

ADCL: Acceleration Driven Clause Learning for Constrained Horn Clauses

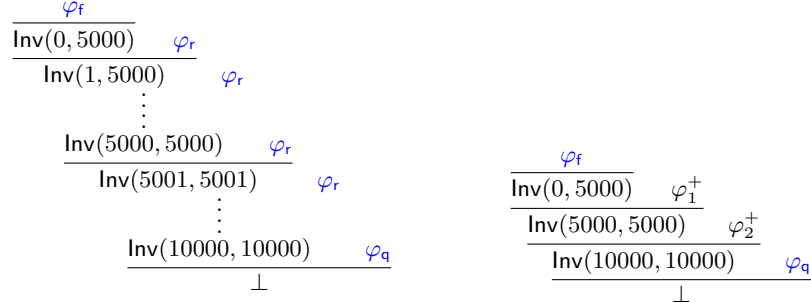
Abstract. *Constrained Horn Clauses* (CHCs) are widely used in automated program verification. Thus, techniques for (dis-)proving satisfiability of CHCs are a very active field of research. On the other hand, *acceleration techniques* for computing formulas that characterize the N -fold closure of loops have successfully been used for static program analysis. We show how to use acceleration to avoid repeated derivations with recursive CHCs in resolution proofs, which reduces the length of the proofs drastically. This idea gives rise to a novel calculus for (dis)proving satisfiability of CHCs, called *Acceleration Driven Clause Learning* (ADCL). We implemented this new calculus in the tool LoAT and evaluate it empirically in comparison to other state-of-the-art tools.

1 Introduction

Constrained Horn Clauses (CHCs) are widely used for expressing verification conditions in automated program verification. Prominent tools based on CHCs include SeaHorn for verifying C and C++ programs [19], JayHorn for Java programs [23], HornDroid for Android apps [9], RustHorn for Rust programs [28], and SmartACE [33] and SolCMC [2] for Solidity. Consequently, techniques for (dis-)proving satisfiability of CHCs (CHC-SAT) are a very active field of research, resulting in powerful tools like Spacer [24], Eldarica [22], FreqHorn [13], Golem [5], Ultimate [11], and RInGEN [25].

On the other hand, *loop acceleration techniques* have been used successfully for static program analyses during the last years, resulting in tools like Flata [7, 16] and LoAT [15]. Essentially, such techniques compute quantifier-free first-order formulas that characterize the N -fold closure of the transition relation of loops without branching in their body. Thus, acceleration techniques can be used when generating verification conditions in order to replace such loops with the closure of their transition relation.

In this paper, we apply acceleration techniques to CHC-SAT, where we restrict ourselves to linear CHCs, i.e., clauses that contain at most one positive and one negative literal with uninterpreted predicates. As our main interest lies in proving *unsatisfiability* of CHCs, our approach does not rely on abstractions, in contrast to most other techniques. Instead, we use acceleration to cut off repeated derivations with recursive CHCs while exploring the state space via resolution. In this way, the number of resolution steps that are required to reach a counterexample can be reduced drastically, as new CHCs that are “learned” via acceleration can simulate arbitrarily many “ordinary” resolution steps at once.



(a) Resolution Proof

(b) Accelerated Resolution Proof

Example 1. Consider the following set of CHCs Φ over the theory of linear integer arithmetic (LIA) with a *fact* φ_f , a *rule* φ_r , and a *query* φ_q , where \top and \perp stand for *true* and *false*:¹

$$\begin{aligned}
 & \top \implies \text{Inv}(0, 5000) & (\varphi_f) \\
 & \text{Inv}(X_1, X_2) \wedge \\
 & ((X_1 < 5000 \wedge Y_2 = X_2) \vee (X_1 \geq 5000 \wedge Y_2 = X_2 + 1)) \implies \text{Inv}(X_1 + 1, Y_2) & (\varphi_r) \\
 & \text{Inv}(10000, 10000) \implies \perp & (\varphi_q)
 \end{aligned}$$

Its unsatisfiability can be proven via resolution and arithmetic simplification as shown in Fig. (a). The proof requires 10001 resolution steps. Using acceleration techniques, we can derive the following two new CHCs from Φ :

$$\begin{aligned}
 & \text{Inv}(X_1, X_2) \wedge N > 0 \wedge X_1 + N < 5001 \implies \text{Inv}(X_1 + N, X_2) & (\varphi_1^+) \\
 & \text{Inv}(X_1, X_2) \wedge N > 0 \wedge X_1 \geq 5000 \implies \text{Inv}(X_1 + N, X_2 + N) & (\varphi_2^+)
 \end{aligned}$$

The first CHC φ_1^+ covers arbitrarily many subsequent resolution steps with φ_r where $X_1 < 5000$. Similarly, the second CHC φ_2^+ covers arbitrarily many steps where $X_1 \geq 5000$. Now we can prove unsatisfiability of Φ with just 3 resolution steps, as shown in Fig. (b).

This idea gives rise to a novel calculus for CHC-SAT, called *Acceleration Driven Clause Learning (ADCL)*. ADCL is refutationally complete and can also prove satisfiability, but it does not necessarily terminate.

So far, our implementation in the tool **LoAT** is restricted to proving unsatisfiability. In program verification (which is one of the most important applications of CHC-SAT), satisfiability usually corresponds to safety, i.e., if an error state is reachable in the original program, then the CHCs derived from the program are unsatisfiable. Hence, **LoAT** can be used to show reachability of error states in program verification. The “witness” of reachability is a resolution proof ending in a conditional empty clause $\psi \implies \perp$ (where the condition ψ is a formula over the signature of some background theory like LIA), together with a model for ψ . Instantiating the variables in the proof according to the model yields a proof

¹ chc-LIA-Lin.052.smt2 from the benchmarks of the CHC Competition ’22 [10]

on ground instances. Then this instantiated proof corresponds to a program run that ends in an error state, i.e., a counterexample.

After introducing preliminaries in Sect. 2, we formalize ADCL in Sect. 3. Next, we discuss how to implement ADCL efficiently in Sect. 4. In Sect. 5, we discuss related work, and we show that our approach is highly competitive with state-of-the-art CHC-SAT solvers by means of an empirical evaluation. All proofs can be found in the appendix.

2 Preliminaries

We assume that the reader is familiar with basic concepts from many-sorted first-order logic. Throughout this paper, Σ denotes a many-sorted first-order signature that just contains predicates, i.e., we do not consider uninterpreted functions. Moreover, \mathcal{V} denotes a countably infinite set of variables and for each entity e , $\mathcal{V}(e)$ denotes the variables occurring in e . We write \vec{x} for sequences and x_i is the i^{th} element of \vec{x} . In the following, we introduce preliminaries regarding *Constrained Horn Clauses* and *acceleration techniques*.

Constrained Horn Clauses are first-order formulas of the form

$$\begin{aligned} \forall \vec{X}_1, \dots, \vec{X}_d, \vec{Y}, \vec{Z}. F_1(\vec{X}_1) \wedge \dots \wedge F_d(\vec{X}_d) \wedge \psi \implies G(\vec{Y}) & \quad \text{or} \\ \forall \vec{X}_1, \dots, \vec{X}_d, \vec{Z}. F_1(\vec{X}_1) \wedge \dots \wedge F_d(\vec{X}_d) \wedge \psi \implies \perp \end{aligned}$$

where $\vec{X}_1, \dots, \vec{X}_d, \vec{Y}, \vec{Z}$ are pairwise disjoint vectors of pairwise different variables, $F_1, \dots, F_d, G \in \Sigma$, $\psi \in \text{QF}(\mathcal{A})$ is a quantifier-free first-order formula over the many-sorted signature $\Sigma_{\mathcal{A}}$ of some theory \mathcal{A} with equality, and $\mathcal{V}(\psi) \subseteq \vec{X}_1 \cup \dots \cup \vec{X}_d \cup \vec{Y} \cup \vec{Z}$.² We assume that Σ and $\Sigma_{\mathcal{A}}$ are disjoint, and we usually omit the leading universal quantifier, i.e., all variables in CHCs are implicitly universally quantified. Moreover, w.l.o.g., we assume that ψ is in negation normal form. In this paper, we restrict ourselves to *linear* CHCs. Thus, we consider CHCs of the following form:

$$\begin{aligned} \psi \implies G(\vec{Y}) \quad (\text{fact}) \quad & F(\vec{X}) \wedge \psi \implies \perp \quad (\text{query}) \\ F(\vec{X}) \wedge \psi \implies G(\vec{Y}) \quad (\text{rule}) \quad & \psi \implies \perp \quad (\text{conditional empty clause}) \end{aligned}$$

The premise and conclusion of a CHC is also called *body* and *head*, a CHC is *recursive* if it is a rule where $F = G$, and it is *conjunctive* if ψ is a conjunction of literals. Throughout this paper, φ and π always denote CHCs. The *condition* of a CHC φ is $\text{cond}(\varphi) := \psi$. We write $\varphi|_{\psi'}$ for the CHC that results from φ by replacing $\text{cond}(\varphi)$ with ψ' . Typically, the original set of CHCs does not contain conditional empty clauses, but in our setting, such clauses result from resolution proofs that start with a fact and end with a query. A conditional empty clause is

² We assume that all arguments of predicates are variables. This is not a restriction, as one can add equations to ψ to identify $\Sigma_{\mathcal{A}}$ -terms with fresh variables. To ease the presentation, we also use $\Sigma_{\mathcal{A}}$ -terms as arguments of predicates in examples (e.g., in Ex. 1 we wrote $\top \implies \text{Inv}(0, 5000)$ instead of $Y_1 = 0 \wedge Y_2 = 5000 \implies \text{Inv}(Y_1, Y_2)$).

called a *refutation* if its condition is satisfiable. We also refer to sets of CHCs as *CHC problems*, denoted Φ or Π .

We call σ an \mathcal{A} -*interpretation* if it is a model of \mathcal{A} whose carrier only contains ground terms over $\Sigma_{\mathcal{A}}$, extended with interpretations for Σ and \mathcal{V} . Given a first-order formula η over $\Sigma \cup \Sigma_{\mathcal{A}}$, an \mathcal{A} -interpretation σ is a *model* of η (written $\sigma \models_{\mathcal{A}} \eta$) if it satisfies η . If such a model exists, then η is *satisfiable*. As usual, $\models_{\mathcal{A}} \eta$ means that η is valid (i.e., we have $\sigma \models_{\mathcal{A}} \eta$ for all \mathcal{A} -interpretations σ) and $\eta \equiv_{\mathcal{A}} \eta'$ means $\models_{\mathcal{A}} \eta \iff \models_{\mathcal{A}} \eta'$. The *ground instances* of a CHC $\eta \wedge \psi \implies \eta'$ are:

$$\text{grnd}(\eta \wedge \psi \implies \eta') := \{\eta\sigma \implies \eta'\sigma \mid \sigma \models_{\mathcal{A}} \psi\}$$

Thus, we have $\sigma \models_{\mathcal{A}} \varphi$ iff $\sigma \models_{\mathcal{A}} \text{grnd}(\varphi)$.

In the following, we use “ $::$ ” for the concatenation of sequences, where we identify sequences of length 1 with their elements, i.e., we sometimes write $x :: xs$ instead of $[x] :: xs$ or $x :: y$ instead of $[x, y]$. As usual, $\text{mgu}(s, t)$ is the most general unifier of s and t . The following definition formalizes resolution (where we disregard the underlying theory and just use ordinary syntactic unification). If the corresponding literals of two clauses φ, φ' do not unify, then we define their resolvent to be $\perp \implies \perp$, so that resolution is defined for pairs of arbitrary CHCs. Note that in our setting, the mgu θ is always a variable renaming.

Definition 2 (Resolution). *Let φ and φ' be CHCs, where we rename the variables in φ and φ' such that they are disjoint. If*

$$\varphi = (\eta \wedge \psi \implies F(\vec{s})), \quad \varphi' = (F(\vec{t}) \wedge \psi' \implies \eta'), \quad \text{and} \quad \theta = \text{mgu}(F(\vec{s}), F(\vec{t})),$$

$$\text{then} \quad \text{res}(\varphi, \varphi') := (\eta \wedge \psi \wedge \psi' \implies \eta')\theta.$$

$$\text{Otherwise,} \quad \text{res}(\varphi, \varphi') := (\perp \implies \perp).$$

Here, η can also be \top and η' can also be \perp . We lift res to non-empty sequences of CHCs by defining

$$\text{res}([\varphi_1, \varphi_2] :: \vec{\varphi}) := \text{res}(\text{res}(\varphi_1, \varphi_2) :: \vec{\varphi}) \quad \text{and} \quad \text{res}(\varphi_1) := \varphi_1.$$

We implicitly lift notations and terminology for CHCs to sequences of CHCs via resolution. So for example, we have $\text{cond}(\vec{\varphi}) := \text{cond}(\text{res}(\vec{\varphi}))$ and $\text{grnd}(\vec{\varphi}) := \text{grnd}(\text{res}(\vec{\varphi}))$.

Example 3. Consider a variation Φ' of the CHC problem Φ from [Ex. 1](#) where φ_{f} is replaced by

$$X_1 \leq 0 \wedge X_2 \geq 5000 \implies \text{Inv}(X_1, X_2) \quad (\varphi'_{\text{f}})$$

To prove its unsatisfiability, one can consider the resolvent of the sequence $\vec{\varphi} := [\varphi'_{\text{f}}, \varphi_1^+, \varphi_2^+, \varphi_{\text{q}}]$:

$$\begin{aligned} & \text{res}([\varphi'_{\text{f}}, \varphi_1^+, \varphi_2^+, \varphi_{\text{q}}]) \\ &= \text{res}([\psi \implies \text{Inv}(X_1 + N, X_2), \varphi_2^+, \varphi_{\text{q}}]) \\ &= \text{res}([\psi \wedge N' > 0 \wedge X_1 + N \geq 5000 \implies \text{Inv}(X_1 + N + N', X_2 + N'), \varphi_{\text{q}}]) \quad (\dagger) \\ &\equiv_{\mathcal{A}} \text{res}([X_1 \leq 0 \wedge X_2 \geq 5000 \wedge N' > 0 \implies \text{Inv}(5000 + N', X_2 + N'), \varphi_{\text{q}}]) \quad (\ddagger) \\ &= (X_1 \leq 0 \wedge X_2 \geq 5000 \wedge N' > 0 \wedge 5000 + N' = X_2 + N' = 10000 \implies \perp) \end{aligned}$$

Here, we have

$$\psi := \text{cond}([\varphi'_f, \varphi_1^+]) = X_1 \leq 0 \wedge X_2 \geq 5000 \wedge N > 0 \wedge X_1 + N < 5001.$$

In the step marked with (†), the variable N' results from renaming N in φ_2^+ . In the step marked with (‡), we simplified $X_1 + N$ to 5000 for readability, as $\psi \wedge X_1 + N \geq 5000$ implies $X_1 + N = 5000$. If $\sigma(X_1) = 0$ and $\sigma(X_2) = \sigma(N') = 5000$, then $\sigma \models_{\mathcal{A}} \text{res}(\vec{\varphi})$ and thus $\text{res}(\vec{\varphi})$ is a refutation, so Φ is unsatisfiable. By instantiating the variables in the proof according to σ (and setting N to 5000, as we had $X_1 + N = 5000$), we obtain an accelerated resolution proof on ground instances that is analogous to Fig. (b) and serves as a “witness” of unsatisfiability, i.e., a “counterexample” to Φ' .

Acceleration Techniques are used to compute the N -fold closure of the transition relation of a loop in program analysis. In the context of CHCs, applying an acceleration technique to a recursive CHC φ yields another CHC φ' which, for any instantiation of a dedicated fresh variable $N \in \mathcal{V}(\varphi')$ with a positive integer, has the same ground instances as $\text{res}(\varphi^N)$. Here, φ^N denotes the sequence consisting of N repetitions of φ . In the following definition, we restrict ourselves to conjunctive CHCs, since many existing acceleration techniques do not support disjunctions [6], or have to resort to approximations in the presence of disjunctions [14].

