Spectral Analysis and Structural Dynamics: The Role of Eigenvalues in Spacecraft Launch Stability

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Abstract*—The structural integrity and stability of launch vehicles during the atmospheric ascent phase are critical factors in the success of space exploration missions. Due to their necessarily low structural mass and high flexibility, large boosters are susceptible to complex dynamic phenomena, including aeroelastic coupling and self-excited vibrations such as "Pogo" oscillations. This paper analyzes NASA's methodology in employing spectral analysis—specifically the derivation of eigenvalues and eigenvectors—to predict and reduce the risks. Eigenvalues are used to identify the natural frequencies and damping ratios of the vehicle, while eigenvectors define the physical mode shapes of the structure. Through the integration of these parameters into high-fidelity linear state-space models, NASA engineers can simulate the interaction between rigid-body dynamics, propellant slosh, and structural vibrations. This study examines historical case studies, such as the Pogo instability encountered during the Apollo 13 mission, and modern applications in the Ares I-X pathfinder vehicle, where the Eigensystem Realization Algorithm (ERA) was employed for in-flight modal identification. Furthermore, the paper discusses the use of the NASTRAN (NASA Structural Analysis) software and the implementation of ground vibration testing (GVT) to validate Finite Element Models (FEM). The results demonstrate that precise spectral characterization allows for the strategic separation of structural and propulsion frequencies, the optimization of damping mechanisms, and the insurance of closed-loop stability. Ultimately, the application of these algebraic frameworks transforms complex physical oscillations into predictable mathematical models, ensuring the safety of both crew and payload in extreme flight environments.

***Keywords*— Ares I-X, Aeroelasticity, Eigenvalues, Launch Vehicle Stability, Modal Analysis, NASA, Pogo Oscillation, Structural Dynamics.**

I. INTRODUCTION

Space exploration missions has become an essential aspect for humanity to find new things in the observable universe. Spacecraft such as rocket s serve as the primary means of transporting payloads, scientific instruments, and humans beyond Earth's atmosphere. During atmospheric ascent, these vehicles are subjected to extreme dynamic loads, including aerodynamic pressure, thrust fluctuations, and engine-induced vibrations. These loads can trigger complex and potentially dangerous interactions among the vehicle structure, the propellant system, and the flight control architecture.

One of the most significant challenges in vehicle launching is the phenomenon known as *pogo oscillation*. Pogo oscillations are longitudinal vibrations caused by a feedback loop between the elastic deformation of the launch vehicle structure and oscillatory behavior in the propulsion system. When the natural frequency of the vehicle structure coincides with characteristic frequencies of the propulsion system, resonance may develop. Such resonance can lead to excessive vibration levels capable of damaging payloads, degrading control performance, causing premature engine shutdown, or, in extreme cases, resulting in total vehicle failure.

To prevent such failures, NASA use mathematical models to analyze the physical behavior of launch vehicles. The complex dynamics of the vehicle are represented using linearized state-space models, where the system behavior is represented by matrices whose spectral properties determine stability. They use eigenvalues and eigenvectors to determine the relation and analyze the matrices. Eigenvalues are directly related to the natural frequencies and damping characteristics of the vehicle, while the corresponding eigenvectors describe the *mode shapes*, which represent the physical deformation patterns of the structure during vibration.

Finally, this paper demonstrates how linear algebraic principles, particularly spectral analysis, are applied to assess and evaluate launch vehicle stability through both theoretical and computational approaches. The study focuses on the formulation of state-space models derived from structural dynamics equations and the use of eigenvalue distributions to determine system stability based on the Hurwitz criterion. In addition, a frequency separation analysis is implemented to assess the risk of pogo oscillations by comparing structural modal frequencies with propulsion system excitation frequencies. Through numerical evaluation and visualization in the complex plane, this paper illustrates how eigenvalues and eigenvectors serve as practical tools for identifying unstable modes, assessing damping characteristics, and supporting engineering decisions in launch vehicle stability analysis.

II. Theoretical Basis

A. Eigenvalues and Eigenvectors

Let A be an n × n matrix. A nonzero vector $x \in \mathbb{R}^n$ is called an eigenvector of matrix A if it satisfies the equation

$$\boxed{\mathbf{Ax} = \lambda\mathbf{x}}$$

Fig 2.1 eigenvector equation
Source:
https://informatika.stei.itb.ac.id/~rinaldi.munir/AljabarGeometri/2025-2026/Algeo-19-Nilai-Eigen-dan-Vektor-Eigen-Bagian1-2025.pdf

Where λ is a scalar known as the eigenvalue corresponding to the eigenvector x.

