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The advantage of the initiative

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Abstract

Recently, the list of solved two-person zero-sum games with perfect information has increased. The state of current knowledge is that many games are a win for the first player, some games are draws, and only a few games are a win for the second player. For games with three outcomes (won, drawn, lost) a game is commonly defined as *fair* if the theoretical value of the game is drawn. For these games as well as for games with two outcomes (won, lost) we were tempted to examine which concepts characterize the outcome of a game.

In this paper, we distinguish two main concepts valid for many two-person games, namely *initiative* and *zugzwang*. The initiative is defined as an action of the first player. The notion of zugzwang is adopted from the game of chess. To investigate the impact of the initiative we determine the game-theoretic values of a large number of k-in-a-row games and over 200 Domineering games as a function of the board size. The results indicate that having the initiative is a clear advantage under the condition that the board size is sufficiently large. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Solving games; Domineering; k-in-a-row games; Initiative; Zugzwang

1. Problem statement

During the last decade, several two-person zero-sum games with perfect information have been solved. We mention Connect-Four [1,2,29], Qubic

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[3,22], Nine Men's Morris [16], Go-Moku [4,5], and Domineering [10,11]. Four games have in common that the first player to move wins the game; only Nine Men's Morris results in a draw. In all five games the players move alternately and hence it seems to be an advantage to be the first player to move. Here we would like to remark that there is a striking difference between games in which a player must play a move (if it is his/her move) and games in which a player has a choice between playing a move and passing. For instance, in Go there are no second-player wins due to the "stealing-strategy" argument: whatever strategy would guarantee a second-player win, a pass of the first player at his/her first move would transfer the win to the first player, contradicting the assumption.

In the literature on games several exhaustive computer analyses of various "little-brother" board games are given, such as small versions of Go [27] and Othello [14]. The analyses of the small Go boards were only performed for *very* small boards, and gave draws on the 1×1, 1×2, and 1×5 boards, and first-player wins on the 1×3, 1×4, 2×4, 3×3, and 3×4 boards. In addition, some conjectures for other board sizes were given, mostly first-player wins, and a few draws. The analyses and conjectures given show that in Go the importance of being the first player grows with the board size. For Othello we are aware of an analysis of the 6×6 game [14], showing that the second player has a forced win. Since no data for other board sizes are available, we cannot give any educated guess whether this is prototypical for Othello™ or all other generalized Othello games.

In 1981, David Singmaster [25,26] proved a rather conclusive and elegant theorem why first-player wins *should* abound over second-player wins. The positions in a game with two outcomes (won, lost) are split up into *P*-positions (from which the previous player can force a win) and *N*-positions (from which the next player can force a win). For the first player to have a forced win, just *one* of the moves needs to lead to a *P*-position. For the second player to have a forced win, *all* of the moves must lead to *N*-positions. For games with three outcomes, possible draws can be easily included in this line of reasoning, stating that first-player wins should abound over draws and second-player wins. From the literature cited above, we see that in relatively many games on small boards the second player yet is able to draw or even win. We assume that Singmaster's theorem has limited value when the board size is small. To investigate the practical research question whether being the first player is such a great advantage as announced, we start defining the concept of *initiative* (Section 2). Then we analyse two series of zero-sum games with varying board sizes. In Section 3 we examine *k*-in-a-row games, and in Section 4 Domineering games. In Section 5 we provide a detailed analysis when and why the initiative fails. In Section 6 we give our conclusions. Section 7 outlines future research.

2. Terminology

Below we distinguish two concepts which occur in many two-person zero-sum games with perfect information, viz. *initiative* and *zugzwang*. We have adopted them from the game of chess. The initiative is defined as an action of the first player, as used in "to take the initiative" (i.e., to be the first to take action). We note that this meaning differs from "to have the initiative" as is used in the military terminology and in chess. It means that one is able to control the enemy's movements. In chess the common strategy for White is to exploit the initiative in such a way that it grows to a winning advantage. As against this it is usually Black's task to attempt to take the initiative. Confusingly as it is, we see a shift from the original meaning of initiative (beginning, a first move) to controlling the opponent's moves. In our definition we stick to "the right to move first".