Definition 4 (Acceleration). *An acceleration technique is a function accel that maps a recursive conjunctive CHC φ to a recursive conjunctive CHC such that $\text{grnd}(\text{accel}(\varphi)) = \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}(\varphi^n)$.*

Example 5. In the CHC problem from Ex. 1, φ_r entails

$$\text{Inv}(X_1, X_2) \wedge X_1 < 5000 \implies \text{Inv}(X_1 + 1, X_2).$$

From this CHC, an acceleration technique would compute φ_1^+ .

Note that most theories are not “closed under acceleration”. For example, consider the left clause below, which only uses linear arithmetic.

$$\text{F}(X, Y) \implies \text{F}(X + Y, Y) \qquad \text{F}(X, Y) \wedge N > 0 \implies \text{F}(X + N \cdot Y, Y)$$

Accelerating it yields the clause on the right, which is not expressible with linear arithmetic due to the sub-expression $N \cdot Y$. Moreover, if there is no sort for integers in the background theory \mathcal{A} , then an additional sort for the range of N is required in the formula that results from applying accel . For that reason, we consider *many-sorted* first-order logic and theories.

3 Acceleration Driven Clause Learning

In this section, we introduce our novel calculus ADCL for (dis)proving satisfiability of CHC problems. Since ADCL relies on acceleration techniques, an important property of ADCL is that it only applies resolution to conjunctive CHCs, even if the analyzed CHC problem is not conjunctive. To obtain conjunctive CHCs from non-conjunctive CHCs, we use *syntactic implicants*.

Definition 6 (Syntactic Implicant Projection). *Let $\psi \in \text{QF}(\mathcal{A})$ be in negation normal form. We define:*

$$\begin{aligned}
\text{sip}(\psi, \sigma) &:= \bigwedge \{ \ell \text{ is a literal of } \psi \mid \sigma \models_{\mathcal{A}} \ell \} && \text{if } \sigma \models_{\mathcal{A}} \psi \\
\text{sip}(\psi) &:= \{ \text{sip}(\psi, \sigma) \mid \sigma \models_{\mathcal{A}} \psi \} \\
\text{sip}(\varphi) &:= \{ \varphi|_{\psi} \mid \psi \in \text{sip}(\text{cond}(\varphi)) \} && \text{for CHCs } \varphi \\
\text{sip}(\Phi) &:= \bigcup_{\varphi \in \Phi} \text{sip}(\varphi) && \text{for sets of CHCs } \Phi
\end{aligned}$$

Here, sip abbreviates syntactic implicant projection.

In contrast to the usual notion of implicants (which just requires that the implicants entail ψ), syntactic implicants are restricted to literals from ψ to ensure that $\text{sip}(\psi)$ is finite. We call such implicants *syntactic* since [Def. 6](#) does not take the semantics of literals into account. For example, we have

$$(X > 1) \notin \text{sip}(X > 0 \wedge X > 1) = \{X > 0 \wedge X > 1\}.$$

It is easy to show that $\psi \equiv_{\mathcal{A}} \bigvee \text{sip}(\psi)$, and thus we also have $\Phi \equiv_{\mathcal{A}} \text{sip}(\Phi)$.

Example 7. In the CHC problem of [Ex. 1](#), we have

$$\text{sip}(\text{cond}(\varphi_r)) = \{(X_1 < 5000 \wedge Y_2 = X_2), (X_1 \geq 5000 \wedge Y_2 = X_2 + 1)\}.$$

Since $\text{sip}(\varphi)$ is worst-case exponential in the size of $\text{cond}(\varphi)$, we do not compute it explicitly: When resolving with φ , we conjoin $\text{cond}(\varphi)$ to the condition of the resulting resolvent and search for a model σ . This ensures that we do not continue with resolvents that have unsatisfiable conditions. Then we replace $\text{cond}(\varphi)$ by $\text{sip}(\text{cond}(\varphi), \sigma)$ in the resolvent. This corresponds to a resolution step with a conjunctive variant of φ whose condition is satisfied by σ . In other words, our calculus constructs $\text{sip}(\text{cond}(\varphi), \sigma)$ “on the fly” when resolving $\vec{\varphi}$ with φ , where $\sigma \models_{\mathcal{A}} \text{cond}(\vec{\varphi} :: \varphi)$, see [Sect. 4](#) for details. In this way, the exponential blowup that results from constructing $\text{sip}(\varphi)$ explicitly can often be avoided.

As ADCL learns new clauses via acceleration, it is important to prefer more general (learned) clauses over more specific clauses in resolution proofs. To this end, we use the following *redundancy relation* for CHCs.

Definition 8 (Redundancy Relation). *For two CHCs φ and π , we say that φ is (strictly) redundant w.r.t. π , denoted $\varphi \sqsubseteq \pi$ ($\varphi \sqsubset \pi$), if $\text{grnd}(\varphi) \subseteq \text{grnd}(\pi)$ ($\text{grnd}(\varphi) \subset \text{grnd}(\pi)$). For a set of CHCs Π , we define $\varphi \sqsubseteq \Pi$ ($\varphi \sqsubset \Pi$) if $\varphi \sqsubseteq \pi$ ($\varphi \sqsubset \pi$) for some $\pi \in \Pi$.*

In the remainder of this section, we assume that we have oracles for checking redundancy, for satisfiability of $\text{QF}(\mathcal{A})$ -formulas, and for acceleration. In practice, we have to resort to incomplete techniques instead. In [Sect. 4](#), we will explain how our implementation takes that into account.

A *state* of ADCL consists of a CHC problem Π , containing the original CHCs and all *learned* clauses that were constructed by acceleration, the *trace* $[\varphi_i]_{i=1}^k$, representing a resolution proof, and a sequence $[B_i]_{i=0}^k$ of sets of *blocking clauses*. Clauses $\varphi \sqsubseteq B_i$ must not be used for the $(i+1)^{\text{th}}$ resolution step. In this way, blocking clauses prevent ADCL from visiting the same part of the search space more than once. ADCL blocks a clause φ after proving that \perp (and thus *unsat*)

cannot be derived after adding φ to the current trace, or if the current trace $\vec{\varphi} :: \vec{\varphi}'$ ends with φ and there is another “more general” trace $\vec{\varphi} :: \vec{\pi}$ such that $\vec{\varphi}' \sqsubseteq \vec{\pi}$ and $|\vec{\varphi}'| \geq |\vec{\pi}|$, where one of the two relations is strict. In the following, Φ denotes the original CHC problem whose satisfiability is analyzed with ADCL.

Definition 9 (State). A state is a triple

$$(\Pi, [\varphi_i]_{i=1}^k, [B_i]_{i=0}^k)$$

where $\Pi \supseteq \Phi$ is a CHC problem, $B_i \subseteq \text{sip}(\Pi)$ for each $0 \leq i \leq k$, and $[\varphi_i]_{i=1}^k \in \text{sip}(\Pi)^*$. The clauses in $\text{sip}(\Phi)$ are called original clauses and all clauses in $\text{sip}(\Pi) \setminus \text{sip}(\Phi)$ are called learned clauses. A clause $\varphi \sqsubseteq B_k$ is blocked, and φ is active if it is not blocked and $\text{cond}([\varphi_i]_{i=1}^k :: \varphi)$ is satisfiable.

Now we are ready to introduce our novel calculus.

Definition 10 (ADCL). Let the “backtrack function” bt be defined as

$$\text{bt}(\Pi, [\varphi_i]_{i=1}^k, [B_0, \dots, B_k]) := (\Pi, [\varphi_i]_{i=1}^{k-1}, [B_0, \dots, B_{k-1} \cup \{\varphi_k\}]).$$

Our calculus is defined by the following rules.

$$\frac{}{\Phi \rightsquigarrow (\Phi, [], [\emptyset])} \quad (\text{INIT})$$

$$\frac{\varphi \in \text{sip}(\Pi) \text{ is active}}{(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow (\Pi, \vec{\varphi} :: \varphi, \vec{B} :: \emptyset)} \quad (\text{STEP})$$

$$\frac{\vec{\varphi}^\circ \text{ is recursive} \quad |\vec{\varphi}^\circ| = |\vec{B}^\circ| \quad \text{accel}(\vec{\varphi}^\circ) = \varphi}{(\Pi, \vec{\varphi} :: \vec{\varphi}^\circ, \vec{B} :: \vec{B}^\circ) \rightsquigarrow (\Pi \cup \{\varphi\}, \vec{\varphi} :: \varphi, \vec{B} :: \{\varphi\})} \quad (\text{ACCELERATE})$$

$$\frac{\vec{\varphi}' \sqsubseteq \text{sip}(\Pi) \quad \text{or} \quad \vec{\varphi}' \sqsubseteq \text{sip}(\Pi) \wedge |\vec{\varphi}'| > 1}{s = (\Pi, \vec{\varphi} :: \vec{\varphi}', \vec{B}) \rightsquigarrow \text{bt}(s)} \quad (\text{COVERED})$$

$$\frac{\text{all rules and queries from } \text{sip}(\Pi) \text{ are inactive} \quad \varphi \text{ is not a query}}{s = (\Pi, \vec{\varphi} :: \varphi, \vec{B}) \rightsquigarrow \text{bt}(s)} \quad (\text{BACKTRACK})$$

$$\frac{\vec{\varphi} \text{ is a refutation}}{(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow \text{unsat}} \quad (\text{REFUTE})$$

$$\frac{\text{all facts and conditional empty clauses from } \text{sip}(\Pi) \text{ are inactive}}{(\Pi, [], [B]) \rightsquigarrow \text{sat}} \quad (\text{PROVE})$$

We write $\overset{\text{I}}{\rightsquigarrow}, \overset{\text{S}}{\rightsquigarrow}, \dots$ to indicate that **INIT**, **STEP**, \dots was used for a \rightsquigarrow -step. All derivations start with **INIT**. **STEP** adds an active CHC φ to the trace. Due to the linearity of CHCs, we can restrict ourselves to proofs that start with a fact or a conditional empty clause, but such a restriction is not needed for the correctness of our calculus and thus not enforced.

As soon as $\vec{\varphi}$ has a recursive suffix $\vec{\varphi}^\circ$ (i.e., a suffix $\vec{\varphi}^\circ$ such that $\text{res}(\vec{\varphi}^\circ)$ is recursive), **ACCELERATE** can be used. Then the suffix $\vec{\varphi}^\circ$ is replaced by the

accelerated clause φ and the suffix \vec{B}^\cup of sets of blocked clauses that corresponds to $\vec{\varphi}^\cup$ is replaced by $\{\varphi\}$. The reason is that for learned clauses, we always have $\text{res}(\varphi, \varphi) \sqsubseteq \varphi$, and thus applying φ twice in a row is superfluous. So in this way, clauses that were just learned are not used for resolution several times in a row. As mentioned in [Sect. 2](#), the condition of the learned clause may not be expressible in the theory \mathcal{A} . Thus, when [ACCELERATE](#) is applied, we may implicitly switch to a richer theory \mathcal{A}' (e.g., from linear to non-linear arithmetic).

If a suffix $\vec{\varphi}'$ of the trace is redundant w.r.t. $\text{sip}(\Pi)$, we can backtrack via [COVERED](#), which removes the last element from $\vec{\varphi}$ (but not the rest of $\vec{\varphi}'$, since this sequence could now be continued in a different way) and blocks it, such that we do not revisit the corresponding part of the search space. So here the redundancy check allows us to use more general (learned) clauses, if available. Here, it is important that we do not backtrack if $\vec{\varphi}'$ is a single, *weakly* redundant clause. Otherwise, [COVERED](#) could always be applied after [STEP](#) or [ACCELERATE](#) and block the last clause from the trace. Thus, we might falsely “prove” satisfiability.

If no further [STEP](#) is possible since all CHCs are inactive, then we [BACKTRACK](#) as well and block the last clause from $\vec{\varphi}$ to avoid performing the same [STEP](#) again.

If we started with a fact and the last CHC in $\vec{\varphi}$ is a query, then $\text{res}(\vec{\varphi})$ is a refutation and [REFUTE](#) can be used to prove *unsat*.

Finally, if we arrive in a state where $\vec{\varphi}$ is empty and all facts and conditional empty clauses are inactive, then [PROVE](#) is applicable as we have exhausted the entire search space without proving unsatisfiability, i.e., Φ is satisfiable. Note that we always have $|\vec{B}| = |\vec{\varphi}| + 1$, since we need one additional set of blocking clauses to block facts. While B_0 is initially empty (see [INIT](#)), it can be populated via [BACKTRACK](#) or [COVERED](#). So eventually, all facts may become blocked, such that *sat* can be proven via [PROVE](#).

Example 11. Using our calculus, unsatisfiability of the CHC problem Φ in [Ex. 1](#) can be proven as follows:

$$\begin{aligned}
\Phi &\xrightarrow{\text{I}} (\Phi, [], [\emptyset]) \\
&\xrightarrow{\text{S}} (\Phi, [\varphi_{\text{f}}], [\emptyset, \emptyset]) && \top \implies \text{Inv}(0, 5000) \\
&\xrightarrow{\text{S}} (\Phi, [\varphi_{\text{f}}, \varphi_{\text{r}}|_{\psi_1}], [\emptyset, \emptyset, \emptyset]) && \text{Inv}(0, 5000) \implies \text{Inv}(1, 5000) \\
&\xrightarrow{\text{A}} (\Pi_1, [\varphi_{\text{f}}, \varphi_1^+], [\emptyset, \emptyset, \{\varphi_1^+\}]) && \text{Inv}(0, 5000) \implies \text{Inv}(1, 5000) \\
&\xrightarrow{\text{S}} (\Pi_1, [\varphi_{\text{f}}, \varphi_1^+, \varphi_{\text{r}}|_{\psi_2}], [\emptyset, \emptyset, \{\varphi_1^+, \}, \emptyset]) && \text{Inv}(5000, 5000) \implies \text{Inv}(5001, 5001) \\
&\xrightarrow{\text{A}} (\Pi_2, [\varphi_{\text{f}}, \varphi_1^+, \varphi_2^+], [\emptyset, \emptyset, \{\varphi_1^+\}, \{\varphi_2^+\}]) && \text{Inv}(5000, 5000) \implies \text{Inv}(5001, 5001) \\
&\xrightarrow{\text{S}} (\Pi_2, [\varphi_{\text{f}}, \varphi_1^+, \varphi_2^+, \varphi_{\text{q}}], [\emptyset, \emptyset, \{\varphi_1^+\}, \{\varphi_2^+\}, \emptyset]) && \text{Inv}(10000, 10000) \implies \perp \\
&\xrightarrow{\text{R}} \text{unsat}
\end{aligned}$$

Here, we have:

$$\begin{aligned}
\Pi_1 &:= \Phi \cup \{\varphi_1^+\} && \psi_1 := X_1 < 5000 \wedge Y_2 = X_2 \\
\Pi_2 &:= \Pi_1 \cup \{\varphi_2^+\} && \psi_2 := X_1 \geq 5000 \wedge Y_2 = X_2 + 1
\end{aligned}$$

Beside the state of our calculus, we show a ground instance of the last element φ of $\bar{\varphi}$ which results from applying a model for $\text{cond}(\bar{\varphi})$ to φ . In our implementation, we always maintain such a model. In general, these models are not unique: For example, after the first acceleration step, we might use $\text{Inv}(0, 5000) \implies \text{Inv}(X_1, 5000)$ for any $X_1 \in [1, 5000]$. The reason is that φ_1^+ can simulate arbitrarily many resolution steps with $\varphi_r|_{\psi_1}$, depending on the choice of N .

After starting the derivation with **INIT**, we apply the only fact φ_f via **STEP**. Next, we apply φ_r , projected to the case $X_1 < 5000$. Since φ_r is recursive, we may apply **ACCELERATE** afterwards, resulting in the new clause φ_1^+ .

Then we apply φ_r , projected to the case $X_1 \geq 5000$. Note that the current model (resulting in the ground head-literal $\text{Inv}(1, 5000)$) cannot be extended to a model for $\varphi_r|_{\psi_2}$ (which requires $X_1 \geq 5000$). However, as the model is not part of the state, we may choose a different one at any point, which is important for implementing ADCL via *incremental* SMT, see [Sect. 4](#). Hence, we can apply $\varphi_r|_{\psi_2}$ nevertheless.

