sigma_g \sigma_f - \frac{1}{2}\sigma_f^2\right) dt + \sigma_f\, dZ_t$$

$$d(\ln g) = \left(r + \frac{1}{2}\sigma_g^2\right) dt + \sigma_g\, dZ_t$$

then

$$d\left(\ln\left(\frac{f}{g}\right)\right) = d\ln(f) - d\ln(g)$$

$$= \left[\cancel{r} + \sigma_g\sigma_f - \frac{1}{2}\sigma_f^2 - \cancel{r} - \frac{1}{2}\sigma_g^2\right] dt + (\sigma_f - \sigma_g)\, dZ_t$$

$$= -\frac{1}{2}(\sigma_f - \sigma_g)^2\, dt + (\sigma_f - \sigma_g)\, dZ_t$$

Use Itô again with $G = e^x$

Then $\left(\frac{f}{g}\right)$ has no drift term ∴ Martingale

Risk Neutral Density / State Price Deflector

$$D(t, T) = \frac{D(T)}{D(t)}$$

$$D(t) = D(0, t)$$

Change of Probability Measure

$\mathbb{P}$: Real World

$\mathbb{Q}$: Risk neutral world

- Only changes the drift
- Martingale will remain martingale ?

FINM3003

Week 9

Martingales & Risk Measures

Proving Martingales

Expected Value proof

$E(S_t \mid \mathcal{F}_u) = S_u \qquad u < t$

Drift term proof

For discount stock process

$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)\,dt + \sigma\, dZ_t$

Martingale iff $\mu = r$

Proving Martingale tip

$E[e^{\sigma Z_t} \mid \mathcal{F}_s] = E[e^{\sigma Z_s + \sigma(Z_t - Z_s)} \mid \mathcal{F}_s]$ By II

$= e^{\sigma Z_s} \times E[e^{\sigma(Z_t - Z_s)} \cancel{\mid \mathcal{F}_s}]$

$= e^{\sigma Z_s} \times e^{[E(\sigma(Z_t - Z_s))}$

By delta approx? $\left. + \frac{1}{2}\mathrm{Var}(\sigma(Z_t - Z_s))\right]$

F_t = Forward price at time t, for delivery of V at time T

$\bar{E}\left[\frac{S_T}{B_T} \mid \mathcal{F}_0\right] = \frac{S_0}{B_0}$?

Show

$\bar{p} = \frac{u(1 - de^{-rT})}{u - d}$

Binomial Model

$\bar{E}[\tilde{B}_T] = \frac{e^{-rT}}{S_0} = p\left(\frac{B_0}{S_0 u}\right) + (1-p)\left(\frac{B_0}{S_0 d}\right)$

$\frac{e^{-rT}}{S_0} = p\left(\frac{1}{S_0 u}\right) + (1-p)\left(\frac{1}{S_0 d}\right)$

$e^{-rT} = \frac{p}{u} + \frac{1-p}{d}$

$= \frac{p}{u} + \frac{1}{d} - \frac{p}{d}$

$= \frac{1}{d} + p\left(\frac{1}{u} - \frac{1}{d}\right) \qquad \frac{d-u}{ud}$

$= \frac{1}{d} + p\left(\frac{d}{du} - \frac{u}{du}\right)$

$e^{-rT} - \frac{1}{d} = p\left(\frac{d-u}{du}\right)$

$du\,e^{-rT} - u = p(d-u)$

$-u(1 - de^{-rT}) = p(d-u)$

$p = \frac{-u(1 - de^{-rT})}{d-u}$

$= \frac{u(1 - de^{-rT})}{u-d}$

FINM3003 Week 10

Interest Rate Derivatives

Euro Bond Options

$$c = e^{-rT}\left[F_B N(d_1) - K N(d_2)\right] \qquad d_1 = \frac{\ln\frac{F_B}{K} + \frac{1}{2}\sigma_B^2 T}{\sigma_B\sqrt{T}}$$

$$p = e^{-rT}\left[K N(-d_2) - F_B N(-d_1)\right] \qquad d_2 = d_1 - \sigma_B\sqrt{T}$$

$$F_B = \frac{B_0 - I}{P(0,T)}$$

F_B = Forward bond price at $t=0$
σ_B = Volatility of forward bond price
B_0 = Bond price at $t=0$
I = PV of coupons paid over option life
☆ Par Value = \$100

$$P(0,T) = e^{-rT}$$

Where
B_0, F_B & K are cash price
Cash Price = Quoted Price + Accrued interest since last coupon date

$$\sigma_B = \frac{SD[\ln(\text{Bond price at option maturity})]}{\sqrt{T}}$$

$$E[(1+r)^2] = \sum \mathbb{P}(r=r_i) \times (1+r)^2$$

$$Var\left[a \times \tilde{S}(b)\right] = a^2\left(E^b[(1+i)^2] - E^{2b}[(1+i)]\right)$$

$$\delta = \ln(1+\bar{i})$$

$$E(\delta^2) = \sum \mathbb{P}(\bar{i} = \bar{i}_x) \times \ln^2(1+\bar{i}) \times \delta_{x}^2$$