An eigenvector x is a column vector that, when multiplied by an n × n matrix A, results in a new vector that is a scalar multiple of the original vector. Geometrically, the operation $Ax = \lambda x$ implies that the vector x is stretched or compressed by a factor of λ. If $\lambda < 0$, the vector's direction is reversed.

B. Equations of Motion

Launch vehicle structures are modeled as multi-degree-of-freedom (MDOF) systems due to their inherent flexibility and complex mass distribution. The fundamental equation of the dynamic behavior of such structure is expressed as

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

where M is the mass matrix that represents the distribution of the vehicle mass, C is the damping matrix accounting for energy dissipation mechanisms, K is the stiffness matrix represents the structural rigidity. The $\ddot{x}, \dot{x}, and\ x$ are the acceleration, velocity, and the displacement vectors, respectively.

This matrix formulation transforms the continuous elastic structure into a discrete system that can be analyzed using linear algebra. For launch vehicle, these matrices are usually large and sparse. The coupling between structural flexibility, rigid-body motion, and propellant dynamics is captured through-off diagonal terms in these matrices.

C. Generalized Eigenvalue Problem & Modal Analysis

To identify the characteristics of the vehicle structure, we analyze the free vibration case by setting the F(t) = 0 and temporarily neglecting damping (C = 0). Then the equation of motion reduces to the generalized eigenvalue problem:

$$Kv = \lambda Mv$$

or equivalently

$$(K - \lambda M)v = 0$$

This system achieve non-trivial solutions only when

$$det(K - \lambda M) = 0$$

The solutions to this characteristics equation yield:

1. Eigenvalues(λ). These represent the square of natural frequencies ($\lambda = \omega^2$). Each eigenvalue corresponds to a distinct vibration mode of the structure
2. Eigenvectors(v). These define the mode shapes. Each eigenvectors describe how different paths of the vehicle move relative to each other when oscillating in certain mode.

The physical interpretation is crucial: when the vehicle is oscillating at a frequency matching an eigenvalue, the structure resonates and deforms according to the corresponding eigenvector pattern.

D. State-Space Representation

To integrate structural dynamics with flight control systems, the second order equation of motion is converted into a first-order system by defining an augmented state vector:

$$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

The system then expressed in the standard state-space form:

$$\dot{z} = Az + Bu$$

Where A is the system matrix that encapsulates rigid-body dynamics and structural flexibility. B is the input matrix, and u represents control inputs. The system matrix A has the structure:

$$A = \begin{bmatrix} 0 & 1 \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

This formula enables the application of linear systems analysis techniques. The eigenvalues of A directly determine system stability, while the eigenvectors provide information about the observability and controllability of different modes.

E. Stability Criteria

The stability of the coupled system is determined by the eigenvalues of the system matrix A. A linear time invariant system is asymptotically stable if and only if all eigenvalues satisfy the Hurwitz criterion :

$$Re(\lambda_i) < 0, \forall i$$

When eigenvalues are complex ($\lambda = \sigma \pm i\omega$), they represent oscillatory modes where the real part (σ)

determines the damping and the imaginary part (ω) represents the oscillation frequency in rad/s.

The damping ratio for each mode can be extracted directly from the eigenvalue such as,

$$\zeta = -\frac{\mathrm{Re}(\lambda)}{|\lambda|}$$

For launch vehicle control design, engineers must ensure adequate stability margins. Marginal stability ($Re(\lambda) = 0$) is unacceptable as it leads to sustained oscillations that can be excited by disturbances. Typical design requirements mandate damping ratios $\zeta > 0.03$ for structural modes to provide robustness against parameter uncertainties and modeling errors. For aeroelastic stability in the presence of aerodynamic and thrust forces, a sufficient condition is that the augmented flexibility matrix A_f remains Hurwitz,

$$Re[\lambda(\mathrm{A_f})] < 0$$

This condition predicts aeroelastic stability of free vibration as a function of dynamic pressure and velocity, independent of the remaining dynamics and attitude control system.

