AE 339 : High speed aerodynamics

(Module $\overline{\underline{IV}}$: Variable Area Flows)

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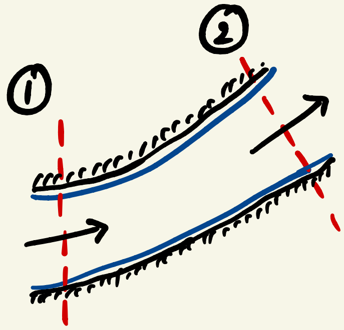
Quasi - 1D flow



$\frac{dA}{dx}$ is small

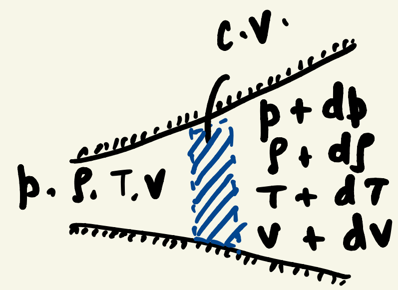
Assumptions

1. Flow is steady
2. Neglect effects of friction and heat transfer (isentropic flow)

Effects of dA on dV (or dM)

$\boxed{\text{Mass}}$ $\rho A V$ = mass flow rate = constant

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \quad \text{— (1)}$$



$\boxed{\text{Energy}}$ $C_p T + \frac{V^2}{2} = \text{constant}$

$$C_p dT + V dV = 0 \quad \text{— (2)}$$

$\boxed{\text{State}}$ $p = \rho R T$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad \text{— (3)}$$

$\boxed{\text{Isentropic}}$ $p/\rho^\gamma = \text{constant}$

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad \text{— (4)}$$

Eqn. 2 can be written as

$$\frac{dT}{T} + \frac{V^2}{C_pT}\frac{dV}{V} = 0 \quad — \ 5$$

Substitute 4 and 5 in 3

$$(\gamma-1)\frac{d\rho}{\rho} + \frac{V^2}{C_pT}\frac{dV}{V} = 0$$

$$\frac{V^2}{C_pT} = (\gamma-1)M^2 \Rightarrow \frac{dT}{T} + (\gamma-1)M^2 = 0 \quad — \ 6$$

$$\Rightarrow \cancel{(\gamma-1)}\frac{d\rho}{\rho} + \cancel{(\gamma-1)}M^2\frac{dV}{V} = 0$$

$$\frac{d\rho}{\rho} = -M^2\frac{dV}{V} \quad — \ 7$$

Substitute 7 in 1

$$-M^2\frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\boxed{\frac{dA}{A} = (M^2-1)\frac{dV}{V}}$$

or

$$\boxed{\frac{dA}{dV} = (M^2-1)\frac{A}{V}}$$

Cases

1. $M<1$, dA and dV have opposite signs

$$dA > 0 \Leftrightarrow dV < 0$$
$$dA < 0 \Leftrightarrow dV > 0$$

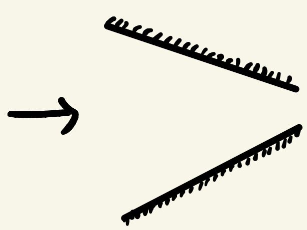
2. $M>1$, dA & dV have the same sign

$$dA < 0 \Leftrightarrow dV < 0$$
$$dA > 0 \Leftrightarrow dV > 0$$

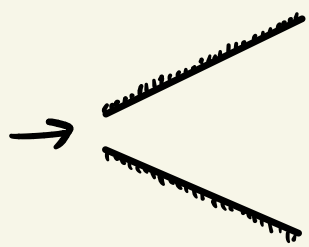
3. $M=1$, sonic flow, $\frac{dA}{dV} = 0$

$\Rightarrow$ A reaches an extremum (minimum)

$M=1$ can only exist at A_{min}



subsonic nozzle / supersonic diffuser



subsonic diffuser / supersonic nozzle

$$V = Ma$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

$$\frac{dT}{T} = -(\gamma-1)M^2\frac{dV}{V}$$

From eqn. 6

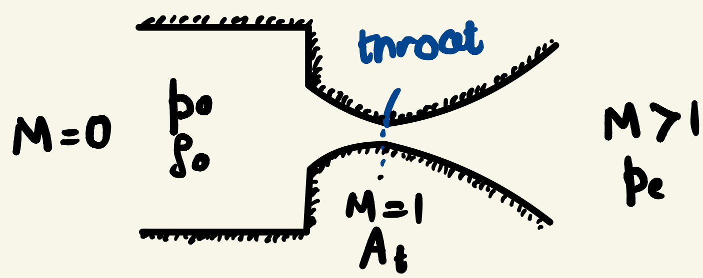
$$\frac{dM}{M} = \left[1+\frac{(\gamma-1)}{2}M^2\right]\frac{dV}{V}$$

$$\boxed{\frac{dA}{A} = \frac{(M^2-1)}{\left[1+\frac{\gamma-1}{2}M^2\right]}\frac{dM}{M}}$$

