



Longitudinal Model Example

$$X_u = -0.045/s, \quad X_w = 0.036/s, \quad Z_u = -0.369/s$$

$$Z_w = -2.02/s, \quad Z_{\dot{\alpha}} = -2.6 ft/s, \quad M_w = -0.05/s$$

$$M_{\dot{w}} = -0.0051/ft; \quad M_q = -2.03/s; \quad \theta_0 = 0$$

$$g = 32.2 ft/s^2; \quad u_0 = 176 \text{ ft/s}$$

(Rest Derivative Terms are '0').

$$[A] = \begin{bmatrix} -0.045 & 0.036 & 0.00000 & -32.2 \\ -0.369 & -2.02 & 176 & 0.0000 \\ 0.0019 & -0.0396 & -2.948 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

Eigenvalues: $-0.0171 \pm j0.213$, $-2.489 \pm j2.59$
(Slow) (Fast)

[A] matrix elements show that ' Δu ' is primarily influenced only by ' $\Delta \theta$ ', while both ' Δw ' and ' Δq ' are influenced only by ' Δq '.

Thus, we can group ' Δu ' and ' $\Delta \theta$ ' as one set of motion variables and ' Δw ' and ' Δq ' as the other set of variables.

In this manner, we can examine the longitudinal motion as combination of two separate, 2^{nd} order motions.



Dominant Solution Extraction

Eigenvectors:

$$\begin{bmatrix} -0.0297 - 0.0011i & -0.0297 + 0.0011i & -0.9983 & -0.9983 \\ -0.9994 & -0.9994 & 0.0584 + 0.0013i & 0.0584 - 0.0013i \\ 0.0026 - 0.0148i & 0.0026 + 0.0148i & -0.0014 + 0.0001i & -0.0014 - 0.0001i \\ -0.0035 + 0.0023i & -0.0035 - 0.0023i & 0.0009 + 0.0066i & 0.0009 - 0.0066i \end{bmatrix}$$

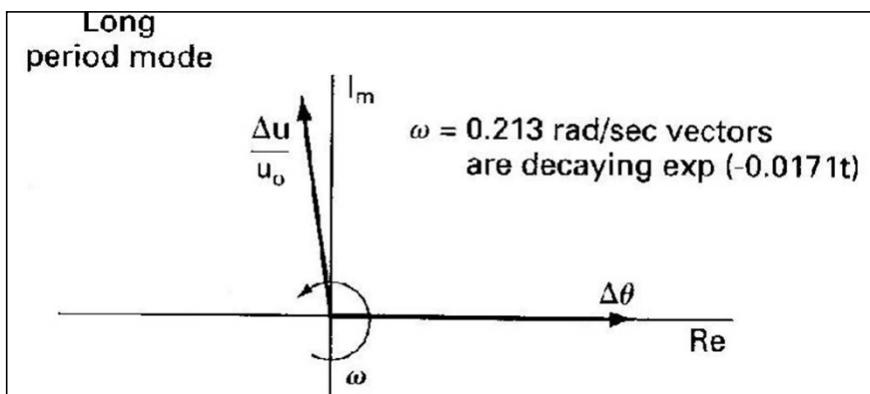
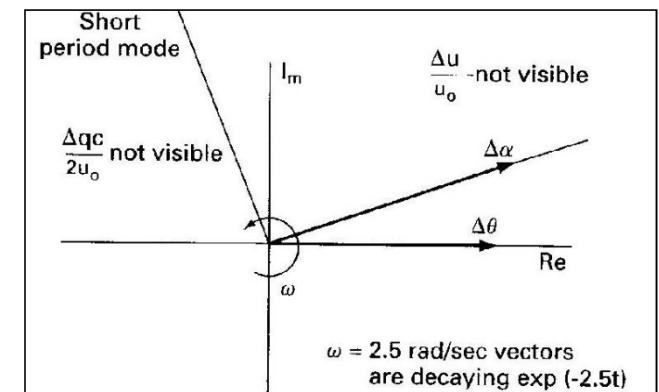
Eigenvector	Long period	Short period
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$$\lambda = -0.0171 \pm 0.213i \quad \lambda = -2.5 \pm 2.59i$$

$$\frac{\Delta u/u_0}{\Delta \theta} \quad -0.114 \pm 0.837i \quad 0.034 \pm 0.025i$$