The notion of zugzwang has a limited meaning beyond chess. Most authors do not elaborate on this concept. As an exception we mention Allis' [2,5] analysis of Connect-Four. Initially he believed that the game was a second-player win due to the fact that in the end the first player must weaken his/her position (since a pass is not allowed). Zugzwang means "the obligation to move in one's turn even though every move the player to move can legally make would lead to a decisive deterioration of his position (e.g., the player must concede a draw, or even observes that his position is lost)" [24]. More interestingly is the concept of mutual zugzwang, which is coined by Nunn [21] as reciprocal zugzwang. Below we quote Nunn: "In a normal zugzwang position it doesn't matter much who is to move, because the superior side usually has a waiting move with which he can pass if it is his turn to move. This is not so in a reciprocal zugzwang, which may be defined as a position in which whoever moves first has to weaken his position". It describes precisely the position of the first player when facing the fact that every move played results in a loss of the game, and a pass when allowed should turn the tables (as is the case in 5×5 Domineering, see Section 5).

Reformulating and simplifying the problem statement, we investigate the research question whether taking/having the initiative is such a great advantage that games without the possibility of passing are in a majority of the cases first-player wins.

3. *k*-In-a-row games

Go-Moku games can be characterized as *mnk*-games, with m and n denoting the number of rows and columns of the board, and k giving the length of the straight chain to be obtained, i.e., an *mnk*-game is an abbreviation for a k-in-a-row game on an $m \times n$ board [28]. The standard 5-in-a-row Go-Moku game can thus be characterized as the 19,19,5-game, [1] whereas the children's game TicTacToe is the 333-game.

3.1. Experimental set-up

To analyse *mnk*-games we have designed the program TTT (for TicTacToe-like games). The program combines domain knowledge with a straightforward search procedure. The framework of TTT is an αβ-search [19], using standard transposition tables [20]. During the search process, every node is extensively investigated to see if the game-theoretic value of the corresponding position can be derived. For that purpose, many general knowledge rules are defined in TTT. The generality of the rules means that they are defined as a function of the board size and the length of the chain to be obtained, i.e., dependent on m, n, and k. The simplest example is as follows: if the player to move has a straight chain of $k-1$ stones within a group of k squares with the kth square empty, the search terminates, returning a game-theoretic win for that position. This rule, called the DirectThreat rule, can be characterized as a 1-ply rule, since it guarantees a win in 1 ply (half move). Some of the rules are instantiations of the *deep-fork* rules as used by Epstein [13] in her program HOYLE. For instance, the win for a player is guaranteed by the combination of several threats; not all threats can be remedied simultaneously. The difference between HOYLE and TTT is that the rules in HOYLE are advice rules, with the characteristic that the applicability of a rule for a particular board position has to be established by search. In TTT, n-ply rules incorporate perfect knowledge for a position. At present the program TTT contains knowledge incorporated in rules with a scope of up to nine plies.

3.2. Results

The 333-game (TicTacToe) is a game-theoretic draw, the first player having insufficient "space" to exploit the initiative. Berlekamp et al. [6] showed that any larger board size (even the addition of a single square) is sufficient to classify the game as a first-player win. Hence, all 3-in-a-row $mn3$-games with $m \geqslant 3$ and $n \geqslant 3$ are game-theoretic wins, except the 333-game. This leads to the following partial results for *mnk*-games. The trivial 1-in-a-row game is a win on any board size of at least 1×1. The 2-in-a-row game is a win on any board size of at least 2×2. The 3-in-a-row game is a win on any board size larger than 3×3. The question now arises: how large a board is needed for the 4-in-a-row game to be a win for the first player? The only clue Berlekamp et al. [6] give is that according to an observation of C.Y. Lee the second player can

[1] Obviously, if m, n, or $k > 9$ the notation will have included two comma's to avoid ambiguity.

tie on a 5×5 board. Moreover, they report that Lustenberger has used a computer to show that 4-in-a-row is a win for the first player on a 4×30 board. Using the knowledge/search-based approach sketched above we have analysed a large series of *mn*4-games. We found that all *m*44-games with $m \leqslant 8$ are game-theoretic draws. Increasing the board size at both edges to at least 5, we found that the 554-game is a draw. The 654-game turned out to be a win for the first player, and hence it is the smallest board size on which the first player always can win. An optimal variation is shown in Fig. 1. Disregarding symmetries, we still remark that the variation given is not unique.

