Incremental Dynamics

Equations given below describe the incremental **dynamics** and require incremental force/**moment** models for their solution.

Δu Equation: $\Delta X = \Delta X_g + \Delta X_T + \Delta X_A = m\Delta\dot{u} + mw_0\Delta q$

Δv Equation: $\Delta Y = \Delta Y_g + \Delta Y_T + \Delta Y_A = m\Delta\dot{v} + mu_0\Delta r + w_0\Delta p$

Δw Equation: $\Delta Z = \Delta Z_g + \Delta Z_T + \Delta Z_A = m\Delta\dot{w} - mu_0\Delta q$

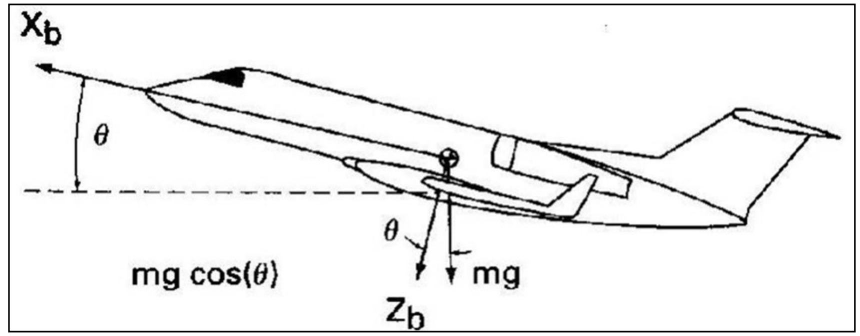
Δp Equation: $\Delta L = \Delta L_T + \Delta L_A = I_{xx}\Delta\dot{p} - I_{zx}\Delta\dot{r}$

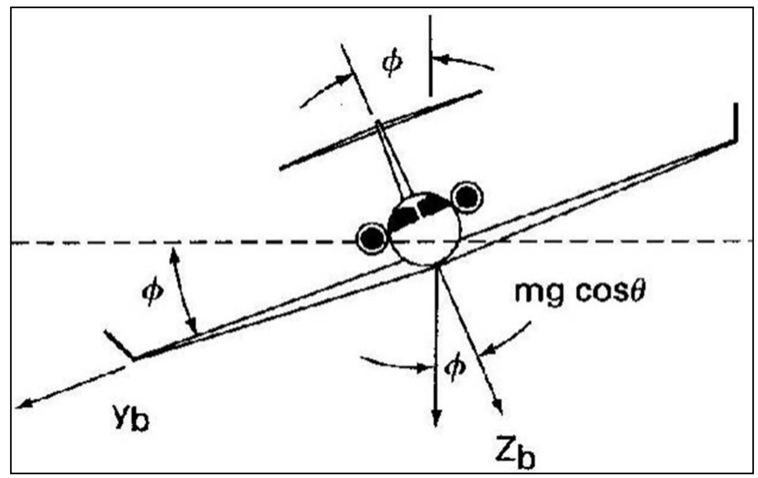
Δq Equation: $\Delta M = \Delta M_T + \Delta M_A = I_{yy}\Delta\dot{q}$

Δr Equation: $\Delta N = \Delta N_T + \Delta N_A = I_{zz}\Delta\dot{r} - I_{zx}\Delta\dot{p}$

Incremental Gravity Forces

The **forces** due to gravity can be obtained through the **projection** of gravity vector along the body axes, as **shown** in figures alongside and given in equations **below**.





$$\Delta\vec{F}_g = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ (-c\phi s\psi + s\phi s\theta c\psi) & (c\phi c\psi + s\phi s\theta s\psi) & s\phi c\theta \\ (s\phi s\psi + c\phi s\theta c\psi) & (-s\phi c\psi + c\phi s\theta s\psi) & c\phi c\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ mg \end{Bmatrix}$$

$$= mg \begin{Bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{Bmatrix}; \quad \Delta X_g = -mg\sin\Delta\theta$$

$$\Delta Y_g = mg\sin\Delta\phi\cos\Delta\theta; \quad \Delta Z_g = mg\cos\Delta\phi\cos\Delta\theta$$

Incremental Thrust Forces in Body Frame

Thrust force, which is a direct body force, is **normally** a steady force in body frame, **including** the moments caused by it (**exception** is thrust vectoring for control).