Now we apply **ACCELERATE** again, resulting in the new clause φ_2^+ . Finally, we apply the only query φ_q via **STEP**, resulting in a conditional empty clause with a satisfiable condition, such that we can finish the proof via **REFUTE**.

Later (in [Def. 14](#)), we will define *reasonable strategies* for applying the rules of our calculus, which ensure that we use **ACCELERATE** instead of applying **STEP** 10001 times in our example.

To see how our calculus proves satisfiability, assume that we replace φ_q with

$$\text{Inv}(10000, X_2) \wedge X_2 \neq 10000 \implies \perp.$$

Then resolution with φ_q via the rule **STEP** is no longer applicable and our derivation continues as follows after the second application of **ACCELERATE**:

$$\begin{aligned} & (\Pi_2, [\varphi_f, \varphi_1^+, \varphi_2^+], [\emptyset, \emptyset, \{\varphi_1^+\}, \{\varphi_2^+\}]) \\ & \xrightarrow{\text{B}} (\Pi_2, [\varphi_f, \varphi_1^+], [\emptyset, \emptyset, \{\varphi_1^+, \varphi_2^+\}]) \\ & \xrightarrow{\text{B}} (\Pi_2, [\varphi_f], [\emptyset, \{\varphi_1^+\}]) \\ & \xrightarrow{\text{B}} (\Pi_2, [], [\{\varphi_f\}]) \\ & \xrightarrow{\text{P}} \text{sat} \end{aligned}$$

For all three **BACKTRACK**-steps, φ_q is clearly inactive, as adding it to $\bar{\varphi}$ results in a resolvent with an unsatisfiable condition. The first **BACKTRACK**-step is possible since $\varphi_r|_{\psi_1}$ and φ_1^+ are inactive, as they require $X_1 < 5000$ for the first argument X_1 of their body-literal, but φ_2^+ ensures $Y_1 > 5000$ for the first argument Y_1 of its head-literal. Moreover, $\varphi_r|_{\psi_2}$ and φ_2^+ are blocked, as $\varphi_r|_{\psi_2} \sqsubseteq \varphi_2^+$. The second **BACKTRACK**-step is performed since $\varphi_r|_{\psi_1}$, $\varphi_r|_{\psi_2}$, φ_1^+ , and φ_2^+ are blocked (as $\varphi_r|_{\psi_1} \sqsubseteq \varphi_1^+$ and $\varphi_r|_{\psi_2} \sqsubseteq \varphi_2^+$). The third **BACKTRACK**-step is possible since $\varphi_r|_{\psi_1}$ and φ_1^+ are blocked, and $\varphi_r|_{\psi_2}$ and φ_2^+ cannot be applied without applying φ_1^+ first, so they are inactive. Thus, we reach a state where the only fact φ_f is blocked and hence **PROVE** applies.

To see an example for **COVERED**, assume that we apply $\varphi_r|_{\psi_1}$ *twice* before using **ACCELERATE**. Then the following derivation yields the trace that we obtained after the first acceleration step above:

$$\begin{aligned}
& (\Phi, [\varphi_f, \varphi_r|_{\psi_1}], [\emptyset, \emptyset, \emptyset]) \\
& \xrightarrow{S} (\Phi, [\varphi_f, \varphi_r|_{\psi_1}, \varphi_r|_{\psi_1}], [\emptyset, \emptyset, \emptyset]) \\
& \xrightarrow{A} (\Pi_1, [\varphi_f, \varphi_r|_{\psi_1}, \varphi_1^+], [\emptyset, \emptyset, \emptyset, \{\varphi_1^+\}]) \\
& \xrightarrow{C} (\Pi_1, [\varphi_f, \varphi_r|_{\psi_1}], [\emptyset, \emptyset, \{\varphi_1^+\}]) \quad (\text{as } [\varphi_r|_{\psi_1}, \varphi_1^+] \sqsubset \varphi_1^+) \\
& \xrightarrow{C} (\Pi_1, [\varphi_f], [\emptyset, \{\varphi_r|_{\psi_1}\}]) \quad (\text{as } \varphi_r|_{\psi_1} \sqsubset \varphi_1^+) \\
& \xrightarrow{S} (\Pi_1, [\varphi_f, \varphi_1^+], [\emptyset, \{\varphi_r|_{\psi_1}\}, \emptyset]) \quad (\dagger)
\end{aligned}$$

As one can see in the example above, our calculus uses *forward reasoning*, i.e., it starts with a fact and resolves it with rules until a query applies. Alternatively, one could use *backward reasoning* by starting with a query and resolving it with rules until a fact applies, as in logic programming. While our calculus can easily be adapted for backward reasoning, such an adaption turned out to be fruitless in our experiments, so that we discarded it.

Theorem 12 (Soundness). *If $\Phi \rightsquigarrow^* \text{sat}$, then Φ is satisfiable. If $\Phi \rightsquigarrow^* \text{unsat}$, then Φ is unsatisfiable.*

*Proof (Sketch).*³ For **unsat**, we have $\Phi \rightsquigarrow^* (\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow \text{unsat}$ where $\Pi \equiv_{\mathcal{A}} \Phi$ and $\vec{\varphi} \in \text{sip}(\Pi)^*$ is a refutation. For **sat**, assume that Φ is unsatisfiable, but $\Phi \rightsquigarrow s = (\Phi, [], [\emptyset]) \rightsquigarrow^* (\Pi, [], [B]) = s' \rightsquigarrow \text{sat}$. Then there is a refutation $\vec{\varphi} \in \text{sip}(\Pi)^*$ that is minimal in the sense that $\varphi_i \not\sqsubset \text{sip}(\Pi)$ for all $1 \leq i \leq |\vec{\varphi}|$ and $\vec{\varphi}' \not\sqsubseteq \text{sip}(\Pi)$ for all infixes $\vec{\varphi}'$ of $\vec{\varphi}$ whose length is at least 2. We say that $\vec{\varphi}$ is *disabled* by a state $(\Pi', \vec{\varphi}', \vec{B}')$ if $\vec{\varphi}'$ has a prefix $[\varphi'_i]_{i=1}^k$ such that $\varphi_i \equiv_{\mathcal{A}} \varphi'_i$ for all $1 \leq i \leq k$ and $\varphi_{k+1} \equiv_{\mathcal{A}} \varphi$ for some $\varphi \in B'_k$. Then $\vec{\varphi}$ is disabled by s' , but not by s . Let $s^{(i)}$ be the last state in the derivation $\Phi \rightsquigarrow^* \text{sat}$ where $\vec{\varphi}$ is enabled. Then $s^{(i)} \xrightarrow{S} s^{(i+1)}$ would imply that $\vec{\varphi}$ is enabled in $s^{(i+1)}$; $s^{(i)} \xrightarrow{A} s^{(i+1)}$ would imply that two consecutive clauses in $\vec{\varphi}$ are both equivalent to the newly learned clause, contradicting minimality of $\vec{\varphi}$; $s^{(i)} \xrightarrow{C} s^{(i+1)}$ would imply that the trace of $s^{(i)}$ is not minimal, which also contradicts minimality of $\vec{\varphi}$; and $s^{(i)} \xrightarrow{B} s^{(i+1)}$ would imply that an element of $\vec{\varphi}$ is strictly redundant w.r.t. the last set of blocking clauses in $s^{(i)}$, which again contradicts minimality of $\vec{\varphi}$. Hence, we derived a contradiction. \square

Another important property of our calculus is that it cannot get “stuck” in states other than **sat** or **unsat**.

Theorem 13 (Normal Forms). *If $\Phi \rightsquigarrow^+ s$ where s is in normal form w.r.t. \rightsquigarrow , then $s \in \{\text{sat}, \text{unsat}\}$.*

Clearly, our calculus admits many unintended derivations, e.g., by applying **STEP** over and over again with recursive CHCs instead of accelerating them. To prevent such derivations, a *reasonable strategy* is required.

³ For reasons of space, we only present some proof sketches in the paper and refer to the appendix for all full proofs.

Definition 14 (Reasonable Strategy). We call a strategy for \rightsquigarrow reasonable if the following holds:

- (1) If $(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow^+ (\Pi, \vec{\varphi} :: \vec{\varphi}', \vec{B}')$ for some $\vec{\varphi}'$ as in the definition of **COVERED**, then **COVERED** is used.
 - (2) **ACCELERATE** is used with higher preference than **STEP**.
 - (3) **ACCELERATE** is only applied to the shortest recursive suffix $\vec{\varphi}^\circ$ such that $\text{accel}(\vec{\varphi}^\circ)$ is not redundant w.r.t. $\text{sip}(\Pi)$.
 - (4) If $\vec{\varphi} = []$, then **STEP** is only applied with facts or conditional empty clauses.
- We write \rightsquigarrow_{rs} for the relation that results from \rightsquigarrow by imposing a reasonable strategy.

Def. 14 (1) ensures that we backtrack if we added a redundant sequence $\vec{\varphi}'$ of CHCs to the trace. However, for refutational completeness (**Thm. 15**), it is important that the application of **COVERED** is only enforced if no new clauses have been learned while constructing $\vec{\varphi}'$ (i.e., Π remains unchanged in the derivation $(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow^+ (\Pi, \vec{\varphi} :: \vec{\varphi}', \vec{B}')$). The reason is that after applying **ACCELERATE**, the trace might have the form $\vec{\varphi} = \vec{\varphi}_1 :: \vec{\varphi}_2 :: \text{accel}(\varphi^\circ)$ where $\vec{\varphi}_2 :: \text{accel}(\varphi^\circ) \sqsubset \text{accel}(\varphi^\circ)$ even if $\vec{\varphi}_2 :: \varphi^\circ$ was non-redundant before learning $\text{accel}(\varphi^\circ)$. If we enforced backtracking via **COVERED** in such situations (which would yield the trace $\vec{\varphi}_1 :: \vec{\varphi}_2$), then to maintain refutational completeness, we would have to ensure that we eventually reach a state with the trace $\vec{\varphi}_1 :: \text{accel}(\varphi^\circ) \sqsubset \vec{\varphi}$. However, this cannot be guaranteed, since our calculus does not terminate in general (see **Thm. 18**).

Def. 14 (2) ensures that we do not “unroll” recursive derivations more than once via **STEP**, but learn new clauses that cover arbitrarily many unrollings via **ACCELERATE** instead.

Def. 14 (3) has two purposes: First, it prevents us from learning redundant clauses, as we must not apply **ACCELERATE** if $\text{accel}(\vec{\varphi}^\circ)$ is redundant. Second, it ensures that we accelerate “short” recursive suffixes first. The reason is that if $\vec{\varphi} = \vec{\varphi}_1 :: \vec{\varphi}_2 :: \vec{\varphi}_3$ where $\vec{\varphi}_2 :: \vec{\varphi}_3$ and $\vec{\varphi}_3$ are recursive, then

$$\begin{aligned} \text{grnd}(\text{accel}(\vec{\varphi}_2 :: \vec{\varphi}_3)) &\stackrel{\text{Def. 4}}{=} \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}((\vec{\varphi}_2 :: \vec{\varphi}_3)^n) \\ &\subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \bigcup_{m \in \mathbb{N}_{\geq 1}} \text{grnd}((\vec{\varphi}_2 :: \vec{\varphi}_3^m)^n) \stackrel{\text{Def. 4}}{=} \text{grnd}(\text{accel}(\vec{\varphi}_2 :: \text{accel}(\vec{\varphi}_3))), \end{aligned}$$

but the other direction (“ \supseteq ”) does not hold in general. So in this way, we learn more general clauses.

Def. 14 (4) ensures that the first element of $\vec{\varphi}$ is always a fact or a conditional empty clause. For unsatisfiable CHC problems, the reason is that **REFUTE** will never apply if $\vec{\varphi}$ starts with a rule or a query. For satisfiable CHC problems, **PROVE** only applies if all facts and conditional empty clauses are blocked. But in order to block them eventually, we have to add them to the trace via **STEP**, which is only possible if $\vec{\varphi}$ is empty.

Despite the restrictions in **Def. 14**, our calculus is still refutationally complete.

Theorem 15 (Refutational Completeness). If Φ is unsatisfiable, then

$$\Phi \rightsquigarrow_{rs}^* \text{unsat}.$$

Proof (Sketch). Given a refutation $\vec{\varphi}$, one can inductively define a derivation $\Phi \rightsquigarrow_{rs}^* \text{unsat}$ where each step applies **ACCELERATE** or **STEP**. For the latter, it is crucial to choose the next clause in such a way that it corresponds to as many steps from $\vec{\varphi}$ as possible, and that it is maximal w.r.t. \sqsubset , to avoid the necessity to backtrack via **COVERED**. \square

However, in general our calculus does not terminate, even with a reasonable strategy. To prove non-termination, we extend our calculus by one additional component: A mapping $\mathcal{L} : \text{sip}(\Pi) \rightarrow \mathcal{P}(\text{sip}(\Phi)^*)$ from $\text{sip}(\Pi)$ to regular languages over $\text{sip}(\Phi)$, where $\mathcal{P}(\text{sip}(\Phi)^*)$ denotes the power set of $\text{sip}(\Phi)^*$. We will show that this mapping gives rise to an alternative characterization of the ground instances of $\text{sip}(\Pi)$, which will be exploited in our non-termination proof (**Thm. 18**). Moreover, this mapping is also used in our implementation to check redundancy, see **Sect. 4**. To extend our calculus, we lift \mathcal{L} from $\text{sip}(\Pi)$ to $\text{sip}(\Pi)^*$ as follows:

$$\mathcal{L}(\varepsilon) := \varepsilon \quad \mathcal{L}(\vec{\pi} :: \pi) := \mathcal{L}(\vec{\pi}) :: \mathcal{L}(\pi)$$

Here, “ $::$ ” is also used to denote language concatenation, i.e., we have

$$\mathcal{L}_1 :: \mathcal{L}_2 := \{\vec{\pi}_1 :: \vec{\pi}_2 \mid \vec{\pi}_1 \in \mathcal{L}_1, \vec{\pi}_2 \in \mathcal{L}_2\}.$$

Definition 16 (ADCL with Regular Languages). We extend states (see **Def. 9**) by a fourth component $\mathcal{L} : \text{sip}(\Pi) \rightarrow \mathcal{P}(\text{sip}(\Phi)^*)$. The rules **INIT** and **ACCELERATE** of the ADCL calculus (see **Def. 10**) are adapted as follows:

$$\frac{\mathcal{L}(\varphi) = \{\varphi\} \text{ for all } \varphi \in \text{sip}(\Phi)}{\Phi \rightsquigarrow_{rs} (\Phi, [], [\emptyset], \mathcal{L})} \quad (\text{INIT})$$

$$\frac{\vec{\varphi}^\circ \text{ is recursive} \quad |\vec{\varphi}^\circ| = |\vec{B}^\circ| \quad \text{accel}(\vec{\varphi}^\circ) = \varphi \quad \mathcal{L}' = \mathcal{L} \uplus (\varphi \mapsto \mathcal{L}(\vec{\varphi}^\circ)^+)}{(\Pi, \vec{\varphi} :: \vec{\varphi}^\circ, \vec{B} :: \vec{B}^\circ, \mathcal{L}) \rightsquigarrow_{rs} (\Pi \cup \{\varphi\}, \vec{\varphi} :: \varphi, \vec{B} :: \{\varphi\}, \mathcal{L}')} \quad (\text{ACCELERATE})$$

All other rules from **Def. 10** leave the last component of the state unchanged.

Here, $\mathcal{L}(\pi)^+$ denotes the “Kleene plus” of $\mathcal{L}(\pi)$, i.e., we have

$$\mathcal{L}(\pi)^+ := \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathcal{L}(\pi)^n.$$

Note that **Def. 16** assumes a reasonable strategy (indicated by the notation \rightsquigarrow_{rs}). Hence, when **ACCELERATE** is applied, we may assume $\varphi \notin \text{sip}(\Pi) = \text{dom}(\mathcal{L})$. Otherwise, φ would be redundant and hence a reasonable strategy would not allow the application of **ACCELERATE**. For this reason, we may write “ \uplus ” in the definition of \mathcal{L}' .