For $\ln(1+\bar{i})$

$$Var(\ln(S(30))) \rightarrow \ln S(30) = \ln\left[\prod_{j=1}^{30}(1+\bar{i}_j)\right]$$
$$= \sum^{30} \ln(1+\bar{i}_j)$$
$$= 30 \ln(1+\bar{j}_1)$$
$$= 30\ \delta$$

$$= 30\ Var[\delta]$$

FINM3003 — Week 11

Term Structure → PV of ZCB with Par = $1

$$P(t,T) = e^{-\int_t^T r_s\, ds} = e^{-r(T-t)} \text{ for constant } r$$

↳ $R(t,T) = \frac{-1}{T-t} \ln[P(t,T)]$ → Calculated for period (t,T), the interest rate per $(T-t)$ cont compounded

Short Rate (Instantaneous) ↳ $= r_t(T)$

$$r_t = \lim_{\Delta t \to 0} r_t(t, t+\Delta_t)$$

$$= r_t(T) = -\frac{\delta}{\delta T} \ln P(t,T)\Big|_{T=t}$$

Bond

$$B = \sum_{i=1}^{n} c_i e^{-y t_i}$$

When Bond is a function of yield

C_t = CF of $C at time t
y = Bond yield

Duration

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-y t_i}$$

Modified

$$D = \frac{-1}{B} \frac{\delta B}{\delta y}$$

Convexity

$$C = \frac{1}{B} \frac{d^2 B}{dy^2} = \frac{1}{B} \sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}$$

For a small change in the yield: Δy

$$\Delta B \approx -BD\Delta y = \frac{dB}{dy}\Delta y = -\Delta y \sum_{i=1}^{n} t_i c_i e^{-y t_i}$$

$$\frac{\Delta B}{B} \approx -D\Delta y$$

Duration under Taylor Expansion

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}(\Delta y)^2 = -BD\Delta y + \frac{1}{2} BC\Delta y^2$$

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2} C(\Delta y)^2$$

Vasicek (Mean Reversion)

Risk neutral process

$$dr_t = a(b - r_t)dt + \sigma dZ_t$$

a = Rate of mean reversion
b = Mean level

Solution via Itô

$$r_t = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dZ_s$$

If $a > 0 \wedge \sigma = 0$

$$\lim_{t\to\infty} r_t = b$$

If $\sigma > 0$

$$\lim_{t\to\infty} r_t = b + \sigma \int_0^\infty e^{-at} dZ_t \sim N\left[b, \frac{\sigma^2}{2a}\right]$$

ZCB Derivation

Given

$$r_t = b + (r_0 - b)e^{-at} + y_t$$

$$y_t \sim N\left[0, \frac{\sigma^2(1 - e^{-2at})}{2a}\right]$$

Derive

$$P(t,T) = A(t,T)e^{-B(t,T)r_t} = \hat{E}\left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right] \quad P(T,T) = 1$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \quad B(T,T) = 0$$

$$A(t,T) = e^\wedge\left[\frac{(B(t,T) - T + t)(a^2 b - \frac{\sigma^2}{2})}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}\right] \quad A(T,T) = 1$$

$$R(t,T) = \frac{-1}{T-t} \ln[A(t,T)] + \frac{1}{T-t} B(t,T) r_t = \frac{1}{T-t} \ln \hat{E}\left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right]$$

CIR

Process

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\, dZ_t$$

Solution (Also use xe^{at})

$$r_T = r_0 e^{-aT} + b(1 - e^{-aT}) + \sigma \int_0^T e^{-as}\sqrt{r_s}\, dZ_s$$

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

$$A(t,T) = \left(\frac{2\gamma e^{(a+\gamma)(T-t)\frac{1}{2}}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)^{\frac{2ab}{\sigma^2}}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

Bond Pricing PDE (ZCB Process)

Terminal condition $F(T,x) = 1$

If $F(t, r_t) = P(t,T)$

$$xF(t,x) = \frac{\delta F(t,x)}{\delta t} + \mu(t, r_t)\frac{\delta F(t,x)}{\delta x} + \frac{1}{2}\sigma^2(t, r_t)\frac{\delta^2 F(t,x)}{\delta x^2} \quad t \geq 0$$