F. Modal Decomposition

The state-space formulation enables modal decomposition, where the complete system response is expressed as a superposition of individual modal contributions,

$$x(t) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \eta_{\mathrm{i}}(t)\mathrm{v_i}$$

where $\eta_{\mathrm{i}}(t)$ are the generalized modal coordinates, also called modal amplitudes. This transformation decouples the system equations when modes are orthogonal, allowing each mode to be analyzed independently:

$$\ddot{\eta}_{\mathrm{i}} + 2\zeta_{\mathrm{i}}\omega_{\mathrm{i}}\dot{\eta}_{\mathrm{i}} + \omega_{\mathrm{i}}{}^2\eta_{\mathrm{i}} = \mathrm{f_i}(t)$$

where $f_i(t)$ is the generalized modal force, obtained by projecting the external forces onto the mode shape:

$$\mathrm{f_i}(t) = \mathrm{v_i}^{\mathrm{T}}F(t)$$

This modal expansion is particularly powerful because it reduces a large system of coupled equations to a set of independent single-degree-of-freedom oscillators. In practice, often only the first few modes contribute significantly to the response, enabling model reduction.

III. ANALYSIS

A. NASA's Frequency Separation Strategy for Pogo Mitigation

One of the most critical applications of eigenvalue analysis in launch vehicle design is the prevention of Pogo oscillations—longitudinal vibrations caused by feedback coupling between structural elasticity and propulsion system dynamics. The phenomenon derives its name from the similarity to a pogo stick's bouncing motion.

Pogo oscillations occur when the natural frequency of the vehicle structure coincides with characteristic frequencies of the propulsion system, particularly in the propellant feed lines. The coupling mechanism involves:

1. Structural vibration creates pressure fluctuations in propellant lines
2. Pressure fluctuations modulate thrust
3. Thrust variations excite structural vibration
4. The cycle amplifies if frequencies align

Then, NASA employs a strategic separation principle to prevent resonance using method named Frequency Separation Criterion,

$$|\omega_{\mathrm{structural}} - \omega_{\mathrm{propulsion}}| > \Delta_{\omega_{\mathrm{min}}}$$

where $\Delta_{\omega_{\mathrm{min}}}$ is a safety margin set at 20% of the lower frequency. This separation ensures that structural modes cannot be excited by propulsion system oscillations, and vice versa.

During the Apollo 13 mission, Pogo oscillations in the center engine of the S-II stage reached levels that damaged fuel lines and caused premature engine shutdown. Eigenvalue analysis of the coupled structure-propulsion system revealed that the second longitudinal mode (≈16 Hz) was coincident with a propellant feed line resonance. The solution implemented for subsequent missions included helium-filled accumulators in the LOX lines to detune the propulsion system frequency, effectively implementing the frequency separation strategy.

B. Ground Vibration Testing and Modal Identification

Before flight, launch vehicles undergo extensive Ground Vibration Testing (GVT) to validate finite element models and identify actual modal parameters. The Eigensystem Realization Algorithm (ERA) is NASA's primary tool for extracting modal information from test data.

Hankel Matrix Construction is a time-series data from accelerometers and strain gauges distributed across the vehicle are arranged into a Hankel matrix H that captures temporal correlation structure such as,

$$H = \begin{bmatrix} \mathrm{Y_1} & \cdots & \mathrm{Y_n} \\ \vdots & \ddots & \vdots \\ \mathrm{Y_m} & \cdots & \mathrm{Y_{m+n-1}} \end{bmatrix}$$

where $\mathrm{Y_i}$ represents system output (sensor measurements) at discrete time step i. The Hankel matrix structure embeds the Markov parameters of the system, which are related to the impulse response. NASA’s tool, ERA has some processing steps such as,

1. Apply controlled excitation (using shakers)
2. Measure response at sensor locations
3. Construct Hankel matrix from free-decay response
4. Perform SVD ($H = U\Sigma V^{\mathrm{T}}$)
5. Truncate at appropriate rank based on singular value magnitude
6. Extract system matrices A, B, C from truncated SVD
7. Compute the eigenvalues and eigenvectors of matrix A
8. Transform to physical modal parameters (frequencies, damping ratios, mode shapes)

C. Aeroelastic Coupling and Stability Analysis

NASA's analysis extends beyond free-free vibration to include aeroelastic effects—the interaction between aerodynamic forces and structural deformation. This coupling can either stabilize or destabilize the structure depending on flight conditions. The local angular displacement in the pitch plane of the x" discrete section of the vehicle with respect to the freestream velocity, with x an integer varying from 1 to I , is given by

$$\alpha_x = \varphi - \frac{\dot{z}}{V} - \frac{l_x}{V}\dot{\varphi} + \frac{w_x}{V} + \sum_{i=1}^{k}\left(\psi'_{ix}\eta_i - \frac{1}{V}\psi_{ix}\dot{\eta}_i\right).$$

Fig 3.1 Local angular displacement
Source : NASA research center

The aerodynamic forces create additional coupling terms that augment the structural matrices such as

$$\mathrm{K_h} = qS\sum_{\mathrm{x}=1}^{\mathrm{I}} \mathrm{d_x}$$

$$\mathrm{D_h} = \frac{qS}{\mathrm{V}}\sum_{\mathrm{x}=1}^{\mathrm{I}} \phi_{\mathrm{x}}$$

where q is dynamic pressure, S is reference area, and $\mathrm{d_x}$, ϕ_{x} are matrices formed from dyadic products of mode shapes weighted by line load distribution at station x. The diagonal matrices $\mathrm{K_h}$, $\mathrm{D_h}$ are the stiffness and damping matrices for the second-order flex dynamics.

