Determinacy in a Canonical New Keynesian Model under a k-rule

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Abstract

Until now, the canonical New Keynesian model under an exogenous path for the growth rate of the money supply, has been shown to have a stationary and unique solution through numerical analysis of the eigenvalues under a broad range of calibrations for the model's parameter values. In what follows I provide an analytical proof for such result.

Proof

We have the following dynamic linear system from Galí (2015), Chapter 3:

$$\mathbf{A_{M,0}} \begin{bmatrix} \tilde{y}_t \\ \pi_t \\ \hat{l}_{t-1} \end{bmatrix} = \mathbf{A_{M,1}} \begin{bmatrix} E_t\{\tilde{y}_t\} \\ E_t\{\pi_{t+1}\} \\ \hat{l}_t \end{bmatrix} + \mathbf{B_M} \begin{bmatrix} \hat{r}_t^n \\ \hat{y}_t^n \\ \Delta m_t \end{bmatrix}$$

where:

$$\mathbf{A_{M,0}} = \begin{bmatrix} 1+\sigma\eta & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \mathbf{A_{M,1}} = \begin{bmatrix} \sigma\eta & \eta & 1 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \mathbf{B_M} = \begin{bmatrix} \eta & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Furthermore, the system can be rewritten in the following way:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \\ \hat{l}_{t-1} \end{bmatrix} = \mathbf{A_M} \begin{bmatrix} E_t\{\tilde{y}_t\} \\ E_t\{\pi_{t+1}\} \\ \hat{l}_t \end{bmatrix} + \mathbf{C_M} \begin{bmatrix} \hat{r}_t^n \\ \hat{y}_t^n \\ \Delta m_t \end{bmatrix}$$

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where $\mathbf{A_M} = \mathbf{A_{M,0}}^{-1}\mathbf{A_{M,1}}$; and we ignore $\mathbf{C_M}$ as it's not needed in what follows. In such a dynamic system, a stationary solution will exist and be unique, if and only if, $\mathbf{A_M}$ has two eivengalues inside the unit circle and one eigenvalue outside (or on) the unit circle.

After some linear algebra:

$$\mathbf{A_M} = \begin{bmatrix} \frac{\sigma\eta}{1+\sigma\eta} & \frac{\eta}{1+\sigma\eta} & \frac{1}{1+\sigma\eta} \\ \frac{\kappa\sigma\eta}{1+\sigma\eta} & \frac{\kappa\eta}{1+\sigma\eta}+\beta & \frac{\kappa}{1+\sigma\eta} \\ \frac{\kappa\sigma\eta}{1+\sigma\eta} & \frac{\kappa\eta}{1+\sigma\eta}+\beta & \frac{\kappa}{1+\sigma\eta}+1 \end{bmatrix}$$

whose eigenvalues are given by the following polynomial:

$$p(\lambda) = -x^3 + \underbrace{\left(\frac{\sigma\eta+\kappa(1+\eta)}{1+\sigma\eta}+1+\beta\right)}_{b} x^2 - \underbrace{\left((1+\beta)\frac{\sigma\eta}{1+\sigma\eta}+\frac{\kappa\eta}{1+\sigma\eta}+\beta\right)}_{c} x + \underbrace{\frac{\beta\sigma\eta}{1+\sigma\eta}}_{d}$$

where $\sigma > 0$; $\eta > 0$; $\kappa > 0$ and $\beta \in (0,1)$. A condition for the inversion of $\mathbf{A_{M,0}}$ is: $1+\sigma\eta \neq 0$; but this condition will never be violated due to the previous assumptions on σ and η.

Since $p(1) = \kappa/(1+\sigma\eta) > 0$ and $p(\infty) = -\infty$ by continuity and Bolzano's theorem there must be at least one (real) root larger than 1 (λ_1) and therefore outside the unit circle. This root is also the only root larger than 1, see the Appendix.

Thus, since it's trivial to see that $p(x) > 0\ \forall x \leq 0$ it must be the case that if the two other roots (λ_2 and λ_3) are real they must be in the inverval (0,1) and therefore inside the unit circle.