Cases

1. $M<1$, dA and dM have opposite signs
2. $M>1$, dA and dM have the same sign
3. $M=1$, $dA=0$, A is a minimum (A_t)

To accelerate a subsonic flow to a supersonic flow, we need a convergent-divergent nozzle (c-d nozzle) or a de Laval nozzle



Features of the flow at ideal operation:

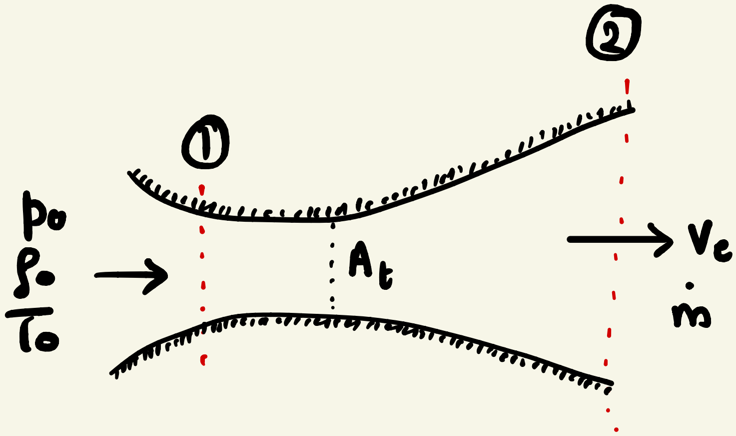
(a) Pressure decreases across the nozzle

(b) $M=1$ at the throat

Supersonic flow is achieved only if p_o/p_e is 'large enough'. If p_o/p_e is not large, v increases till flow reaches throat ($M<1$) and then decreases like a venturi; ie, flow is subsonic throughout.

$$\dot{m} = \rho_1 V_1 A_1 = \text{constant}$$

$$V_1^2 + \frac{2a_1^2}{(\gamma-1)} = \frac{2a_0^2}{\gamma-1} \quad —① $$



Flow isentropic

$$\frac{a_1}{a_0} = \left(\frac{T_1}{T_0}\right)^{1/2} = \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{\rho_1}{\rho_0}\right)^{\frac{\gamma-1}{2}}$$

$$V_1 = \left\{\left(\frac{2}{\gamma-1}\right) a_0^2 \left[1 - \left(\frac{a_1}{a_0}\right)^2\right]\right\}^{1/2}$$

$$\boxed{V_1 = \left\{\left(\frac{2\gamma}{\gamma-1}\right) \frac{p_0}{\rho_0} \left[1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}} \quad — (\star)$$

$$\dot{m} = \rho_0 V_1 A_1 \frac{\rho_1}{\rho_0}$$

$$\boxed{\dot{m} = \rho_0 A_1 \left(\frac{p_1}{p_0}\right)^{\frac{1}{\gamma}} \left\{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p_0}{\rho_0}\right)\left[1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}} \quad (\star\star)$$

Critical conditions (p^*, T^*, ρ^*)

A convenient reference point in presenting the flow equations in a variable area duct.

Point in the flow at which $M=1$ (may not actually exist)

Energy equation at critical condition gives ($V^* = a^*$)

$$\boxed{V^{*2} = \frac{2}{\gamma+1} a_0^2}$$

Also, $\left(\frac{a^*}{a_0}\right)^2 = \frac{T^*}{T_0} = \frac{2}{\gamma+1}$

$$\Rightarrow \frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}, \quad \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

For air ($\gamma = 1.4$) Remember these !!!

$$\frac{T^*}{T^0} = 0.833 \qquad \frac{p^*}{p_0} = 0.528 \qquad \frac{\rho^*}{\rho_0} = 0.634$$

Now (**) at critical condition gives

$$\dot{m} = \rho_0 A^* \left(\frac{p^*}{p_0}\right)^{\frac{1}{\gamma}} \left\{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p_0}{\rho_0}\right)\left[1 - \left(\frac{p^*}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}$$

$$= \rho_0 A^* \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left\{\left(\frac{2\gamma}{\cancel{\gamma-1}}\right)\left(\frac{p_0}{\rho_0}\right)\left[\frac{\cancel{\gamma-1}}{\gamma+1}\right]\right\}^{1/2}$$

$$\Rightarrow \boxed{A^* = \frac{\dot{m}}{\sqrt{\gamma p_0 \rho_0}} \left(\frac{2}{\gamma+1}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}} \quad \text{— (***)}$$