Aeroelastic stability requires $\mathrm{Re}[\lambda(\mathrm{A_f})] < 0$ for all eigenvalues. This condition is evaluated across the flight envelope (Mach number, altitude, dynamic pressure) to identify potential instability regions.

NASA uses NASTRAN (NASA Structural Analysis) software to perform aeroelastic flutter analysis. This process consist of several steps :

1. Import FEM with structural modes
2. Define unsteady aerodynamic theory (double lattice method)
3. Compute generalized aerodynamic forces
4. Form aeroelastic equations in modal coordinates
5. Solve for flutter speeds and frequencies using V-g or p-k methods
6. Generate flutter boundaries on V-ρ diagrams

D. Closed-Loop Stability

A complete stability assessment of a launch vehicle cannot be limited to its structural or aeroelastic characteristics in open mode. In operational flight, the vehicle operates as a closed dynamic system, in which the flight control system continuously interacts with rigid body motions, flexible structural modes, and propulsion dynamics. These interactions introduce additional coupling paths through sensor measurements, actuator dynamics, and control law feedback, which can significantly alter the spectral properties of the system. The fully coupled system includes,

- Rigid-body dynamics
- Structural modes
- Propellant slosh
- Actuator Dynamics
- Control law Dynamics

These come with some design considerations such as sensor placement that affect observability of critical modes, system is conditionally stable, and bending filters must attenuate structural modes without introducing excessive phase lag. (1)

Be sure that the symbols in your equation have been defined before the equation appears or immediately following. Italicize symbols (*T* might refer to temperature, but T is the unit tesla). Refer to “(1),” not “Eq. (1)” or “equation (1),” except at the beginning of a sentence: “Equation (1) is ... .”

E. Computational Implementation in Linear Algebra

NASA implementation use advanced sparse linear algebra techniques optimized for high-performance computing. The sparse system matrices use BSR (Block Sparse Row), also the linearized systems at each nonlinear iteration are solved using preconditioned GMRES (Generalized Minimum Residual).

IV. IMPLEMENTATION

A. Eigenvalue distribution in Stability Analysis

Author wants to evaluate the system stability using plot complex plane that uses the eigenvalues. Python programming language is used to visualize how the Eigenvalue distribution to analysis the stability of a system.

We already given the original equation of motion that is a second order differential equation,

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

To analyze stability using eigenvalues, we need to convert this into a first-order system. This is done by defining an augmented state vector that explained in the theorethical basis.

```python
def check_stability(M, C, K):
    """
    Check Hurwitz stability criterion: Re(λ) < 0 for all eigenvalues
    """
    n = M.shape[0]
    M_inv = np.linalg.inv(M)

    # State-space matrix
    A = np.block([
        [np.zeros((n, n)), np.eye(n)],
        [-M_inv @ K, -M_inv @ C]
    ])

    eigenvalues = np.linalg.eigvals(A)
    real_parts = np.real(eigenvalues)

    is_stable = np.all(real_parts < 0)

    return is_stable, eigenvalues, real_parts
```

Fig 4.1 Check system stability
Source :
Author's source code

First we construct the system matrix A as seen in fig 4.1, where we use the state vector to build state space matrix. The top half of matrix A, is the velocity an the bottom half of A is the acceleration from dynamics. Specifically,

- Top-left block (0×n): Zero matrix - position doesn't directly affect velocity rate
- Top-right block (I_n): Identity matrix - velocity is the time derivative of position
- Bottom-left block ($-M^{-1}K$) : Stiffness effect on acceleration
- Bottom-right block ($-M^{-1}C$: Damping effect on acceleration

After that, author compute the eigenvalues using python libraries. Also the real parts need to be extracted. This is because the real part (σ) determines the stability behavior.

1. $\sigma < 0$, exponential decay, system status stable
2. $\sigma = 0$, sustained oscillation, marginally stable
3. $\sigma > 0$, exponential growth, unstable

The last step is to check the Hurwitz Criterion. It is known that, a linear time-invariant (LTI) system is asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$ forall i.