To adress the case of two (λ_2 and λ_3) complex conjugates, we write our polynomial in a more general form:

$$p(x) = -(x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = -x^3+bx^2+cx+d$$

Where $d = \lambda_1\lambda_2\lambda_3$ is the independent component. Since $\lambda_2 = |\lambda_3|$ and $\lambda_2\lambda_3 = d\frac{1}{\lambda_1} \rightarrow |\lambda_3|^2$ (complex module) $= d\frac{1}{\lambda_1}$.

$$d\frac{1}{\lambda_1} = \underbrace{\beta}_{\in(0,1)} \underbrace{\frac{\sigma\eta}{\lambda_1+\lambda_1\sigma\eta}}_{<1\ due\ to\ \lambda_1>1} < 1$$

Therefore, one root is outside the unit circle (λ_1) and two (λ_2 and λ_3) inside the unit circle. The dynamic linear system has a unique and stable solution under any finite $\sigma > 0$; $\eta > 0$; $\kappa > 0$; and $\beta \in (0,1)$. $\square$

Discussion

The fact that an exogenous path for the growth rate of the money supply is always determinate is a desirable property of such rule. This implies that under this rule, no fluctuations due to agents trying to coordinate between different equilibria arise in this economy. Furthermore, if policy makers were unsure about the fundamental parameters of the economy (say risk aversion), or suspect that monetary policy could influence them is some way, the model would still be determinate, which is not the case for other simple interest rate rules explored in Bullard and Mitra (2002). The rule, obviously, is not without any downside, see Galí (2015) Chapter 3, Table 4.1 and 4.2.

References

Bullard, James, and Kaushik Mitra. "Learning about monetary policy rules." *Journal of monetary economics* 49.6 (2002): 1105-1129.

Galí, Jordi. *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications.* Princeton University Press, 2015.

Appendix

Proof that λ_1 is the only root of $p(x)$ for $x \in [1, +\infty)$:

The slope of the polynomial $p(x)$ is given by:

$$\frac{\partial p(x)}{\partial x} = -3x^2 + 2\left(\frac{\sigma\eta + \kappa(1+\eta)}{1+\sigma\eta} + 1 + \beta\right)x - \left((1+\beta)\frac{\sigma\eta}{1+\sigma\eta} + \frac{\kappa\eta}{1+\sigma\eta} + \beta\right) \quad \clubsuit$$

We will now explore two cases:

- *Case 1*: $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$

Since $\frac{\partial p(x)}{\partial x}|_{x=0} < 0$; $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$; and $\frac{\partial p(x)}{\partial x}|_{x=+\infty} < 0$; by continuity and Bolzano's theorem there is a root of $\clubsuit$ between 0 and 1 and another root between 1 and $+\infty$. Since $\clubsuit$ is a polynomial of second degree, it has only two roots. Since we found that one root must between 0 and 1 and another one between 1 and $+\infty$, there is one and only one root for equation 1 between 1 and $+\infty$. This implies that there is only one sign change in the slope of $p(x)$ for the interval $(1, +\infty)$. Therefore, due to $p(1) > 0$, $\frac{\partial p(x)}{\partial x}|_{x=1} > 0$ and there being only one sign change for $x \geq 1$, λ_1 is the only finite root of $p(x)$ for $x \in [1, +\infty)$.

- *Case 2*: $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$

For $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$; it must be the case that:

$$\beta + (1-\beta)\frac{\sigma\eta}{1+\sigma\eta} + \frac{2\kappa}{1+\sigma\eta} + \frac{\kappa\eta}{1+\sigma\eta} \leq 1 \quad \diamond$$

Now, the second derivative of $p(x)$ is:

$$\frac{\partial^2 p(x)}{\partial x^2} = -6x + 2\left(\frac{\sigma\eta + \kappa(1+\eta)}{1+\sigma\eta} + 1 + \beta\right)$$

Note that if $\frac{\partial^2 p(x)}{\partial x^2}|_{x=1} < 0 \rightarrow \frac{\partial^2 p(x)}{\partial x^2}|_{x\geq 1} < 0$. If the second derivative is negative at $x = 1$ the function will be concave for all $x \geq 1$. For $\frac{\partial^2 p(x)}{\partial x^2}|_{x=1} < 0$ to be the case it must be that:

$$\frac{\sigma\eta}{1+\sigma\eta} + \frac{\kappa}{1+\sigma\eta} + \frac{\kappa\eta}{1+\sigma\eta} + \beta - 1 < 1 \quad \star$$

If $\diamond$ is meet, $\star$ must also be met and thefore $p(x)$ is concave for all $x \geq 1$, this implies that since $p(1) > 0$ and $\frac{\partial p(x)}{\partial x}|_{x=1} \leq 0$, λ_1 is the only real root of $p(x)$ for $x \in [1, +\infty)$.