$$dP(t,T) = r_t P(t,T)dt + \sigma(t, r_t)\frac{\delta F(t,x)}{\delta x} dZ_t$$

FINM3003

$\kappa = a$
$\theta = b$

Solving Vasicek

Given Dynamics

$$dr_t = a(b - r_t)\,dt + \sigma\,dZ_t$$

Non-Itô Method

$$dr_t = ab\,dt - ar_t\,dt + \sigma\,dZ_t$$

$$dr_t + ar_t\,dt = ab\,dt + \sigma\,dZ_t$$

$$e^{at}dr_t + ae^{at}r_t\,dt = e^{at}ab\,dt + e^{at}\sigma\,dZ_t$$

$\times\, e^{at}$

Because we want the LHS to equal $d(\ldots r_t)$

$$d(e^{at}r_t) = e^{at}dr_t + ae^{at}r_t\,dt$$

$$d(e^{at}r_t) = e^{at}ab\,dt + e^{at}\sigma\,dZ_t$$

$$\int_s^T d(e^{as}r_s) = ab\int_0^T e^{as}\,ds + \sigma\int_0^T e^{as}\,dZ_s$$

$$\rightarrow \int_0^T d(e^{at}r_t) = e^{aT}r_T - e^{at}r_t$$

$$e^{aT}r_T - r_0 = ab\,\frac{1}{a}(e^{aT}-1) + \sigma\int_0^T e^{as}\,dZ_s$$

$$e^{aT}r_T = r_0 + b(e^{aT}-1) + \sigma\int_0^T e^{as}\,dZ_s$$

$$r_T = r_0e^{-aT} + be^{-aT}(e^{aT}-1) + \sigma\int_0^T e^{-aT}e^{as}\,dZ_s$$

$$= r_0e^{-aT} + b(1-e^{-aT}) + \sigma\int_0^T e^{-a(T-s)}\,dZ_s$$

Vasicek Duration

Duration

$$D = \frac{\sum t_i C_i e^{-yt_i}}{B} = t_i \sum\left(\frac{C_i e^{-yt_i}}{B}\right) = -\frac{1}{B}\frac{\delta B}{\delta y}$$

Consider a small change in the yield: Δy

$$\frac{\Delta B}{\Delta y} \approx \frac{\delta B}{\delta y}$$

$$\Delta B \approx \frac{\delta B}{\delta y}\times\Delta y$$

$$\Delta B \approx -\Delta y \sum C_i t_i e^{-yt_i}$$

Apply to Vasicek, for a bond with price: Q & yield r

$$\frac{\delta Q}{\delta r} = -\hat{D}Q$$

When

$$Q = ZCB$$

$$\frac{\delta Q}{\delta r} = \frac{\delta P}{\delta r} = \frac{\delta}{\delta r}A(t,T)\,e^{-B(t,T)r_t}$$

$$= -B(t,T)\,P(t,T) \qquad = -\hat{D}Q$$

$$B(t,T)\,P(t,T) = \hat{D}\,P(t,T)$$

$$\hat{D} = B(t,T)$$

FINM3003 Lecture Workings

Week 2

PCP Euro

$C_t - P_t = S_t - Ke^{-r(T-t)}$

Suppose

$C_t - P_t < S_t - Ke^{-r(T-t)}$

$C_t + Ke^{-r(T-t)} < P_t + S_t$

$$\underbrace{C_t + Ke^{-rT}}_{\text{Sell now}} < \underbrace{P_t - S_t}_{\text{Buy now}}$$

At $t=0$
- Sell put
- Sell share
- Buy call
- Invest (Ke^{-rT}) in bonds

	Value	
Portfolio	$t=0$	$t=T$
Put	$-P_t$	$-\max(K-S_T, 0)$
Share	$-S_t$	$-S_T$
Call	C_t	$\max(S_T-K, 0)$
Cash	Ke^{-rT}	K

Value at T (Not Payoff)

$$\underbrace{-\max(K-S_T, 0)}_{\text{Put}} \underbrace{- S_T}_{\text{Share}} + \underbrace{\max(S_T-K, 0)}_{\text{Call}} + \underbrace{K}_{\text{Invest}}$$

If $S_T > K$

$0 - S_T + S_T - K + K = 0$

If $S_T < K$

$-(K-S_T) - S_T + K$

$-K + S_T - S_T + K = 0$

No liability at $t=T$
But positive income at $t=0$

Euro Bounds

Euro Call

$\max(S_T - Ke^{-rT}, 0) \leq C_t \leq S_t$

If $C_t > S_t$ at $t=0$
- Sell call
- Buy Stock

$\rightarrow +C_t - S_t > 0$

At $t=T$

$\pi_T = -\max(S_T-K, 0) + S_T$

$= \min(S_T, K)$ → See exam study for min/max conversion

If $S_T > K \rightarrow \pi_T = -(S_T-K) + S_T = K$

$S_T < K \rightarrow \pi_T = 0 + S_T = S_T$

Since

$\min(S_T, K) \geq 0$

$\pi_T \geq 0$ → Arbitrage profit

American Early Exercise

Non-div

If

$\pi_0 = 1$ US call

At $t=t$

$S_t > K$ → Tempted to exercise

Portfolio 1
- Borrow K & exercise

$\pi_T^1 = S_T - Ke^{r(T-t)}$

where possibly $\pi_T < 0$

$\therefore \pi^1 \leq \pi^2$

Portfolio 2
- Hold until T

$\pi_T^2 = \max(S_T - K, 0)$

Week 5

Solution

$y_t = e^{\mu t + \sigma Z_t}$

Find SDE

Underlying process $= Z_t$

Shortcuts

If

$$S_T = S_t e^{[(r+\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)]}$$

then

$$S_T^2 = S_t^2 e^{2\times[(r+\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)]}$$

$$W_T - W_t = \sqrt{T-t}\, Z = Z_{T-t}$$

Practice Shwy

Po using

Itô with $(T-t)$

If

$$x = -(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3$$

For

$\frac{dx}{dt} \to$ Expand First

$$= \frac{dx}{dt}\left(-Tr_t + tr_t + \frac{\sigma^2}{6}(T-t)^3\right)$$

$$= r_t + \left(\frac{\sigma^2}{2}(T-t)^2 \times -1\right)$$

$$= r_t - \frac{\sigma^2}{2}(T-t)^2$$

100BP = 1%

For duration $\hat{D}$,

A "1"++ on the short rate decreases the bond price by $\hat{D}$%

Check

- Dividend stock is result at rate q

  $e^{qT} S$?

