

ANDRÉ-QUILLEN HOMOLOGY OF REES ALGEBRAS AND EXTENDED REES ALGEBRAS

TONY J. PUTHENPURAKAL

ABSTRACT. Let (A, \mathfrak{m}) be an excellent local complete intersection ring and let $I = (a_1, \dots, a_r)$ be an ideal of positive height. Let $\mathcal{R}(I) = A[It]$ be the Rees algebra of I . Consider the map $\psi: S = A[X_1, \dots, X_r] \rightarrow \mathcal{R}(I)$ which maps $X_i \rightarrow a_i t$ for all i . Let $J = \ker \psi$ and let $H_*(J)$ be the Koszul homology of J . We prove that the following assertions are equivalent:

- (i) $\text{Proj } \mathcal{R}(I)$ is a complete intersection.
- (ii) (a) $D_3(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$.
(b) For $P \in \text{Proj } \mathcal{R}(I)$ we have $H_1(J)_P$ is a free $\mathcal{R}(I)_P$ -module.

Here $D_3(\mathcal{R}(I)|A, \mathcal{R}(I))$ is the third André-Quillen homology of $\mathcal{R}(I)$ with respect to $A \rightarrow \mathcal{R}(I)$. We prove an analogous result for the extended Rees algebra $\widehat{\mathcal{R}}(I) = A[It, t^{-1}]$. When A is a Cohen-Macaulay domain (not necessarily a complete intersection) we compute that rank of $H_1(J)$ and hence compute its free locus.

1. INTRODUCTION

1.1. Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring and let $I = (a_1, \dots, a_r)$ (minimally) be an ideal. Assume height $I \geq 1$. Let $\mathcal{R}(I) = A[It]$ be the Rees algebra of I . Consider the map $\epsilon: S = A[X_1, \dots, X_r] \rightarrow \mathcal{R}(I)$ which maps $X_i \rightarrow a_i t$ for all i . Let $J = \ker \epsilon$. We give the standard grading to S . The ideal J is called the defining ideal of the Rees algebra of I and has been extensively studied. Analogously let $\widehat{\mathcal{R}}(I) = A[It, t^{-1}]$ be the extended Rees algebra of I . Consider the map $\widehat{\epsilon}: \widehat{S} = A[X_1, \dots, X_r, T] \rightarrow \widehat{\mathcal{R}}(I)$ which maps $X_i \rightarrow a_i t$ for all i and T is mapped to t^{-1} . Let $\widehat{J} = \ker \widehat{\epsilon}$. We give the following grading to \widehat{S} ; set $\deg A = 0$, $\deg X_i = 1$ for all i and $\deg T = -1$. The ideal \widehat{J} is the defining ideal of the extended Rees algebra of I . It can be shown that $\widehat{J} = J\widehat{S} + (TX_i - a_i \mid 1 \leq i \leq r)$, see [8, 5.5.7].

Let $H_*(J)$ ($H_*(\widehat{J})$) denote the Koszul homology of J (respectively of \widehat{J}). In this paper we investigate some Koszul homology of J (respectively of \widehat{J}) and decode its impact on properties of $\mathcal{R}(I)$ (respectively of $\widehat{\mathcal{R}}(I)$). The main tool for this analysis is the André-Quillen homology $D_*(\mathcal{R}(I)|A, -)$ and $D_*(\widehat{\mathcal{R}}(I)|A, -)$. We note that André-Quillen homology has been earlier studied in the context of Rees algebras by André [2] and by Planas-Vilanova [12], [13], [14] and [15].

If $T = \bigoplus_{n \in \mathbb{Z}} T_n$ is a graded ring then by T_+ we denote the ideal generated by $\bigoplus_{n \geq 1} T_n$. By $\text{Proj}(T)$ we mean the scheme consisting of homogeneous prime ideals P with $P \not\supseteq T_+$. André-Quillen homology is quite effective when dealing with complete intersections. We show

Date: November 19, 2025.

2020 Mathematics Subject Classification. Primary 13D03, 13A30; Secondary 13H10, 14M10.

Key words and phrases. André-Quillen homology, complete intersections, Proj of a graded ring, Koszul homology.

Theorem 1.2. *Let (A, \mathfrak{m}) be a local complete intersection and let I be an ideal of positive height. Assume either A is excellent or A is a quotient of a regular local ring. The following assertions are equivalent:*

- (i) $\text{Proj } \mathcal{R}(I)$ is a complete intersection.
- (ii) $\text{Proj } \widehat{\mathcal{R}}(I)$ is a complete intersection.
- (iii) (a) $D_3(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$.
(b) $H_1(J)_P$ is a free $\mathcal{R}(I)_P$ -module for every $P \in \text{Proj}(\mathcal{R}(I))$.
- (iv) (a) $D_3(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_n = 0$ for $n \gg 0$.
(b) $H_1(\widehat{J})_P$ is a free $\widehat{\mathcal{R}}(I)_P$ -module for every $P \in \text{Proj}(\widehat{\mathcal{R}}(I))$.

Bountiful examples of rings satisfying the assumptions of Theorem 1.2 arise from Hironaka's resolution of singularities, see 6.3.

Remark 1.3. We consider 0 to be a free module over the ambient ring.

1.4. We note that $D_j(\mathcal{R}(I)|A, -) = D_j(\mathcal{R}(I)|S, -)$ for $j \geq 2$ and $D_j(\widehat{\mathcal{R}}(I)|A, -) = D_j(\widehat{\mathcal{R}}(I)|\widehat{S}, -)$ for $j \geq 2$. If A contains a field of characteristic zero then we have an exact sequence (see [6, 4.6])

$$D_4(\mathcal{R}(I)|S, \mathcal{R}(I)) \rightarrow \bigwedge^2 H_1(J) \rightarrow H_2(J) \rightarrow D_3(\mathcal{R}(I)|S, \mathcal{R}(I)) \rightarrow 0,$$

and an exact sequence

$$D_4(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \rightarrow \bigwedge^2 H_1(\widehat{J}) \rightarrow H_2(\widehat{J}) \rightarrow D_3(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \rightarrow 0.$$

Here $\bigwedge^2 H_1(-) \rightarrow H_2(-)$ is the natural multiplication map induced in the algebra $H_*(-)$. Thus the condition $D_3(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$ (respectively $D_3(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_n = 0$ for $n \gg 0$) can be made entirely in terms of Koszul homology of J (and \widehat{J} respectively).

In view of Theorem 1.2 we might wonder what happens when $\mathcal{R}(I)$ or $\widehat{\mathcal{R}}(I)$ is a complete intersection. There is a paucity of examples when $\mathcal{R}(I)$ is a complete intersection. However there are bountiful examples of $\widehat{\mathcal{R}}(I)$ being a complete intersection, see [16, 1.5]. Our result is

Theorem 1.5. *Let (A, \mathfrak{m}) be a local complete intersection and let I be an ideal of height ≥ 1 . The following assertions are equivalent:*

- (i) $\widehat{\mathcal{R}}(I)$ is a complete intersection.
- (ii) $H_1(\widehat{J})$ is a free $\widehat{\mathcal{R}}(I)$ -module and $D_3(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I)) = 0$.

In view of the above results it is necessary to understand the free locus of $H_1(J)$ and $H_1(\widehat{J})$. When T is local (or $*$ -local) for a finitely generated module M (graded if T is $*$ -local) we denote the number of minimal generators of M by $\mu(M)$. When A is a domain we prove:

Theorem 1.6. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local domain of dimension $d \geq 1$ and let I be a non-zero ideal. Then we have*

- (1) $\text{rank } H_1(\widehat{J}) = \mu(\widehat{J}) - \mu(I)$.
- (2) $\text{rank } H_1(J) = \mu(J) - \mu(I) + 1$.

If M is an T -module then let $\text{Fitt}_j(M)$ denote the j^{th} Fitting ideal of M . The following result is well-known and easy to prove:

Proposition 1.7. *Let T be a Noetherian domain and let M be a finitely generated T -module of rank $r \geq 1$. Let P be a prime ideal in T . Then the following assertions are equivalent:*

- (1) M_P is free T_P -module.
- (2) $P \not\subseteq \text{Fitt}_r(M)$.

By Theorem 1.6 and Proposition 1.7 we can determine the free locus of $H_1(\hat{J})$ and $H_1(J)$. When A is a regular local ring we show

Proposition 1.8. *Let A be a regular local ring and let I be a non-zero ideal of A . We have*

- (I) *Let P be a prime ideal in $\mathcal{R}(I)$. Then the following assertions are equivalent:*
 - (a) $H_1(J)_P$ is a free $\mathcal{R}(I)_P$ -module.
 - (b) $\mathcal{R}(I)_P$ is a complete intersection.
- (II) *Let P be a prime ideal in $\hat{\mathcal{R}}(I)$. Then the following assertions are equivalent:*
 - (a) $H_1(\hat{J})_P$ is a free $\hat{\mathcal{R}}(I)_P$ -module.
 - (b) $\hat{\mathcal{R}}(I)_P$ is a complete intersection.

Proposition 1.8 follows easily from a result of Gulliksen cf., [5, 1.4.9]. We give bounds on the rank of minimal generators of $H_1(\hat{J})$ when I is equi-multiple and $\hat{\mathcal{R}}(I)$ is a complete intersection. We prove

Theorem 1.9. *Let (A, \mathfrak{m}) be a complete intersection and let I be an equi-multiple of height $r \geq 1$. Assume the residue field of A is infinite. Let Q be a minimal reduction of I . If $\hat{\mathcal{R}}(I)$ is a complete intersection then we have*

$$\mu(H_1(\hat{J})) \leq \text{embdim } A/Q + r - d.$$

We give an example (see 9.1) which shows that equality can occur in the above bound. We also note that $\text{embdim } A/Q \leq \text{embdim } A$. So $\text{embdim } A + r - d$ is an upper bound for minimal number of generators of $H_1(\hat{J})$ which is independent of I .

Recall if T is a ring of finite Krull dimension and E is a finitely generated T -module then

- (1) E is said to be unmixed if all associate primes of E are minimal.
- (2) E is said to be equi-dimensional if $\dim T/P = \dim E$ for all minimal primes P of E .

We prove:

Theorem 1.10. *(with hypotheses as in 1.1). Further assume A is Cohen-Macaulay and height $I > 0$. We have*

- (I) *The following assertions are equivalent:*
 - (a) $H_1(J)$ is unmixed and equi-dimensional.
 - (b) $D_2(\mathcal{R}(I)|A, \mathcal{R}(I)) = 0$.
- (II) *The following assertions are equivalent:*
 - (a) $H_1(\hat{J})$ is unmixed and equi-dimensional.
 - (b) $D_2(\hat{\mathcal{R}}(I)|A, \hat{\mathcal{R}}(I)) = 0$.

We note the modules $D_*(\mathcal{R}(I)|A, \mathcal{R}(I))$ and $D_*(\hat{\mathcal{R}}(I)|A, \hat{\mathcal{R}}(I))$ are graded. So we may consider vanishing of these modules for high degrees. If L is an ideal in a ring T then by $V(L)$ we denote the primes in T containing L . We show

Theorem 1.11. *(with hypotheses as in 1.1). Further assume A is Cohen-Macaulay and height $I > 0$. The following assertions are equivalent:*

- (i) $\text{Ass } H_1(J) \subseteq \text{Ass } \mathcal{R}(I) \cup V(\mathcal{R}(I)_+)$.
- (ii) $\text{Ass } H_1(\widehat{J}) \subseteq \text{Ass } \widehat{\mathcal{R}}(I) \cup V(\widehat{\mathcal{R}}(I)_+)$.
- (iii) $D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$.
- (iv) $D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_n = 0$ for $n \gg 0$.

Remark 1.12. Theorems 1.10, 1.11 are not only interesting in its own regard but it is also an essential ingredient in the proofs of Theorems 1.2 and 1.5.

1.13. Let (A, \mathfrak{m}) be a Noetherian local ring and let I be an \mathfrak{m} -primary ideal. Let $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ be the Rees algebra of I . It is easily shown that for all $j \geq 1$ the A -module $D_j(\mathcal{R}(I)|A, E)_n$ has finite length for all n (here E is a finitely generated graded $\mathcal{R}(I)$ -module); see 10.2. Thus the function $n \rightarrow \ell(D_j(\mathcal{R}(I)|A, E)_n)$ is of polynomial type of degree $\leq d - 1$. Here $d = \dim A$. We assume $d \geq 1$. It is of some interest to find general bounds on this polynomial. Let $S = A[X_1, \dots, X_l]$ be a graded polynomial algebra mapping onto $\mathcal{R}(I)$ and let $\epsilon: S \rightarrow \mathcal{R}(I)$ be this map. We set the degree of the zero polynomial to be -1 .