The following lemma allows us to characterize the ground instances of elements of $\text{sip}(\Pi)$ via \mathcal{L} . Here, we lift grnd to sets by defining $\text{grnd}(X) := \bigcup_{x \in X} \text{grnd}(x)$, where X may be a set of CHCs or a language over CHCs. Thus, $\text{grnd}(\mathcal{L}(\pi))$ is the set of all ground instances of the final resolvents of the sequences in $\mathcal{L}(\pi)$.

Lemma 17. *If $\Phi \rightsquigarrow_{rs}^* (\Pi, \vec{\varphi}, \vec{B}, \mathcal{L})$ and $\pi \in \text{sip}(\Pi)$, then $\text{grnd}(\pi) = \text{grnd}(\mathcal{L}(\pi))$.*

Now we are ready to prove that, even with a reasonable strategy, ADCL does not terminate.

Theorem 18 (Non-Termination). *There exists a satisfiable CHC problem Φ such that $\Phi \not\vdash_{rs}^* \text{sat}$. Thus, \rightsquigarrow_{rs} does not terminate.*

Proof (Sketch). One can construct a satisfiable CHC problem Φ such that all (infinitely many) resolution sequences with Φ are *square-free*, i.e., they do not contain a non-empty subsequence of the form $\vec{\varphi} :: \vec{\varphi}$. For example, this can be achieved by encoding the differences between subsequent numbers of the Thue-Morse sequence [30, 31]. As an invariant of our calculus, $\mathcal{L}(\Pi)$ just contains finitely many square-free words for any reachable state $(\Pi, \vec{\varphi}, \vec{B}, \mathcal{L})$. As $\text{grnd}(\Pi) = \text{grnd}(\mathcal{L}(\Pi))$, this means that Π cannot cover all resolution sequences with Φ . Thus, the assumption $\Phi \rightsquigarrow_{rs}^* \text{sat}$ results in a contradiction. \square

The construction from the proof of Thm. 18 can also be used to show that there are non-terminating derivations $\Phi \rightsquigarrow_{rs} s_1 \rightsquigarrow_{rs} s_2 \rightsquigarrow_{rs} \dots$ where Φ is unsatisfiable. However, in this case there is also another derivation $\Phi \rightsquigarrow_{rs}^* \text{unsat}$ due to refutational completeness (see Thm. 15).

4 Implementing ADCL

We now explain how we implemented ADCL efficiently in the tool LoAT. Here we focus on proving unsatisfiability. The reason is that our implementation cannot prove **sat** at the moment, since it uses certain approximations that are incorrect for **sat**, as detailed below. Thus, when applying **PROVE**, our implementation returns **unknown** instead of **sat**. Our implementation uses Yices [12] and Z3 [29] for SMT solving. Moreover, it is based on the acceleration technique from [14], whose implementation solves recurrence relations with PURRS [3].

Checking Redundancy To check redundancy in **ACCELERATE** (as required for reasonable strategies in Def. 14), we use the fourth component \mathcal{L} of states introduced in Def. 16. More precisely, for **ACCELERATE**, we check if $\mathcal{L}(\vec{\varphi}^\circ)^+ \subseteq \mathcal{L}(\varphi)$ holds for some learned clause φ . In that case, $\text{accel}(\vec{\varphi}^\circ)$ is redundant due to Lemma 17. Since $\mathcal{L}(\vec{\varphi}^\circ)^+ \subseteq \mathcal{L}(\varphi)$ is simply an inclusion check for regular languages, it can be implemented efficiently using finite automata. Our implementation uses the automata library libFAUDES [26].

However, this is just a sufficient criterion for redundancy. For example, a learned clause might be redundant w.r.t. an original clause, but such redundancies cannot be detected using \mathcal{L} . To see this, note that we have $|\mathcal{L}(\varphi)| = 1$ if φ is an original clause, but $|\mathcal{L}(\varphi)| = \infty$ if φ is a learned clause.

For **COVERED**, we also check redundancy via \mathcal{L} , but if $\vec{\varphi}' = \varphi'$, i.e., if $|\vec{\varphi}'| = 1$, then we only apply **COVERED** if φ' is an original clause. Then $\mathcal{L}(\varphi') \subseteq \mathcal{L}(\varphi)$ for some $\varphi \neq \varphi'$ implies that φ is a learned clause. Hence, we have $\mathcal{L}(\varphi') \subset \mathcal{L}(\varphi)$, as $|\mathcal{L}(\varphi')| = 1 < |\mathcal{L}(\varphi)| = \infty$. This is just a heuristic, as even $\mathcal{L}(\varphi') \subset \mathcal{L}(\varphi)$ just implies $\varphi' \sqsubseteq \varphi$, but not $\varphi' \sqsubset \varphi$. To see this, consider an original clause $\varphi = (F(X) \implies F(0))$. Then $\mathcal{L}(\varphi) = \{\varphi\}$, $\text{accel}(\varphi) \equiv_{\mathcal{A}} \varphi$ (but not necessarily $\text{accel}(\varphi) = \varphi$, as $\text{accel}(\varphi)$ and φ might differ syntactically), and $\mathcal{L}(\text{accel}(\varphi)) = \mathcal{L}(\varphi)^+$. So we have $\mathcal{L}(\varphi) \subset \mathcal{L}(\text{accel}(\varphi))$ and $\varphi \sqsubseteq \text{accel}(\varphi)$, but $\varphi \not\sqsubset \text{accel}(\varphi)$. This is uncritical for proving **unsat**, but a potential soundness issue for proving **sat**, which is one reason why our current implementation cannot prove **sat**.

Implementing STEP and Blocked Clauses To find an active clause in STEP, we proceed as described before Def. 8, i.e., we search for a suitable element of $\text{sip}(\Pi)$ “on the fly”. So we search for a clause $\varphi \in \Pi$ whose body-literal unifies with the head-literal of $\text{res}(\vec{\varphi})$ using an mgu θ . Then we use an SMT solver to check whether

$$\theta(\text{cond}(\text{res}(\vec{\varphi}))) \wedge \theta(\text{cond}(\varphi)) \wedge \bigwedge_{\pi \in B \cap \text{sip}(\varphi)} \neg \theta(\text{cond}(\pi)) \quad (\text{STEP-SMT})$$

is satisfiable, where B is the last element of \vec{B} . Here, we assume that $\text{res}(\vec{\varphi})$ and φ are variable disjoint (and thus the mgu θ exists). If we find a model σ for STEP-SMT, then we apply STEP with $\varphi|_{\text{sip}(\text{cond}(\varphi), \sigma)}$. So to exclude blocked clauses, we do not use the redundancy check based on \mathcal{L} explained above, but we conjoin the negated conditions of certain blocked clauses to STEP-SMT. To see why we only consider blocked clauses from $\text{sip}(\varphi)$, consider the case that $B = \{\pi\}$ is a singleton. Note that both $\theta(\text{cond}(\varphi))$ and $\theta(\text{cond}(\pi))$ might contain variables that do not occur as arguments of predicates in the (unified) head- or body-literals. So if

$$\begin{aligned} \varphi &\equiv_{\mathcal{A}} \forall \vec{X}, \vec{Y}_{\varphi}, \vec{X}'. F(\vec{X}) \wedge \psi_{\varphi}(\vec{X}, \vec{Y}_{\varphi}, \vec{X}') \implies G(\vec{X}'), \\ \varphi' &\equiv_{\mathcal{A}} \forall \vec{X}, \vec{Y}_{\varphi'}, \vec{X}'. F(\vec{X}) \wedge \psi_{\varphi'}(\vec{X}, \vec{Y}_{\varphi'}, \vec{X}') \implies G(\vec{X}'), \quad \text{and} \\ \pi &\equiv_{\mathcal{A}} \forall \vec{X}, \vec{Y}_{\pi}, \vec{X}'. F(\vec{X}) \wedge \psi_{\pi}(\vec{X}, \vec{Y}_{\pi}, \vec{X}') \implies G(\vec{X}'), \end{aligned}$$

for some $\varphi' \in \text{sip}(\varphi)$, then $\varphi' \sqsubseteq \pi$ iff

$$\models_{\mathcal{A}} \psi_{\varphi'} \implies \exists \vec{Y}_{\pi}. \psi_{\pi}. \quad (\sqsubseteq\text{-EQUIV})$$

Thus, to ensure that we only find models σ such that $\text{sip}(\text{cond}(\varphi), \sigma)$ is not blocked by π , we would have to conjoin

$$\neg(\psi_{\varphi} \implies \exists \vec{Y}_{\pi}. \psi_{\pi}) \equiv_{\mathcal{A}} \psi_{\varphi} \wedge \forall \vec{Y}_{\pi}. \neg \psi_{\pi}$$

to the SMT problem. Unfortunately, as SMT solvers have limited support for quantifiers, such an encoding is impractical. Hence, we again use a sufficient criterion for redundancy: If

$$\models_{\mathcal{A}} \psi_{\varphi'} \implies \psi_{\pi}, \quad (\sqsubseteq\text{-SUFFICIENT})$$

then $\sqsubseteq\text{-EQUIV}$ trivially holds as well. So to exclude conjunctive variants φ' of φ where $\sqsubseteq\text{-SUFFICIENT}$ is valid, we add

$$\neg(\psi_{\varphi} \implies \psi_{\pi}) \equiv_{\mathcal{A}} \psi_{\varphi} \wedge \neg \psi_{\pi} \quad (\not\sqsubseteq\text{-SUFFICIENT})$$

to the SMT problem. If $\vec{Y}_{\pi} \not\subseteq \vec{Y}_{\varphi}$, then satisfiability of $\not\sqsubseteq\text{-SUFFICIENT}$ is usually trivial. Thus, to avoid increasing the size of the SMT problem unnecessarily, we only add $\not\sqsubseteq\text{-SUFFICIENT}$ to the SMT problem if $\pi \in \text{sip}(\varphi)$. Instead, we could try to rename variables from \vec{Y}_{π} to enforce $\vec{Y}_{\pi} \subseteq \vec{Y}_{\varphi}$. However, it is difficult to predict which renaming is the “right” one, i.e., which renaming would allow us to prove redundancy.

If $B \cap \text{sip}(\varphi)$ contains several clauses π_1, \dots, π_{ℓ} , then $\sqsubseteq\text{-SUFFICIENT}$ becomes

$$\models_{\mathcal{A}} \psi_{\varphi'} \implies \text{cond}(\pi_1) \text{ or } \dots \text{ or } \models_{\mathcal{A}} \psi_{\varphi'} \implies \text{cond}(\pi_{\ell}) \quad (\sqsubseteq\text{-SUFFICIENT}^+)$$

Instead, our encoding excludes syntactic implicants φ' of φ where

$$\models_{\mathcal{A}} \psi_{\varphi'} \implies \text{cond}(\pi_1) \vee \dots \vee \text{cond}(\pi_\ell) \quad (\sqsubseteq\text{-INSUFFICIENT}^+)$$

which is a necessary, but not a sufficient condition for $\sqsubseteq\text{-SUFFICIENT}^+$. To see why this is not a problem, first note that $\sqsubseteq\text{-SUFFICIENT}^+$ trivially holds if $\psi_{\varphi'} \in \{\text{cond}(\pi_i) \mid 1 \leq i \leq \ell\}$. Otherwise, we have

$$\models_{\mathcal{A}} (\text{cond}(\pi_1) \vee \dots \vee \text{cond}(\pi_\ell)) \implies \bigvee \text{sip}(\psi_\varphi) \setminus \{\psi_{\varphi'}\}$$

because we assumed $\psi_{\varphi'} \notin \{\text{cond}(\pi_i) \mid 1 \leq i \leq \ell\}$, which implies $\{\text{cond}(\pi_i) \mid 1 \leq i \leq \ell\} \subseteq \text{sip}(\psi_\varphi) \setminus \{\psi_{\varphi'}\}$. Together with $\sqsubseteq\text{-INSUFFICIENT}^+$, this implies

$$\models_{\mathcal{A}} \psi_{\varphi'} \implies \bigvee \text{sip}(\psi_\varphi) \setminus \{\psi_{\varphi'}\}.$$

Therefore, we have $\psi_{\varphi'} \equiv_{\mathcal{A}} \bigvee \text{sip}(\psi_\varphi) \setminus \{\psi_{\varphi'}\}$. Thus, we may assume that $\sqsubseteq\text{-INSUFFICIENT}^+$ implies redundancy without loss of generality. The reason is that we could analyze the following equivalent CHC problem instead of Π , otherwise:

$$(\Pi \setminus \{\varphi\}) \cup (\text{sip}(\varphi) \setminus \{\varphi'\})$$

Hence, in **STEP-SMT**, we add (a variable-renamed variant of)

$$\begin{aligned} \neg(\psi_\varphi \implies \text{cond}(\pi_1) \vee \dots \vee \text{cond}(\pi_\ell)) &\equiv_{\mathcal{A}} \psi_\varphi \wedge \neg \text{cond}(\pi_1) \wedge \dots \wedge \neg \text{cond}(\pi_\ell) \\ &\equiv_{\mathcal{A}} \text{cond}(\varphi) \wedge \bigwedge_{\pi \in B \cap \text{sip}(\varphi)} \neg \text{cond}(\pi) \end{aligned}$$

to the SMT problem.

Example 19. Consider the state \dagger from [Ex. 11](#). First applying **STEP** with $\varphi_r|_{\psi_1}$ and then applying **COVERED** yields

$$(\Pi_1, \vec{\varphi}, [\emptyset, \{\varphi_r|_{\psi_1}\}, \{\varphi_r|_{\psi_1}\}])$$

where $\vec{\varphi} = [\varphi_f, \varphi_1^+]$. When attempting a **STEP** with an element of $\text{sip}(\varphi_r)$, we get:

$$\begin{aligned} \theta(\text{cond}(\vec{\varphi})) &\equiv_{\mathcal{A}} X_1 = 0 \wedge X_2 = 5k \wedge N > 0 \wedge X'_1 < 5001 \wedge X'_1 = X_1 + N \wedge X'_2 = X_2 \\ \theta(\text{cond}(\varphi_r)) &\equiv_{\mathcal{A}} ((X'_1 < 5k \wedge Y_2 = X'_2) \vee (X'_1 \geq 5k \wedge Y_2 = X'_2 + 1)) \\ \bigwedge_{\pi \in B \cap \text{sip}(\varphi_r)} \neg \theta(\text{cond}(\pi)) &= \neg \theta(\text{cond}(\varphi_r|_{\psi_1})) \equiv_{\mathcal{A}} X'_1 \geq 5k \vee Y_2 \neq X'_2 \end{aligned}$$

Here, $5k$ abbreviates 5000. Then **STEP-SMT** is equivalent to

$$X_1 = 0 \wedge X_2 = N = X'_1 = X'_2 = 5k \wedge Y_2 = 5001.$$

Hence, we have $\sigma \models_{\mathcal{A}} X'_1 \geq 5k \wedge Y_2 = X'_2 + 1$ for the unique model σ of **STEP-SMT**, i.e., σ satisfies the second disjunct of $\text{cond}(\varphi_r)$. Thus, we add $\varphi_r|_{\text{sip}(\text{cond}(\varphi_r), \sigma)} = \varphi_r|_{\psi_2}$ to the trace.

Leveraging Incremental SMT The search for suitable models can naturally be implemented via incremental SMT solving: When trying to apply **STEP**, we construct θ in such a way that $\theta(\text{cond}(\text{res}(\vec{\varphi}))) = \text{cond}(\text{res}(\vec{\varphi}))$. This is easily possible, as θ just needs to unify predicates whose arguments are duplicate free and pairwise disjoint vectors of variables. Then we push

$$\theta(\text{cond}(\varphi)) \wedge \bigwedge_{\pi \in B \cap \text{sip}(\varphi)} \neg \theta(\text{cond}(\pi)) \quad (\text{INCREMENTAL})$$

to the SMT solver. If the model from the previous resolution step can be extended to satisfy **INCREMENTAL**, then the SMT solver can do so, otherwise it searches for

another model. If it fails to find a model, we pop **INCREMENTAL**, i.e., we remove it from the current SMT problem. **ACCELERATE** can be implemented similarly by popping $\theta(\text{cond}(\bar{\varphi}^\circ))$ and pushing $\theta(\text{cond}(\varphi))$ instead.