- Units

  σ = p.a

  $r \wedge q$ p.a.c.c

No arbitrage model → Drift term is function of time

$\theta(t)$ = The average direction r moves in time t

Sample Itô

$$dS_t = \mu S_t\, dt + \sigma S_t\, dZ_t$$

$$\frac{dS_t^n}{S_t} = \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)dt + n\sigma\, dZ_t$$

$$S_T^n = S_t^n e^{(\mu_n - \frac{\sigma_n^2}{2})(T-t) + \sigma_n(Z_T - Z_t)} \qquad \sigma_n = n\sigma$$

$$e^{-r(T-t)}\hat{E}(S_t^n)$$

Week 8 Workshop

Q1) Δ of euro gap call

S_T G K $\qquad$ S_T K G

$$C_T = (S_T - G)\,\mathbb{1}_{[S_T > K]}$$

$$= S_T - K + K - G\,\mathbb{1}$$

$$= (S_T - K)\,\mathbb{1}_{[S_T > K]} + (K - G)\,\mathbb{1}_{[S_T > K]}$$

$$= \text{Call} + \text{CoN}$$

$$\Delta_C = e^{-qT} N(d_1)$$

$$\Delta_{CoN} = \frac{\delta}{\delta S}\,\text{CoN}$$

$$\text{CoN}_T = y\,\mathbb{1}_{[S_T > K]}$$

$$E(\text{CoN}_T) = y \times N(d_2)$$

$$E(\text{CoN}_0) = e^{-rT} y N(d_2)$$

$$\frac{\delta}{\delta S} e^{-rT} y N(d_2) = e^{-rT} y \frac{\delta}{\delta S} N(d_2) \times \frac{\delta}{\delta S} d_2$$

$$\left| \frac{\delta}{\delta S} N(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \times \frac{S}{K} e^{rT} \right.$$

$$\left| \frac{\delta}{\delta S} d_2 = \frac{\delta}{\delta S} \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right.$$

$$= \frac{1}{S\sigma\sqrt{T}}$$

$$\frac{\delta}{\delta S} e^{-rT} y N(d_2) = \cancel{e^{-rT}}\, y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \frac{\cancel{S}}{K} \cancel{e^{rT}} \frac{1}{\cancel{S}\sigma\sqrt{T}}$$

$$= y e^{-\frac{1}{2}d_1^2} \times \frac{1}{\sigma\sqrt{2\pi T}}$$

Week 7 Workshop

Q2) Solution of BSM Euro Call

a) $Z \sim N(0,1)$

$$\left.\begin{array}{l} a>0 \\ b>0 \\ c>0 \end{array}\right\} \text{Constants}$$

Calculate in terms of Φ:

$$E[\max(ae^{bz}-c,0)]$$

$$\int_{-\infty}^{\infty} \max[ae^{bz}-c,0] \times f(z)\,dz \quad = \quad ae^{\frac{b^2}{2}} \times \Phi\left[b+\frac{1}{2}\ln\frac{a}{c}\right]-c\,\Phi\left[\frac{1}{b}\ln\frac{a}{c}\right]$$

Week 7 Workshop

Q1) $S_t = 40$ Euro Call
$\sigma = 0.3$ $K = 40$
$r = 0.08$ $T = 91/365$
$q = 0$
Find $C_t, \Delta_t, \Gamma_t, \Theta_t$

$d_1 = 0.208048$
$d_2 = 0.058253$
$C_t = 2.780398$
$\Delta_t = 0.582404$
$\Gamma_t = 0.066156$
$\Theta_t = -6.3191$

Q2) Solution of BSM Euro Call

a) $Z \sim N(0,1)$

$\left.\begin{array}{l} a > 0 \\ b > 0 \\ c > 0 \end{array}\right\}$ Constants

Calculate in terms of Φ:

$$E[\max(ae^{bZ} - c, 0)]$$

Assume stock price process follows GBM(μ, σ)

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t$$

Risk neutral distribution $\hat{p}$ requires

$$\ln S_T \sim \text{Norm}\left[\ln S_0 + (r + \tfrac{1}{2}\sigma^2)T \, , \, \sigma^2 T\right]$$

Value of Euro Call

$$c = e^{-rT} \times E[\max(S_T - K, 0)]$$

Let

$$\left.\begin{array}{l} u = \ln S_0 + (r + \frac{1}{2}\sigma^2)T \\ v = \sigma\sqrt{T} \end{array}\right\} \rightarrow \ln S_T \sim \text{Norm}[u, v^2]$$