The area at any section of the duct, in terms of the critical area A^* is

$$\frac{A}{A^*} = \left(\frac{p^*}{p}\right)^{\frac{1}{\gamma}} \left\{\frac{1 - \left(\frac{p^*}{p_0}\right)^{\frac{\gamma-1}{\gamma}}}{1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}}\right\}^{1/2}$$

$$\Rightarrow \boxed{\frac{A}{A^*} = \frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}}}{\left\{\left(\frac{p}{p_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p}{p_0}\right)^{\frac{\gamma+1}{\gamma}}\right\}^{\frac{1}{2}}}} \quad - (\ast\ast\ast\ast)$$

Let us say, a nozzle is designed for a given $\dot{m}$, with an overall pressure ratio of p_e/p_0

Discharge velocity is obtained as

$$V_e = \left\{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p_0}{\rho_0}\right)\left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad - (1)$$

Exit area A_e is found from

$$\dot{m} = \rho_0 A_e \left(\frac{p_e}{p_0}\right)^{\frac{1}{\gamma}}\left\{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p_0}{\rho_0}\right)\left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} \quad - (2)$$

Throat area A^* is found from

$$\frac{A_e}{A^*} = \frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}}}{\left\{\left(\frac{p_e}{p_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma+1}{\gamma}}\right\}^{\frac{1}{2}}} \quad - (3)$$

A/A^* in terms of M

Energy equation gives

$$V = M a_0 \left[1 + \frac{\gamma-1}{2} M^2\right]^{-\frac{1}{2}}$$

$$\frac{\rho}{\rho_0} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{-1}{\gamma-1}}$$

$$\frac{\dot{m}}{A} = \rho V = \rho_0 V \cdot \frac{\rho}{\rho_0}$$

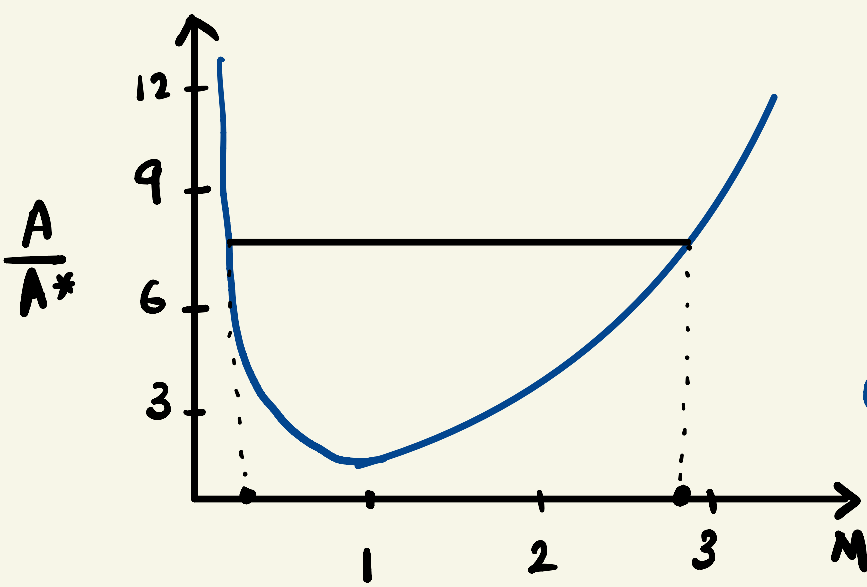
$$= \frac{\rho_0 a_0 M}{\left\{1 + \frac{\gamma-1}{2} M^2\right\}^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$\Rightarrow \boxed{\frac{A_2}{A_1} = \left(\frac{M_1}{M_2}\right)\left\{\frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2}\right\}^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$\dot{m}$ constant

For $A_1 = A^*$, $A_2 = A$, $M_2 = M$, $M_1 = 1$, we have

$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left\{\left(\frac{2}{\gamma+1}\right)\left[1 + \frac{\gamma-1}{2} M^2\right]\right\}^{\frac{\gamma+1}{2(\gamma-1)}}} \quad - \quad (*****)$$



(a) $\frac{A}{A^*}$ reaches a minimum at $M=1$

(b) For a given A/A^* there are two M

Operating characteristics of nozzles

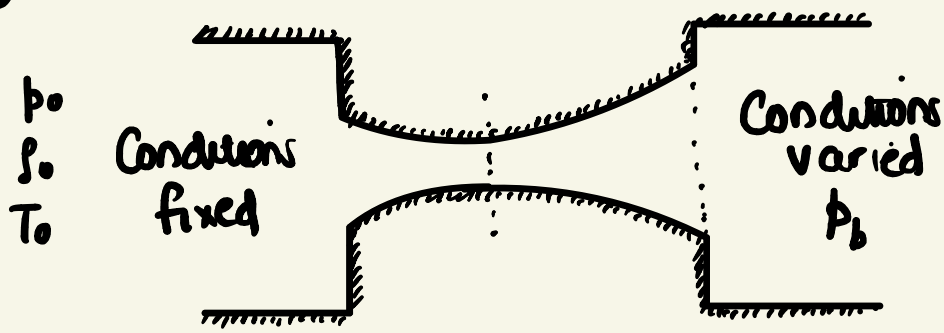
Nozzle – a shape that accelerates gas flow