Theorem 1.14. *(with hypotheses as in 1.13). Also assume A is Cohen-Macaulay. Let $1 \leq i \leq d$. The following are equivalent:*

- (i) *For any finitely generated graded $\mathcal{R}(I)$ -module E the function $n \rightarrow \ell(D_2(\mathcal{R}(I)|A, E)_n)$ is of polynomial type of degree $\leq d - i - 1$.*
- (ii) *For every prime $P \in \text{Proj}(\mathcal{R}(I))$ with height $P \leq i$ the map $\epsilon_P: S_{\epsilon^{-1}(P)} \rightarrow \mathcal{R}(F)_P$ is a complete intersection.*

Furthermore if any of the above conditions hold then for any finitely generated graded $\mathcal{R}(I)$ -module and $j \geq 2$ the function $n \rightarrow \ell(D_j(\mathcal{R}(I)|A, E)_n)$ is of polynomial type of degree $\leq d - i - 1$.

We need to find a single module which can determine a bound on the degree of the above polynomials. We are able to do this when (A, \mathfrak{m}) is regular local. We prove

Corollary 1.15. *(with hypotheses as in 1.13). Further assume that A is regular local. Let $J = \ker \epsilon$. Let $1 \leq i \leq d$. The following are equivalent:*

- (i) *For any finitely generated graded $\mathcal{R}(I)$ -module E the function $n \rightarrow \ell(D_2(\mathcal{R}(I)|A, E)_n)$ is of polynomial type of degree $\leq d - i - 1$.*
- (ii) *For every prime $P \in \text{Proj}(\mathcal{R}(I))$ with height $P \leq i$ the module $H_1(J)_P$ is free.*
- (iii) *$\mathcal{R}(I)_P$ is a complete intersection for every prime $P \in \text{Proj}(\mathcal{R}(I))$ with height $P \leq i$.*

We now describe in brief the contents of this paper. In section two we discuss some preliminaries that we need. In section three we prove some basic results in Andr -Quillen homology of Rees algebras. In section four we prove Theorems 1.10 and 1.11. In section five we prove Theorem 1.5. In section six we prove Theorem 1.2. In the next section we give a proof of Theorem 1.6. In section eight we give a proof of Proposition 1.8. In the next section we give a proof of Theorem 1.9. In section ten we give proofs of Theorem 1.14 and Corollary 1.15. Finally in the appendix we give a proof of some results which we believe is already known. However we do not have a reference.

2. PRELIMINARIES

In this section we discuss some preliminary facts that we need.

2.1. Let $\psi: R \rightarrow S$ be a homomorphism of Noetherian rings. The André-Quillen homology $D_n(S|R, N)$ of the R -algebra S with coefficients in an S -module N is the n^{th} homology module of $L(S|R) \otimes_S N$, where $L(S|R)$ is the cotangent complex of ψ , uniquely defined in the derived category of S -modules $D(S)$, see [1] and [18]. We follow the exposition in [9]. For $n = 0, 1, 2$ a nice treatment has been given in [10]. If R and S are (\mathbb{Z}) -graded and ψ is degree preserving then $L(S|R)$ is a complex of graded S -modules and if N is a graded S -module then $D_n(S|R, N)$ is a graded S -module for all $n \geq 0$.

2.2. If $\psi: R \rightarrow S$ is essentially of finite type then $L(S|R)$ is homotopic to a complex

$$\cdots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow 0,$$

where each L_i is a finitely generated free S -module, see [9, 6.11]. So if N is a finitely generated S -module then so is $D_n(S|R, N)$ for all $n \geq 0$.

2.3. Let (A, \mathfrak{m}) be a Noetherian local ring. Let $T = \bigoplus_{n \in \mathbb{Z}} T_n$ be a \mathbb{Z} -graded ring with $T_0 = A$. Assume $\mathfrak{n} = (\bigoplus_{n \leq -1} T_n) \oplus \mathfrak{m} \oplus (\bigoplus_{n \geq 1} T_n)$ is a maximal ideal of T . Let I be a graded ideal of T . If T is Cohen-Macaulay then all non-zero Koszul homology modules of I have Krull dimension equal to $\dim T/I$. This is proved for local rings in [19, 4.2.2]. The same proof works for $*$ -local rings.

2.4. (with hypotheses as in 2.3). Assume $K = (u_1, \dots, u_s)$ where u_i are homogeneous. Let M be a graded T/K -module. (Here T need not be Cohen-Macaulay). Then there exists an exact sequence

$$\begin{aligned} 0 \rightarrow D_2(T/K|T, M) \rightarrow H_1(u_1, \dots, u_s) \otimes_{T/K} M \\ \rightarrow (T/K)^s \otimes_{T/K} M \rightarrow K/K^2 \otimes_{T/K} M \rightarrow 0. \end{aligned}$$

This proof is proved for local rings in [10, 2.5.1]. The same proof works for $*$ -local rings.

3. SOME PRELIMINARY RESULTS

In this section we first prove:

Lemma 3.1. *Let (A, \mathfrak{m}) be a Noetherian local ring and let $I \subseteq \mathfrak{m}$ be an ideal of A . Let M be a finitely generated A -module with an I -stable filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{Z}}$. Let $\widehat{\mathcal{R}}(\mathcal{F}, M) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$ be the extended Rees module of M with respect to \mathcal{F} . Then for $j \geq 1$ there exists positive integers s_j (depending on j) such that*

$$t^{-s_j} D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) = 0.$$

In particular $D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))_n = 0$ for $n \ll 0$.

Proof. Fix $j \geq 1$. We note that $D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))$ is a finitely generated graded $\widehat{\mathcal{R}}(I)$ -module. So it suffices to show $D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))_{t^{-1}} = 0$. We have an exact sequence of $\widehat{\mathcal{R}}(I)$ -modules

$$0 \rightarrow \widehat{\mathcal{R}}(\mathcal{F}, M) \rightarrow M[t, t^{-1}] \rightarrow L_{\mathcal{F}}(M) = \bigoplus_{n \in \mathbb{Z}} M/\mathcal{F}_n \rightarrow 0.$$

We note that $L_{\mathcal{F}}(M)_n = 0$ for $n \ll 0$. In particular $L_{\mathcal{F}}(M)_{t^{-1}} = 0$. So $\widehat{\mathcal{R}}(\mathcal{F}, M)_{t^{-1}} \cong M[t, t^{-1}]$. We also have $\widehat{\mathcal{R}}(I)_{t^{-1}} = A[t, t^{-1}]$ a smooth A -algebra. We have for $j \geq 1$

$$\begin{aligned} D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))_{t^{-1}} &\cong D_j(\widehat{\mathcal{R}}(I)_{t^{-1}}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)_{t^{-1}}) \\ &= D_j(A[t, t^{-1}]|A, M[t, t^{-1}]) = 0. \end{aligned}$$

The vanishing in the last line occurs since $A[t, t^{-1}]$ is a smooth A -algebra. \square

3.2. Let M be a finitely generated A -module. If $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ is an I -stable filtration then set $G_{\mathcal{F}}(M) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n / \mathcal{F}_{n+1}$ the associated graded module of M with respect to \mathcal{F} . We note $G_{\mathcal{F}}(M)_n = 0$ for $n \ll 0$. We have the following:

Corollary 3.3. *(with hypotheses as above). If for some $j \geq 2$ we have $D_j(\widehat{\mathcal{R}}(I)|A, G_{\mathcal{F}}(M)) = 0$ then $D_{j-1}(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) = 0$.*

Proof. We have an exact sequence of graded $\widehat{\mathcal{R}}(I)$ -modules

$$0 \rightarrow \widehat{\mathcal{R}}(\mathcal{F}, M)(+1) \xrightarrow{t^{-1}} \widehat{\mathcal{R}}(\mathcal{F}, M) \rightarrow G_{\mathcal{F}}(M) \rightarrow 0.$$

By considering the long exact sequence in homology we obtain an injective map

$$0 \rightarrow D_{j-1}(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))(+1) \xrightarrow{t^{-1}} D_{j-1}(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M)).$$

By 3.1 we know that $D_{j-1}(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))$ is t^{-1} -torsion. The result follows. \square

Next we prove:

Lemma 3.4. *Let (A, \mathfrak{m}) be a local Noetherian ring and let $I \subseteq \mathfrak{m}$ be an ideal. Let M be an A -module and let $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ be an I -stable filtration on M with $\mathcal{F}_i = M$ for $i \leq 0$ and $\mathcal{F}_1 \neq F_0$. Let $\widehat{\mathcal{R}}(\mathcal{F}, M) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$ be the extended Rees module of M with respect to \mathcal{F} (considered as an $\widehat{\mathcal{R}}(I)$ -module). Let $\mathcal{R}_{\mathcal{F}}(M) = \bigoplus_{n \geq 0} \mathcal{F}_n$ be the Rees module of M with respect to \mathcal{F} (considered as an $\mathcal{R}(I)$ -module). For each $j \geq 0$ there exists $n(j)$ (depending on j) such that we have an isomorphism of A -modules*

$$D_j(\mathcal{R}(I)|A, \mathcal{R}_{\mathcal{F}}(M))_n \cong D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))_n \quad \text{for } n \geq n(j).$$

Proof. We have an exact sequence of $\mathcal{R}(I)$ -modules

$$0 \rightarrow \mathcal{R}_{\mathcal{F}}(M) \rightarrow \widehat{\mathcal{R}}(\mathcal{F}, M) \rightarrow E \rightarrow 0.$$

We note that the cotangent complex of $\mathcal{R}(I)$ with respect to A is homotopic to a complex of finitely generated graded free $\mathcal{R}(I)$ -modules. As $E_n = 0$ for $n \gg 0$ it follows that given $j \geq 0$ there exists $n(j)'$ depending on j such that $D_j(\mathcal{R}(I)|A, E)_n = 0$ for all $n \geq n(j)'$. Set $n(j)^* = \max\{n(j)', n(j+1)'\}$. Therefore for all $j \geq 0$ we have an isomorphism of A -modules

$$D_j(\mathcal{R}(I)|A, \mathcal{R}_{\mathcal{F}}(M))_n \cong D_j(\mathcal{R}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M))_n \quad \text{for all } n \geq n(j)^*.$$

Consider the Jacobi-Zariski sequence to $A \rightarrow \mathcal{R}(I) \rightarrow \widehat{\mathcal{R}}(I)$ for $j \geq 0$

$$D_j(\mathcal{R}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) \rightarrow D_j(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) \rightarrow D_j(\widehat{\mathcal{R}}(I)|\mathcal{R}(I), \widehat{\mathcal{R}}(\mathcal{F}, M)).$$

Let $I = (a_1, \dots, a_s)$. Set $X_i = a_i t$. Note $\mathcal{R}(I)_{X_i} = \widehat{\mathcal{R}}(I)_{X_i}$. We then have for $j \geq 1$

$$D_j(\widehat{\mathcal{R}}(I)|\mathcal{R}(I), \widehat{\mathcal{R}}(\mathcal{F}, M))_{X_i} \cong D_j(\widehat{\mathcal{R}}(I)_{X_i}|\mathcal{R}(I)_{X_i}, \widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}) = 0.$$

The last equality holds as $\mathcal{R}(I)_{X_i} = \widehat{\mathcal{R}}(I)_{X_i}$. We note that $D_j(\widehat{\mathcal{R}}(I)|\mathcal{R}(I), \widehat{\mathcal{R}}(\mathcal{F}, M))$ is a finitely generated graded $\widehat{\mathcal{R}}(I)$ -module. By the above calculation we have that $D_j(\widehat{\mathcal{R}}(I)|\mathcal{R}(I), \widehat{\mathcal{R}}(\mathcal{F}, M))$ is $\widehat{\mathcal{R}}(I)_+$ -torsion. Therefore $D_j(\widehat{\mathcal{R}}(I)|\mathcal{R}(I), \widehat{\mathcal{R}}(\mathcal{F}, M))_n = 0$ for $n \gg 0$. The result follows. \square

4. PROOFS OF THEOREMS 1.10 AND 1.11

In this section we give proofs of Theorem 1.10 and 1.11. We first state a few preliminary results first.

4.1. We need to change base to the case when A has infinite residue field. If the residue field of A is finite then let $B = A[Y]_{\mathfrak{m}_A[Y]}$ with maximal ideal $\mathfrak{n} = \mathfrak{m}B$. We note that the residue field of B is $k(Y)$ which is infinite. Let $I = (a_1, \dots, a_r)$ be an ideal in A . Set $T = B[IBt] = \mathcal{R}(I) \otimes_A B$ and $\widehat{T} = B[IB, t^{-1}] = \widehat{\mathcal{R}}(I) \otimes_A B$.