CDF of $\ln S_T$

$$\mathbb{P}(\ln S_T \leq w) = \int_{-\infty}^{w} \frac{1}{v \times \sqrt{2\pi}} e^{\frac{-(y-u)^2}{2v^2}} dy$$

Let $s = e^w$

$$\mathbb{P}(S_T \leq s) = \frac{1}{v \times \sqrt{2\pi}} \times \int_{-\infty}^{\ln s} e^{\frac{-(y-u)^2}{2v^2}} dy$$

PDF of S_T

$$\mathbb{P}(S_T = s) = f_{S_T}(s)$$
$$= \frac{1}{s v \sqrt{2\pi}} e^{\frac{-(\ln s - u)^2}{2v^2}} \qquad s \geq 0$$

Let

$w = \ln s$

$s = e^w$

$\frac{ds}{s} = dw$

Expectation

$$\hat{E}[\max(S_T - K, 0)] = \int_K^{\infty} (s - K) \times f_{S_T}(s) \, ds$$
$$= \frac{1}{v\sqrt{2\pi}} \int_K^{\infty} (s - K) \frac{1}{s} e^{-\frac{(\ln s - u)^2}{2v^2}} ds$$
$$= \frac{1}{v\sqrt{2\pi}} \int_{\ln K}^{\infty} (e^w - K) e^{-\frac{(w-u)^2}{2v^2}} dw$$

Very important
Continue later

Week 7
Hedging
Delta

$\Delta C = N(d_1)\,\Delta S$

7.11) $S_0 = 100$

$C_0 = 10$

$\Delta = 0.6$

Sell call on 2000 shares

Hedge:

Replicating Portfolio

Hold Δ_t shares & $B_t = C_t - \Delta S_t$ Bonds at t

$\Delta_p \approx N(d_1) - 1$ $\qquad \Delta_c = N(d_1)$?

$\approx \Delta_c - 1$

Gamma

$$\Gamma_c = \Gamma_p = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2} = \frac{1}{\sigma\sqrt{2\pi}} \times \frac{e^{-\frac{1}{2}d_1^2}}{S\sqrt{T}} > 0$$

Week 6

6.21) $S_0 = 30$

$u = 1.2$
$d = 0.8$
$q = 0.03$
$r = 0.05$

6 month

Find put value

$K = 28$

CRR Model

$$P = \frac{e^{(r-q)\Delta t} - d}{u - d} = 0.5251$$

Workshop

Q1) $S_0 = 100$

$K = 95$
$\sigma = 0.3$
$r = 0.08$
$q = 0.03$
$T = 0.75$

BSM euro call

$$d_1 = \frac{\ln\left(\frac{100}{95}\right) + \left(0.08 - 0.03 + \frac{1}{2}0.3^2\right) \times 0.75}{0.3 \times \sqrt{0.75}} = 0.47$$

$$d_2 = d_1 - 0.3 \times \sqrt{0.75} = 0.21$$

$$C_0 = 14.3863$$

9 month Futures price

$$f_0 = S_0 e^{(r-q) \times T} = 103.8212$$

BSM Euro Call

T = 9 months
$K = 95$
Asset = Futures, with same maturity

C_0 = same as before

Q2) Replicate strategy

Target: Euro call
with: Futures & Bond
2 period CRR

Week 5 Workshop

Q1) $dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dZ_t$

$Z_t \sim SBM$
$0 \leq u < t$
$0 < b < c$
$0 < a$
$S_u = a$

Find $\mathbb{P}(b \leq S_t \leq c)$

Given

$dS_t = \mu S_t \, dt + \sigma S_t \, dZ_t$

Solution

$\ln\left(\frac{S_t}{S_0}\right) = [\mu - \frac{1}{2}\sigma^2] t + \sigma Z_t$

For $0 \leq u < t$

$\ln\left(\frac{S_t}{S_u}\right) = [\mu - \frac{1}{2}\sigma^2] \times (t-u) + \sigma(Z_t - Z_u)$

Note that

1) $Z_t \perp\!\!\!\perp Z_u$
2) $Z_t - Z_u \sim N(0, (t-u))$

So

1) $\ln\left(\frac{S_t}{S_u}\right) \perp\!\!\!\perp \ln S_u$
2) $\ln\left(\frac{S_t}{S_u}\right) = \ln\left(\frac{S_{t-u}}{S_0}\right) \sim N\left((\mu + \frac{1}{2}\sigma^2)(t-u), \sigma_t^2 (t-u)\right)$