Diffuser – a shape that decelerates gas flow

Assumptions

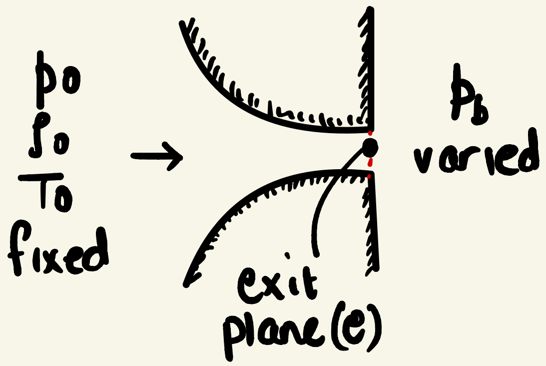
(1) Upstream conditions are stagnation conditions

(2) Only conditions downstream of nozzle varied



Downstream pressure is called **back pressure** (p_b)

Case 1: convergent nozzle



$p_b = p_0 \Rightarrow$ no flow ($M_e = 0$)

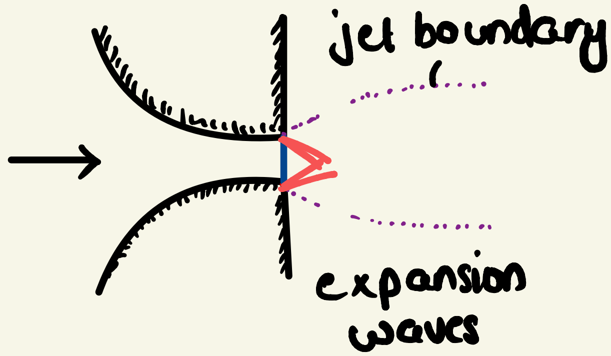
$p_b < p_0 \Rightarrow$ flow ($M_e \neq 0$)

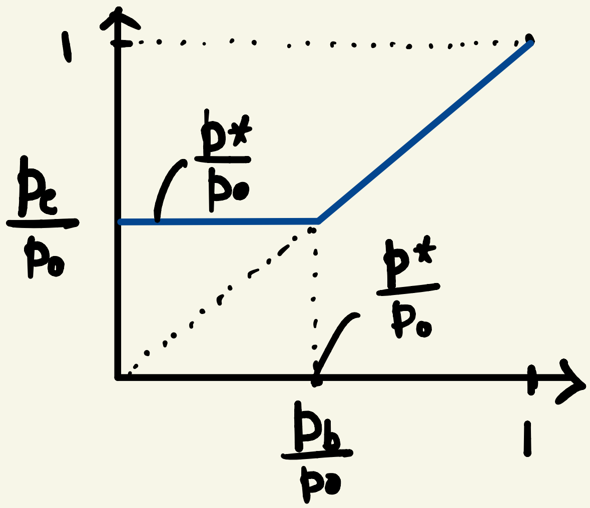
When $p_b = p^* = p_0 \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$. $M_e = 1$

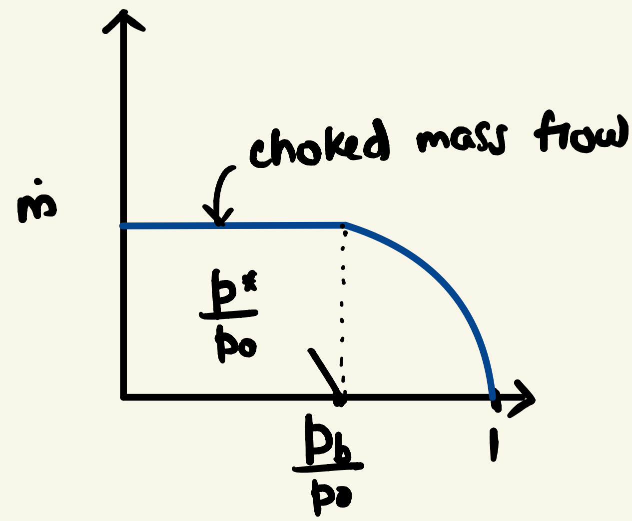
For $p_b > p^*$, reduction in p_b increases the mass flow