Consider the map $\epsilon: S = A[X_1, \dots, X_r] \rightarrow \mathcal{R}(I)$ which maps $X_i \rightarrow a_i t$ for all i . Let $J = \ker \epsilon$. Consider $\epsilon \otimes_A B: U \rightarrow T$ where $U = B[X_1, \dots, X_r] = S \otimes_A B$. We note that $J_B := \ker \epsilon \otimes B = J \otimes_A B$.

Consider the map $\widehat{\epsilon}: \widehat{S} = A[X_1, \dots, X_r, V] \rightarrow \widehat{\mathcal{R}}(I)$ which maps $X_i \rightarrow a_i t$ for all i and V is mapped to t^{-1} . Let $\widehat{J} = \ker \widehat{\epsilon}$. Consider $\widehat{\epsilon} \otimes_A B: \widehat{U} \rightarrow \widehat{T}$ where $\widehat{U} = B[X_1, \dots, X_r, V] = \widehat{S} \otimes_A B$. We note that $\widehat{J}_B := \ker \widehat{\epsilon} \otimes B = \widehat{J} \otimes_A B$.

Lemma 4.2. (with hypotheses as in 4.1). Further assume A is Cohen-Macaulay and height $I > 0$. We have

- (I) The following assertions are equivalent:
 - (a) $H_1(J)$ is an unmixed and equi-dimensional $\mathcal{R}(I)$ -module.
 - (b) $H_1(J_B)$ is an unmixed and equi-dimensional T -module.
- (II) The following assertions are equivalent:
 - (a) $H_1(\widehat{J})$ is an unmixed and equi-dimensional $\widehat{\mathcal{R}}(I)$ -module.
 - (b) $H_1(\widehat{J}_B)$ is an unmixed and equi-dimensional \widehat{T} -module.
- (III) Fix $n \in \mathbb{Z}$. The following assertions are equivalent:
 - (a) $D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$
 - (b) $D_2(T|B, T)_n = 0$.
- (IV) Fix $n \in \mathbb{Z}$. The following assertions are equivalent:
 - (a) $D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_n = 0$
 - (b) $D_2(\widehat{T}|B, \widehat{T})_n = 0$.
- (V) The following assertions are equivalent:
 - (a) $\text{Ass } H_1(J) \subseteq \text{Ass } \mathcal{R}(I) \cup V(\mathcal{R}(I)_+)$.
 - (b) $\text{Ass } H_1(J_B) \subseteq \text{Ass } T \cup V(T_+)$.
- (VI) The following assertions are equivalent:
 - (a) $\text{Ass } H_1(\widehat{J}) \subseteq \text{Ass } \widehat{\mathcal{R}}(I) \cup V(\widehat{\mathcal{R}}(I)_+)$.
 - (b) $\text{Ass } H_1(\widehat{J}_B) \subseteq \text{Ass } \widehat{T} \cup V(\widehat{T}_+)$.

To prove this result we need Theorem 23.3 from [11]. Unfortunately there is a typographical error in the statement of Theorem 23.3 in [11]. So we state it here.

Theorem 4.3. Let $\varphi: A \rightarrow B$ be a homomorphism of Noetherian rings, and let E be an A -module and G a B -module. Suppose that G is flat over A ; then we have the following:

- (i) if $\mathfrak{p} \in \text{Spec } A$ and $G/\mathfrak{p}G \neq 0$ then

$${}^a\varphi(\text{Ass}_B(G/\mathfrak{p}G)) = \text{Ass}_A(G/\mathfrak{p}G) = \{\mathfrak{p}\}.$$

$$(ii) \text{ Ass}_B(E \otimes_A G) = \bigcup_{\mathfrak{p} \in \text{Ass}_A(E)} \text{Ass}_B(G/\mathfrak{p}G).$$

Remark 4.4. In [11] $\text{Ass}_A(E \otimes G)$ is typed instead of $\text{Ass}_B(E \otimes G)$. Also note that ${}^a\varphi(\mathfrak{P}) = \mathfrak{P} \cap A$ for $\mathfrak{P} \in \text{Spec } B$.

We now give

Proof of Lemma 4.2. (I) We note that T is faithfully flat over R and $H_1(J_B) = H_1(J) \otimes_R T$. To prove the result we may assume $H_1(J) \neq 0$. By Theorem 4.3 we have

$$\text{Ass}_T H_1(J_B) = \bigcup_{\mathfrak{p} \in \text{Ass}_{\mathcal{R}(I)} H_1(J)} \text{Ass}_T T/\mathfrak{p}T.$$

We also have

$$\text{Ass}_T T = \bigcup_{\mathfrak{p} \in \text{Ass}_{\mathcal{R}(I)} \mathcal{R}(I)} \text{Ass}_T T/\mathfrak{p}T.$$

We also note that $\dim H_1(J) = \dim \mathcal{R}(I)$ and $\dim H_1(J_B) = \dim T$; see 2.3. We also have $\dim \mathcal{R}(I) = \dim T$.

(a) \implies (b) We note that as $H_1(J)$ is an unmixed and equi-dimensional $\mathcal{R}(I)$ -module and $\dim H_1(J) = \dim \mathcal{R}(I)$; it follows that $\text{Ass } H_1(J) \subseteq \text{Ass } \mathcal{R}(I)$. By the above equality we have $\text{Ass}_T H_1(J_B) \subseteq \text{Ass}_T T$. As T is unmixed and equi-dimensional; see 11.2; the result follows.

(b) \implies (a) We note that as $H_1(J_B)$ is an unmixed and equi-dimensional T -module and $\dim H_1(J_B) = \dim T$; it follows that $\text{Ass } H_1(J_B) \subseteq \text{Ass } T$. Let $\mathfrak{p} \in \text{Ass } H_1(J)$. As T is faithfully flat over $\mathcal{R}(I)$ we get that $T/\mathfrak{p}T \neq 0$. Let $Q \in \text{Ass}_T T/\mathfrak{p}T$. Then note that as $Q \in \text{Ass } T$ we get that $\mathfrak{p} = Q \cap \mathcal{R}(I) \in \text{Ass } \mathcal{R}(I)$. So $\text{Ass } H_1(J) \subseteq \text{Ass } \mathcal{R}(I)$. As $\mathcal{R}(I)$ is unmixed and equi-dimensional; see 11.2; the result follows.

(II) This follows as in (I).

(III) By [9, 6.3] we have for all $j \geq 0$ we have a graded isomorphism

$$D_j(T|B, T) = D_j(\mathcal{R}(I)|A, T)$$

We note that $D_j(\mathcal{R}(I)|A, T) = D_j(\mathcal{R}(I)|A, \mathcal{R}(I)) \otimes B$. So for all $n \in \mathbb{Z}$

$$D_j(T|B, T)_n = D_j(\mathcal{R}(I)|A, \mathcal{R}(I))_n \otimes B.$$

The result follows as B is faithfully flat over A .

(IV) This follows as in (III).

(V) We note that if $\mathfrak{p} \in \text{Spec } \mathcal{R}(I)$ and $Q \in \text{Ass}_T T/\mathfrak{p}T$ then

- (1) $Q \cap \mathcal{R}(I) = \mathfrak{p}.$
- (2) $\mathfrak{p} \supseteq \mathcal{R}(I)_+$ if and only if $Q \supseteq T_+.$
- (3) $\mathfrak{p} \in \text{Ass } \mathcal{R}(I)$ if and only if $Q \in \text{Ass } T.$

(a) \implies (b) Let $Q \in \text{Ass}_T H_1(J_B)$. Then $Q \in \text{Ass } T/\mathfrak{p}T$ for some $\mathfrak{p} \in \text{Ass}_{\mathcal{R}(I)} H_1(J)$. If $\mathfrak{p} \supseteq \mathcal{R}(I)_+$ then $Q \supseteq T_+.$ If $\mathfrak{p} \in \text{Ass } \mathcal{R}(I)$ then $Q \in \text{Ass } T.$

(b) \implies (2). Let $\mathfrak{p} \in \text{Ass } H_1(J)$. The map $\mathcal{R}(I) \rightarrow T$ is faithfully flat. So $T/\mathfrak{p}T \neq 0$. Let $Q \in \text{Ass}_T T/\mathfrak{p}T$. Then $Q \in \text{Ass}_T H_1(J_B)$ and $Q \cap \mathcal{R}(I) = \mathfrak{p}.$ If $Q \supseteq T_+$ then $\mathfrak{p} \supseteq \mathcal{R}(I)_+.$ If $Q \in \text{Ass } T$ then $\mathfrak{p} \in \text{Ass } \mathcal{R}(I)$. The result follows.

(VI) This is as in (V). \square

We give

Proof of Theorem 1.10. .

(II) We note that $D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I)) = D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I))$. We also have $\widehat{\mathcal{R}}(I)$ is unmixed and equi-dimensional (see 11.2). By 2.4 we have an exact sequence

$$0 \rightarrow D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \rightarrow H_1(\widehat{J}) \rightarrow \widehat{\mathcal{R}}(I)^s \dots$$

(II) (b) \implies (a) follows as $\widehat{\mathcal{R}}(I)$ is unmixed and equi-dimensional, see 11.2.

(a) \implies (b). Suppose if possible $D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \neq 0$. By Lemma 3.1 we have $D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I))$ is t^{-1} -torsion. So $\dim D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \leq d$. But $H_1(I)$ is unmixed and equi-dimensional; and has dimension $= \dim \widehat{\mathcal{R}}(I) = d + 1$. This is a contradiction.

(I) We note that $D_2(\mathcal{R}(I)|A, \mathcal{R}(I)) = D_2(\widehat{\mathcal{R}}(I)|S, \mathcal{R}(I))$. We also have $\widehat{\mathcal{R}}(I)$ is unmixed and equi-dimensional (see 11.2). By 2.4 we have an exact sequence

$$0 \rightarrow D_2(\mathcal{R}(I)|S, \mathcal{R}(I)) \rightarrow H_1(J) \rightarrow \mathcal{R}(I)^l \dots$$

(I) (b) \implies (a) follows as $\mathcal{R}(I)$ is unmixed and equi-dimensional.

(a) \implies (b). By 4.2 we may assume that the residue field k of A is infinite. Let $I = (a_1, \dots, a_s)$. By 11.5, we may assume that each a_i is A -regular. Set $X_i = a_i t$ for $i = 1, \dots, s$. Suppose if possible $E = D_2(\mathcal{R}(I)|S, \mathcal{R}(I)) \neq 0$.

Claim-1 E is $\mathcal{R}(I)_+$ -torsion.

Suppose if possible this is not true. then $E_{X_i} \neq 0$ for some i . But

$$\begin{aligned} E_{X_i} &= D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_{X_i} = D_2(\mathcal{R}(I)_{X_i}|A, \mathcal{R}(I)_{X_i}) \\ &= D_2(\widehat{\mathcal{R}}(I)_{X_i}|A, \widehat{\mathcal{R}}(I)_{X_i}) \\ &= D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_{X_i}. \end{aligned}$$

So E_{X_i} is t^{-1} -torsion (by 3.1). By hypothesis $H_1(J)$ is unmixed, equi-dimensional graded $\mathcal{R}(I)$ -module of dimension $\dim \mathcal{R}(I)$. We have $\text{Ass } H_1(J)_{X_i} \subseteq \text{Ass } \mathcal{R}(I)_{X_i}$. But $\mathcal{R}(I)_{X_i} = \widehat{\mathcal{R}}(I)_{X_i}$. We have $\text{Ass } \mathcal{R}(I)_{X_i} = \{Q_{X_i} \mid Q \in \text{Ass } \widehat{\mathcal{R}}(I)\}$.

Claim-2: $t^{-1} \notin Q_{X_i}$ for every $Q \in \text{Ass } \widehat{\mathcal{R}}(I)$.

Suppose Claim-2 is true. Then t^{-1} is $H_1(J)_{X_i}$ -regular. But E_{X_i} is t^{-1} -torsion and a submodule of $H_1(J)_{X_i}$. This is a contradiction. So Claim-1 follows.