However, its value can be changed through throttle, **which** modifies all components, as **per** the engine axis**.**

In most cases of aircraft, the engine axis generally **passes** through ‘cg’ and is also **along** the aircraft axis, so that **thrust** contributes only to the ‘Δu’ equation.

Unsteady Aerodynamic Forces and Moments

Among the many effects that are usually present in **an** unsteady flow, following are **considered** to be significant and, **hence**, are usually included in most force models.

Change in forward speed, ‘Δu’

Presence of pitch rate ‘Δq’

Time rate of change of angle of attack ‘$d(\Delta\alpha)/dt$’

Presence of roll rate ‘Δp’

Presence of yaw rate ‘Δr’

Forward Speed (u) Effect

All aerodynamic forces change with change in **forward** speed 'u', as it not only **changes** the dynamic pressure but **also** the Mach number and, hence, the flow regime itself.

However, for an initial assessment, we assume that **it** significantly influences only **lift** 'L', drag 'D' and pitching **moment** 'M', as shown below.

$$\Delta X = \frac{\partial X}{\partial u}\Delta u = -\frac{\partial D}{\partial u}\Delta u + \frac{\partial T}{\partial u}\Delta u$$

$$\Delta Z = \frac{\partial Z}{\partial u}\Delta u = -\frac{\partial L}{\partial u}\Delta u; \quad \Delta M = \frac{\partial M}{\partial u}\Delta u$$

Forward Speed (u) Derivative for ΔX_A

$$\frac{\partial X}{\partial u} = X_u = QS\frac{\partial C_X}{\partial u} = \frac{QS}{u_0} \times \frac{\partial C_X}{\partial \bar{u}} = \frac{QS}{u_0} \times C_{X\bar{u}}; \quad \bar{u} = \frac{u}{u_0}; \quad u_0 \to \text{Trim Value}$$

$$\Delta X = \frac{\partial X}{\partial u}\Delta u = -\frac{\partial D}{\partial u}\Delta u + \frac{\partial T}{\partial u}\Delta u; \quad C_{D\bar{u}} = \frac{\partial C_D}{\partial \bar{u}}; \quad C_T = \frac{T}{QS}; \quad C_{T\bar{u}} = \frac{\partial C_T}{\partial \bar{u}}$$

$$\frac{\partial X}{\partial u} = -\frac{\rho S}{2} \times \left(u_0^2 \frac{\partial C_D}{\partial u} + 2u_0 C_{D0} \right) + \frac{\partial T}{\partial u}, \quad C_{X\bar{u}} = \frac{\partial C_X}{\partial \bar{u}} = -\left(C_{D\bar{u}} + 2C_{D0} \right) + C_{T\bar{u}}$$

It should be noted that while, $C_{D\bar{u}}$ is primarily a **function** of Mach number, $C_{T\bar{u}}$ is **assumed** to be ‘0’ for jet engines, and is **approximated** as $-C_{D0}$ for propeller aircraft.

Forward Speed (u) Derivatives for ΔZ_A, ΔM_A

$$\frac{\partial Z}{\partial u} = Z_u = -\frac{\partial L}{\partial u} = -\frac{\rho S}{2} \times \frac{\partial\left(u^2 C_L\right)}{\partial u} = -\frac{\rho S}{2}\left(u_0^2 \frac{\partial C_L}{\partial u} + 2u_0 C_{L0}\right)$$