For $p_b \leq p^*$, reduction in p_b have no effect on the flow in the nozzle, mass flow rate remains constant and $M_e = 1 \Rightarrow$ Nozzle is **choked**

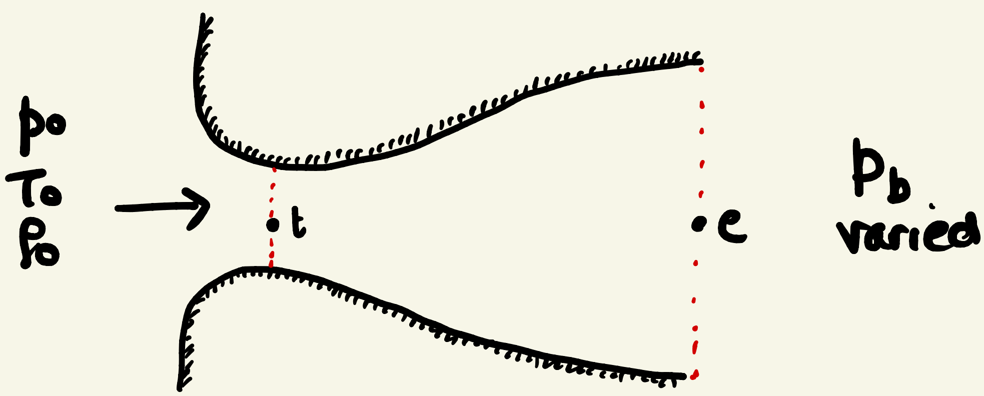
When $p_b < p^*$, expansion from p_e to p_b takes place outside the nozzle through a series of expansion waves



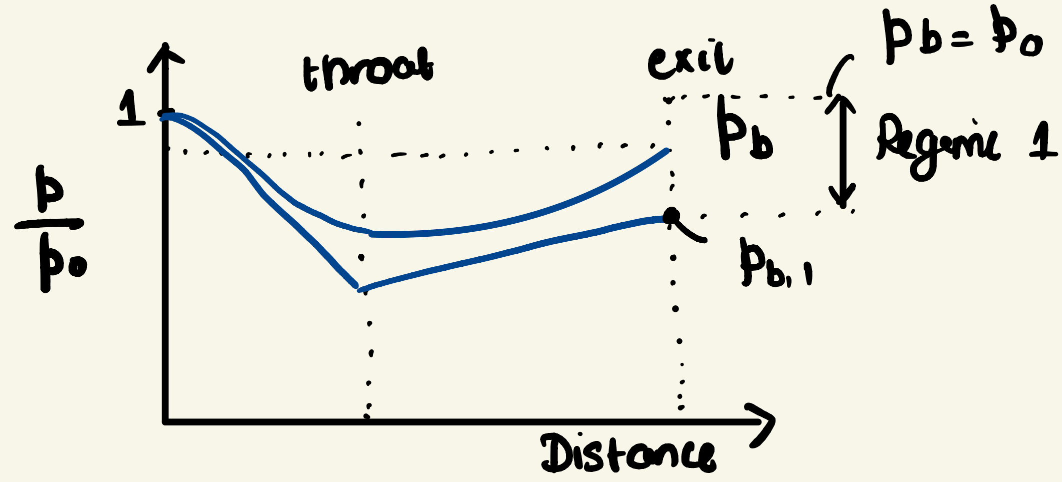




$\boxed{\text{Case 2: Convergent-divergent nozzle}}$



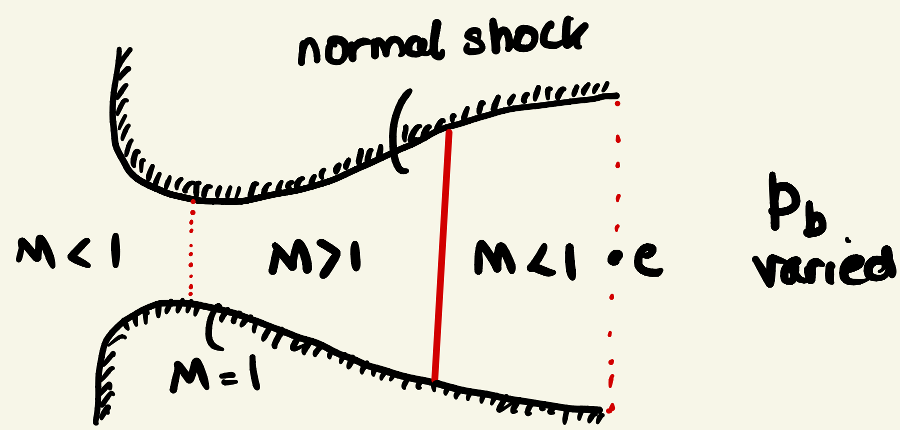
Regime 1 : $p_t > p^*$. Flow subsonic throughout