We prove Claim-2. Suppose $t^{-1} \in Q_{X_i}$ for some $Q \in \text{Ass } \widehat{\mathcal{R}}(I)$. As $\widehat{\mathcal{R}}(I)$ is unmixed, by [4, 4.5.5] we have $Q = PA[t, t^{-1}] \cap \widehat{\mathcal{R}}(I)$ for some minimal prime P of A . We have $X_i^l t^{-1} \in Q$ for some $l \geq 0$. Note that $l = 0$ is not possible. Also note that if $l \geq 1$ then $a_i \in P$. This is a contradiction as a_i is A -regular and $P \in \text{Ass } A$.

So $E_{X_i} = 0$ for all i . Thus E is a finitely generated $\mathcal{R}(I)/\mathcal{R}(I)_+^m$ module for some $m \geq 1$. So E is a finitely generated A -module. Therefore $\dim E \leq d$. But $\dim H_1(J)$ is $d + 1$ and it is unmixed and equi-dimensional. This is a contradiction. So $E = 0$. \square

Next we give

Proof of Theorem 1.11. The assertion (iii) \Leftrightarrow (iv) follows from Lemma 3.4. For the rest of the assertions we first note that $D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I)) = D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I))$ and $D_2(\mathcal{R}(I)|A, \mathcal{R}(I)) = D_2(\widehat{\mathcal{R}}(I)|S, \mathcal{R}(I))$.

By 2.4 we have an exact sequence

$$0 \rightarrow D_2(\widehat{\mathcal{R}}(I)|\widehat{S}, \widehat{\mathcal{R}}(I)) \rightarrow H_1(\widehat{J}) \rightarrow \widehat{\mathcal{R}}(I)^s \dots$$

(iv) \implies (ii): We note that $D_2(\widehat{\mathcal{R}}(I)|S, \mathcal{R}(I))$ is a finitely generated graded $\mathcal{R}(I)$ -module. As $D_2(\widehat{\mathcal{R}}(I)|S, \mathcal{R}(I))_n = 0$ for $n \gg 0$ it follows that $D_2(\widehat{\mathcal{R}}(I)|S, \mathcal{R}(I))$ is $\mathcal{R}(I)_+$ -torsion. The result follows from the above exact sequence.

(iii) \implies (i): This is similar to (iv) \implies (ii).

(i) \implies (iii) and (ii) \implies (iv): By 4.2 we may assume that the residue field k of A is infinite. Let $I = (a_1, \dots, a_s)$. By 11.5 we may assume that each a_i is A -regular. Set $X_i = a_i t$ for $i = 1, \dots, s$. We also note that

$$\begin{aligned} E &= D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_{X_i} \cong D_2(\mathcal{R}(I)_{X_i}|A, \mathcal{R}(I)_{X_i}) \\ &= D_2(\widehat{\mathcal{R}}(I)_{X_i}|A, \widehat{\mathcal{R}}(I)_{X_i}) \\ &= D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_{X_i}. \end{aligned}$$

Suppose if possible $E \neq 0$. By 3.1 we get that E is t^{-1} -torsion. By our hypotheses $\text{Ass } H_1(J)_{X_i} \subseteq \text{Ass } \mathcal{R}(I)_{X_i}$ and $\text{Ass } H_1(\widehat{J})_{X_i} \subseteq \text{Ass } \widehat{\mathcal{R}}(I)_{X_i}$. We note that $\mathcal{R}(I)_{X_i} = \widehat{\mathcal{R}}(I)_{X_i}$. By an argument similar to that of Claim-2 in proof of Theorem 1.10, we get that t^{-1} is $H_1(J)_{X_i}$ and $H_1(\widehat{J})_{X_i}$ -regular. Also t^{-1} is E -torsion. This is a contradiction. So E is zero. It follows that $D_2(\mathcal{R}(I)|A, \mathcal{R}(I))$ (and $D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))$) is supported at $V(\mathcal{R}(I)_+)$ (respectively $V(\widehat{\mathcal{R}}(I)_+)$). The result follows. \square

5. PROOF OF THEOREM 1.5

In this section we first show that while proving 1.5 we can assume that A is complete.

Lemma 5.1. *(with hypotheses as in 1.5) Let A^* be the completion of A . Set $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(I)$ and $T = A^*[IA^*t, t^{-1}] = \widehat{\mathcal{R}} \otimes_A A^*$. The natural map $\widehat{S} \otimes_A A^* \rightarrow T$ has kernel $\widehat{J}^* = \widehat{J} \otimes_A A^*$. We note that $H_1(\widehat{J}^*) = H_1(\widehat{J}) \otimes_{\widehat{\mathcal{R}}} T$. We have*

- (I) *The following assertions are equivalent:*
 - (a) $\widehat{\mathcal{R}}$ is a complete intersection.
 - (b) T is a complete intersection.
- (II) *The following assertions are equivalent:*
 - (a) $H_1(\widehat{J})$ is a free $\widehat{\mathcal{R}}$ -module.
 - (b) $H_1(\widehat{J}^*)$ is a free T -module.
- (III) *The following assertions are equivalent:*
 - (a) $D_3(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}) = 0$.
 - (b) $D_3(T|A^*, T) = 0$.

Proof. (I) The map $\widehat{\mathcal{R}} \rightarrow T$ is flat by fiber k . The result follows from [10, 4.3.8].

(II) We have $\text{Tor}_1^T(k, H_1(\widehat{J}^*)) \cong \text{Tor}_1^{\widehat{\mathcal{R}}}(k, H_1(\widehat{J})) \otimes_{\widehat{\mathcal{R}}} T$. The result follows as T is a faithfully flat $\widehat{\mathcal{R}}$ -algebra.

(III) As A^* is a flat A -algebra we have $D_3(T|A^*, T) = D_3(\widehat{\mathcal{R}}|A, T)$, see [9, 6.3]. Also note that $D_3(\widehat{\mathcal{R}}|A, T) = D_3(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}) \otimes_A A^*$. The result follows as A^* is a faithfully flat A -algebra. \square

In this section we first give

Proof of Theorem 1.5. : By 5.1 we may assume that A is complete. Let Q be a complete regular local ring with a surjection $\psi: Q \rightarrow A$. We have $\ker \psi$ is generated by a Q -regular sequence $\mathbf{f} = f_1, \dots, f_c$. Let $k = A/\mathfrak{m}$. Set $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(I)$.

(i) \implies (ii):

Consider $A \rightarrow \widehat{\mathcal{R}} \rightarrow k$. The Jacobi-Zariski sequence yields for $j \geq 3$

$$D_{j+1}(k|\widehat{\mathcal{R}}, k) \rightarrow D_j(\widehat{\mathcal{R}}|A, k) \rightarrow D_j(k|A, k).$$

As A is a complete intersection we have $D_j(k|A, k) = 0$ for $j \geq 3$. Also as $\widehat{\mathcal{R}}$ is a complete intersection we have $D_j(k|\widehat{\mathcal{R}}, k) = 0$ for $j \geq 3$. Therefore $D_j(\widehat{\mathcal{R}}|A, k) = 0$ for $j \geq 3$. From [9, 8.7] it follows that $D_j(\widehat{\mathcal{R}}|A, E) = 0$ for $j \geq 3$ for any $\widehat{\mathcal{R}}$ -module E . In particular this holds when $E = G_{\mathcal{F}}(M)$ the associated graded module of an I -stable filtration on an A -module M . By 3.3 it follows that $D_j(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) = 0$ for $j \geq 2$.

Recall we have a right exact complex $C: 0 \rightarrow H_1(\widehat{J}) \xrightarrow{\theta} \widehat{\mathcal{R}}^s \rightarrow \widehat{J}/\widehat{J}^2 \rightarrow 0$ where $\ker \theta = D_2(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}})$. The later module is zero. So C is exact. It follows that $\text{Tor}_1^{\widehat{\mathcal{R}}}(\widehat{J}/\widehat{J}^2, \widehat{\mathcal{R}}(\mathcal{F}, M)) = D_2(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)) = 0$. Consider the exact sequence $0 \rightarrow \mathfrak{M} \rightarrow \widehat{\mathcal{R}} \rightarrow k \rightarrow 0$. As t^{-1} is \mathfrak{M} -regular we get that $\mathfrak{M} = \widehat{\mathcal{R}}(\mathcal{F}, N)$ for some A -module N and a I -stable filtration on N , see [17, 3.1]. It follows that $\text{Tor}_2^{\widehat{\mathcal{R}}}(\widehat{J}/\widehat{J}^2, k) = 0$. So $\text{projdim}_{\widehat{\mathcal{R}}} \widehat{J}/\widehat{J}^2 \leq 1$. As C is exact it follows that $H_1(\widehat{J})$ is free $\widehat{\mathcal{R}}$ -module.

(ii) \implies (i):

As $H_1(J)$ is free $\widehat{\mathcal{R}}$ -module it is unmixed and equ-dimensional (as $\widehat{\mathcal{R}}$ is). So by Theorem 1.10 (II) we get

$D_2(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}) = 0$. So the complex $C: 0 \rightarrow H_1(\widehat{J}) \xrightarrow{\theta} \widehat{\mathcal{R}}^s \rightarrow \widehat{J}/\widehat{J}^2 \rightarrow 0$ is exact. This implies that $\text{projdim}_{\widehat{\mathcal{R}}} \widehat{J}/\widehat{J}^2 \leq 1$. So we have

$$D_2(\widehat{\mathcal{R}}|A, \mathfrak{M}) = \text{Tor}_1^{\widehat{\mathcal{R}}}(\mathfrak{M}, \widehat{J}/\widehat{J}^2) = \text{Tor}_2^{\widehat{\mathcal{R}}}(k, \widehat{J}/\widehat{J}^2) = 0.$$

We consider the exact sequence $0 \rightarrow \mathfrak{M} \rightarrow \widehat{\mathcal{R}} \rightarrow k \rightarrow 0$. As $D_3(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}) = 0$ and $D_2(\widehat{\mathcal{R}}|A, \mathfrak{M}) = 0$ it follows from the long exact sequence of homology that $D_3(\widehat{\mathcal{R}}|A, k) = 0$. By the Jacobi-Zariski sequence for $Q \rightarrow A \rightarrow \widehat{\mathcal{R}}$ it follows that $D_3(\widehat{\mathcal{R}}|Q, k) = 0$. It follows that $D_3(\widehat{\mathcal{R}}|Q, -) = 0$. Now $\widehat{\mathcal{R}}$ has finite flat dimension over Q . By [3, 1.4], it follows that $\widehat{\mathcal{R}}$ is a complete intersection. \square

6. PROOF OF THEOREM 1.2

6.1. We first prove that it suffices to assume that A is a quotient of a regular local ring. Let (A, \mathfrak{m}) be an excellent complete intersection and let I be an ideal in A of positive height. Let A^* be the completion of A . Let $T = A^*[IA^*t] = \mathcal{R}(I) \otimes_A A^*$ and let $\widehat{T} = A^*[IA^*t, t^{-1}] = \widehat{\mathcal{R}}(I) \otimes_A A^*$. The natural map $S \otimes_A A^* \rightarrow T$ has kernel $J^* = J \otimes_A A^*$. We note that $H_1(J^*) = H_1(J) \otimes_{R(I)} T$. The natural map $\widehat{S} \otimes_A A^* \rightarrow \widehat{T}$ has kernel $\widehat{J}^* = \widehat{J} \otimes_A A^*$. We note that $H_1(\widehat{J}^*) = H_1(\widehat{J}) \otimes_{\widehat{\mathcal{R}}(I)} \widehat{T}$.

Lemma 6.2. *(with hypotheses as in 6.1 We have*

- (I) *The following assertions are equivalent:*
 - (a) $\text{Proj } \mathcal{R}(I)$ *is a complete intersection.*
 - (b) $\text{Proj } T$ *is a complete intersection.*
- (II) *The following assertions are equivalent:*
 - (a) $\text{Proj } \widehat{\mathcal{R}}(I)$ *is a complete intersection.*
 - (b) $\text{Proj } \widehat{T}$ *is a complete intersection.*

- (III) *The following assertions are equivalent:*
 (a) $D_3(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$.
 (b) $D_3(T|A^*, T)_n = 0$ for $n \gg 0$.
 (IV) *The following assertions are equivalent:*
 (a) $D_3(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_n = 0$ for $n \gg 0$.
 (b) $D_3(\widehat{T}|A^*, \widehat{T})_n = 0$ for $n \gg 0$.
 (V) *The following assertions are equivalent:*
 (a) $H_1(J)_P$ is a free $\mathcal{R}(I)_P$ -module for every $P \in \text{Proj}(\mathcal{R}(I))$.
 (b) $H_1(J^*)_P$ is a free T_Q -module for every $Q \in \text{Proj}(T)$.
 (VI) *The following assertions are equivalent:*
 (a) $H_1(\widehat{J})_P$ is a free $\widehat{\mathcal{R}}(I)_P$ -module for every $P \in \text{Proj}(\widehat{\mathcal{R}}(I))$.
 (b) $H_1(\widehat{J}^*)_P$ is a free \widehat{T}_Q -module for every $Q \in \text{Proj}(\widehat{T})$.