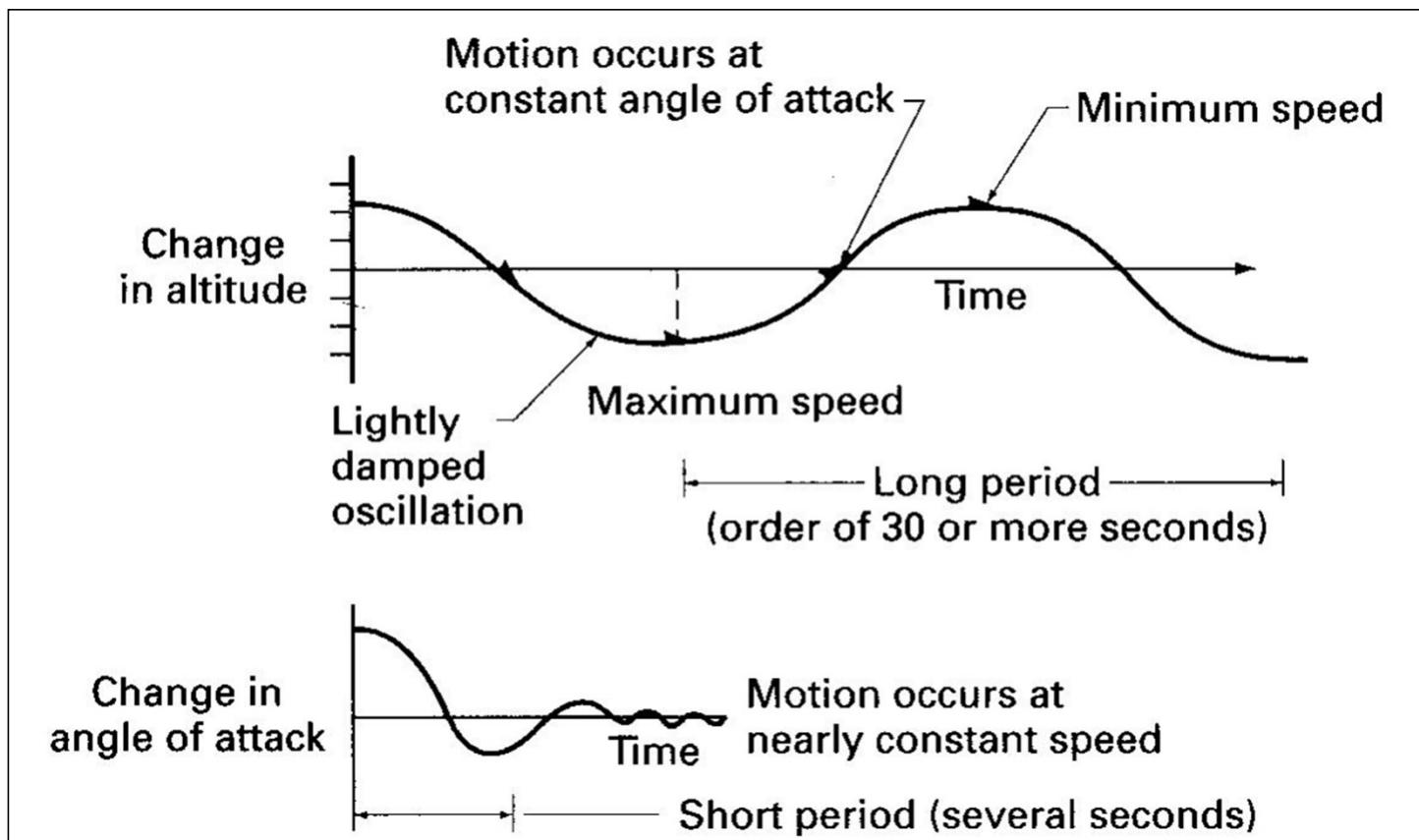
$$\frac{\Delta w/u_0}{\Delta \theta} = \frac{\Delta \alpha}{\Delta \theta} \quad 0.008 \pm 0.05i \quad 1.0895 \pm 0.733i$$

$$\frac{\Delta [qc/(2u_0)]}{\Delta \theta} \quad -0.000027 \pm 0.00347i \quad -0.039 \pm 0.041i$$





Characteristic Longitudinal Motion





Longitudinal Approximations - Phugoid

Long period motion is also called ‘phugoid’, in **which**, angle of attack remains **nearly** constant, while ‘ $\Delta\theta$ ’, ‘ Δh ’ (or Δz) and ‘ Δu ’ change.