Note that satisfiability of $\bar{\varphi}$ is an invariant of ADCL. Hence, as soon as the last element of $\bar{\varphi}$ is a query, **REFUTE** can be applied without further SMT checks. Otherwise, if **STEP** cannot be applied with any clause, then **BACKTRACK** or **PROVE** can be applied without further SMT queries.

Dealing with Incompleteness As mentioned in Sect. 3, we assumed that we have oracles for checking redundancy, satisfiability of $\text{QF}(\mathcal{A})$ -formulas, and acceleration when we formalized ADCL. As this is not the case in practice, we now explain how to proceed if those techniques fail or approximate.

As explained above, SMT is needed for checking activity in **STEP**. If the SMT solver fails, we assume inactivity. Thus, we do not exhaust the entire search space if we falsely classify active clauses as inactive. Hence, we may miss refutations, which is another reason why our current implementation cannot prove **sat**.

Regarding acceleration, our implementation of **accel** may return under-approximations, i.e., we just have $\text{grnd}(\text{accel}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}(\varphi^n)$. While this is uncritical for correctness by itself (as learned clauses are still entailed by Φ), it weakens our heuristic for redundancy via \mathcal{L} , as we no longer have $\text{grnd}(\varphi) = \text{grnd}(\mathcal{L}(\varphi))$, but just $\text{grnd}(\varphi) \subseteq \text{grnd}(\mathcal{L}(\varphi))$ for learned clauses φ .

Another pitfall when using under-approximating acceleration techniques is that we may have $\bar{\varphi}^\circ \not\sqsubseteq \text{accel}(\bar{\varphi}^\circ)$. In this case, applying **ACCELERATE** can result in an inconsistent trace where $\text{cond}(\bar{\varphi})$ is unsatisfiable. To circumvent this problem, we only add $\text{accel}(\bar{\varphi}^\circ)$ to the trace after removing $\bar{\varphi}^\circ$ if doing so results in a consistent trace. Here, we could do better by taking the current model σ into account when accelerating $\bar{\varphi}^\circ$ in order to ensure $\sigma \models_{\mathcal{A}} \text{cond}(\text{accel}(\bar{\varphi}^\circ))$. We leave that to future work.

Restarts When testing our implementation, we noticed that several instances “jiggled”, i.e., they were solved in some test runs, but failed in others. The reason is a phenomenon that is well-known in SAT solving, called “heavy-tail behavior”. Here, the problem is that the solver sometimes gets “stuck” in a part of the state space whose exploration is very expensive, even though finding a solution in another part of the search space is well within the solver’s capabilities. This problem also occurs in our implementation, due to the depth-first strategy of our solver (where derivations may even be non-terminating, see Thm. 18). To counter this problem, SAT solvers use restarts [17], where one of the most popular approaches has been proposed by Luby et al. [27]. For SAT solving, the idea is to restart the search after a certain number of conflicts, where the number of conflicts for the next restart is determined by the *Luby sequence*, scaled by a parameter u . When restarting, randomization is used to avoid revisiting the same part of the search space again. We use the same strategy with $u = 10$, where we count the number of learned clauses instead of the number of conflicts. To restart the search, we clear the trace, change the seed of the SMT solver (which may result in different models such that we may use different syntactic implicants),

and shuffle the vectors of clauses (to change the order in which clauses are used for resolution).

5 Related Work and Experiments

We presented the novel ADCL calculus for (dis)proving satisfiability of CHCs. Its distinguishing feature is its use of acceleration for learning new clauses. For unsatisfiability, these learned clauses often enable very short resolution proofs for CHC problems whose original clauses do not admit short resolution proofs. For satisfiability, learned clauses often allow for covering the entire (usually infinite) search space by just considering finitely many resolution sequences.

Related Work The most closely related work is [21], where acceleration is used in two ways: (1) as preprocessing and (2) to generalize interpolants in a CEGAR loop. In contrast to (1), we use acceleration “on the fly” to accelerate resolvents. In contrast to (2), we do not use abstractions, so our learned clauses can directly be used in resolution proofs. Moreover, [21] only applies acceleration to conjunctive clauses, whereas we accelerate conjunctive variants of arbitrary clauses. Both (1) and (2) are implemented in *Eldarica*, but according to its authors, (2) is just supported for transition systems, but not yet for CHCs. Hence, we only considered (1) in our evaluation (named *Eld. Acc. below*).

Transition power abstraction (TPA) [5] computes a sequence of over-approximations for transition systems where the n^{th} element captures 2^n instead of just n steps of the transition relation. So like ADCL, TPA can help to find long refutations quickly, but in contrast to ADCL, TPA relies on over-approximations.

Some leading techniques for CHC-SAT like GPDR [20] and, in particular, the *Spacer* algorithm [24], are adaptations of the IC3 algorithm [8] from transition systems to CHCs. IC3 computes a sequence of abstractions of reachable states, aiming to find an abstraction that is inductive w.r.t. the transition relation and implies safety.

Other approaches for CHC-SAT are based on interpolation [11, 22], CEGAR and predicate abstraction [18, 22], automata [11], machine learning [13, 34], bounded model checking (BMC) [4], or combinations thereof.

Experiments So far, our implementation of ADCL in *LoAT* is restricted to integer arithmetic. Thus, to evaluate our approach, we used the examples from the category LIA-Lin (linear CHCs with linear integer arithmetic) from the CHC competition ’22 [10]. Somewhat surprisingly, these examples contain additional features like variables of type `Bool` and the operators `div` and `mod`.

Since variables of type `Bool` are used in the majority of the examples, we extended our implementation with rudimentary support for `Bools`. In particular, we implemented a simplistic acceleration technique for `Bools` that only applies if their values stabilize after a few iterations. We excluded the 72 examples that use `div` or `mod`, as those operators are not supported by our implementation.

Using the remaining 427 examples, we compared our implementation with the leading CHC-SAT solvers *Spacer* [24] (which is part of *Z3* [29]), *Eldarica* [22],

		unsat		sat	Runtime overall			Runtime unsat		
		solved	unique	solved	average	median	SD ⁴	average	median	SD
Int & Bool	LoAT	78	11 (5)	0	107.1 s	2.4 s	137.8 s	10.0 s	0.1 s	39.1 s
	Spacer	78	2 (2)	176	119.9 s	5.0 s	141.9 s	15.1 s	0.1 s	52.5 s
	Z3 BMC	76	5 (5)	23	198.7 s	300.0 s	140.6 s	10.8 s	0.1 s	33.0 s
	Golem TPA	56	0 (0)	72	125.6 s	1.5 s	146.4 s	7.1 s	0.1 s	35.5 s
	Eldarica	53	0 (–)	165	155.3 s	177.7 s	144.5 s	13.4 s	2.6 s	33.4 s
	Eld. Acc.	56	– (4)	142	124.5 s	18.2 s	139.9 s	12.0 s	3.4 s	28.8 s
Int only	LoAT	30	5 (5)	0	71.4 s	0.2 s	124.4 s	0.5 s	0.1 s	2.1 s
	Spacer	23	0 (0)	91	136.0 s	5.2 s	146.7 s	11.4 s	0.1 s	45.6 s
	Z3 BMC	24	1 (1)	23	232.3 s	300.0 s	124.7 s	9.7 s	0.1 s	32.9 s
	Golem TPA	15	0 (0)	45	110.8 s	0.3 s	143.1 s	4.3 s	0.1 s	6.3 s
	Eldarica	23	0 (–)	122	100.6 s	3.8 s	136.6 s	8.3 s	3.3 s	9.8 s
	Eld. Acc.	21	– (0)	105	106.2 s	106.2 s	4.1 s	10.3 s	6.1 s	10.5 s

and Golem [5]. Additionally, we compared with Z3’s implementation of BMC. As mentioned above, Eldarica supports acceleration as preprocessing. Thus, besides Eldarica’s default configuration (which does not use acceleration), we also compared with a configuration Eld. Acc. where we enabled this feature. By default, Golem uses the Spacer algorithm, but the Spacer implementation in Z3 performed better in our tests. Thus, we used Golem’s implementation of TPA instead, which targets similar classes of examples like ADCL, as explained above. We used Z3 4.11.2, Eldarica 2.0.8, and Golem 0.3.0 and ran our experiments on StarExec [32] with a wallclock timeout of 300s, a cpu timeout of 1200s, and a memory limit of 128GB per example.

The results can be seen in the table above. We evaluated all tools on the entire benchmark set (Int & Bool) and on the 209 examples that do not use Booleans (Int only). The results show that LoAT is very competitive w.r.t. proving *unsat* in terms of solved instances as well as runtime. The entries in the column “unique” give the number of examples where the respective tool succeeds and all others fail. Here we disregard Eld. Acc., as it would be pointless to consider several variants of the same algorithm in such a comparison. If we consider Eld. Acc. instead of Eldarica, then the numbers change according to the values given in parentheses.

The numbers indicate that LoAT is particularly powerful on examples that operate on Ints only, whereas it is on par with Spacer when proving *unsat* for examples that may also operate on Booleans. This is not surprising, as the core of LoAT’s approach are its acceleration techniques, which have been designed for integers. In contrast, Spacer’s algorithm is similar to GPDR [20], which generalizes the IC3 algorithm [8] from transition systems over Booleans to transition systems over theories (like integers).

Our implementation is open-source and available on Github. For the sources, a pre-compiled binary, and more information on our evaluation, we refer to [1]. In future work, we plan to extend our implementation to also prove *sat*. Moreover, we want to add support for further theories by developing specialized acceleration techniques. Furthermore, we intend to lift ADCL to non-linear CHCs.

⁴ Standard Deviation

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A Additional Definitions

Here, we provide two additional definitions that are used in the proofs.

Definition 20 (Resolution with Ground Instances). *Let φ and φ' be ground instances of CHCs.⁵ If*

⁵ Note that in our setting, in general a ground instance of a CHC is not a CHC, because here the predicates are not applied to variables anymore.

$$\varphi = (\eta \implies F(\vec{s})) \quad \text{and} \quad \varphi' = (F(\vec{s}) \implies \eta'),$$

$$\begin{array}{ll} \text{then} & \text{res}(\varphi, \varphi') := \eta \implies \eta'. \\ \text{Otherwise,} & \text{res}(\varphi, \varphi') := (\perp \implies \perp). \end{array}$$

Here, η can also be \top and η' can also be \perp . For non-empty sequences of ground instances, **res** is defined analogously to sequences of CHCs.

Definition 21 (Liftings of Resolution). We define:

$$\begin{aligned} \text{res}(\mathcal{L}) &:= \{\text{res}(\vec{\pi}) \mid \vec{\pi} \in \mathcal{L}\} && \text{for sets of sequences of CHCs } \mathcal{L} \\ \text{res}(\Phi_1, \Phi_2) &:= \{\text{res}(\varphi_1, \varphi_2) \mid \varphi_1 \in \Phi_1, \varphi_2 \in \Phi_2\} && \text{for sets of CHCs } \Phi_1, \Phi_2 \\ \text{res}(\mathcal{L}_1, \mathcal{L}_2) &:= \text{res}(\text{res}(\mathcal{L}_1), \text{res}(\mathcal{L}_2)) && \text{for sets of sequences of CHCs } \mathcal{L}_1, \mathcal{L}_2 \end{aligned}$$

For sets of (sequences of) ground instances of CHCs, we lift **res** analogously.

B Auxiliary Lemmas

This appendix contains a number of auxiliary lemmas that are needed for the proofs of our main results. We start with a lemma which essentially corresponds to the classical lifting lemma of resolution.

Lemma 22 (Resolution Distributes over grnd). For two CHCs φ_1, φ_2 we have:

$$\text{grnd}(\text{res}(\varphi_1, \varphi_2)) = \{\varphi \in \text{res}(\text{grnd}(\varphi_1), \text{grnd}(\varphi_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\}$$

Proof. If $\text{res}(\varphi_1, \varphi_2) = (\perp \implies \perp)$, then $\text{res}(\pi_1, \pi_2) = (\perp \implies \perp)$ for all ground instances π_1 of φ_1 and π_2 of φ_2 . Thus,

$$\text{grnd}(\text{res}(\varphi_1, \varphi_2)) = \{\varphi \in \text{res}(\text{grnd}(\varphi_1), \text{grnd}(\varphi_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} = \emptyset.$$

Assume $\text{res}(\varphi_1, \varphi_2) \neq (\perp \implies \perp)$. Then we have $\varphi_1 = (\eta_1 \wedge \psi_1 \implies \eta'_1)$, $\varphi_2 = (\eta_2 \wedge \psi_2 \implies \eta'_2)$, and $\theta = \text{mgu}(\eta'_1, \eta_2)$. Then

$$\begin{aligned} & \{\varphi \in \text{res}(\text{grnd}(\varphi_1), \text{grnd}(\varphi_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} \\ &= \{\text{res}(\pi_1, \pi_2) \mid \pi_1 \in \text{grnd}(\varphi_1), \pi_2 \in \text{grnd}(\varphi_2), \models_{\mathcal{A}} \text{cond}(\varphi)\} && (\text{Def. 21}) \\ &= \{\text{res}(\varphi_1\sigma_1, \varphi_2\sigma_2) \mid \sigma_1 \models_{\mathcal{A}} \psi_1, \sigma_2 \models_{\mathcal{A}} \psi_2, \eta'_1\sigma_1 = \eta_2\sigma_2\} && (\text{def. of grnd}) \\ &= \{\eta_1\sigma_1 \implies \eta'_2\sigma_2 \mid \sigma_1 \models_{\mathcal{A}} \psi_1, \sigma_2 \models_{\mathcal{A}} \psi_2, \eta'_1\sigma_1 = \eta_2\sigma_2\} && (\text{Def. 20}) \\ &= \{\eta_1\theta\sigma \implies \eta'_2\theta\sigma \mid \sigma \models_{\mathcal{A}} \psi_1\theta, \sigma \models_{\mathcal{A}} \psi_2\theta\} && (\dagger) \\ &= \{\eta_1\theta\sigma \implies \eta'_2\theta\sigma \mid \sigma \models_{\mathcal{A}} \psi_1\theta \wedge \psi_2\theta\} \\ &= \text{grnd}((\eta_1 \wedge \psi_1 \wedge \psi_2 \implies \eta'_2)\theta) && (\text{by def. of grnd}) \\ &= \text{grnd}(\text{res}(\varphi_1, \varphi_2)) && (\text{Def. 2}) \end{aligned}$$

The step marked with (\dagger) holds as θ is the most general unifier of η'_1 and η_2 and as φ_1 and φ_2 are assumed to be variable disjoint in [Def. 2](#). Therefore, there exists a model σ such that $x\sigma_1 = x\theta\sigma$ for all $x \in \mathcal{V}(\varphi_1)$ and $x\sigma_2 = x\theta\sigma$ for all $x \in \mathcal{V}(\varphi_2)$. \square

Lemma 23 (Associativity of Resolution).

$$\text{grnd}(\text{res}(\text{res}(\varphi_1, \varphi_2), \varphi_3)) = \text{grnd}(\text{res}(\varphi_1, \text{res}(\varphi_2, \varphi_3)))$$