Proof. Let P be a prime ideal in $\mathcal{R}(I)$ and Q a prime ideal in T with $Q \cap \mathcal{R}(I) = P$. Then $P \supseteq \mathcal{R}(I)_+$ if and only if $Q \supseteq T_+$. The map $A \rightarrow A^*$ is regular. As $\mathcal{R}(I)$ is a finitely generated A -algebra it follows that the map $\mathcal{R}(I) \rightarrow \mathcal{R}(I) \otimes_A A^* = T$ is also regular (and faithfully flat).

(I) First assume that $\text{Proj } \mathcal{R}(I)$ is a complete intersection. If $Q \in \text{Proj}(T)$ then note that $P = \mathcal{R}(I) \cap Q$ is in $\text{Proj}(\mathcal{R}(I))$. The map $\mathcal{R}(I)_P \rightarrow T_Q$ is regular. In particular its fiber is a regular local ring. So by a result of Avramov it follows that T_Q is also a complete intersection, see [10, 4.3.8].

Conversely assume that $\text{Proj } T$ is a complete intersection. Let $P \in \text{Proj } \mathcal{R}(I)$. As the map $\mathcal{R}(I) \rightarrow T$ is faithfully flat it follows that there exists prime ideal Q in T with $Q \cap \mathcal{R}(I) = P$. If Q is not homogeneous then if Q^* is the prime ideal in T generated by homogeneous elements in Q we get that $Q^* \supseteq P$. Thus $Q^* \cap \mathcal{R}(I) = P$. Note necessarily $Q^* \in \text{Proj } T$. Thus we can assume there exists $Q \in \text{Proj } T$ such that $Q \cap \mathcal{R}(I) = P$. The map $\mathcal{R}(I)_P \rightarrow T_Q$ is flat. So by a result of Avramov it follows that $\mathcal{R}(I)_P$ is also a complete intersection, see [10, 4.3.8]. The result follows.

(II) This follows from a similar argument as in (I).

(III) As A^* is a flat A -algebra we have $D_3(T|A^*, T) = D_3(\mathcal{R}(I)|A, T)$, see [9, 6.3]. Also note that $D_3(\mathcal{R}(I)|A, T) = D_3(\mathcal{R}(I)|A, \widehat{\mathcal{R}}) \otimes_A A^*$. This induces an isomorphism $D_3(T|A^*, T)_n = D_3(\mathcal{R}(I)|A, \widehat{\mathcal{R}})_n \otimes_A A^*$ for all $n \in \mathbb{Z}$. The result follows as A^* is a faithfully flat A -algebra.

(IV) This follows from a similar argument as in (III).

(V) Let $I = (a_1, \dots, a_m)$. Set $X_i = a_i t \in \mathcal{R}(I)_1$ and $Y_i = a_i t \in T_1$. The map $\mathcal{R}(I)_{X_i} \rightarrow T_{Y_i}$ is faithfully flat for all i . It suffices to show that if M is a graded $\mathcal{R}(I)$ -module then M is a projective $\mathcal{R}(I)_{X_i}$ -module if and only if $(M \otimes_{\mathcal{R}(I)} T)_{Y_i}$ is a projective T_{Y_i} -module for all i .

If M_{X_i} is projective $\mathcal{R}(I)_{X_i}$ -module then clearly $M_{X_i} \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i}$ is a projective T_{Y_i} -module. Observe that

$$(M \otimes_{\mathcal{R}(I)} T)_{Y_i} = M_{X_i} \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i}.$$

Conversely assume $M_{X_i} \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i}$ is a projective T_{Y_i} -module. Let $0 \rightarrow N \rightarrow F \rightarrow M_{X_i} \rightarrow 0$ be an exact sequence where F is a free $\mathcal{R}(I)_{X_i}$ -module. So N is also graded. Then notice that

$$\text{Ext}_{\mathcal{R}(I)_{X_i}}^1(M_{X_i}, N) \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i} = \text{Ext}_{T_{Y_i}}^1(M_{X_i} \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i}, N \otimes_{\mathcal{R}(I)_{X_i}} T_{Y_i}) = 0$$

As the map $\mathcal{R}(I)_{X_i} \rightarrow T_{Y_i}$ is faithfully flat we get $\text{Ext}_{\mathcal{R}(I)_{X_i}}^1(M_{X_i}, N) = 0$. So the map $F \rightarrow M_{X_i}$ splits. Thus M_{X_i} is a projective $\mathcal{R}(I)_{X_i}$ -module.

(VI) This follows from a similar argument as in (V). \square

Proof of Theorem 1.2. By 6.2 we may assume A is a quotient of a regular local ring (Q, \mathfrak{n}) . Let $A = Q/(\mathbf{f})$ where $\mathbf{f} = f_1, \dots, f_c \subseteq \mathfrak{n}^2$ is a Q -regular sequence.

(i) \Leftrightarrow (ii): This follows as $\text{Proj } \mathcal{R}(I) \cong \text{Proj } \widehat{\mathcal{R}}(I)$.

(i) \Rightarrow (iii): Let M be a finitely generated A -module and let \mathcal{F} be an I -stable filtration on M . Set $\mathcal{R}(\mathcal{F}, M)$ be the Rees module of M with respect to \mathcal{F} and let $\widehat{\mathcal{R}}(\mathcal{F}, M)$ be the extended Rees module associated to M with respect to \mathcal{F} . Let E be a finitely generated graded $\mathcal{R}(I)$ -module.

Claim-1: For $j \geq 2$ there exists $n(j)$ depending on j and $\mathcal{R}(\mathcal{F}, M)$ such that $D_j(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_n = 0$ for $n \geq n(j)$.

Let Q' be a polynomial algebra over Q mapping onto $\mathcal{R}(I)$. We note that $D_j(\mathcal{R}(I)|Q, E) = D_j(\mathcal{R}(I)|Q', E)$ for $j \geq 2$. Let $I = (a_1, \dots, a_r)$. Set $X_i = a_i t$ and let Y_i be an inverse image of X_i in Q' . The surjective map $Q_{Y_i} \rightarrow \mathcal{R}(I)_{X_i}$ is locally a complete intersection as both $\mathcal{R}(I)_{X_i}$ is a complete intersection and Q_{Y_i} is a regular ring. So we have $D_j(\mathcal{R}(I)|Q', E)_{X_i} = 0$ for all i . It follows that $D_j(\mathcal{R}(I)|Q', E)$ is $\mathcal{R}(I)_+$ -torsion. It follows that for $j \geq 2$ there exists $n(j)$ depending on j and E such that $D_j(\mathcal{R}(I)|Q', E)_n = 0$ for all $n \geq n(j)$.

We consider the Jacobi-Zariski sequence for $Q \rightarrow A \rightarrow \mathcal{R}(I)$. We have $D_j(A|Q, -) = 0$ for $j \geq 2$. So for $j \geq 3$ we have $D_j(\mathcal{R}(I)|Q, E) \cong D_j(\mathcal{R}(I)|A, E)$. Thus $D_j(\mathcal{R}(I)|A, E)_n = 0$ for $n \gg 0$, for $j \geq 3$.

For $j = 2$ we set $E = \mathcal{R}(\mathcal{F}, M)$. We have an exact sequence

$$\begin{aligned} 0 &= D_2(A|Q, \mathcal{R}(\mathcal{F}, M)) \rightarrow D_2(\mathcal{R}(I)|Q, \mathcal{R}(\mathcal{F}, M)) \rightarrow D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M)) \\ &\rightarrow D_1(A|Q, \mathcal{R}(\mathcal{F}, M)) = (\mathbf{f})/(\mathbf{f})^2 \otimes_A \mathcal{R}(\mathcal{F}, M) = \mathcal{R}(\mathcal{F}, M)^c. \end{aligned}$$

We have $D_2(\mathcal{R}(I)|Q, \mathcal{R}(\mathcal{F}, M))_{X_i} = 0$. So we have $D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_{X_i}$ is a $\mathcal{R}(I)_{X_i}$ submodule of $\mathcal{R}(\mathcal{F}, M)_{X_i}^c = \widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}^c$. Note $\mathcal{R}(I)_{X_i} = \widehat{\mathcal{R}}(I)_{X_i}$. We also have

$$\begin{aligned} D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_{X_i} &\cong D_2(\mathcal{R}(I)_{X_i}|A, \mathcal{R}(\mathcal{F}, M)_{X_i}) \\ &\cong D_2(\widehat{\mathcal{R}}(I)_{X_i}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}). \end{aligned}$$

If $\widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}^c = 0$ then $D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_{X_i} = 0$. If $\widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}^c \neq 0$ then t^{-1} is $\widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}^c$ -regular. However by 3.1 we get that $D_2(\widehat{\mathcal{R}}(I)_{X_i}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i})$ is t^{-1} -torsion. It follows that $D_2(\widehat{\mathcal{R}}(I)_{X_i}|A, \widehat{\mathcal{R}}(\mathcal{F}, M)_{X_i}) = 0$.

Thus $D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))$ is supported on $V(\mathcal{R}(I)_+)$. It follows that $D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_n = 0$ for $n \gg 0$.

Notice Claim-1 proves the assertion (a). We prove (b).

Consider the right exact complex $C: 0 \rightarrow H_1(J) \xrightarrow{\theta} \mathcal{R}(I)^s \rightarrow J/J^2 \rightarrow 0$. Note $\ker \theta = D_2(\mathcal{R}(I)|A, \mathcal{R}(I))$ which vanishes in high degrees. Let $W = \text{image } \theta$. We note that if \mathcal{F} is an I -stable filtration on M then $(H_1(J) \otimes \mathcal{R}(\mathcal{F}, M))_n \cong (W \otimes \mathcal{R}(\mathcal{F}, M))_n$ for $n \gg 0$. It follows that

$$\text{Tor}_1^{\mathcal{R}(I)}(J/J^2, \mathcal{R}(\mathcal{F}, M))_n \cong D_2(\mathcal{R}(I)|A, \mathcal{R}(\mathcal{F}, M))_n = 0 \text{ for } n \gg 0.$$

So $\text{Tor}_1^{\mathcal{R}(I)}(J/J^2, \mathcal{R}(\mathcal{F}, M))$ is $\mathcal{R}(I)_+$ -torsion.

Let $P \in \text{Proj}(\mathcal{R}(I)) = \text{Proj}(\widehat{\mathcal{R}}(I))$. So $\mathcal{R}(I)_P \cong \widehat{\mathcal{R}}(I)_P$. Let $\kappa(P)$ be the residue field of $\widehat{\mathcal{R}}(I)_P$. We have an exact sequence $0 \rightarrow E \rightarrow \widehat{\mathcal{R}}(I) \rightarrow \widehat{\mathcal{R}}(I)/P \rightarrow 0$. As t^{-1} is E -regular, by [17, 3.1] we get that $E = \widehat{\mathcal{R}}(\mathcal{F}, N)$ for some module N and I -stable filtration \mathcal{F} on N . We note that $\widehat{\mathcal{R}}(\mathcal{F}, N)_P = \mathcal{R}(\mathcal{F}, N)_P$ and $\widehat{\mathcal{R}}(I)_P = \mathcal{R}(I)_P$. As $\text{Tor}_1^{\mathcal{R}(I)}(J/J^2, \mathcal{R}(\mathcal{F}, N))$ is $\mathcal{R}(I)_+$ -torsion we get $\text{Tor}_1^{\mathcal{R}(I)_P}((J/J^2)_P, \mathcal{R}(\mathcal{F}, N)_P) = 0$. So we have $\text{Tor}_2^{\mathcal{R}(I)_P}((J/J^2)_P, \kappa(P)) = 0$. Thus $\text{projdim}_{\mathcal{R}(I)_P}(J/J^2)_P \leq 1$. We note that as $D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_n = 0$ for $n \gg 0$ we get C_P is an exact complex. So $H_1(J)_P$ is free $\mathcal{R}(I)_P$ -module.

(ii) \implies (iv): This is similar to (i) \implies (iii).