We can obtain a reasonable approximation of the **above** motion by neglecting the **pitching** moment equation and **assuming** that $(d\alpha/dt)$ is zero, so that we can write,

$$\Delta\alpha = \frac{\Delta w}{u_0}, \quad \Delta\alpha \approx 0 \rightarrow \Delta w \approx 0$$



Homogeneous Phugoid Mode Model

$$\begin{Bmatrix} \Delta \dot{u} \\ \Delta \dot{\theta} \end{Bmatrix} = \begin{bmatrix} X_u & -g \cos \theta_0 \\ -\frac{Z_u}{u_0} & 0 \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta \theta \end{Bmatrix}; \quad \det[\lambda I - A] = 0$$
$$\lambda^2 - X_u \lambda + \frac{g \cos \theta_0 Z_u}{u_0} \rightarrow \lambda = 0.5 \left[X_u \pm \sqrt{X_u^2 + 4 \frac{Z_u g \cos \theta_0}{u_0}} \right]$$
$$\omega_{np} = \sqrt{\frac{-Z_u g \cos \theta_0}{u_0}}; \quad \zeta_p = \frac{-X_u}{2\omega_{np}} = \frac{-X_u \sqrt{u_0}}{2\sqrt{-Z_u g \cos \theta_0}} \approx \infty \frac{1}{(L/D)}$$



Phugoid Extraction Example

$$X_u = -0.045; \quad Z_u = -0.369; \quad u_0 = 176$$

$$\omega_{np} = \sqrt{-\frac{Z_u g}{u_0}} = \left[\frac{-(-0.369)(32.21)}{176} \right]^{1/2} \\ = 0.260 (\text{Exact: } 0.214)$$

$$\zeta_p = -\frac{X_u}{2\omega_{np}} = -\frac{(-0.045)}{2(0.26)} = 0.087 (\text{Exact: } 0.080)$$



Homogeneous Short Period Mode

An approximation to short period **mode** can be obtained by assuming ' Δu ' = 0 and dropping X-force equation, as shown alongside.

$$\begin{aligned}\begin{Bmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{Bmatrix} &= \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 \end{bmatrix} \begin{Bmatrix} \Delta w \\ \Delta q \end{Bmatrix} \\ \Delta \alpha &= \frac{\Delta w}{u_0}; \quad M_\alpha = \frac{u_0}{I_{yy}} \times \frac{\partial M}{\partial w}; \quad Z_\alpha = u_0 Z_w; \quad M_{\dot{\alpha}} = u_0 M_{\dot{w}} \\ \begin{Bmatrix} \Delta \dot{\alpha} \\ \Delta \dot{q} \end{Bmatrix} &= \begin{bmatrix} \left(\frac{Z_\alpha}{u_0} \right) & 1 \\ M_\alpha + M_{\dot{\alpha}} \left(\frac{Z_\alpha}{u_0} \right) & M_q + M_{\dot{\alpha}}u_0 \end{bmatrix} \begin{Bmatrix} \Delta \alpha \\ \Delta q \end{Bmatrix}\end{aligned}$$



Longitudinal Approximation - Short Period

$$\lambda^2 - \left(\frac{Z_\alpha}{u_0} + M_q + M_{\dot{\alpha}} u_0 \right) \lambda - \left(M_\alpha + M_{\dot{\alpha}} \frac{Z_\alpha}{u_0} \right) = 0$$

$$\lambda_{sp} = \frac{\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0} \right)}{2} \pm \sqrt{\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0} \right)^2 - 4 \left(M_q \frac{Z_\alpha}{u_0} - M_\alpha \right)}^{1/2}$$

$$\omega_{nsp} = \left(M_q \frac{Z_\alpha}{u_0} - M_\alpha \right)^{1/2}, \quad \zeta_{sp} = - \frac{\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0} \right)}{(2\omega_{nsp})}$$



Short-period Extraction Example

$$\omega_{np} = \sqrt{\frac{Z_\alpha M_q}{u_0} - M_\alpha} = [(-2.02)(-2.05) - (-0.05)(176)]^{1/2} = 3.6 \text{ (Exact: 3.6)}$$

$$\zeta_p = -\frac{\left(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0} \right)}{(2\omega_{nsp})} = -\frac{[(-2.05) + (-0.88) + (-2.02)]}{2(3.6)} = 0.69 \text{ (Exact: 0.69)}$$