$$C_{Z\bar{u}} = \frac{\partial C_Z}{\partial \bar{u}} = u_0 \frac{\partial C_Z}{\partial u} = \frac{2Z_u}{\rho S u_0} = -\left(u_0 C_{Lu} + 2C_{L0}\right) = -\left(C_{L\bar{u}} + 2C_{L0}\right)$$

$$C_{L\bar{u}} = \frac{\partial C_L}{\partial \bar{u}} = \frac{u_0}{a}\frac{\partial C_L}{\partial \mathrm{M}} = \frac{\mathrm{M}^2}{1-\mathrm{M}^2} C_{L0}; \quad \frac{\partial M}{\partial u} = M_u = \frac{\rho S \bar{c}}{2} \times \frac{\partial\left(u^2 C_m\right)}{\partial u}$$

$$M_u = \frac{\rho S \bar{c}}{2}\left(u_0^2 \frac{\partial C_m}{\partial u}\right); \quad C_{m\bar{u}} = \frac{\partial C_m}{\partial \bar{u}} = u_0 \frac{\partial C_m}{\partial u}$$

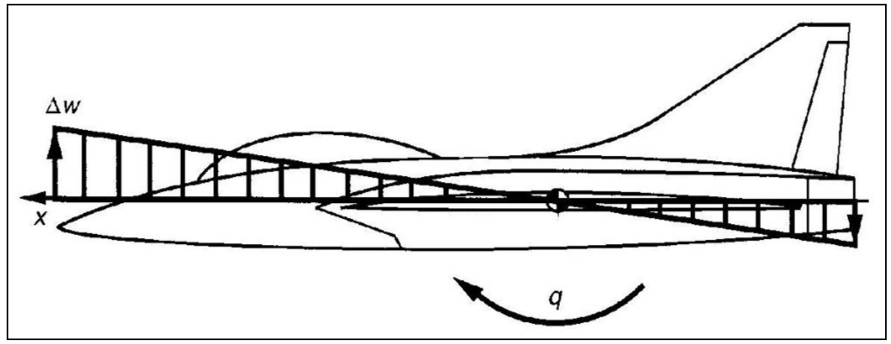
$$C_{m\bar{u}} = \frac{2M_u}{\rho \bar{c} S u_0} = \left(u_0 C_{mu}\right) = \mathrm{M}\frac{\partial C_m}{\partial \mathrm{M}}$$

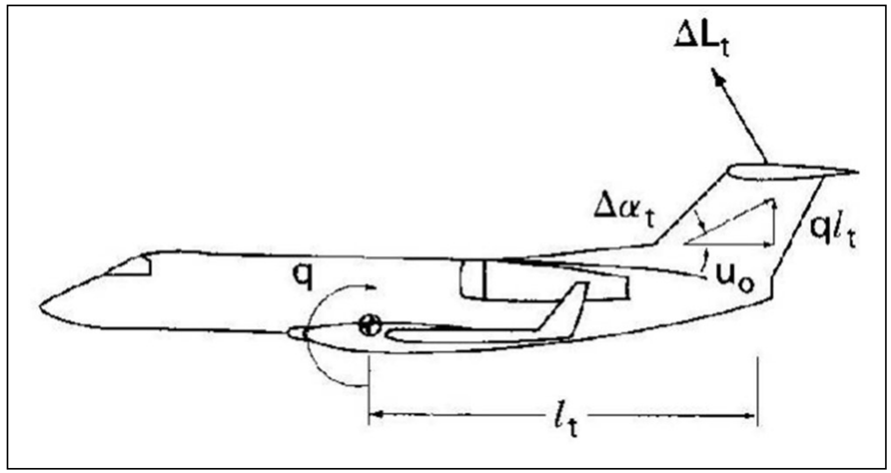
Pitch Rate (q) Effect

Given alongside are the schematics of an aircraft **experiencing** a pitch rate, along with the associated **aerodynamic** forces**.**

It is seen that pitch rate 'q', induces a normal **velocity** (positive downwards) at HT, which is **proportional** to the distance 'l_T' from 'cg'.