(iv) \implies (ii). Set $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(I)$. We first note that as $H_1(\widehat{J})_P$ is free for all $P \in \text{Proj}(\widehat{\mathcal{R}})$ we get $\text{Ass } H_1(\widehat{J}) \subseteq V(\widehat{\mathcal{R}}(I)_+) \cup \text{Ass } \widehat{\mathcal{R}}$. It follows from Theorem 1.11 that $D_2(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}})_n = 0$ for $n \gg 0$. Let P be a prime in $\text{Proj}(\widehat{\mathcal{R}})$. Consider the exact sequence

$$(\dagger) \quad 0 \rightarrow E \rightarrow \widehat{\mathcal{R}} \rightarrow \widehat{\mathcal{R}}/P \rightarrow 0$$

Consider the right exact complex $C: 0 \rightarrow H_1(\widehat{J}) \xrightarrow{\theta} \widehat{\mathcal{R}}^s \rightarrow \widehat{J}/\widehat{J}^2 \rightarrow 0$. We note that $\ker \theta = D_2(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}})_n = 0$ for $n \gg 0$. So $\ker \theta_L = 0$ for every $L \in \text{Proj } \mathcal{R}(I)$. Thus $\text{projdim}(\widehat{J}/\widehat{J}^2)_L \leq 1$ for every $L \in \text{Proj}(\widehat{\mathcal{R}})$. So $\text{Tor}_1^{\widehat{\mathcal{R}}_L}(E_L, (\widehat{J}/\widehat{J}^2)_L) = 0$. It follows that $\text{Tor}_1^{\widehat{\mathcal{R}}}(E, \widehat{J}/\widehat{J}^2)$ is $\widehat{\mathcal{R}}_+$ -torsion. So $\text{Tor}_1^{\widehat{\mathcal{R}}}(E, \widehat{J}/\widehat{J}^2)_n = 0$ for $n \gg 0$. Let $W = \text{image } \theta$. We note that $(H_1(\widehat{J}) \otimes E)_n \cong (W \otimes E)_n$ for $n \gg 0$. It follows that $D_2(\widehat{\mathcal{R}}|A, E)_n \cong \text{Tor}_1^{\mathcal{R}(I)}(\widehat{J}/\widehat{J}^2, E)_n = 0$ for $n \gg 0$. So $D_2(\widehat{\mathcal{R}}|A, E)$ is $\widehat{\mathcal{R}}_+$ -torsion. By the exact sequence (\dagger) we get an exact sequence

$$D_3(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}) \rightarrow D_3(\widehat{\mathcal{R}}|A, \widehat{\mathcal{R}}/P) \rightarrow D_2(\widehat{\mathcal{R}}|A, E).$$

Let $\kappa(P)$ be the residue field of $\widehat{\mathcal{R}}_P$. Then by the above exact sequence we get $D_3(\widehat{\mathcal{R}}_P|A, \kappa(P)) = 0$. So by [9, 8.7] it follows that $D_3(\widehat{\mathcal{R}}_P|A, -) = 0$. As $A = Q/(\mathbf{f})$ we get that $D_3(\widehat{\mathcal{R}}_P|Q, -) = 0$. Now $\widehat{\mathcal{R}}_P$ has finite flat dimension over Q . It follows from [3, 1.4] that $\widehat{\mathcal{R}}_P$ is a complete intersection.

(iii) \implies (i): This is similar to (iv) \implies (ii). \square

6.3. Resolution of singularities:

Let k be a field of characteristic zero and let (A, \mathfrak{m}) be a reduced local ring essentially of finite type over k . Hironaka, see [7, Main Theorem, p. 132], proved that there exists an ideal $I \subseteq A$ such that:

- (1) $V(I) = \text{Sing}(A)$.
- (2) The natural projection morphism $\pi: \text{Proj } \mathcal{R}(I) \rightarrow \text{Spec}(A)$ is a resolution of singularities of $\text{Spec}(A)$, i.e., $\text{Proj } \mathcal{R}(I)$ is non-singular and π induces a k -scheme isomorphism

$$\pi: \text{Proj}(\mathcal{R}(I)) \setminus \pi^{-1}(V(I)) \rightarrow \text{Spec}(A) \setminus \text{Sing}(A).$$

Example: Further assume that A is a reduced complete intersection of dimension $d \geq 1$. Let I be the ideal as constructed above. As A is reduced note height $I > 0$. By construction $\text{Proj } \mathcal{R}(I)$ is non-singular. In particular $\text{Proj } \mathcal{R}(I)$ is a complete intersection.

7. PROOF OF THEOREM 1.6

7.1. Setup: In this section (A, \mathfrak{m}) is a Cohen-Macaulay domain. Let $I \subseteq \mathfrak{m}$ be a non-zero ideal. Let J and \widehat{J} denote the defining ideals of the Rees algebra (and extended Rees algebra respectively) of I .

We first prove

Lemma 7.2. *(with hypotheses as in 7.1) We have*

$$\text{rank}_{\widehat{\mathcal{R}}(I)} \widehat{J}/\widehat{J}^2 = \mu(I).$$

The proof of Lemma 7.2 requires a few preparatory results.

Proposition 7.3. *(with hypotheses as in 7.1). Assume $I = (a_1, \dots, a_l)$ minimally and $l \geq 2$. Let $Q = (a_1, \dots, a_{l-1})$. We have an inclusion $\epsilon: \widehat{\mathcal{R}}(Q) \rightarrow \widehat{\mathcal{R}}(I)$. Consider the map $\tilde{\epsilon}: \widehat{\mathcal{R}}(Q)[X] \rightarrow \widehat{\mathcal{R}}(I)$ defined by mapping X to a_l . Clearly $\tilde{\epsilon}$ is surjective and let \mathfrak{p} be its kernel. Then*

$$\text{rank}_{\widehat{\mathcal{R}}(I)} \mathfrak{p}/\mathfrak{p}^2 = 1.$$

Proof. We note that $\epsilon_{t-1}: \widehat{\mathcal{R}}(Q)_{t-1} \rightarrow \widehat{\mathcal{R}}(I)_{t-1}$ is an isomorphism. We also note that $t^{-1} \notin \mathfrak{p}$. So we have an exact sequence

$$0 \rightarrow \mathfrak{p}_{t-1} \rightarrow \widehat{\mathcal{R}}(Q)[X]_{t-1} = A[t, t^{-1}][X] \xrightarrow{\tilde{\epsilon}_{t-1}} \widehat{\mathcal{R}}(I)_{t-1} = A[t, t^{-1}] \rightarrow 0.$$

We have that $f = X - a_l t \in \ker \tilde{\epsilon}_{t-1}$.

Claim: $\mathfrak{p}_{t-1} = (f)$.

Let $g(X) \in \mathfrak{p}_{t-1}$. As f is monic we have $g(X) = fq + u$ where $u \in A[t, t^{-1}]$. Then note that $\tilde{\epsilon}_{t-1}(u) = \epsilon_{t-1}(u) = 0$. But ϵ_{t-1} is an isomorphism. So $u = 0$. Thus Claim is proved.

We note that \mathfrak{p} is a height one prime ideal in $\widehat{\mathcal{R}}(Q)[X]$ and as $t^{-1} \notin \mathfrak{p}$ we have that $\mathfrak{p}_{\mathfrak{p}}$ is principal. So $\widehat{\mathcal{R}}(Q)[X]_{\mathfrak{p}}$ is a DVR. The result follows. \square

We will also need the following well-known result. We give a proof for the convenience of the reader.

Proposition 7.4. *Let $B \subseteq T$ be Noetherian domains. Let M be a finitely generated B -module. Then*

$$\text{rank}_B M = \text{rank}_T M \otimes_B T.$$

Proof. If M is a torsion B -module then clearly $M \otimes_B T$ is a torsion T -module. Now assume $\text{rank}_B M = r > 0$. Then there exists an exact sequence $0 \rightarrow B^r \rightarrow M \rightarrow E \rightarrow 0$ where E is a torsion B -module. Say $bE = 0$ and $b \in B$ is non-zero. So we have an exact sequence of T -modules

$$\text{Tor}_1^B(E, T) \xrightarrow{\delta} T^r \rightarrow M \otimes_B T \rightarrow E \otimes_B T \rightarrow 0.$$

We note that $b \in B \subseteq T$ annihilates $E \otimes_B T$ and $\text{Tor}_1^B(E, T)$ (and hence image δ). It follows that $\text{rank}_T M \otimes_B T = r$. The result follows. \square

We now give

Proof of Lemma 7.2. We prove the result by induction on $\mu(I)$.

We first consider the case when $I = (a)$. As A is a domain we get that a is A -regular. So $\widehat{J} = (TX_1 - a)$; see [8, 5.5.9]. So $S_{\widehat{J}}$ is a DVR. In this case the result holds trivially.

Now assume that $\mu(I) = r \geq 2$ and the result holds for all ideals Q such that $\mu(Q) = r - 1$. Let $I = (a_1, \dots, a_r)$ and assume $Q = (a_1, \dots, a_{r-1})$. Let $\theta: \hat{S} \rightarrow \hat{\mathcal{R}}(Q)$ be a minimal presentation. Let $U = \ker \theta$. By induction hypothesis $\text{rank}_{\hat{\mathcal{R}}(Q)} U/U^2 = r - 1$. Consider $\theta': \hat{S}[X] \rightarrow \hat{\mathcal{R}}(Q)[X]$ induced by θ and mapping X to X . We note that $\ker \theta' = U\hat{S}[X]$. Let $\epsilon: \hat{\mathcal{R}}(Q) \rightarrow \hat{\mathcal{R}}(I)$ be the inclusion and let $\tilde{\epsilon}: \hat{\mathcal{R}}(Q)[X] \rightarrow \hat{\mathcal{R}}(I)$ be the map induced by ϵ and by mapping X to $a_r t$. We note that $\tilde{\epsilon}$ is surjective. Furthermore $\eta = \tilde{\epsilon} \circ \theta': \hat{S}[X] \rightarrow \hat{\mathcal{R}}(I)$ is a minimal presentation. Let $\hat{J} = \ker \eta$ and let $\mathfrak{p} = \ker \tilde{\epsilon}$.

We apply the Jacobi-Zariski sequence to $\hat{S}[X] \xrightarrow{\theta'} \hat{\mathcal{R}}(Q)[X] \xrightarrow{\tilde{\epsilon}} \hat{\mathcal{R}}(I)$. We have an exact sequence

$$(*) \quad D_2(\hat{\mathcal{R}}(I)|\hat{\mathcal{R}}(Q)[X], \hat{\mathcal{R}}(I)) \rightarrow D_1(\hat{\mathcal{R}}(Q)[X]|\hat{S}[X], \hat{\mathcal{R}}(I)) \rightarrow \hat{J}/\hat{J}^2 \rightarrow \mathfrak{p}/\mathfrak{p}^2 \rightarrow 0.$$

We have

$$D_2(\hat{\mathcal{R}}(I)|\hat{\mathcal{R}}(Q)[X], \hat{\mathcal{R}}(I))_{t^{-1}} = D_2(A[t, t^{-1}]|A[t, t^{-1}][X], A[t, t^{-1}]) = 0.$$

So $D_2(\hat{\mathcal{R}}(I)|\hat{\mathcal{R}}(Q)[X], \hat{\mathcal{R}}(I))$ is t^{-1} -torsion. We also have

$$\begin{aligned} D_1(\hat{\mathcal{R}}(Q)[X], \hat{S}[X]) &= \frac{U\hat{S}[X]}{U^2\hat{S}[X]} \otimes_{\hat{\mathcal{R}}(Q)[X]} \hat{\mathcal{R}}(I) \\ &= \left(\frac{U}{U^2} \otimes_S S[X] \right) \otimes_{\hat{\mathcal{R}}(Q)[X]} \hat{\mathcal{R}}(I) \\ &= \left(\frac{U}{U^2} \otimes_{\hat{\mathcal{R}}(Q)} \hat{\mathcal{R}}(Q) \otimes_S S[X] \right) \otimes_{\hat{\mathcal{R}}(Q)[X]} \hat{\mathcal{R}}(I) \\ &= \left(\frac{U}{U^2} \otimes_{\hat{\mathcal{R}}(Q)} \hat{\mathcal{R}}(Q)[X] \right) \otimes_{\hat{\mathcal{R}}(Q)[X]} \hat{\mathcal{R}}(I) \\ &= \frac{U}{U^2} \otimes_{\hat{\mathcal{R}}(Q)} \hat{\mathcal{R}}(I). \end{aligned}$$

By 7.4 it follows that

$$\text{rank}_{\hat{\mathcal{R}}(I)} D_1(\hat{\mathcal{R}}(Q)[X], \hat{S}[X]) = \text{rank}_{\hat{\mathcal{R}}(Q)} U/U^2 = r - 1.$$

By (*) and 7.3, it follows that

$$\text{rank}_{\hat{\mathcal{R}}(I)} \hat{J}/\hat{J}^2 = r.$$

The result follows. \square

We now give

Proof of Theorem 1.6. (1) Let $\hat{S} \rightarrow \hat{\mathcal{R}}(I)$ be a minimal presentation. By 2.4 we have an exact sequence

$$0 \rightarrow D_2(\hat{\mathcal{R}}(I)|\hat{S}, \hat{\mathcal{R}}(I)) \rightarrow H_1(\hat{J}) \rightarrow \hat{\mathcal{R}}(I)^{\mu(\hat{J})} \rightarrow \hat{J}/\hat{J}^2 \rightarrow 0.$$

We now $D_2(\hat{\mathcal{R}}(I)|\hat{S}, \hat{\mathcal{R}}(I))$ is t^{-1} -torsion. So by Lemma 7.2 the result follows.