⋮

Q2) $t: \forall \ 0 \leq t \leq T$

① $S_T = S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}$

② $F_t = S_t e^{r(T-t)}$

Prove that

③ $C_0 = e^{-r(T-t)} \times [F_t N(d_1(t)) - K N(d_2(t))]$

From ②

$F_t = S_t e^{r(T-t)}$

$S_T = F_t e^{r(T-t)}$

Sub into BS call

$C_0 = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$

$= F_t e^{r(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$

Q3) In BSM

$X = \max[K - (S_T)^2, 0]$

$K =$ Strike
$T =$ Exercise date
$0 \leq t \leq T$

$S_T = S_t e^{(r + \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}$

$W_{t, t>0} \sim$ SBM under RNM

Find value

$F_t = e^{-r(T-t)} \times E[\max(K - S_T^2, 0) \mid S_t]$

$= e^{-r(T-t)} \times [K - S_T^2] \times I_{K > S_T^2}]$

⋮

531) Cash or Nothing Call

$$CON_0 = e^{-rt} \times Q \times N(d_2)$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Find Value

$$CON_0 = PV\left[\hat{E}\left(Q \times I_{(K<S_t)}\right)\right]$$

$$= e^{-rt} \times \hat{E}\left[Q \times I_{(K<S_t)}\right]$$

$$= e^{-rt} \times Q \times \hat{P}(S_t > K)$$

where

$$\ln S_t \sim N\left[\ln S_0 + \left(r + \tfrac{1}{2}\sigma^2\right)T,\ \sigma^2 T\right]$$

$\therefore$

$$\hat{P}(S_T > K) = \hat{P}(\ln S_T > \ln K)$$

Normalise, let

$$z = \left[S_T - \left(\ln S_0 + \left(r + \tfrac{1}{2}\sigma^2\right)T\right)\right] \times \frac{1}{\sigma\sqrt{T}}$$

$$z \sim N(0,1)$$

$$\hat{P}\left(z > \frac{\ln K - \left(\ln S_0 + \left(r + \frac{1}{2}\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right) = \Phi(d_2)$$

Week 5

Lectures

5(2) Stock $\sim GBM(\mu, \sigma)$

Find SDE of forwards price

Forward price

$$F_t = S_0 e^{r\Delta t}$$

SDE for GBM

$$dS_t = \mu S_t\, dt + \sigma S_t\, dZ_t \quad \rightarrow \text{Ito with} \quad \begin{aligned} a(S,t) &= \mu S_t \\ b(S,t) &= \sigma S_t \end{aligned}$$

From F_t,

$$\frac{\partial F}{\partial S} = e^{r\Delta t}$$

$$\frac{\partial^2 F}{\partial S^2} = 0$$

$$\frac{\partial F}{\partial T} = -r S e^{r\Delta t} = -r F_t$$

Ito's lemma

$$dF_t = \Big(\underbrace{\mu S_t}_{a} \times \underbrace{e^{r\Delta t}} + \underbrace{-rF_t} + \underbrace{\tfrac{1}{2}\sigma^2 S_t^2 \times 0} \Big) dt + \underbrace{\sigma S_t}_{b} \underbrace{e^{r\Delta t}} \; dZ_t$$

5(3) $\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t$

By EMM

$$\ln S_t \sim N\left(\ln S_0 + (r + \tfrac{1}{2}\sigma^2)T, \sigma^2 T\right)$$

Value of call

$$C_0 = e^{-rT} \times \hat{E}[\max(0, S_T - K)]$$

Let

$$u = \ln S_0 + (r + \tfrac{1}{2}\sigma^2)T$$

$$v = \sigma\sqrt{T}$$

Then

$$\ln S_T \sim N(u, v^2) \qquad \Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$$

Then

$$\mathbb{P}(\ln S_T \leq w) = \int_{-\infty}^{w} \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2} \times \left(\frac{y}{v} - u\right)^2} dy$$

$$= \frac{1}{v \times \sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-u)^2}{2v^2}} dy$$

Exponentiate both

Let $s = e^w \rightarrow w = \ln(s)$

$$\mathbb{P}(\ln S_t \leq \ln(s)) = \frac{1}{v\sqrt{2\pi}} \int_{-\infty}^{\ln s} e^{-\frac{(y-u)^2}{2v^2}} dy$$

Exponentiate to get CDF

$$\mathbb{P}(S_t \leq s) = \frac{1}{v\sqrt{2\pi}} \int_{-\infty}^{\ln s} e^{-\frac{(y-u)^2}{2v^2}} dy$$

to get PDF, differentiate

$$\mathbb{P}(S_t = s) = f_{S_t}(s) = \frac{1}{s} \times \frac{1}{v\sqrt{2\pi}} e^{-\frac{(\ln s - u)^2}{2v^2}} \qquad s \geq 0$$

Let

$$w = \ln s$$

$$s = e^w$$

$$\frac{ds}{s} = dw$$

⋮

Week 4 Workshop

Q1) $\{B_t : t > 0\}$ $B \perp\!\!\!\perp t$
$\{W_t : t > 0\}$

$X_t = \frac{B_t + W_t}{\sqrt{2}}$

Show that X_t is BM

3 properties

i) $X_t = \frac{B_t}{\sqrt{2}} + \frac{W_t}{\sqrt{2}} \rightarrow X_0 = 0 \rightarrow X_t \sim N(0, \frac{t}{2} + \frac{t}{2}) \rightarrow \sim N(0, t)$

ii) Increment process for $s < t$

$X_t - X_s = \frac{B_t - W_t}{\sqrt{2}} + \frac{B_s - W_s}{\sqrt{2}}$

$= \frac{B_t - B_s}{\sqrt{2}} + \frac{W_t - W_s}{\sqrt{2}}$

$\rightarrow$ $B_s \perp\!\!\!\perp B_t - B_s$
$W_s \perp\!\!\!\perp W_t - W_s$ $\rightarrow$ Scaling $\sqrt{2}$ does not influence independence