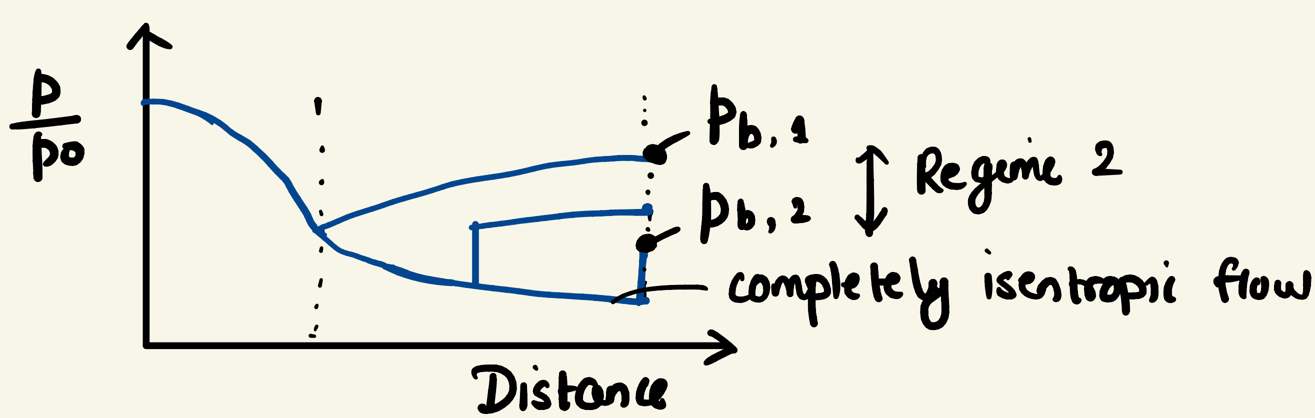


Nozzle operates like a venturi until p_b decreases enough to make $M_t = 1$

$p_{b,1} = p_b$ at which $p_t = p^*$

Regime 2: From the end of regime 1 until there is a normal shock at the exit ($p_b = p_{b,2}$)





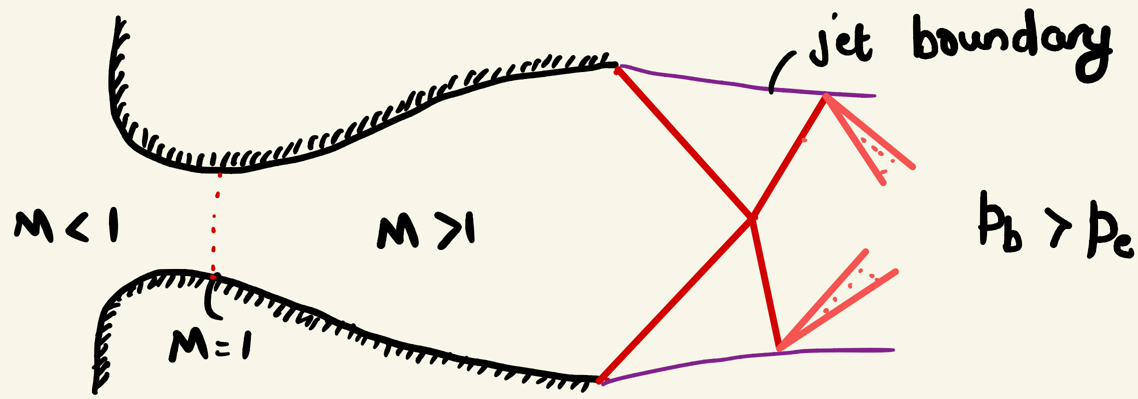
For $p_b < p_{b,1}$ there will be a normal shock in the divergent portion and this shock will move downstream as p_b is decreased until it reaches the exit at $p_b = p_{b,2}$