Claim-1: $\hat{S}_{\hat{J}}$ is a regular local ring.

As $\hat{S}/\hat{J} = \hat{\mathcal{R}}(I)$ it follows that $\text{height } \hat{J} = d + \mu(I) + 1 - (d + 1) = \mu(I)$. As $\text{rank}_{\hat{\mathcal{R}}(I)} \hat{J}/\hat{J}^2 = \mu(I)$ it follows that $\hat{S}_{\hat{J}}$ is a regular local ring.

(2) We first assert that

Claim-2 S_J is a regular ring.

We note that $S \subseteq \widehat{S} = S[T]$ is a flat extension. We have by [8, 5.5.7] that $J \subseteq \widehat{J} \cap S$. Set $Q = \widehat{J} \cap S$. The extension $S_Q \rightarrow \widehat{S}_{\widehat{J}}$ is flat. As $\widehat{S}_{\widehat{J}}$ is a regular local ring it follows that S_Q is a regular local ring. Thus S_J is a regular local ring.

So we have that $\text{rank}_{\mathcal{R}(I)} J/J^2 = \text{height } J$. As $S/J = \mathcal{R}(I)$ it follows that $\text{height } J = \mu(I) - 1$. By 2.4 we have an exact sequence

$$0 \rightarrow D_2(\mathcal{R}(I)|A, \mathcal{R}(I)) \rightarrow H_1(J) \rightarrow \mathcal{R}(I)^{\mu(J)} \rightarrow J/J^2 \rightarrow 0.$$

Set $X_1 = a_1 t$ and localize the above sequence with respect to X_1 . We note that as $\mathcal{R}(I)_{X_1} = \widehat{\mathcal{R}}(I)_{X_1}$

$$D_2(\mathcal{R}(I)|A, \mathcal{R}(I))_{X_1} = D_2(\widehat{\mathcal{R}}(I)|A, \widehat{\mathcal{R}}(I))_{X_1}.$$

The later module is t^{-1} -torsion. It follows that

$$\text{rank}_{\mathcal{R}(I)} H_1(J) = \text{rank}_{\mathcal{R}(I)_{X_1}} H_1(J)_{X_1} = \mu(J) - \text{rank } J/J^2 = \mu(J) - \mu(I) + 1.$$

□

8. PROOF OF PROPOSITION 1.8

In this section we give:

Proof of Proposition 1.8. (I) We note that $S = A[X_1, \dots, X_n]$ is regular $*$ -local ring and J is a graded ideal of S . We have $\text{projdim } S/J$ is finite.

If $H_1(J)_P$ is free $\mathcal{R}(I)_P$ -module then by a result Gulliksen, see [5, 1.4.9], it follows that J_P is generated by a regular sequence. It follows that $\mathcal{R}(I)_P$ is a complete intersection.

Conversely if $\mathcal{R}(I)_P$ is a complete intersection then J_P is generated by a regular sequence. It follows that $H_1(J)_P$ is free $\mathcal{R}(I)_P$ -module.

(II) This is similar to (I).

□

Remark 8.1. We note that even if we choose minimal generators of J they might not remain minimal in J_P

9. PROOF OF THEOREM 1.9

In this section we give

Proof of Theorem 1.9. Let Q be a minimal reduction of I . So $Q = (x_1, \dots, x_r)$ as I is equi-multiple. We note that x_1, \dots, x_r is an A -regular sequence. Let $\widehat{S}_L = A[X_1, \dots, X_r, T]$ be a minimal presentation of $\widehat{\mathcal{R}}(Q)$ and let $\eta: \widehat{S}_Q \rightarrow \widehat{\mathcal{R}}(Q)$ be the corresponding map. Then by [8, 5.5.9] we get that $\ker \eta$ is generated by a regular sequence. So $D_2(\widehat{\mathcal{R}}(Q)|A, -) = 0$. Applying the Jacobi-Zariski sequence to $A \rightarrow \widehat{\mathcal{R}}(Q) \rightarrow \widehat{\mathcal{R}}(I)$ we obtain an inclusion $D_2(\widehat{\mathcal{R}}(I)|A, k) \hookrightarrow D_2(\widehat{\mathcal{R}}(I)|\widehat{\mathcal{R}}(Q), k)$. It follows that $\mu(H_1(\widehat{J})) \leq \text{rank}_k D_2(\widehat{\mathcal{R}}(I)|\widehat{\mathcal{R}}(Q), k)$.

As $\widehat{\mathcal{R}}(I)$ is a complete intersection we obtain $D_3(k|\widehat{\mathcal{R}}(I), k) = 0$. We apply the Jacobi-Zariski sequence to $\widehat{\mathcal{R}}(Q) \rightarrow \widehat{\mathcal{R}}(I) \rightarrow k$. So we get an inclusion $D_2(\widehat{\mathcal{R}}(I)|\widehat{\mathcal{R}}(Q), k) \hookrightarrow D_2(k|\widehat{\mathcal{R}}(Q), k)$. We note that as $\widehat{\mathcal{R}}(Q)$ is a complete intersection $\text{rank}_k D_2(k|\widehat{\mathcal{R}}(Q), k) = \text{embdim}(\widehat{\mathcal{R}}(Q)) - \dim \widehat{\mathcal{R}}(Q)$. We note that $\text{embdim } \widehat{\mathcal{R}}(Q) = 1 + \text{embdim } G_Q(A)$. We have $G_Q(A) \cong A/Q[x_1^*, \dots, x_r^*]$. So $\text{embdim } G_Q(A) = \text{embdim } A/Q + r$. By the previous inequality it follows that $\mu(H_1(\widehat{J})) \leq \text{embdim } A/Q + r - d$.

□

We give an example which proves that the bound in Theorem 1.9 is strict.

Example 9.1. Let (R, \mathfrak{n}) be a regular local ring with infinite residue field and let $(A, \mathfrak{m}) = (R/(f), \mathfrak{n}/(f))$ for some $f \in \mathfrak{n}^2$. Take $I = \mathfrak{m}$. We first note that $\text{projdim}_{\widehat{S}} \widehat{\mathcal{R}}(\mathfrak{m}) = \infty$ (otherwise $\text{projdim}_A \mathfrak{m} < \infty$, which is false). So \widehat{J} is not generated by a regular sequence. So $H_1(\widehat{J}) \neq 0$. So $\mu(H_1(\widehat{J})) \geq 1$. Let Q be a minimal reduction of \mathfrak{m} . Then notice that $A/Q = \overline{R}/(\overline{f})$ where \overline{R} is a DVR. So $\text{embdim } A/Q = 1$. Thus the bound is attained.

10. PROOF OF THEOREM 1.14 AND COROLLARY 1.15

10.1. Let (A, \mathfrak{m}) be a Noetherian local ring and let I be an \mathfrak{m} -primary ideal. Set $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ be the Rees algebra of I . Assume $\dim A = d \geq 1$. We first prove

Lemma 10.2. (with hypotheses as in 10.1) Let E be a finitely generated $\mathcal{R}(I)$ -module. Then

- (1) Fix $j \geq 2$. We have $\ell(D_j(\mathcal{R}(I)|A, E)_n)$ is finite for all $n \in \mathbb{Z}$.
- (2) Fix $j \geq 2$. The function $n \rightarrow \ell(D_j(\mathcal{R}(I)|A, E)_n)$ is polynomial of degree $\leq d-1$.

Proof. (1) Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal in A . We note that $\mathcal{R}(I)_{\mathfrak{p}} = A_{\mathfrak{p}}[t]$ which is a smooth $A_{\mathfrak{p}}$ -algebra. So $D_j(\mathcal{R}(I)|A, E)_{\mathfrak{p}} = D_j(A_{\mathfrak{p}}[t]|A_{\mathfrak{p}}, E_{\mathfrak{p}}) = 0$. We note that $(D_j(\mathcal{R}(I)|A, E)_n)_{\mathfrak{p}} = (D_j(A_{\mathfrak{p}}[t]|A_{\mathfrak{p}}, E_{\mathfrak{p}})_n)_{\mathfrak{p}} = 0$. We also note that as $D_j(\mathcal{R}(I)|A, E)$ is a finitely generated graded $\mathcal{R}(I)$ -module, each $D_j(\mathcal{R}(I)|A, E)_n$ is a finitely generated A -module. The result follows.

(2) As $D_j(\mathcal{R}(I)|A, E)$ is a finitely generated graded $\mathcal{R}(I)$ -module, and as by (1) we have $\ell(D_j(\mathcal{R}(I)|A, E)_n)$ is finite for all $n \in \mathbb{Z}$ it follows that there exists s such that $\mathfrak{m}^s D_j(\mathcal{R}(I)|A, E) = 0$. It follows that $D_j(\mathcal{R}(I)|A, E)$ is a $\mathcal{R}(I)/\mathfrak{m}^s \mathcal{R}(I)$ -module and the latter has dimension d . The result follows. \square

We now give

Proof of Theorem 1.14. (i) \implies (ii):

We note that $\dim D_2(\mathcal{R}(I)|A, E) \leq d - i$. Let $\mathfrak{p} \in \text{Proj}(\mathcal{R}(I))$ with height $\leq i$. As $\mathcal{R}(I)$ is equi-dimensional and catenary it follows that $\dim \widehat{\mathcal{R}}(I)_{\mathfrak{p}} = \dim \widehat{\mathcal{R}}(I) - \text{height } \mathfrak{p} \geq d + 1 - i$, see [11, Section 31, Lemma 2] for the local case; the same result holds for $*$ -local case. It follows that \mathfrak{p} is not in the support of $D_2(\mathcal{R}(I)|A, E)$ for any graded $\mathcal{R}(I)$ -module E . We now choose $E = \mathcal{R}(I)/\mathfrak{p}$ for a prime ideal of height $\leq i$. We have $D_2(\mathcal{R}(I)|A, E) = D_2(\mathcal{R}(I)|S, E)$. Let Q be the inverse image of \mathfrak{p} in S . So we have

$$0 = D_2(\mathcal{R}(I)|S, \mathcal{R}(I)/\mathfrak{p})_{\mathfrak{p}} = D_2(\mathcal{R}(I)_{\mathfrak{p}}|S_Q, \kappa(\mathfrak{p})).$$

It follows that $H_1(J_Q) = 0$. The result follows.

(ii) \implies (i): Suppose if possible there exists a finitely generated graded $\mathcal{R}(I)$ -module with $\dim D_2(\mathcal{R}(I)|A, E) \geq d - i + 1$. Then there exists a prime ideal \mathfrak{p} of $\mathcal{R}(I)$ in the support of $D_2(\mathcal{R}(I)|A, E)$ with height $\mathfrak{p} \leq i$. As $D_2(\mathcal{R}(I)|A, E)$ is a $\mathcal{R}(I)/\mathfrak{m}^s$ -module for some $s \geq 1$ it follows that $\mathfrak{p} \in \text{Proj}(\mathcal{R}(I))$. But as $S_Q \rightarrow \mathcal{R}(I)_{\mathfrak{p}}$ is a complete intersection we have

$$D_2(\mathcal{R}(I)|S, E)_{\mathfrak{p}} = D_2(\mathcal{R}(I)_{\mathfrak{p}}|S_Q, E_{\mathfrak{p}}) = 0,$$

see [10, 2.5.2]. This is a contradiction. The result follows.

If (i) (and so (ii)) holds then by [9, 8.4] we have for any prime ideal \mathfrak{p} of $\mathcal{R}(I)$ with height $\mathfrak{p} \leq i$ we have

$$D_j(\mathcal{R}(I)|S, E)_{\mathfrak{p}} = D_j(\mathcal{R}(I)_{\mathfrak{p}}|S_Q, E_{\mathfrak{p}}) = 0,$$

for any $j \geq 2$. The result follows. \square

We now give

Proof of Corollary 1.15. (i) \implies (ii):

By the theorem above we obtain that for any prime ideal \mathfrak{p} in $\text{Proj } \mathcal{R}(I)$ of height $\leq i$ the ideal J_Q is a complete intersection. So $H_1(J)_{\mathfrak{p}}$ is free $\mathcal{R}(I)_{\mathfrak{p}}$ -module.