$\hookrightarrow \therefore X_s \perp\!\!\!\perp X_t - X_s$

iii) Show that increments $\sim N(0, \sigma^2)$

$$\mathrm{Var}(X_t - X_s) = \mathrm{Var}\left(\frac{B_t + W_t}{\sqrt{2}} - \frac{B_s + W_s}{\sqrt{2}}\right)$$

$$\frac{1}{2}\mathrm{Var}\left((B_t + W_t) - (B_s + W_s)\right)$$ Var scale constant

$$\frac{1}{2}\left(\mathrm{Var}(B_t - B_s) + \mathrm{Var}(W_t - W_s) - 2\mathrm{Cov}(B_t - B_s, W_t - W_s)\right)$$ Var of sum of RV

$$\frac{1}{2}\left(\mathrm{Var}(B_{t-s}) + \mathrm{Var}(W_{t-s}) - 2\mathrm{Cov}(B_{t-s}, W_{t-s})\right)$$

$$\frac{1}{2}\left((t-s) + (t-s) - 2(0)\right)$$ $B_t \perp\!\!\!\perp W_t$

$$= t - s$$

$\therefore X_t - X_s \sim N(0, t-s)$

X_t is BM

Q3) $S_t \rightarrow dS_t = \mu dt + \sigma dZ_t$
$\{Z_t : t > 0\}$
$t \in \{1, 2, 3\}$
$\mu = 4$ $\sigma = 3$
$t \in \{4, 5, 6\}$
$\mu = 3$ $\sigma = 5$
$S_0 = 3$

Find dist of S_6

$S_3 - S_0 \sim N(4 \times 3, 3^2 \times 3)$
$\sim N(12, 27)$
$S_6 - S_3 \sim N(9, 75)$

$S_6 - S_0 = (S_6 - S_3) + (S_3 - S_0)$ sum of 2 RV $\sim N$
$\sim N(21, 102)$

where

S_0 is strict

$S_6 \sim N(24, 102)$

W5 Workshop

Q 3) $X_T = \max(K - (S_T)^2, 0)$

$$S_T = S_t e^{[(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)]}$$

$$X_0 = PV\left[\hat{E}\left(\max(K - S_T^2, 0)\right)\right]$$

$$= e^{-r(T-t)} \times \hat{E}\left((K - S_T^2) \times \mathbb{1}_{[K > S_T^2]}\right)$$

$$\times \mathbb{1}_{[\sqrt{K} > S_T]}$$

$$\Big| \hat{E}\left[\left(K - S_t^2 e^{2((r - \frac{\sigma^2}{2})(T-t) + 2\sigma(W_T - W_t))}\right) \times \mathbb{1}_{[\sqrt{K} > S_t e^{\cdots}]}\right.$$

$$\times \mathbb{1}_{[\frac{\sqrt{K}}{S_t} > e^{\cdots}]}$$

W4 Workshop

Q1) $\{B_t : t \geq 0\}$, $\{W_t : t \geq 0\}$ $\perp\!\!\!\perp$ BM

Show that

$$X_t = \frac{(B_t + W_t)}{\sqrt{2}} \sim BM$$

① Check that $X_0 = 0$ & Distribution

$E(B_0) = 0$

$E(W_0) = 0$

$$X_t = \frac{B_t}{\sqrt{2}} + \frac{W_t}{\sqrt{2}}$$

$$E(X_0) = \frac{1}{\sqrt{2}} E(B_0) + \frac{1}{\sqrt{2}} E(W_0) = 0$$

$Var(B_t) = t$

$Var(W_t) = t$

$$Var(X_t) = Var\left(\frac{B_t}{\sqrt{2}}\right) + Var\left(\frac{W_t}{\sqrt{2}}\right) = \frac{1}{2} Var(B_t) + \frac{1}{2} Var(W_t) = t$$

$$X_t \sim N(0, t)$$

② $\perp\!\!\!\perp$ increments

$$X_t - X_s = \frac{1}{\sqrt{2}} \underbrace{(B_t - B_s)}_{\perp\!\!\!\perp B_s} + \frac{1}{\sqrt{2}} \underbrace{(W_t - W_s)}_{\perp\!\!\!\perp W_s} \quad \forall s < t$$

③ Increments $\sim N$

$E(X_t - X_s) = 0$

$$Var(X_t - X_s) = \frac{1}{2} Var\left[(B_t + W_t) - (B_s - W_s)\right]$$

$\vdots$ Var .. Var .. Cov ..