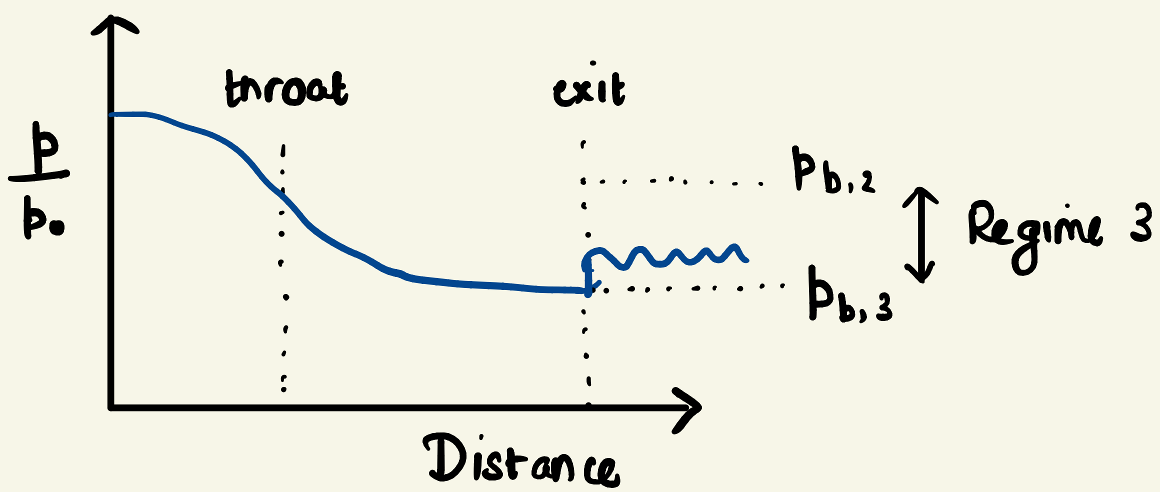
As p_b decreases in this regime, we have stronger normal shocks as M upstream is larger.

When the shock reaches the exit plane, flow in the nozzle is isentropic throughout and p_e/p_0 is the design pressure ratio and M_e is the design Mach number

Once shock reaches the exit, further reductions in p_b will not affect the nozzle

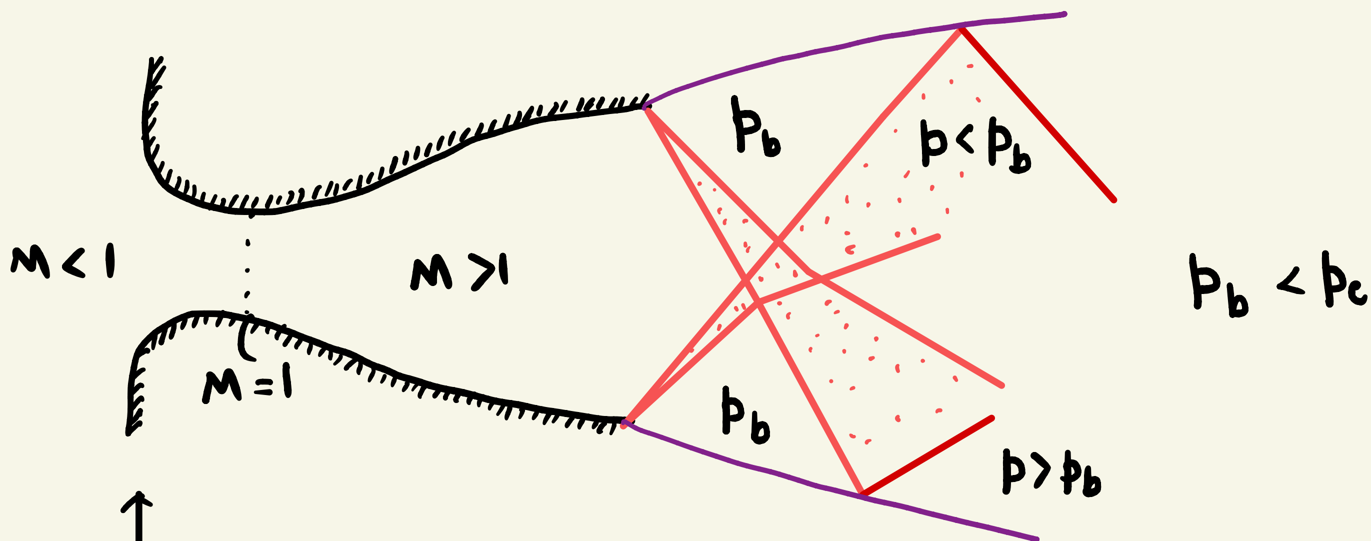
Regime 3 (overexpanded regime): p_b at which oblique shocks are present at the nozzle exit