Below we provide some explanations in terms of taking/having the initiative. We assume White to be the first player and Black to be the second. The best reply to the opening move at c3 is the diagonal response by Black at d2. Then optimal play might proceed as indicated in Fig. 1. The third move at d3 contains a threat at b3 or e3. From then every move is almost forced, i.e., deviations of the variations given are treated analogously. The threat by White at d3 can be countered at e3, after which White builds a new threat at c2. This time Black has seemingly a strong counter at c1, since White is forced to play at f4. However, since White's move at c2 simultaneously had another threat, Black still is forced to respond to it, presumably at e4. But now White can play at c4 and after the forced reply at c5, White can play at e2, with a double threat that Black is unable to parry. Obviously White has succeeded to exploit the initiative and to take advantage out of it.

To show the power of the initiative we briefly analyse another strong reply after the first move, namely at d3 (see Fig. 2).

Again, the remainder of the game is forced and can be easily explained by knowledge rules. White's move at d2 forces Black to play at b4 (or e1). Next White plays at b2. If Black plays at c2 (as in Fig. 2) d4 creates a double threat (at a1 and e5), of which only one can be prevented. Had Black played at d4 at move 6, then c2 also creates a double threat (at a2 and e2), of which again only one can be prevented.

Since it is easy to show that a win on some $m \times n$ board is easily transferable to a win on any board including an $m \times n$ subboard, it follows that all *mn*4

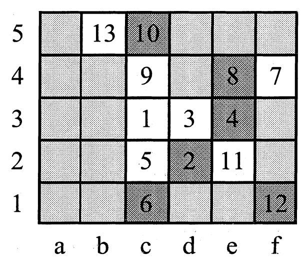

Fig. 1. An optimal variation in the 654-game.

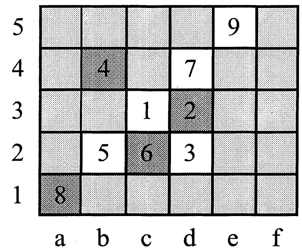

Fig. 2. Another optimal variation in the 654-game.

games with $m \geqslant 6$ and $n \geqslant 5$ are game-theoretic wins for the first player. The only question remaining with respect to 4-in-a-row is: at what size (value of m) game-theoretic draws change into wins for the first player in m44-games? The answer lies between 9 and 29, both inclusive.

Turning to 5-in-a-row games, we start demonstrating that on small boards the second player (Black) can achieve a draw. A suitable instrument for such a demonstration is the so-called Hales–Jewett [18] pairing strategy. In this strategy, (most) squares of a board are marked. Fig. 3 provides an easy-to-grasp example, taken from [6].

In Fig. 3 any possible straight chain of five squares contains two "paired" squares (indicated by a marker in the direction of the chain), e.g., for the diagonal group from the left under to the right upper corner the two corner squares (marked '/') are paired. Black can guarantee the draw by always playing the second square of a pair as soon as White plays the first (unless the second square was already played before, in which case a free move can be made). At the end there will be at least one black stone in every conceivable winning chain. Since White can follow the same strategy after a first arbitrary move, it demonstrates that the 555-game is a draw. With TTT we have established that the 5-in-a-row game also is drawn on 6×5 and 6×6 boards. Larger boards were not investigated yet. Allis et al. [4,5] demonstrated that Go-Moku (i.e., the 19,19,5-game) is a first-player win. Again, where the borderline lies between a draw and a first-player win in mn5-games is a topic of further research.

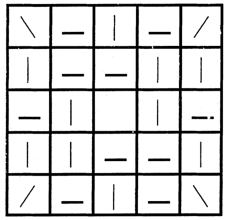

Fig. 3. A Hales-Jewett pairing for the 555-game.

Jumping from 5-in-a-row to 8-in-a-row and 9-in-a-row, we mention some old results. In 1954, Henry Oliver Pollak and Claude Elwood Shannon showed that 9-in-a-row (and hence m-in-a-row for $m \geqslant 9$) is drawn on an infinite board and as a consequence on any finite board [6]. Later, T.L.G. Zetters demonstrated the same for 8-in-a-row [6]. Whether 6-in-a-row and 7-in-a-row can always been drawn or have a winning region has to be demonstrated yet.