Change in HT angle of attack, so caused, produces **both** a small lift and a significant pitching moment at **'cg'**.





Pitch Rate (q) Derivatives for ΔZ_A, ΔM_A

$$\Delta\alpha_T \approx \frac{ql_T}{u_0}; \quad \Delta C_{LT} = \frac{a_T q l_T}{u_0}; \quad \Delta L_T = \Delta C_{LT} Q_T S_T = \frac{a_T q l_T Q_T S_T}{u_0}$$

$$\Delta Z = -\frac{a_T q Q_T l_T S_T}{u_0} \rightarrow Z_q = -\frac{a_T l_T Q_T S_T}{u_0}; \quad C_{Zq} = \frac{Z_q}{Q_w S_w} = -\frac{a_T \bar{c} Q_T l_T S_T}{u_0 Q_w S_w \bar{c}}$$

$$C_{Zq} = -\eta_T V_T \frac{a_T \bar{c}}{u_0}; \quad \bar{q} = \frac{q\bar{c}}{2u_0}; \quad C_{Z\bar{q}} = -2\eta_T V_T a_T; \quad \Delta M_{cg} = -l_T \times \Delta L_T$$

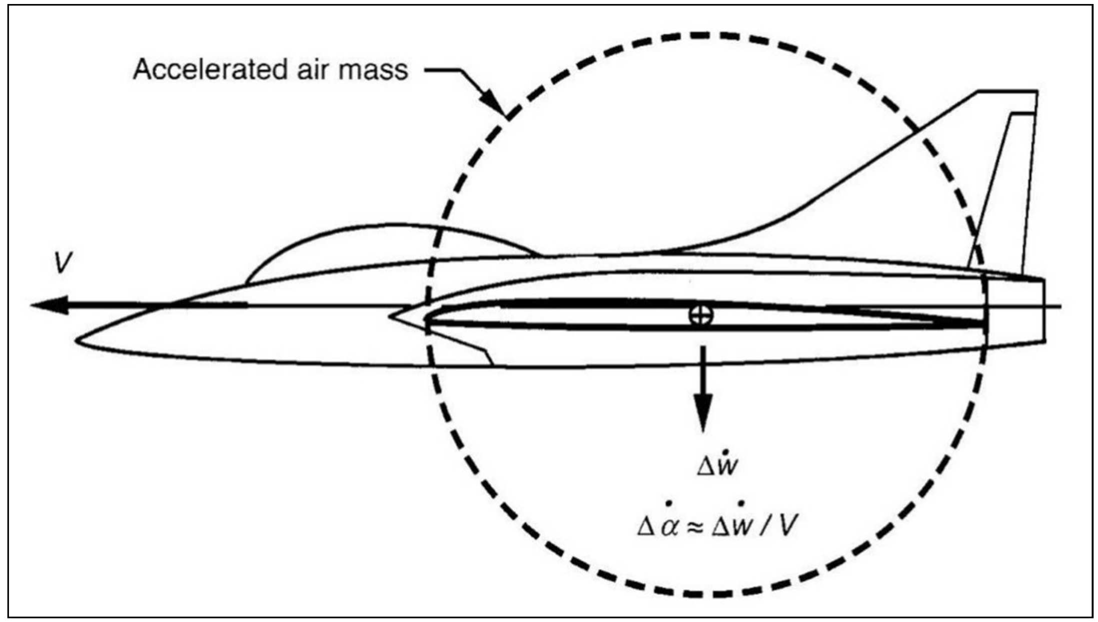
$$\Delta M_{cg} = -\frac{a_T q l_T^2 Q_T S_T}{u_0}; \quad M_q = -\frac{a_T l_T^2 Q_T S_T}{u_0}; \quad C_{mq} = \frac{M_q}{Q_w S_w \bar{c}} = -\frac{a_T l_T Q_T l_T S_T}{u_0 Q_w S_w \bar{c}}$$

$$C_{mq} = -\eta_T V_T \frac{a_T l_T}{u_0}; \quad \bar{q} = \frac{q\bar{c}}{2u_0}; \quad C_{m\bar{q}} = -2\eta_T V_T a_T \bar{l}_T$$

Angle-of-Attack Rate ($d\alpha/dt$) Vs. Apparent Mass

A change in 'α' is associated with change in '**w**', so that '$d\alpha/dt$' is actually 'dw/dt', or the **vertical** (heave) acceleration.