Proof. Let $\varphi \in \text{grnd}(\text{res}(\text{res}(\varphi_1, \varphi_2), \varphi_3))$. Due to Lemma 22, there are ground instances $\varphi'_1, \varphi'_2, \varphi'_3$ of $\varphi_1, \varphi_2, \varphi_3$ such that $\varphi = \text{res}(\text{res}(\varphi'_1, \varphi'_2), \varphi'_3)$. Since $\varphi'_1, \varphi'_2, \varphi'_3$ are ground, there exist ground atoms η_1, \dots, η_4 such that

$$\begin{aligned} & \text{res}(\text{res}(\varphi'_1, \varphi'_2), \varphi'_3) \\ &= \text{res}(\text{res}(\eta_1 \implies \eta_2, \eta_2 \implies \eta_3), \eta_3 \implies \eta_4) \\ &= \text{res}(\eta_1 \implies \eta_3, \eta_3 \implies \eta_4) \\ &= (\eta_1 \implies \eta_4) \\ &= \text{res}(\eta_1 \implies \eta_2, \eta_2 \implies \eta_4) \\ &= \text{res}(\eta_1 \implies \eta_2, \text{res}(\eta_2 \implies \eta_3, \eta_3 \implies \eta_4)) \\ &= \text{res}(\varphi'_1, \text{res}(\varphi'_2, \varphi'_3)) \end{aligned}$$

□

Lemma 24 (Composition of Ground Instances). *If $\text{grnd}(\vec{\varphi}_1) \subseteq \text{grnd}(\vec{\pi}_1)$ and $\text{grnd}(\vec{\varphi}_2) \subseteq \text{grnd}(\vec{\pi}_2)$, then*

$$\text{grnd}(\vec{\varphi}_1 :: \vec{\varphi}_2) \subseteq \text{grnd}(\vec{\pi}_1 :: \vec{\pi}_2).$$

Proof.

$$\begin{aligned} & \text{grnd}(\vec{\varphi}_1 :: \vec{\varphi}_2) \\ &= \text{grnd}(\text{res}(\vec{\varphi}_1), \text{res}(\vec{\varphi}_2)) && \text{(Lemma 23)} \\ &= \{\varphi \in \text{res}(\text{grnd}(\vec{\varphi}_1), \text{grnd}(\vec{\varphi}_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} && \text{(Lemma 22)} \\ &\subseteq \{\varphi \in \text{res}(\text{grnd}(\vec{\pi}_1), \text{grnd}(\vec{\varphi}_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} && (\text{grnd}(\vec{\varphi}_1) \subseteq \text{grnd}(\vec{\pi}_1)) \\ &\subseteq \{\varphi \in \text{res}(\text{grnd}(\vec{\pi}_1), \text{grnd}(\vec{\pi}_2)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} && (\text{grnd}(\vec{\varphi}_2) \subseteq \text{grnd}(\vec{\pi}_2)) \\ &= \text{grnd}(\text{res}(\vec{\pi}_1), \text{res}(\vec{\pi}_2)) && \text{(Lemma 22)} \\ &= \text{grnd}(\vec{\pi}_1 :: \vec{\pi}_2) && \text{(Lemma 23)} \end{aligned}$$

□

Lemma 25 (Resolution Distributes over \mathcal{L}). *If $\Phi \rightsquigarrow^* (II, \vec{\varphi}, \vec{B}, \mathcal{L})$, $\vec{\pi} \in \text{sip}(II)^*$, and $\pi \in \text{sip}(II)$, then*

$$\text{grnd}(\text{res}(\mathcal{L}(\vec{\pi} :: \pi))) = \text{grnd}(\text{res}(\mathcal{L}(\vec{\pi}), \mathcal{L}(\pi))).$$

Proof.

$$\begin{aligned} & \text{grnd}(\text{res}(\mathcal{L}(\vec{\pi}), \mathcal{L}(\pi))) \\ &= \text{grnd}(\text{res}(\text{res}(\mathcal{L}(\vec{\pi})), \text{res}(\mathcal{L}(\pi)))) && \text{(Def. 21)} \\ &= \text{grnd}(\{\text{res}(\pi_1, \pi_2) \mid \pi_1 \in \text{res}(\mathcal{L}(\vec{\pi})), \pi_2 \in \text{res}(\mathcal{L}(\pi))\}) && \text{(Def. 21)} \\ &= \text{grnd}(\{\text{res}(\vec{\pi}_1 :: \vec{\pi}_2) \mid \vec{\pi}_1 \in \mathcal{L}(\vec{\pi}), \vec{\pi}_2 \in \mathcal{L}(\pi)\}) && \text{(Lemma 23)} \\ &= \text{grnd}(\{\text{res}(\vec{\pi}) \mid \vec{\pi} \in \mathcal{L}(\vec{\pi}) :: \mathcal{L}(\pi)\}) \\ &= \text{grnd}(\{\text{res}(\vec{\pi}) \mid \vec{\pi} \in \mathcal{L}(\vec{\pi} :: \pi)\}) && \text{(by definition of } \mathcal{L} \text{)} \\ &= \text{grnd}(\text{res}(\mathcal{L}(\vec{\pi} :: \pi))) && \text{(Def. 21)} \end{aligned}$$

□

C Missing Proofs

C.1 Proof of Thm. 12

The following easy lemmas and definitions are needed to prove soundness of our approach. We first prove that adding learned clauses results in an equivalent CHC problem.

Lemma 26 (\rightsquigarrow Preserves Equivalence). *If $(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow^* (\Pi', \vec{\varphi}', \vec{B}')$, then $\Pi \equiv_{\mathcal{A}} \Pi'$.*

Proof. We only consider the case $(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow (\Pi', \vec{\varphi}', \vec{B}')$. Then the claim also follows for $(\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow^* (\Pi', \vec{\varphi}', \vec{B}')$ by a straightforward induction.

ACCELERATE adds $\text{accel}(\vec{\varphi}^\circ)$ to $\vec{\varphi}$ and Π . Hence, we have to prove that $\sigma \models_{\mathcal{A}} \Pi$ implies $\sigma \models_{\mathcal{A}} \text{accel}(\vec{\varphi}^\circ)$. We have:

$$\begin{aligned}
& \sigma \models_{\mathcal{A}} \Pi \\
& \iff \sigma \models_{\mathcal{A}} \text{sip}(\Pi) && (\text{since } \text{sip}(\Pi) \equiv_{\mathcal{A}} \Pi) \\
& \implies \sigma \models_{\mathcal{A}} \text{res}(\vec{\varphi}^\circ) && (\text{by soundness of resolution, as } \vec{\varphi}^\circ \in \text{sip}(\Pi)^*) \\
& \iff \sigma \models_{\mathcal{A}} \text{grnd}(\vec{\varphi}^\circ) && (\text{as } \sigma\text{'s carrier only contains ground terms over } \Sigma_{\mathcal{A}}) \\
& \iff \sigma \models_{\mathcal{A}} \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}((\vec{\varphi}^\circ)^n) && (\text{by soundness of resolution}) \\
& \iff \sigma \models_{\mathcal{A}} \text{grnd}(\text{accel}(\vec{\varphi}^\circ)) && (\text{by Def. 4}) \\
& \iff \sigma \models_{\mathcal{A}} \text{accel}(\vec{\varphi}^\circ) && (\text{as } \sigma\text{'s carrier only contains ground terms over } \Sigma_{\mathcal{A}})
\end{aligned}$$

All other rules do not modify Π . \square

The following relation is needed to prove the soundness of our calculus.

Definition 27 (\succ_{Π}). *Given a CHC problem Π , $\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3 \in \text{sip}(\Pi)^*$, and $\varphi \in \text{sip}(\Pi)$, we define*

$$\vec{\varphi}_1 :: \vec{\varphi}_2 :: \vec{\varphi}_3 \succ_{\Pi} \vec{\varphi}_1 :: \varphi :: \vec{\varphi}_3 \quad \text{if} \quad \vec{\varphi}_2 \sqsubset \varphi \quad \text{or} \quad \vec{\varphi}_2 \sqsubseteq \varphi \wedge |\vec{\varphi}_2| > 1.$$

Note that the definition of \succ_{Π} is closely related to the definition of **COVERED**. In particular, the trace $\vec{\varphi} :: \vec{\varphi}'$ is not minimal w.r.t. \succ_{Π} whenever **COVERED** is applicable.

Lemma 28. \succ_{Π} is well founded.

Proof. Assume that there is an infinite chain

$$\vec{\varphi}^{(1)} \succ_{\Pi} \vec{\varphi}^{(2)} \succ_{\Pi} \dots$$

Since $\vec{\varphi}^{(i)} \succ_{\Pi} \vec{\varphi}^{(i+1)}$ implies $|\vec{\varphi}^{(i)}| \geq |\vec{\varphi}^{(i+1)}|$ and $>$ is well founded on \mathbb{N} , there is an i_0 such that $|\vec{\varphi}^{(i)}| = |\vec{\varphi}^{(i+1)}|$ for all $i \geq i_0$. Thus, for each $i \geq i_0$, there is a unique j_i such that $\varphi_{j_i}^{(i)} \sqsubset \varphi_{j_i}^{(i+1)}$ by the definition of \succ_{Π} . Hence, there is at least one j such that $j = j_i$ for infinitely many values i_1, i_2, \dots of i . Therefore, we have

$$\varphi_j^{(i_1)} \sqsubset \varphi_j^{(i_2)} \sqsubset \dots$$

Since \sqsubset is transitive and $\varphi_j^{(i_k)} \sqsubset \varphi_j^{(i_{k'})}$ implies $\varphi_j^{(i_k)} \neq \varphi_j^{(i_{k'})}$, $\{\varphi_j^{(i_k)} \mid k \in \mathbb{N}_{>0}\}$ is infinite. However, $\{\varphi_j^{(i_k)} \mid k \in \mathbb{N}_{>0}\}$ is a subset of $\text{sip}(\Pi)$, which is finite, so we derived a contradiction. \square

Now we are ready to prove soundness of our approach.

Theorem 12 (Soundness). *If $\Phi \rightsquigarrow^* \text{sat}$, then Φ is satisfiable. If $\Phi \rightsquigarrow^* \text{unsat}$, then Φ is unsatisfiable.*

Proof. For **unsat**, assume $\Phi \rightsquigarrow^* (\Pi, \vec{\varphi}, \vec{B}) \rightsquigarrow \text{unsat}$. Then $\vec{\varphi} \in \text{sip}(\Pi)^*$ is a refutation by definition of **REFUTE**. Thus, by soundness of resolution, $\text{sip}(\Pi)$ and hence also Π is unsatisfiable. By [Lemma 26](#), we have $\Phi \equiv_{\mathcal{A}} \Pi$, and thus Φ is unsatisfiable, too.

For **sat**, we use proof by contradiction. Assume that Φ is unsatisfiable and

$$\Phi \rightsquigarrow (\Pi^{(1)}, \vec{\varphi}^{(1)}, \vec{B}^{(1)}) \rightsquigarrow \dots \rightsquigarrow (\Pi^{(m)}, \vec{\varphi}^{(m)}, \vec{B}^{(m)}) \rightsquigarrow \text{sat}. \quad (1)$$

By [Lemma 26](#), we have $\Phi \equiv_{\mathcal{A}} \Pi^{(m)}$, so $\Pi^{(m)}$ is also unsatisfiable. Thus, $\text{sip}(\Pi^{(m)})$ is unsatisfiable as well. By completeness of binary input resolution for Horn clauses, $\text{sip}(\Pi^{(m)})^*$ contains at least one refutation. Thus, due to [Lemma 28](#), $\text{sip}(\Pi^{(m)})^*$ contains at least one $\succ_{\Pi^{(m)}}$ -minimal refutation $\vec{\varphi}$.

Let $k_i = |\vec{\varphi}^{(i)}|$ and $s_i = (\Pi^{(i)}, \vec{\varphi}^{(i)}, \vec{B}^{(i)})$ for all $1 \leq i \leq m$. We say that $\vec{\varphi}$ is *disabled* in s_i if there is a $0 \leq \ell \leq k_i$ and a $\varphi \in B_\ell^{(i)}$ such that $\varphi_j^{(i)} \equiv_{\mathcal{A}} \varphi_j$ for all $1 \leq j \leq \ell$ and $\varphi \equiv_{\mathcal{A}} \varphi_{\ell+1}$. Then $\vec{\varphi}$ is disabled in s_m . To see this, note that we have $\varphi_1 \sqsubseteq B_0^{(m)}$ by definition of **PROVE**, and $\varphi_1 \not\sqsubseteq B_0^{(m)}$ as $\vec{\varphi}$ is $\succ_{\Pi^{(m)}}$ -minimal. Hence there is a $\varphi \in B_0^{(m)}$ such that $\varphi_1 \equiv_{\mathcal{A}} \varphi$, so $\vec{\varphi}$ is disabled in s_m . Let $1 \leq i < m$ be the largest index such that $\vec{\varphi}$ is enabled in s_i . Note that such an index exists, as $\vec{\varphi}$ is enabled in s_1 , since $\vec{B}^{(1)} = [\emptyset]$ by definition of **INIT**.

If $s_i \xrightarrow{S} s_{i+1}$, then $\vec{\varphi}$ is disabled in s_i iff it is disabled in s_{i+1} , as

$$B_{k_{i+1}}^{(i+1)} = \emptyset \quad \text{and} \quad \forall j \in \{1, \dots, k_{i+1} - 1\}. \varphi_j^{(i+1)} = \varphi_j^{(i)} \wedge B_j^{(i+1)} = B_j^{(i)}$$

by definition of **STEP**. Hence, $s_i \not\xrightarrow{S} s_{i+1}$.

If $s_i \xrightarrow{A} s_{i+1}$, then

$$B_{k_{i+1}}^{(i+1)} = \{\varphi_{k_{i+1}}^{(i+1)}\} \quad \text{and} \quad \forall j \in \{1, \dots, k_{i+1} - 1\}. \varphi_j^{(i+1)} = \varphi_j^{(i)} \wedge B_j^{(i+1)} = B_j^{(i)}$$

by definition of **ACCELERATE**. Hence, since $\vec{\varphi}$ is enabled in s_i , but disabled in s_{i+1} , we get $\varphi_j^{(i+1)} \equiv_{\mathcal{A}} \varphi_j$ for all $1 \leq j \leq k_{i+1}$ and $\varphi_{k_{i+1}}^{(i+1)} \equiv_{\mathcal{A}} \varphi_{k_{i+1}+1}$. So in particular, we have $\varphi_{k_{i+1}} \equiv_{\mathcal{A}} \varphi_{k_{i+1}+1} \equiv_{\mathcal{A}} \varphi_{k_{i+1}}^{(i+1)}$. By definition of **ACCELERATE**, there is a $\vec{\varphi}^\circ \in \text{sip}(\Pi^{(i)})^*$ such that $\varphi_{k_{i+1}}^{(i+1)} = \text{accel}(\vec{\varphi}^\circ)$. Hence, we have $[\varphi_{k_{i+1}}^{(i+1)}, \varphi_{k_{i+1}}^{(i+1)}] \sqsubseteq \varphi_{k_{i+1}}^{(i+1)}$ and thus $[\varphi_{k_{i+1}}, \varphi_{k_{i+1}+1}] \sqsubseteq \varphi_{k_{i+1}}^{(i+1)}$ by [Def. 4](#), contradicting $\succ_{\Pi^{(m)}}$ -minimality of $\vec{\varphi}$.

If $s_i \xrightarrow{C} s_{i+1}$ or $s_i \xrightarrow{B} s_{i+1}$, then we have

$$\begin{aligned} B_{k_{i+1}}^{(i+1)} &= B_{k_{i+1}}^{(i)} \cup \{\varphi_{k_i}^{(i)}\}, \\ \forall j \in \{1, \dots, k_{i+1}\}. \varphi_j^{(i+1)} &= \varphi_j^{(i)}, \quad \text{and} \\ \forall j \in \{1, \dots, k_{i+1} - 1\}. B_j^{(i+1)} &= B_j^{(i)} \end{aligned}$$

by definition of **bt**. Since $\vec{\varphi}$ is enabled in s_i , but disabled in s_{i+1} , we get $\varphi_j^{(i+1)} = \varphi_j^{(i)} \equiv_{\mathcal{A}} \varphi_j$ for all $1 \leq j \leq k_{i+1} = k_i - 1$ and $\varphi_{k_i}^{(i)} \equiv_{\mathcal{A}} \varphi_{k_i}$. So in particular,

we have $\varphi_j^{(i)} \equiv_{\mathcal{A}} \varphi_j$ for all $1 \leq j \leq k_i$. If $s_i \xrightarrow{\mathcal{C}} s_{i+1}$, then the definition of **COVERED** implies that $\vec{\varphi}^{(i)}$ is not $\succ_{\Pi^{(i)}}$ -minimal, contradicting $\succ_{\Pi^{(i)}}$ -minimality and therefore also $\succ_{\Pi^{(m)}}$ -minimality of $\vec{\varphi}$.