Summarizing the results of this section we have seen that for *mnk*-games the effect of the initiative is correlated to the length of the chain to be achieved and the board size. From 1-in-a-row to 5-in-a-row we observe that the first player needs continuously more space to force a win. For 6-in-a-row and 7-in-a-row this is still an open question, but for 8-in-a-row and onwards the second player is able to keep a draw whatever the board size.

4. Domineering games

The Domineering game is also known as crosscram, and (confusingly) as dominoes. It was proposed by Göran Andersson around 1973 [12,15]. In Domineering the players alternately place a domino (2×1 tile) on a board, i.e., on a finite subset of Cartesian boards of any size or shape. The game is usually played on rectangular boards. The two players are denoted by Vertical and Horizontal. In standard Domineering the first player is Vertical, who is only allowed to place its dominoes vertically on the board. Horizontal may play only horizontally. Of course, dominoes are not allowed to overlap. As soon as a player is unable to move the player loses. As a consequence, draws are not possible in Domineering. Although Domineering can be played on any board and with Vertical as well as Horizontal to move first, the original game was played on a (8×8) checkerboard with Vertical to start and this instance has generally been adopted as standard Domineering.

4.1. Experimental set-up

For solving Domineering we have developed the program Domi, which uses a straightforward variant of the $\alpha\beta$ search technique [19], enhanced with a transposition table. The algorithm used is similar to the one used for solving *mnk*-games, except that we do not use perfect domain knowledge for classifying positions as wins, losses, draws, or forced positions. The solution of Domineering games thus can be characterized as a pure brute-force approach. Below we present a few details of the algorithm. Our experience with chess [8,9] and some preliminary experiments on Domineering [10,11] resulted in a transposition-table technique with an appropriate replacement scheme when clashes occurred. The best results were obtained by a replacement scheme called

TwoBig1, a so-called *two-level* scheme [9], in which each entry has two table positions. The newest position is *always* stored, and the less important position of the remaining two positions (in terms of the number of nodes of the search) is overwritten. This implementation of the transposition table made the experiments feasible [10].

4.2. Results

Using the technique described above, we recently solved a broad class of Domineering games [11]. We investigated many rectangular boards sized $m \times n$, with m denoting the number of rows and n the number of columns. In all games, Vertical plays first. Of course the result for an $m \times n$ board with Horizontal to move equals the result of the $n \times m$ board with Vertical to move. The results are summarized in Table 1. A '1' indicates a first-player win, a '2' a second-player win, a blank entry denotes a game not analysed yet.

Some remarks are in order. First, our results fully agree with the results published earlier by Berlekamp and coworkers [6,7,17] as far as investigated by them. They provide complete analyses for boards with a size of $2 \times n$ $(2 \leqslant n \leqslant 7)$, $2 \times (2n+1)$ (n arbitrary), $3 \times n$ $(3 \leqslant n \leqslant 5)$, and 5×5.

Second, of the 212 games analysed, 120 (56.6%) are first-player wins. Of course, one might argue that many of these boards do represent biased games. Therefore, we restrict the investigation to the boards where both players can place the same total number of dominoes on the empty board (if the opponent does not play). These are the $2m \times 2n$ boards plus the square $m \times m$ boards. Of the 46 games resulting 30 (65.2%) are first-player wins. Even then the comparison is not completely unbiased. For instance, on the 2×4 board both players can place maximally 4 dominoes on the board, without any interference by the opponent. However, at the start Vertical has a choice out of 4 possible moves, Horizontal out of 6. To eliminate this inequality we restrict the investigation to $m \times m$ boards. We then see that 6 out of 8 (75.0%) of the games are first-player wins, and only two games are second-player wins, i.e., the 1×1 and the 5×5 board. Disregarding the 1×1 board we see that the initiative only in the 5×5 game is insufficient to force a win. In all other $m \times m$ games $(m \leqslant 8)$, the initiative is predominant.

From Table 1 we make two conjectures:

Conjecture 1. Domineering on $m \times (2n+1)$ boards, $m > 2n+1$, are first-player wins. Equivalently, Domineering on $(2m+1) \times n$ boards, $n > 2m+1$, are second-player wins.