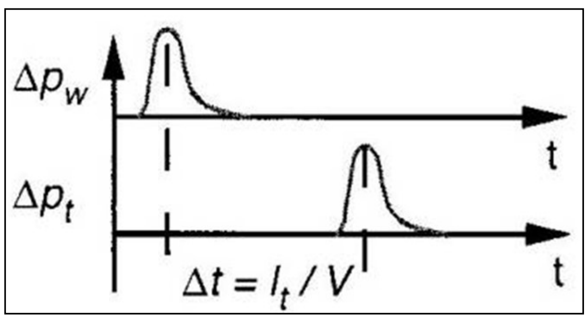
When a wing experiences (dw/dt), mass of **air** immediately above/ below also **faces** the same, acceleration, as shown **alongside**.



Angle-of-Attack Rate (dα/dt) Impact

Air that is accelerated vertically, **creates** ‘Δp’ which takes a finite time to **reach** tail, due to ‘l_T’, causing a time **lag.**

This is shown in the figure below and **expression** alongside.



$$\Delta\varepsilon = \frac{\partial\varepsilon}{\partial\alpha}\Delta t\dot{\alpha} = \frac{\partial\varepsilon}{\partial\alpha}\frac{l_T}{u_0}\dot{\alpha};\quad \Delta C_{LT} = a_T\left(\frac{\partial\varepsilon}{\partial\alpha}\frac{l_T}{u_0}\dot{\alpha}\right)Q_T S_T$$

$$\Delta Z = -\Delta L_T;\quad \Delta Z = -a_T\left(\varepsilon_\alpha\left\{\frac{l_T}{u_0}\right\}\dot{\alpha}\right)Q_T S_T$$

$$Z_{\dot{\alpha}} = -a_T\varepsilon_\alpha\left\{\frac{l_T}{u_0}\right\}Q_T S_T;\quad C_{Z\dot{\alpha}} = \frac{Z_{\dot{\alpha}}}{Q_w S_w};\quad C_{Z\dot{\alpha}} = -\frac{a_T\varepsilon_\alpha\bar{c}Q_T S_T l_T}{\bar{c}Q_w S_w u_0}$$

$$C_{Z\dot{\alpha}} = -a_T\varepsilon_\alpha\bar{c}\eta_T V_T\frac{1}{u_0};\quad \bar{\dot{\alpha}} = \frac{\dot{\alpha}\bar{c}}{2u_0};\quad C_{Z\bar{\dot{\alpha}}} = -2a_T\varepsilon_\alpha\eta_T V_T$$

$$\Delta M = -l_T\times\Delta L_T = -a_T l_T\left(\varepsilon_\alpha\left\{\frac{l_T}{u_0}\right\}\dot{\alpha}\right)Q_T S_T$$

$$M_{\dot{\alpha}} = -a_T\varepsilon_\alpha l_T\left\{\frac{l_T}{u_0}\right\}Q_T S_T;\quad C_{m\dot{\alpha}} = \frac{M_{\dot{\alpha}}}{\bar{c}Q_w S_w} = -\frac{a_T\varepsilon_\alpha l_T Q_T S_T l_T}{\bar{c}Q_w S_w u_0}$$

$$C_{m\dot{\alpha}} = -a_T\varepsilon_\alpha\eta_T V_T\frac{l_T}{u_0};\quad C_{m\bar{\dot{\alpha}}} = -2a_T\varepsilon_\alpha\eta_T V_T\frac{l_T}{\bar{c}}$$