If $s_i \xrightarrow{\mathcal{B}} s_{i+1}$, then *all* rules and queries are inactive in the state s_i by definition of **BACKTRACK**. Thus, we have $\varphi_{k_i+1} \sqsubseteq B_{k_i}^{(i)}$. Moreover, as $\vec{\varphi}$ is enabled in s_i , we have $\varphi_{k_i+1} \not\equiv_{\mathcal{A}} \varphi$ for all $\varphi \in B_{k_i}^{(i)}$. Thus, we get $\varphi_{k_i+1} \sqsubset B_{k_i}^{(i)}$, contradicting $\succ_{\Pi^{(m)}}$ -minimality of $\vec{\varphi}$. \square

C.2 Proof of Thm. 13

Theorem 13 (Normal Forms). *If $\Phi \rightsquigarrow^+ s$ where s is in normal form w.r.t. \rightsquigarrow , then $s \in \{\text{sat}, \text{unsat}\}$.*

Proof. Let $s = (\Pi, \vec{\varphi}, \vec{B})$. If $\vec{\varphi}$ is empty, then **STEP** or **PROVE** is applicable. If $\vec{\varphi}$ is non-empty, then either **STEP** or **BACKTRACK** is applicable. Hence, s is not in normal form. \square

C.3 Proof of Thm. 15

Theorem 15 (Refutational Completeness). *If Φ is unsatisfiable, then*

$$\Phi \rightsquigarrow_{rs}^* \text{unsat}.$$

Proof. Due to completeness of binary input resolution for Horn Clauses, there is a refutation $\vec{\varphi} \in \text{sip}(\Phi)^*$. We inductively define a \rightsquigarrow_{rs} -derivation of the form

$$\begin{aligned} \Phi &\rightsquigarrow_{rs} (\Phi, [], [\emptyset]) = (\Pi_0, \vec{\varphi}_0, \vec{B}_0) = s_0 \\ &\rightsquigarrow_{rs} (\Phi, [\varphi_1], [\emptyset, \emptyset]) = (\Pi_1, \vec{\varphi}_1, \vec{B}_1) = s_1 \\ &\rightsquigarrow_{rs} (\Pi_2, \vec{\varphi}_2, \vec{B}_2) = s_2 \\ &\rightsquigarrow_{rs} \dots \\ &\rightsquigarrow_{rs} (\Pi_k, \vec{\varphi}_k, \vec{B}_k) = s_k \\ &\rightsquigarrow_{rs} \text{unsat} \end{aligned}$$

such that $J_i = \{j \mid 1 \leq j \leq |\vec{\varphi}|, [\varphi_m]_{m=1}^j \sqsubseteq \vec{\varphi}_i\}$ is non-empty for each $1 \leq i \leq k$. Here, φ_m are the elements of $\vec{\varphi}$. Then the claim follows immediately. In the following, j_i denotes $\max(J_i)$. Given the state $s_i = (\Pi_i, \vec{\varphi}_i, \vec{B}_i)$, we distinguish the following cases:

- (1) If $\vec{\varphi}_i$ has a minimal recursive suffix $\vec{\varphi}^{\circ}$ such that $\text{accel}(\vec{\varphi}^{\circ})$ is not redundant, then we apply **ACCELERATE**, resulting in the state s_{i+1} .
- (2) Otherwise, let $c \in \mathbb{N}_{\geq 1}$ be the maximal number such that $[\varphi_m]_{m=j_i+1}^{j_i+c} \sqsubseteq \text{sip}(\Pi_i)$. Then we apply **STEP** with some

$$\pi \in \max_{\sqsubseteq} \left\{ \pi \in \text{sip}(\Pi_i) \mid [\varphi_m]_{m=j_i+1}^{j_i+c} \sqsubseteq \pi \right\}$$

(which may not be unique, as \sqsubseteq is a partial order), resulting in the state s_{i+1} with $j_{i+1} = j_i + c$.

Note that a clause π as in (2) always exists. The reason is that $[\varphi_m]_{m=j_i+1}^{j_i+1} \sqsubseteq \varphi_{j_i+1}$ and $\varphi_{j_i+1} \in \text{sip}(\Pi_i)$. Hence, there always exists a $c \in \mathbb{N}_{\geq 1}$ with $[\varphi_m]_{m=j_i+1}^{j_i+c} \sqsubseteq \text{sip}(\Pi_i)$.

To show $\Phi \rightsquigarrow s_1 \rightsquigarrow \dots \rightsquigarrow s_k$, it remains to show that the clause π in (2) is not blocked. To this end, we prove

If π is blocked in state s_i , then $[\varphi_m]_{m=j_i+1}^{j_i+c} \not\sqsubseteq \pi$ for all $c \geq 1$.

This implies that the clause π in (2) is not blocked and thus **STEP** is applicable. Assume that π is blocked in state s_i . Then π is the last element of the trace by definition of **ACCELERATE**, and we have $\text{res}(\pi, \pi) \sqsubseteq \pi$, since this holds for every accelerated clause π . Moreover, $j_i = j_{i-1} + c'$, where $c' \geq 0$ is the maximal number such that $[\varphi_m]_{m=1}^{j_{i-1}+c'} \sqsubseteq \vec{\varphi}_i$. Hence, there is a suffix $\vec{\pi}$ of $[\varphi_m]_{m=1}^{j_i}$ such that $|\vec{\pi}| \geq c'$, $\vec{\pi} \sqsubseteq \pi$, and

$$\vec{\pi} :: [\varphi_m]_{m=j_i+1}^{j_i+c''} \not\sqsubseteq \pi \text{ for all } c'' > 0 \quad (2)$$

by maximality of c' . Assume that there is a $c \in \mathbb{N}$ with $[\varphi_m]_{m=j_i+1}^{j_i+c} \sqsubseteq \pi$. Then we have

$$\vec{\pi} :: [\varphi_m]_{m=j_i+1}^{j_i+c} \sqsubseteq \text{res}(\pi, \pi) \sqsubseteq \pi$$

and hence (2) implies $c = 0$.

Hence, we have $\Phi \rightsquigarrow s_0 \rightsquigarrow \dots \rightsquigarrow s_k$. Next, we prove that this sequence is reasonable. We handle each item of **Def. 14** individually.

Def. 14 (1): This case never applies. To show this, we prove that $i' < i$, $\Pi_{i'} = \Pi_i$, and $\vec{\varphi}_i = \vec{\varphi}_{i'} :: \vec{\pi}'$ implies $\vec{\pi}' \not\sqsubseteq \text{sip}(\Pi_i)$ if $|\vec{\varphi}_i| = 1$, and $\vec{\pi}' \not\sqsubseteq \text{sip}(\Pi_i)$ if $|\vec{\varphi}_i| > 1$. Assume $\Pi_{i'} = \Pi_i$ and $\vec{\varphi}_i = \vec{\varphi}_{i'} :: \vec{\pi}'$ for some $i' < i$. Then $\Pi_{i'} = \Pi_i$ implies that $s_{i'+1}, \dots, s_i$ were constructed via (2). Thus, if π'_1 is the first CHC of the sequence $\vec{\pi}'$, then we have $[\varphi_m]_{m=j_{i'}+1}^{j_{i'}+c} \sqsubseteq \pi'_1$ for some $c \in \mathbb{N}_{\geq 1}$, which has to be maximal according to (2). If $|\vec{\pi}'| = 1$, then $\vec{\pi}' = \pi'_1 \sqsubset \text{sip}(\Pi_i)$ would contradict the requirement in (2) that π'_1 is maximal w.r.t. \sqsubset . Assume $|\vec{\pi}'| > 1$. As $s_{i'+1}$ was also constructed via (2), $|\vec{\pi}'| > 1$ implies $j_i > j_{i'} + 1$. Hence, we get $[\varphi_m]_{m=j_{i'}+1}^{j_i} \not\sqsubseteq \text{sip}(\Pi_i)$, as $j_i > j_{i'} + 1 = j_{i'} + c$, but c is the maximal natural number such that $[\varphi_m]_{m=j_{i'}+1}^{j_{i'}+c} \sqsubseteq \Pi_i$. Since $[\varphi_m]_{m=j_{i'}+1}^{j_i} \sqsubseteq \vec{\pi}'$, this implies $\vec{\pi}' \not\sqsubseteq \text{sip}(\Pi_i)$.

Def. 14 (2): Trivial, as (1) is applied with higher preference than (2).

Def. 14 (3): Trivial due to (1).

Def. 14 (4): The first **STEP** in the constructed \rightsquigarrow -sequence uses a CHC φ such that we have $\vec{\varphi}' \sqsubseteq \varphi$ for a non-empty prefix $\vec{\varphi}'$ of $\vec{\varphi}$. Since $\vec{\varphi}$ is a refutation, its first element is a fact or a conditional empty clause. Hence, $\text{res}(\vec{\varphi}')$ is a fact or a conditional empty clause, too. Thus, φ must be a fact or conditional empty clause as well.

Next, we prove that each J_i is non-empty, i.e., that for each $1 \leq i \leq k$, there is a $1 \leq j \leq |\vec{\varphi}|$ such that $[\varphi_m]_{m=1}^j \sqsubseteq \vec{\varphi}_i$. We use induction on $1 \leq i \leq k$, where the case $i = 1$ is immediate due to the choice of π in (2). Let $i > 1$. First assume that (1) was applied to the state s_{i-1} and $\vec{\varphi}_{i-1} = \vec{\varphi}'_{i'} :: \vec{\varphi}^{\circ}$ for some $i' \in \mathbb{N}_{\geq 1}$ where $i' < i - 1$. Then by the induction hypothesis, J_{i-1} is non-empty and thus j_{i-1} exists. We get:

$$\begin{aligned}
& \text{grnd}([\varphi_m]_{m=1}^{j_{i-1}}) \\
& \subseteq \text{grnd}(\vec{\varphi}_{i-1}) \tag{IH} \\
& = \text{grnd}(\vec{\varphi}_{i'} :: \vec{\varphi}^\circ) \tag{as } \vec{\varphi}_{i-1} = \vec{\varphi}'_{i'} :: \vec{\varphi}^\circ \\
& \subseteq \text{grnd}(\vec{\varphi}_{i'} :: \text{accel}(\vec{\varphi}^\circ)) \tag{Lemma 24, as } \vec{\varphi}^\circ \sqsubseteq \text{accel}(\vec{\varphi}^\circ) \\
& = \text{grnd}(\vec{\varphi}_i)
\end{aligned}$$

Hence, $j_{i-1} \in J_i$, i.e., J_i is non-empty. If (2) was applied to the state s_{i-1} , then J_i is trivially non-empty due to the choice of π in (2).

Finally, note that the resulting sequence $\Phi \rightsquigarrow s_0 \rightsquigarrow \dots \rightsquigarrow s_k$ is finite: For (2), we have $j_i < j_{i+1}$, and the sequence j_1, j_2, \dots is bounded by $|\vec{\varphi}|$. For (1), we have $j_i \leq j_{i+1}$, but the (finite) number of recursive infixes of $\vec{\varphi}$ that are not yet redundant is decremented. Hence, we get a lexicographic termination argument, which ensures that there is an s_k with $j_k = |\vec{\varphi}|$. This implies $s_k \rightsquigarrow \text{unsat}$. \square

C.4 Proof of Lemma 17

Lemma 17. *If $\Phi \rightsquigarrow_{rs}^* (\Pi, \vec{\varphi}, \vec{B}, \mathcal{L})$ and $\pi \in \text{sip}(\Pi)$, then $\text{grnd}(\pi) = \text{grnd}(\mathcal{L}(\pi))$.*

Proof. We use induction on the length of the \rightsquigarrow_{rs} -derivation. If $\Phi \rightsquigarrow_{rs} (\Pi, \vec{\varphi}, \vec{B}, \mathcal{L})$, then the claim follows from the definition of **INIT**. Assume

$$\Phi \rightsquigarrow_{rs}^* (\Pi', \vec{\varphi}', \vec{B}', \mathcal{L}') \rightsquigarrow_{rs} (\Pi, \vec{\varphi}, \vec{B}, \mathcal{L}).$$

If the last step is not **ACCELERATE**, then the claim follows from the induction hypothesis. Assume that the last step is **ACCELERATE**. Then we have $\Pi = \Pi' \cup \{\varphi\}$ and $\mathcal{L} = \mathcal{L}' \uplus \varphi \mapsto \mathcal{L}'(\vec{\varphi}^\circ)^+$. If $\pi \neq \varphi$, then the claim follows from the induction hypothesis, so assume $\pi = \varphi$. Then it remains to show

$$\text{grnd}(\varphi) = \text{grnd}(\mathcal{L}(\varphi)) = \text{grnd}(\mathcal{L}'(\vec{\varphi}^\circ)^+), \text{ where } \varphi = \text{accel}(\vec{\varphi}^\circ). \tag{3}$$

By the induction hypothesis, we have $\text{grnd}(\varphi') = \text{grnd}(\mathcal{L}'(\varphi'))$ for all $\varphi' \in \vec{\varphi}^\circ$. We first show

$$\text{grnd}(\vec{\varphi}^\circ) = \text{grnd}(\mathcal{L}'(\vec{\varphi}^\circ)). \tag{4}$$

To this end, we show that $\vec{\pi} \in \text{sip}(\Pi')^*$, $\pi \in \text{sip}(\Pi')$, $\text{grnd}(\vec{\pi}) = \text{grnd}(\mathcal{L}'(\vec{\pi}))$ and $\text{grnd}(\pi) = \text{grnd}(\mathcal{L}'(\pi))$ imply $\text{grnd}(\vec{\pi} :: \pi) = \text{grnd}(\mathcal{L}'(\vec{\pi} :: \pi))$. Then (4) follows by a straightforward induction over $|\vec{\varphi}^\circ|$. We have:

$$\begin{aligned}
& \text{grnd}(\vec{\pi} :: \pi) \\
& = \{\varphi \in \text{res}(\text{grnd}(\vec{\pi}), \text{grnd}(\pi)) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} \tag{Lemma 22} \\
& = \{\varphi \in \text{res}(\text{grnd}(\mathcal{L}'(\vec{\pi}), \text{grnd}(\mathcal{L}'(\pi))) \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} \tag{IH} \\
& = \text{grnd}(\text{res}(\mathcal{L}'(\vec{\pi}), \mathcal{L}'(\pi))) \tag{Lemma 22} \\
& = \text{grnd}(\text{res}(\mathcal{L}'(\vec{\pi} :: \pi))) \tag{Lemma 25} \\
& = \text{grnd}(\mathcal{L}'(\vec{\pi} :: \pi)) \tag{by the definition of grnd}
\end{aligned}$$

Now we prove (3):

$$\text{grnd}(\text{accel}(\vec{\varphi}^\circ))$$

$$\begin{aligned}
&= \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}((\bar{\varphi}^\circ)^n) && (\text{Def. 4}) \\
&= \bigcup_{n \in \mathbb{N}_{\geq 1}} \{\varphi \in \text{grnd}(\bar{\varphi}^\circ)^n \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} && (\text{Lemma 22}) \\
&= \bigcup_{n \in \mathbb{N}_{\geq 1}} \{\varphi \in \text{grnd}(\mathcal{L}'(\bar{\varphi}^\circ))^n \mid \models_{\mathcal{A}} \text{cond}(\varphi)\} && (\text{by (4)}) \\
&= \bigcup_{n \in \mathbb{N}_{\geq 1}} \text{grnd}(\mathcal{L}'(\bar{\varphi}^\circ)^n) && (\text{Lemma 22}) \\
&= \text{grnd}(\mathcal{L}'(\bar{\varphi}^\circ)^+) && (\text{by the definition of Kleene plus})
\end{aligned}$$

□

C.5 Proof of Thm. 18

Theorem 18 (Non-Termination). *There exists a satisfiable CHC problem Φ such that $\Phi \not\rightsquigarrow_{rs}^* \text{sat}$. Thus, \rightsquigarrow_{rs} does not terminate.*

Proof.