Conjecture 2. Domineering on $m \times 4n$ boards, $m < 4n$, are second-player wins. Equivalently, Domineering on $4m \times n$ boards, $n < 4m$, are first-player wins.

Table 1
Game-theoretic values of many $m \times n$ Domineering games

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	1	1	1	2	1	1	1	2	1	1	1	2	2	1	1	2	2	1	1	2	2	2	1	2	2	2	1	2	2	
3	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2				
4	1	1	1	1	1	1	1	2	1	2	1	2	2	2	2															
5	1	2	1	2	2	2	2	2	2																					
6	1	1	1	1	1	1	1	2	1																					
7	1	1	1	2	1	2	1	2	2																					
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9	1	2	1	2	1	2	1	2																						
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Of course it is tempting to predict also the outcome of the 9×9 game. The row for $m = 9$ suggests a first-player win, but both the diagonal and the column for $n = 9$ suggest a second-player win. If the latter turns out to hold, the 9×9 game would be another example of a game where the first player finds him/herself at the outset in a reciprocal zugzwang position.

Finally, we remark that the standard 8×8 Domineering game is a first-player win. In our opinion, this is one more sign that for games played by humans having the initiative is usually a clear advantage.

5. The initiative fails

Of all games solved, many are a first-player win. These games show that taking/having the initiative is an advantage. Therefore it is worth investigating what happens if the initiative fails and the game is a second-player win.

Since the 5×5 game is a second-player win, an optimal strategy begins by refuting every first move by Vertical. Below we analyse the game. For notational purposes we have moves indicated by their upper or left square numbers, which run from 0 to 24 in left-to-right and top-to-bottom order. For ease of discussion, we introduce the concept of a savings strategy. In this strategy a player places a domino in such a way that a space is created where at a later move a domino can be placed (in the mean time the opponent cannot use this space for a move, therefore the space is called a *safe* move). For instance, when Vertical places its domino at 1 at the start of a 5×5 game (covering squares 1 and 6) the empty squares 0 and 5 can never be covered by Horizontal, guaranteeing Vertical a free move in the future. As a result, the second and last but one column are favourite columns for Vertical, whereas the second and last but one row are such for Horizontal.

Due to symmetry, Vertical only has six distinct starting moves, i.e., the moves 0, 1, 2, 5, 6, and 7. Below, we give all winning moves for Horizontal after each starting move by Vertical:

- 0: winning moves: $\{1, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18\}$; winning ratio: $12/18 = 66.7\%$.
- 1: winning moves: $\{7, 8\}$; winning ratio: $2/16 = 12.5\%$.
- 2: winning moves: $\{5, 8, 10, 11, 12, 13, 15, 16, 17, 18\}$; winning ratio: $10/16 = 62.5\%$.
- 5: winning moves: $\{1, 6, 7, 8, 11, 13, 15, 16, 17, 18, 20, 21\}$; winning ratio: $12/18 = 66.7\%$.
- 6: winning moves: $\{7, 8\}$; winning ratio: $2/16 = 12.5\%$.
- 7: winning moves: $\{5, 8, 10, 13, 15, 16, 17, 18, 20, 21, 22, 23\}$; winning ratio: $12/16 = 75.0\%$.

So only the starting moves 1 and 6 offer some real resistance, being also the only two moves (apart from their six symmetry-related moves) that obey the savings strategy. Concerning other first moves, it is sufficient to remember that for such a starting move any response by Horizontal in the second or fourth row (and thus also according the savings strategy) wins. After the sequence 1-7 Vertical has 14 possible moves, all losing. We refrain from enumerating all winning moves for Horizontal after them, but only give Horizontal's winning ratio after Vertical's indicated move:

0: 14/14 = 100.0%	11: 2/10 = 20.0%	16: 2/10 = 20.0%
4: 13/13 = 100.0%	12: 4/10 = 40.0%	17: 4/10 = 40.0%
5: 10/13 = 76.9%	13: 2/10 = 20.0%	18: 2/10 = 20.0%
9: 7/13 = 53.8%	14: 9/12 = 75.0%	19: 4/12 = 33.3%
10: 5/12 = 41.7%	15: 5/12 = 41.7%	