(ii) \implies (i):

We note that the S -ideal J has finite projective dimension. So if $H_1(J)_{\mathfrak{p}}$ is free then by a result due to Gulliksen [5, 1.4.9] we obtain that J_Q is a complete intersection.

(i) \Leftrightarrow (iii) This follows from the fact (i) \Leftrightarrow (ii) of the above theorem with the fact that S is regular. \square

11. APPENDIX

In the appendix we prove some results which we believe are already known. However we are unable to find a reference for these results. As it is critical for us we give proofs.

11.1. We will need the following exercise problem from [11, Exercise 6.7]. Let $A \rightarrow B$ be a homomorphism of Noetherian rings and let M be a finitely generated B -module. Then

$$\text{Ass}_A M = \{P \cap A \mid P \in \text{Ass}_B M\}.$$

Theorem 11.2. Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let $I \subseteq \mathfrak{m}$. Then

- (1) $\widehat{\mathcal{R}}(I)$ is unmixed and equi-dimensional.
- (2) If height $I > 0$ then $\mathcal{R}(I)$ is also unmixed and equi-dimensional.

Proof. (1) Claim-1: We have

$$\text{Ass } \widehat{\mathcal{R}}(I) = \{\mathfrak{p}A[t, t^{-1}] \cap \mathcal{R}(I) \mid \mathfrak{p} \in \text{Ass } A\}.$$

Proof of Claim-1: Set $H = \{\mathfrak{p}A[t, t^{-1}] \cap \mathcal{R}(I) \mid \mathfrak{p} \in \text{Ass } A\}$. As A is Cohen-Macaulay all associate primes of A are minimal. So by [4, 4.5.5] H is the set of minimal primes of $\widehat{\mathcal{R}}(I)$. Conversely if $Q \in \text{Ass } \widehat{\mathcal{R}}(I)$ then note that as t^{-1} is $\widehat{\mathcal{R}}(I)$ -regular we have that $t^{-1} \notin Q$. So $Q = \mathfrak{p}A[t, t^{-1}] \cap \widehat{\mathcal{R}}(I)$ for some prime \mathfrak{p} of A . We note that $\mathfrak{p} = Q \cap A \in \text{Ass}_A \widehat{\mathcal{R}}(I) = \bigcup_{n \in \mathbb{Z}} \text{Ass}_A I^n \subseteq \text{Ass}_A A$ (here we are considering I^n as an A -module). So Claim-1 follows.

Let $Q = \mathfrak{p}A[t, t^{-1}] \cap \widehat{\mathcal{R}}(I)$ be an associate prime of $\widehat{\mathcal{R}}(I)$. Then \mathfrak{p} is a minimal prime of A . As A is Cohen-Macaulay we have $\dim A/\mathfrak{p} = d$. So we have a chain of prime ideals in A

$$\mathfrak{p} = P_0 \subseteq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_{d-1} \subsetneq P_d = \mathfrak{m}.$$

So we have a chain of prime ideals in $\widehat{\mathcal{R}}(I)$,

$$Q = P_0A[t, t^{-1}] \cap \widehat{\mathcal{R}}(I) \subsetneq \cdots \subsetneq P_dA[t, t^{-1}] \cap \widehat{\mathcal{R}}(I) = \mathfrak{m}A[t, t^{-1}] \cap \widehat{\mathcal{R}}(I).$$

Finally note that $\mathfrak{m}A[t, t^{-1}] \cap \widehat{\mathcal{R}}(I)$ is properly contained in the maximal homogeneous ideal of $\widehat{\mathcal{R}}(I)$. So $\dim \widehat{\mathcal{R}}(I)/Q = d + 1$. The result follows.

(2) Let $Q \in \text{Ass } \mathcal{R}(I)$. Then $Q \cap A = \mathfrak{p} \in \text{Ass}_{\mathfrak{a}} \mathcal{R}(I) = \bigcup_{n \geq 0} \text{Ass}_A I^n \subseteq \text{Ass}_A A$ (here we are considering I^n as an A -module). As A is Cohen-Macaulay we get that \mathfrak{p} is a minimal prime of A and $\dim A/\mathfrak{p} = d$. As height $I \geq 1$ there exists $x \in I$ which is A -regular and so not contained in any minimal prime of A . We note that $\mathcal{R}(I)_{xt^0} \cong A_x[t]$. It follows that $Q_{xt^0} = \mathfrak{p}A_x[t]$. We note that $\dim(A/\mathfrak{p})_x = d - 1$. So there is a chain of primes in A_x

$$\mathfrak{p} = P_0 \subseteq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_{d-1}.$$

This yields a chain of prime ideals in $\mathcal{R}(I)$

$$Q = P_0 A_x[t] \cap \mathcal{R}(I) \subsetneq \cdots \subsetneq P_{d-1} A_x[t] \cap \mathcal{R}(I) \subsetneq (P_{d-1}, t) A_x[t] \cap \mathcal{R}(I).$$

We note that

$$(P_{d-1}, t) A_x[t] \cap \mathcal{R}(I) \subsetneq (\mathfrak{m}, \mathcal{R}(I)_+).$$

So $\dim \mathcal{R}(I)/Q = d + 1$. It follows that $\mathcal{R}(I)$ is unmixed and equi-dimensional. \square

11.3. Let (A, \mathfrak{m}) be a Noetherian local ring and let $I \subseteq \mathfrak{m}$ be an ideal in A . An element $x \in I$ is said to be I -superficial if there exists non-negative integers c, n_0 such that $(I^{n+1} : x) \cap I^c = I^n$ for all $n \geq n_0$. Superficial elements exist when the residue field k of A is infinite. If grade $I > 0$ then it is not difficult to prove that any I -superficial element is A -regular.

11.4. Sketch of existence of a superficial element: Suppose A/\mathfrak{m} is infinite. Let $G = G_I(A)$ the associated graded ring of A with respect to I . Let $P_1, \dots, P_r, Q_1, \dots, Q_s$ be the associate primes of G such that $P_i \not\subseteq G_+$ for all i and $Q_j \supseteq G_+$ for all j . Set $P_{i,1} = I/I^2 \cap P_i$. We note that $P_{i,1}$ is properly contained in I/I^2 . So by Nakayama's lemma $V_i = (P_{i,1} + \mathfrak{m}I)/\mathfrak{m}I$ is a proper subspace of the k -vector space $U = I/\mathfrak{m}I$. As k is infinite it follows that $U \setminus \bigcup_{i=1}^r V_i$ is non-empty. Then it is not difficult to show that any $x \in I$ such that $\bar{x} \in U \setminus \bigcup_{i=1}^r V_i$ is an I -superficial element.

We need the following result:

Proposition 11.5. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring with infinite residue field. Let $I \subseteq \mathfrak{m}$ be an ideal of positive height. Then there exists $a_1, \dots, a_s \in I$ such that*

- (1) $I = (a_1, \dots, a_s)$ (minimally).
- (2) Each a_i is A -regular.

To prove this result we need the following:

Lemma 11.6. *Let (A, \mathfrak{m}) be a Noetherian local ring with infinite residue field. Let $I \subseteq \mathfrak{m}$ be a non-zero ideal. Then there exists $a_1, \dots, a_s \in I$ such that*

- (1) $I = (a_1, \dots, a_s)$ (minimally).
- (2) Each a_i is I -superficial.

We prove Proposition 11.5 assuming Lemma 11.6.

Proof of Proposition 11.5. As A is Cohen-Macaulay we have grade $Q = \text{height } Q$ for any ideal Q in A . So grade I is positive. By Lemma 11.6 there exists $a_1, \dots, a_s \in I$ such that $I = (a_1, \dots, a_s)$ (minimally) and each a_i is I -superficial. As grade $I > 0$ each I -superficial element a_i is A -regular \square

It remains to give

Proof of Lemma 11.6. Let $U = I/\mathfrak{m}I$. By 11.4 there exists proper k -subspaces V_i of U (with $1 \leq i \leq r$) such that if $x \in I$ with $\bar{x} \in U \setminus \cup_{i=1}^r V_i$ is an I -superficial element.

Let $a_1 \in I$ such that $\overline{a_1} \in U \setminus \cup_{i=1}^r V_i$. If $(a_1) = I$ then we are done. Otherwise $W_1 = ((a_1) + \mathfrak{m}I)/\mathfrak{m}I$ is a proper subspace of U . Choose $a_2 \in I$ such that $\overline{a_2} \in U \setminus (\cup_{i=1}^r V_i) \cup W_1$. Then a_2 is I -superficial and $(a_1) \subsetneq (a_1, a_2) \subseteq I$. Iterating we obtain a sequence

$$(a_1) \subsetneq (a_1, a_2) \subsetneq \cdots \subsetneq (a_1, \dots, a_s) \subseteq I$$

such that each a_i is I -superficial. If $I = (a_1, \dots, a_s)$ then we are done. Otherwise $W_s = ((a_1, \dots, a_s) + \mathfrak{m}I)/\mathfrak{m}I$ is a proper subspace of U . Choose $a_{s+1} \in I$ such that $\overline{a_{s+1}} \in U \setminus (\cup_{i=1}^r V_i) \cup W_s$. Then a_{s+1} is I -superficial and $(a_1, \dots, a_s) \subsetneq (a_1, \dots, a_s, a_{s+1}) \subseteq I$. This process will terminate as A is Noetherian. The result follows. \square

REFERENCES

- [1] M. André, *Homologie des algèbres commutatives*, Die Grundlehren der mathematischen Wissenschaften, Band 206, Springer-Verlag, Berlin-New York, 1974.
- [2] ———, *Algèbres graduées associées et algèbres symétriques plates*, Comment. Math. Helv., 49, 1974, 277–301.
- [3] L. L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*, Ann. of Math. (2), 150, (1999), no. 2, 455–487.
- [4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, revised edition, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, 1998.
- [5] T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen’s Papers in Pure and Appl. Math., No. 20 Queen’s University, Kingston, ON, Canada.
- [6] J. Herzog, B. Briggs and S. B. Iyengar, *Homological properties of the module of differentials*, arXiv:2502.14159.
- [7] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II*, Ann. of Math. 79 (1964) 109–326.
- [8] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Math. Soc. Lecture Note Ser., 336 Cambridge University Press, Cambridge, 2006.
- [9] S. B. Iyengar, *André-Quillen homology of commutative algebras*, Interactions between homotopy theory and algebra, 203–234. Contemp. Math., 436 American Mathematical Society, Providence, RI, 2007.
- [10] J. Majadas and A. G. Rodicio, *Smoothness, regularity and complete intersection*, London Math. Soc. Lecture Note Ser., 373, Cambridge University Press, Cambridge, 2010.
- [11] H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.
- [12] F. Planas-Vilanova, *Vanishing of the André-Quillen homology module $H_2(A, B, G(I))$* , Comm. Algebra, 24, 1996, Vol 8, 2777–2791.
- [13] ———, *The relation type of affine algebras and algebraic varieties*, J. Algebra, 441, (2015), 166–179.
- [14] ———, *Noetherian rings of low global dimension and syzygetic prime ideals*, J. Pure Appl. Algebra 225 (2021), no. 2, Paper No. 106494, 8 pp.
- [15] ———, *Regular local rings of dimension four and Gorenstein syzygetic prime ideals*, J. Algebra, 601 (2022), 105–114.
- [16] T. J. Puthenpurakal, *Itoh’s conjecture for normal ideals*, Math. Z. 307 (2024), no. 1, Paper No. 9, 24 pp.
- [17] T. J. Puthenpurakal, *A solution to Itoh’s conjecture for integral closure filtration*, Math. Z. 311 (2025), no. 3, Paper No. 65, 20 pp.
- [18] D. Quillen, *On the (co)-homology of commutative rings*, in Applications of Categorical Algebra (New York, 1968), Proc. Sympos. Pure Math. 17, A.M.S., Providence, RI, 1970, 65–87.

- [19] W. V. Vasconcelos, *Arithmetic of blowup algebras*, London Math. Soc. Lecture Note Ser., 195
Cambridge University Press, Cambridge, 1994

DEPARTMENT OF MATHEMATICS, IIT BOMBAY, POWAI, MUMBAI 400 076, INDIA
Email address: `tputhen@gmail.com`