Proof Idea:

We will construct a satisfiable CHC problem Φ such that $\Phi \not\rightsquigarrow_{rs}^* \text{sat}$. Since Φ is satisfiable, we also have $\Phi \not\rightsquigarrow_{rs}^* \text{unsat}$ by the soundness of our calculus (Thm. 12). But by Thm. 13, the only normal forms w.r.t. \rightsquigarrow (and also w.r.t. \rightsquigarrow_{rs}) are **sat** and **unsat**. Therefore, \rightsquigarrow_{rs} does not terminate when starting with Φ . In fact, it is not even normalizing (i.e., there does not *exist* any derivation to a normal form).

Construction of Φ :

To construct Φ , we consider the Thue-Morse sequence $[v_i]_{i \in \mathbb{N}}$ [30]. Let $w_i := v_{i+1} - v_i$. The resulting infinite sequence $[w_i]_{i \in \mathbb{N}}$ over the alphabet $\{-1, 0, 1\}$ is well-known to be *square-free*, i.e., it does not contain a non-empty infix $u :: u$ [31]. Let Φ be a satisfiable CHC problem that contains

$$\top \implies \text{ThueMorse}(0, 1), \quad (\varphi_f)$$

the following CHCs $\Phi_{\text{ThueMorse}}$,

$$\begin{aligned}
&\text{ThueMorse}(I, X) \wedge X = -1 \wedge J = I + 1 \implies \text{Next}(J) \\
&\text{ThueMorse}(I, X) \wedge X = 0 \wedge J = I + 1 \implies \text{Next}(J) \\
&\text{ThueMorse}(I, X) \wedge X = 1 \wedge J = I + 1 \implies \text{Next}(J),
\end{aligned}$$

and CHCs Φ_{Next} that are equivalent to

$$\text{Next}(I) \wedge X = w_I \implies \text{ThueMorse}(I, X). \quad (5)$$

Note that Φ_{Next} exists, since $[w_i]_{i \in \mathbb{N}}$ is computable and linear CHCs are Turing complete. W.l.o.g., we assume $\text{sip}(\Phi) = \Phi$ in the sequel. Then for each $n \in \mathbb{N}$, there is a unique sequence

$$\vec{\varphi}_n \in \varphi_f :: (\Phi_{\text{ThueMorse}} :: \Phi_{\text{Next}}^+)^n$$

such that $\text{grnd}(\vec{\varphi}_n) = \{\top \implies \text{ThueMorse}(n, w_n)\}$. Hence, $\langle \vec{\varphi}_n \rangle_{\Phi_{\text{ThueMorse}}}$ (i.e., the sequence that results from $\vec{\varphi}_n$ by omitting all CHCs that are not contained in $\Phi_{\text{ThueMorse}}$) is square-free.

Contradicting $\Phi \rightsquigarrow_{rs}^* \text{sat}$:

Assume

$$\Phi \rightsquigarrow_{rs} s_1 \rightsquigarrow_{rs} \dots \rightsquigarrow_{rs} s_k = (\Pi, [], [B], \mathcal{L}) \rightsquigarrow_{rs} \text{sat}.$$

1. Finitely many square-free words in $\langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}$:

We first prove that any finite union $\mathcal{L} = \bigcup_{i=1}^c \mathcal{L}_i$ of languages \mathcal{L}_i over a finite alphabet that are built from singleton languages, concatenation, and Kleene plus just contains finitely many square-free words. To this end, it suffices to prove that each \mathcal{L}_i contains at most one square-free word. Let $1 \leq i \leq c$ be arbitrary but fix. We use induction on \mathcal{L}_i . If \mathcal{L}_i is a singleton language, then the claim is trivial. If $\mathcal{L}_i = \mathcal{L}' :: \mathcal{L}''$, then the square-free words in \mathcal{L}_i are a subset of

$$\{w' :: w'' \mid w' \in \mathcal{L}', w'' \in \mathcal{L}'', w' \text{ and } w'' \text{ are square-free}\}. \quad (6)$$

By the induction hypothesis, there is at most one square-free word $w' \in \mathcal{L}'$ and at most one square-free word $w'' \in \mathcal{L}''$. Thus, the size of (6) is at most one. If $\mathcal{L} = (\mathcal{L}')^+$, then \mathcal{L} contains the same square-free words as \mathcal{L}' . To see this, let $w \in (\mathcal{L}')^+$ be square-free. Then there are $n \in \mathbb{N}_{\geq 1}$, $v_1, \dots, v_n \in \mathcal{L}'$ such that $v_1 :: \dots :: v_n = w$. Since w is square-free, each v_i must be square-free, too. As \mathcal{L}' contains at most one square-free word by the induction hypothesis, we get $v_1 = \dots = v_n$. Since w is square-free, this implies $n = 1$ (or that w and the v_i are the empty word).

Note that

$$\mathcal{L}(\Pi) = \bigcup_{\pi \in \Pi} \mathcal{L}(\pi)$$

is a finite union of languages over Φ that are built from singleton languages, concatenation, and Kleene plus. Then

$$\langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}} := \{\langle w \rangle_{\Phi_{\text{ThueMorse}}} \mid w \in \mathcal{L}(\Pi)\}$$

is a finite union of languages over $\Phi_{\text{ThueMorse}}$ that are built from singleton languages, concatenation, and Kleene plus. Thus, $\langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}$ just contains finitely many square-free words.

2. Extending Φ to an unsatisfiable CHC problem Φ' :

Thus, there is an $m > 1$ such that $\langle \vec{\varphi}_m \rangle_{\Phi_{\text{ThueMorse}}} \notin \langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}$. To see that, note that the words $\langle \vec{\varphi}_n \rangle_{\Phi_{\text{ThueMorse}}}$ are square-free and pairwise different, but $\langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}$ only contains finitely many square-free words. Let

$$\varphi_q := \text{ThueMorse}(I, X) \wedge I = m \wedge X = w_m \implies \perp,$$

$$\Phi' := \Phi \cup \{\varphi_q\},$$

and

$$s_i := (\Pi_i, \vec{\pi}_i, \vec{B}_i, \mathcal{L}_i)$$

for each $i \in [1, k]$.

3. $\Phi' \rightsquigarrow^* \text{sat}$:

Then we get

$$\Phi' \rightsquigarrow_{rs} s'_1 \rightsquigarrow_{rs} \dots \rightsquigarrow_{rs} s'_k \rightsquigarrow_{rs} \text{sat}$$

where for each $1 \leq i \leq k$, we have

$$\Pi'_i := \Pi_i \cup \{\varphi_q\}, \quad \mathcal{L}'_i := \mathcal{L}_i \uplus \varphi_q \mapsto \{\varphi_q\}, \quad \text{and} \quad s'_i := (\Pi'_i, \vec{\pi}_i, \vec{B}_i, \mathcal{L}'_i).$$

To prove this, it suffices to show $\Phi' \rightsquigarrow_{rs}^k s'_k$, as we trivially have $s'_k \rightsquigarrow \text{sat}$ since s'_k and s_k contain the same facts and no conditional empty clauses. To this end, we prove $\Phi' \rightsquigarrow_{rs}^i s'_i$ for all $1 \leq i \leq k$ by induction on i .

3.1. Induction Base: For $i = 1$, the only applicable rule is [INIT](#), i.e., we have

$$\begin{aligned} \Phi &\rightsquigarrow_{rs}^I (\Phi, [], [\emptyset], \varphi \mapsto \{\varphi\}) = (\Pi_1, \vec{\pi}_1, \vec{B}_1, \mathcal{L}_1) & \text{and} \\ \Phi' &\rightsquigarrow_{rs}^I (\Phi', [], [\emptyset], \varphi \mapsto \{\varphi\}) = (\Pi'_1, \vec{\pi}_1, \vec{B}_1, \mathcal{L}'_1). \end{aligned}$$

3.2. Induction Step – Trivial Cases: For $i > 1$, we perform a case analysis on the rule that is used for the step $s_{i-1} \rightsquigarrow_{rs} s_i$. [INIT](#) does not apply to s_{i-1} . For [STEP](#), if φ was active in the i^{th} step starting from Φ , then it is also active in the i^{th} step starting from Φ' , as \vec{B}_{i-1} is the same in both derivations. Moreover, we get the same set B and thus, the same \vec{B}_i , since the same clauses are learned in the first $i - 1$ (identical) steps of both \rightsquigarrow_{rs} -derivations. For [COVERED](#), we have $\vec{\varphi}' \sqsubseteq \Pi_i$ iff $\vec{\varphi}' \sqsubseteq \Pi'_i$, and $\vec{\varphi}' \sqsubseteq \Pi_i$ iff $\vec{\varphi}' \sqsubseteq \Pi'_i$. The reason is that every model of Φ is also a model of Φ' and thus $\vec{\varphi}' \sqsubseteq \varphi_q$ would imply unsatisfiability of Φ , but Φ is satisfiable. For the same reason, we have $\text{accel}(\vec{\varphi}^\circ) \sqsubseteq \Pi_i$ iff $\text{accel}(\vec{\varphi}^\circ) \sqsubseteq \Pi'_i$ in the case of [ACCELERATE](#).

3.3. Induction Step – Backtrack: For [BACKTRACK](#), we have to show that φ_q is inactive. Since φ_q is clearly not blocked, we have to show that $\text{cond}(\vec{\pi}_{i-1} :: \varphi_q)$ is unsatisfiable. Assume otherwise. Then we have:

$$\begin{aligned} \vec{\varphi}_m &= \varphi_f :: \vec{\varphi} & \text{for some } \vec{\varphi} \in \Phi^* \\ \vec{\pi}_{i-1} &= \varphi_f :: \vec{\pi} :: \pi & \text{for some } \vec{\pi} :: \pi \in \Pi^+ \\ \top &\implies \text{ThueMorse}(m, w_m) \in \text{grnd}(\vec{\pi}_{i-1}) \\ \text{ThueMorse}(0, 1) &\implies \text{ThueMorse}(m, w_m) \in \text{grnd}(\vec{\pi} :: \pi) \end{aligned}$$

To see why the equality $\vec{\pi}_{i-1} = \varphi_f :: \vec{\pi} :: \pi$ holds (i.e., why $|\vec{\pi}_{i-1}| > 1$), note that $\vec{\pi}_{i-1} = [\varphi_f]$ would imply $m = 0$, contradicting $m > 1$ (i.e., then $\text{cond}(\vec{\pi}_{i-1} :: \varphi_q)$ would be unsatisfiable).

Step 3.3.1. We show $\vec{\varphi} \in \mathcal{L}(\Pi)$: Note that some clause from $\Phi_{\text{ThueMorse}}$ is blocked in state s_{i-1} (as [BACKTRACK](#) applies to s_{i-1}). The reason is that the head symbol of the last clause in $\vec{\pi}_{i-1}$ is [ThueMorse](#). Thus, one of the clauses φ' of $\Phi_{\text{ThueMorse}}$ would be applicable (i.e., $\text{cond}(\vec{\pi}_{i-1} :: \varphi')$ is satisfiable). But since the clause φ' is inactive, it must be blocked. Thus, there is a $j < i - 1$ such that $\vec{\pi}_j = \vec{\pi}_{i-1}$

and no element of $\Phi_{\text{ThueMorse}}$ is blocked (right after adding π to the trace via [ACCELERATE](#) or [STEP](#)). We cannot have $j = i - 1$ because when adding π to the trace via [ACCELERATE](#) or [STEP](#), we reach a state where no clause of $\Phi_{\text{ThueMorse}}$ is blocked.

Case 3.3.1.1. $\vec{\pi}$ contains clauses with body symbol ThueMorse: If $\vec{\pi}$ contains $c > 0$ clauses with body symbol **ThueMorse**, then π_1 is one of them, as $\vec{\pi}_j$ starts with φ_f . Thus, $\vec{\pi} :: \pi$ is recursive, i.e., $\vec{\pi}_j$ has a recursive suffix with ground instance

$$\text{ThueMorse}(0, 1) \implies \text{ThueMorse}(m, w_m). \quad (7)$$

If $\text{accel}(\vec{\pi} :: \pi) \sqsubseteq \Pi_j$, then $\Pi_j \subseteq \Pi$ contains a clause with ground instance (7).

Otherwise, a reasonable strategy needs to apply [ACCELERATE](#) in state s_j . Then we get $\text{accel}(\vec{\pi} :: \pi) \sqsubseteq \Pi$ by induction on c : If $c = 1$, then $\text{accel}(\vec{\pi} :: \pi) \in \Pi_{j+1} \subseteq \Pi$. If $c > 1$ and $\text{accel}(\vec{\pi} :: \pi) \not\sqsubseteq \Pi_{j+1}$, then $\vec{\pi}_{j+1}$ has a recursive suffix $\vec{\varphi}^\circ \sqsubseteq \vec{\pi} :: \pi$ (due to [Lemma 23](#)) such that $\text{accel}(\vec{\varphi}^\circ)$ is not redundant, and hence a reasonable strategy again needs to apply [ACCELERATE](#) in state s_{j+1} . Thus, $\text{accel}(\vec{\pi} :: \pi) \sqsubseteq \Pi$ follows from the induction hypothesis

So Π contains a clause with ground instance (7). Thus, we have

$$(7) \in \text{grnd}(\Pi) \stackrel{\text{Lemma 17}}{=} \text{grnd}(\mathcal{L}(\Pi)). \quad (8)$$

Note that for all $\vec{\varphi}' \in \Phi^*$, we have

$$(7) \in \text{grnd}(\vec{\varphi}') \quad \text{iff} \quad \vec{\varphi} = \vec{\varphi}'$$

since otherwise, $\vec{\varphi}_m$ would not be unique. Hence, (8) implies $\vec{\varphi} \in \mathcal{L}(\Pi)$.

Case 3.3.1.2. $\vec{\pi}$ contains no clause with body symbol ThueMorse: Now consider the case that $\vec{\pi}$ does not contain a clause with body symbol **ThueMorse**. As $\vec{\pi}_j$ starts with φ_f , this implies $\vec{\pi} = []$. So we get

$$(7) \in \text{grnd}(\pi) \stackrel{\text{Lemma 17}}{=} \text{grnd}(\mathcal{L}(\pi)) \stackrel{\pi \in \Pi}{\subseteq} \text{grnd}(\mathcal{L}(\Pi)).$$

Hence, $\vec{\varphi} \in \mathcal{L}(\Pi)$ follows as in the previous case.

Step 3.3.2. We show that $\vec{\varphi} \in \mathcal{L}(\Pi)$ implies $\langle \vec{\varphi}_m \rangle_{\Phi_{\text{ThueMorse}}} \in \langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}$: We have shown $\vec{\varphi} \in \mathcal{L}(\Pi)$. Hence, we have

$$\langle \vec{\varphi} \rangle_{\Phi_{\text{ThueMorse}}} \in \langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}.$$

As $\langle \vec{\varphi} \rangle_{\Phi_{\text{ThueMorse}}} = \langle \vec{\varphi}_m \rangle_{\Phi_{\text{ThueMorse}}}$, this contradicts the observation

$$\langle \vec{\varphi}_m \rangle_{\Phi_{\text{ThueMorse}}} \notin \langle \mathcal{L}(\Pi) \rangle_{\Phi_{\text{ThueMorse}}}.$$

4. $\Phi' \rightsquigarrow^* \text{sat}$ contradicts Soundness

This contradicts the soundness of \rightsquigarrow_{rs} , because $\Phi' = \Phi \cup \{\varphi_q\}$ is unsatisfiable. The reason is that $\sigma \models_{\mathcal{A}} \Phi$ implies $\sigma \models_{\mathcal{A}} \text{ThueMorse}(n, w_n)$ for all $n \in \mathbb{N}$. Hence, our assumption was wrong and we have $\Phi \not\rightsquigarrow_{rs}^* \text{sat}$. \square