Hence, we see that Vertical has four strong moves after the sequence 1-7, i.e., 11, 13, 16, and 18. These moves are again obvious, since they are the only moves obeying the savings strategy. The winning moves of Horizontal for these four responses are also straightforward, using the same principle, amounting to $\{17, 18\}$, $\{15, 16\}$, $\{17, 18\}$, and $\{15, 16\}$, respectively. It appears that if Horizontal plays 8 instead of 7 at the second move, the optimal third and fourth moves are identical. Using the principle of toughest resistance as criterion, the optimal variations are 1-$\{7, 8\}$-11-$\{17, 18\}$, 1-$\{7, 8\}$-13-$\{15, 16\}$, 1-$\{7, 8\}$-16-$\{17, 18\}$, 1-$\{7, 8\}$-18-$\{15, 16\}$. A set in a variations indicates that any move from the set can be played in that position. An optimal opening variation is given in Fig. 4(a). After the opening the continuation is straightforward; the game from Fig. 4(a) might continue as in Fig. 4(b). Then Horizontal (to move) has three safe moves, whereas Vertical only has two safe moves. Hence, Horizontal is winning the game. A convincing analysis of why the initiative fails.

In Fig. 5(a), the same principles as above can be applied: Vertical starts with 6, after which Horizontal replies 7. Winning ratio's for Horizontal after all third moves are:

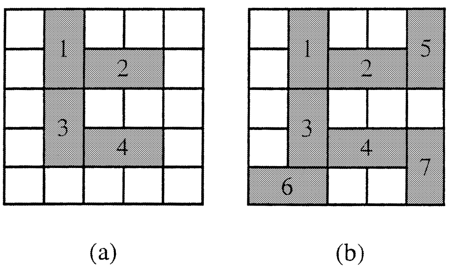
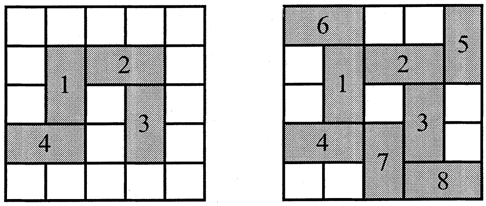


Fig. 4. An optimal opening variant (a) and a plausible continuation (b) in the 5 × 5 Domineering game.



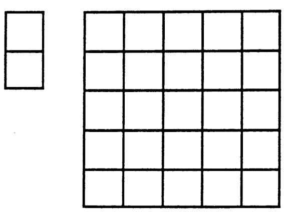
(a) (b)

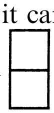
Fig. 5. Another optimal opening variant (a) and a plausible continuation (b) in the Domineering game.

0: 6/13 = 46.2%	12: 4/11 = 36.4%	17: 5/10 = 50.0%
4: 6/13 = 46.2%	13: 2/10 = 20.0%	18: 2/10 = 20.0%
5: 13/14 = 92.9%	14: 8/12 = 66.7%	19: 6/12 = 50.0%
9: 11/13 = 84.6%	15: 6/12 = 50.0%	
10: 10/13 = 76.9%	16: 2/10 = 20.0%	

Hence, Vertical's strongest moves after 6-7 are 13, 16, and 18 (obeying again the same principle), after which $\{15, 16\}$, $\{17, 18\}$, and $\{15, 16\}$ are securing the win for Horizontal. Subsequently, this gives rise to the following additional optimal variations: 6-$\{7, 8\}$-13-$\{15, 16\}$, 6-$\{7, 8\}$-16-$\{17, 18\}$, and 6-$\{7, 8\}$-18-$\{15, 16\}$. An optimal opening variation from this category is given in Fig. 5(a). After the opening the continuation is straightforward; the game from Fig. 5(a) might continue as in Fig. 5(b), after which both players have two safe moves; Vertical to move thus loses the game. This concludes our analysis of why the initiative fails.

The 5×5 Domineering game is interesting not only since the initiative fails, but mainly since our analyses (of which only a part is shown above) have proved that the outcome is governed by reciprocal zugzwang and that the concept of zugzwang only occurs in the initial position. Hence the following sum of two games



is a win for Vertical, since it can carry over the zugzwang to Horizontal. For this reason the general sum  and an $m \times m$ board thus always is a win for Vertical.