A manmade condition is used to test the program., then the results visualized in complex plane

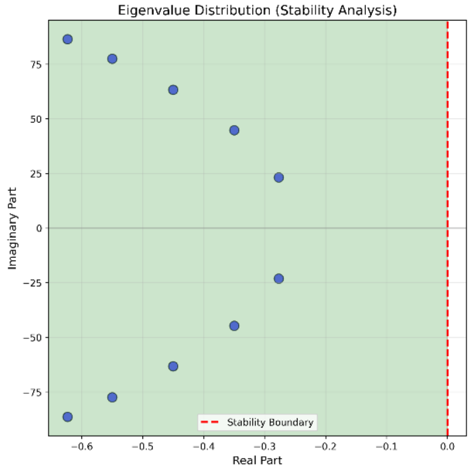

Fig 4.2 Stability analysis with complex plane plot
Source: Author

The horizontal axis represent the real part that determines the system stable or unstable. From the result we found that all of the eigenvalues (blue dot) are in the left of the red line. This indicate that the system stable according to Hurwitz criterion.

B. Pogo Analysis

One of the most critical applications of eigenvalue analysis is the prevention of Pogo oscillations—self-excited longitudinal vibrations resulting from coupling between structural dynamics and propulsion system oscillations. This phenomenon was notably experienced during the Apollo 13 mission, where the center engine of the S-II stage exhibited oscillations at approximately 16 Hz, causing premature shutdown.

Frequency Separation Criterion NASA employs a frequency separation strategy to prevent resonance between structural and propulsion modes. The criterion is defined as (already explained in IIIA):

$$|\omega_{\text{structural}} - \omega_{\text{propulsion}}| > \Delta_{\omega_{\min}}$$

where,

$$\Delta_{\omega_{\min}} = 0.2 \times \min(|\omega_{\text{structural}}, \omega_{\text{propulsion}}|)$$

This 20% safety margin ensures that even with parameter uncertainties and variations during flight, sufficient separation is maintained to prevent coupling.

For this analysis, the propulsion system frequency is set at 16.0 Hz, consistent with the Apollo 13 case study. The frequency separation for each structural mode is evaluated:

- Mode 1: 3.6 Hz – Safe
- Mode 2: 7.2 Hz - Safe
- Mode 3: 10.1 Hz - Safe
- Mode 4: 12.2 Hz - Cautious

- Mode 5: 13.5 Hz – **Dangerous** (near 16 Hz**)**

Also for the safety assesment, Green bar means that the separation frequency is bigger than the minimum required, and the red bar means that the separations is not big enough.The result state that the Mode 5 is most likely to make a Pogo oscillation.

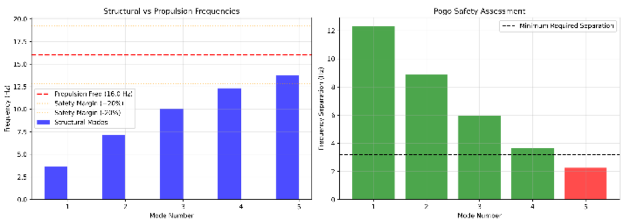

Fig 4.3 Structural vs Propulsion frequencies (left) and Pogo safety Assessment (right)
Source : Author

V. Conclusion

This paper has demonstrated that spectral analysis, particularly eigenvalue and eigenvector theory from linear algebra, plays a fundamental role in ensuring the stability and structural integrity of spacecraft launch vehicles. Through the formulation of structural dynamics using matrix-based equations of motion and state-space representations, complex physical phenomena such as structural vibration, aeroelastic coupling, and propulsion–structure interaction can be systematically analyzed within a unified mathematical framework.

Overall, this study confirms that linear algebra is not merely a theoretical discipline but a critical analytical foundation for modern aerospace engineering. The integration of eigenvalue analysis with numerical computation and experimental validation allows engineers to predict instability, design effective mitigation strategies, and ensure closed-loop stability throughout the launch phase. These methods continue to be essential as launch vehicles evolve toward lighter, more flexible, and increasingly complex architectures.

VI. Appendix

The following is the source for the code that have been analyzed and for implementing the eigenvalue distribution in stability analysis, mode shapes, pogo analysis, and pogo response comparison. Some of the code may be not explained and showed in this paper due to author recent conditions. All of the theoretical basis also written down from the references below.

https://github.com/Lloyd565/Spectral-Analysis-and-Structural-Dynamics-The-Role-of-Eigenvalues-in-Spacecraft-Launch-Stability

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Pernyataan

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

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