$$= \frac{1}{2}\left((t-s) + (t-s)\right) = t - s$$

$$X_t - X_s \sim N(0, t-s)$$

Q2) $0 < s < t < u$

$E(B_s B_t B_u)$

$Var(X) = E(X^2) - E^2(X)$

$E(X^2) = Var(X) - E^2(X)$

Let

	$E(\cdot)$	Var	$E(\cdot^2)$
$X = B_s$	0	s	s
$Y = B_t - B_s$	0	$t-s$	$t-s$
$Z = B_u - B_t$	0	$u-t$	$u-t$

$$E(B_s B_t B_u) = E\left[X \times (Y + X) \times (Z + Y + Z)\right]$$

$$= E(X^3) + 2E(X^2) + E(Y)E(Z) + E(X) + E(Y^2) + E(X^2)E(Z) + E(X)E(Y)E(Z)$$

$$= 0$$

Q3) If S_t follows process

$$dS_t = \mu\, dt + \sigma\, dZ_t$$

$S_0 \_\_\_ S_3 \_\_\_ S_6$

$\left.\begin{matrix}\mu = 4 \\ \sigma = 3\end{matrix}\right\}$ $\forall t = 1, 2, 3$ $\} \rightarrow S_3 - S_0 \sim N(\mu_{3-0} = 3 \times 4,\ \sigma^2_{3-0} = 3^2 \times 3) \sim N(12, 27)$

$\left.\begin{matrix}\mu = 3 \\ \sigma = 5\end{matrix}\right\}$ $\forall t = 4, 5, 6$ $\} \rightarrow S_6 - S_3 \sim N(\mu_{6-3} = 3 \times 3,\ \sigma^2_{6-3} = 5^2 \times 3) \sim N(9, 75)$

$S_0 = 3$

Find distribution of S_6

$$S_6 - S_0 = S_6 - S_3 + S_3 - S_0 = N(12, 27) + N(9, 75) = \sim N(21, 102)$$

$$\therefore S_6 = S_6 - S_0 + S_0$$

$| S_0 \sim N(0,$

Week 3 Workshop

Q2) Euro Call

$T = 3/12$
$S_0 = 50$
$\sigma = 0.4$
$r_f = 0.1$
$K = 50$

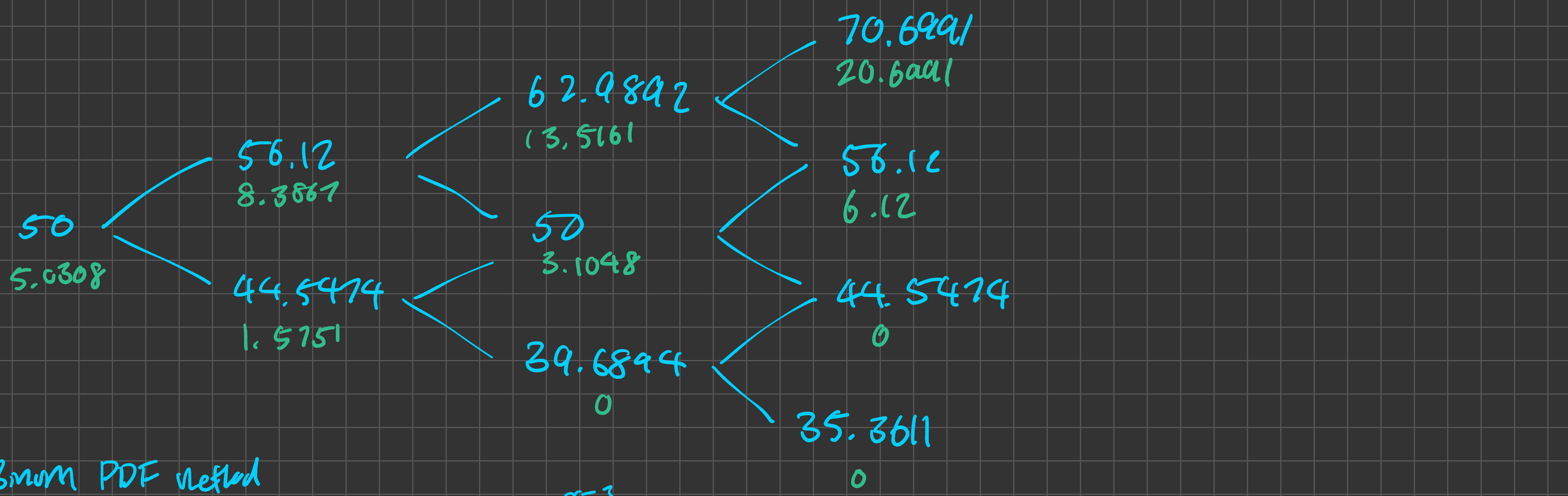
a) CRR $n=3$

Under $\hat{P}$

$u = e^{0.4\sqrt{\frac{0.25}{3}}} = 1.1224$

$d = \frac{1}{u} = 0.8909$

$p = \frac{e^{rT} - d}{u - d} = 0.5073$



Binom PDF method

$C_0 = \sum 20.6991 \times f_{Binom}(x=3, n=3, p=0.5073) = 2.7021$

$6.12 \times f_B(x=2, n=3) = 2.3280$

c) Self replicating
use formula for M_0 & B_0