There are four other board sizes for which reciprocal zugzwang is a predominant concept, viz. the boards 2×13 and 4×13 and their symmetry-related ones (see Table 1 in Section 4). Yet we must admit that reciprocal zugzwang plays a minor part with respect to the initiative.

It is even unclear which role the notion of (reciprocal) zugzwang plays in a general sum of combinatorial games (such as a set of differently-sized Domineering games). If a player will worsen his/her position by playing in a subgame, that player can simply play in another subgame instead.

6. Conclusions

In Table 2, we have collected the main results known today, in which '0' stands for draw, '1' for a first-player win, and '2' for a second-player win. For *mnk*-games we only included games with m and n at least being equal to k, since otherwise the game is trivially drawn. This can easily be shown by the Hales–Jewett pairing strategy. If one dimension (say m) of the board is smaller than k, only horizontal alignments of chains of length k are possible. Dividing the board into horizontal pairs of neighbouring squares, a draw is guaranteed by claiming at least one of the two squares of each pair (i.e., playing the second square of a pair as soon as the opponent plays the first or vice versa). For Domineering we only have included $m \times m$ games.

From Table 2 we see that in a variety of two-person zero-sum games with perfect information the concept of *initiative* seems to be the predominant notion,

Table 2
Some game-theoretic values of games known today

Game	Result
Connect-Four	1
Qubic	1
Nine Men's Morris	0
$1 \times m$ Go $(m = 1, 2, 5)$	0
1×3, 1×4, 2×4, 3×3, 3×4 Go	1
6×6 Othello	2
mnk-games ($k = 1,2$)	1
333-game (TicTacToe)	0
*mn*3-games $(m \geqslant 4,\ n \geqslant 3)$	1
*m*44-games $(m \leqslant 8)$	0
*mn*4-games $(m \leqslant 5,\ n \leqslant 5)$	0
*mn*4-games $(m \geqslant 6,\ n \geqslant 5)$	1
*mn*5-games $(m \leqslant 6,\ n \leqslant 6)$	0
19,19,5-game (Go Moku)	1
mnk-games $(k \geqslant 8)$	0
$m \times m$ Domineering $(m = 1, 5)$	2
$m \times m$ Domineering $(m = 2, 3, 4, 6, 7, 8)$	1

the requirement being that the first player has sufficient space to fulfil his goals. Most games thus are first-player wins; only on small boards the second player can achieve a draw or in exceptional cases force a win. In "all-or-nothing" games such as Domineering, the shape and space of the board counterbalances somewhat, but even then the initiative is the predominant one (see Table 1). We consider the results of our investigations as described in Sections 3–5 as a piece of empirical evidence supporting David Singmaster's theorem [25,26].

7. Future research

The future research can be split into three areas. First, we have the leftovers of our current investigations. These are essentially two groups of problems which are assumed to be solved soon. After adding some additional domain knowledge, faster computers will do the job. For the *mnk*-games the remaining problems are:

(1) determining the draw/win bound of the *m*44-games (*m* between 9 and 29, both inclusive),

(2) determining the draw/win regions of the *mn*5-games,

(3) determining the winning regions of the *mn*6-games and *mn*7-games.

For the Domineering games it comes down to

(4) verifying Conjectures 1 and 2;

(5) solving the 9×9 Domineering. Of course after its solution, the extrapolation to other $m \times n$ Domineering games still remains (see Table 1).

Second, it is planned to investigate the Domineering and *mnk*-games from the viewpoint of sums over games (see, e.g., [6,7,12,17]). From this perspective the outcome of a game is the sum of the outcomes of several subgames on smaller board fragments. A player to move may choose the fragment he wants to play in. This view leads naturally to the notions of a *value* of a fragment, which is an indication of how much the fragment contributes to the complete game, and a *temperature* of the fragments, being related with the importance of a fragment to be chosen first (see also Section 5).

Third, future research will focus on the strategy to be followed by a side in a subgame or even in a sum of games. Having a correct strategy at one's disposal the question of solving a game by a brute-force method is no longer relevant. Hence the question remains: is a long-term strategy computable by a machine? A first research step in this direction has recently been published [23].

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