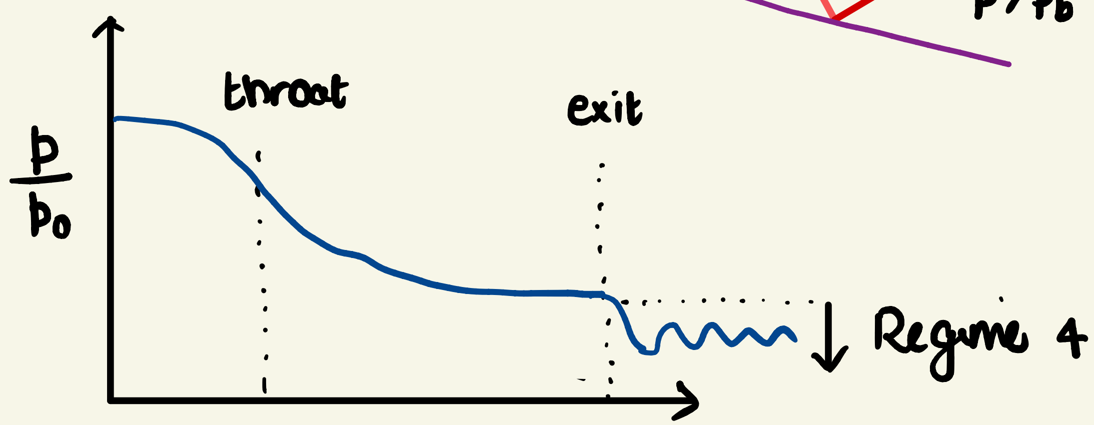




As p_b decreases and approaches p_e, oblique shock becomes weaker until it vanishes at $p_b = p_e$. The nozzle is now operating at design conditions and there are no waves inside or outside the nozzle ($p_b = p_{b,3}$)

Regime 4 (underexpanded regime): p_b at which expansion waves are present at the nozzle exit





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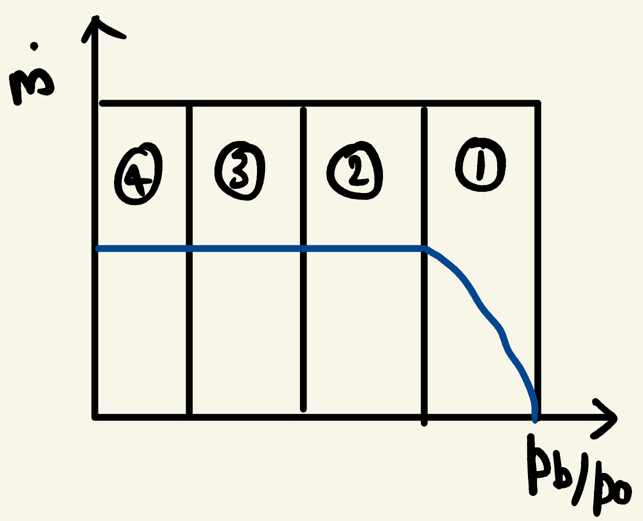
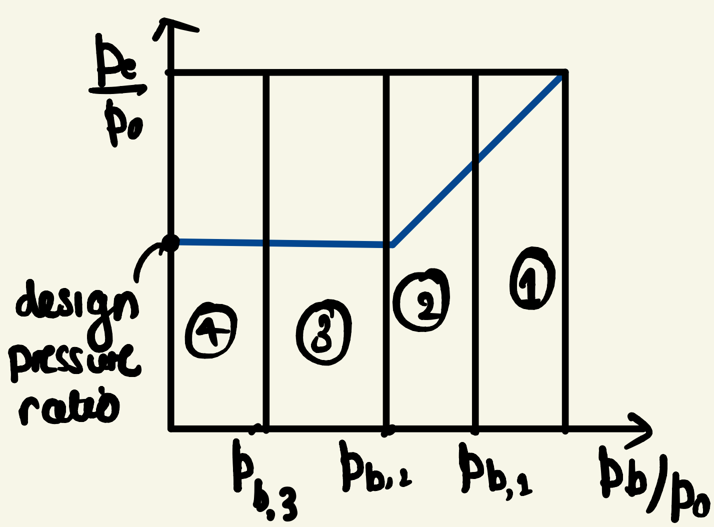
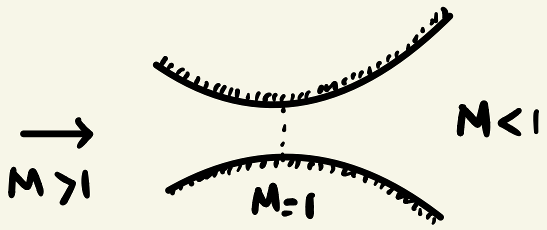




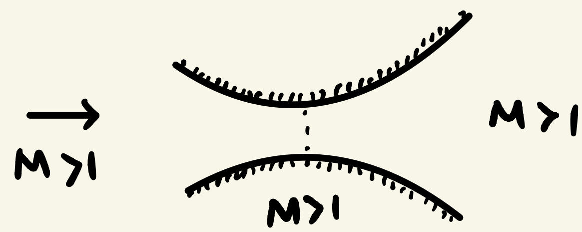
Convergent - Divergent Supersonic Diffusers

If initial flow is supersonic, two things happen

$\boxed{\text{Case 1}}$



$\boxed{\text{Case 2}}$

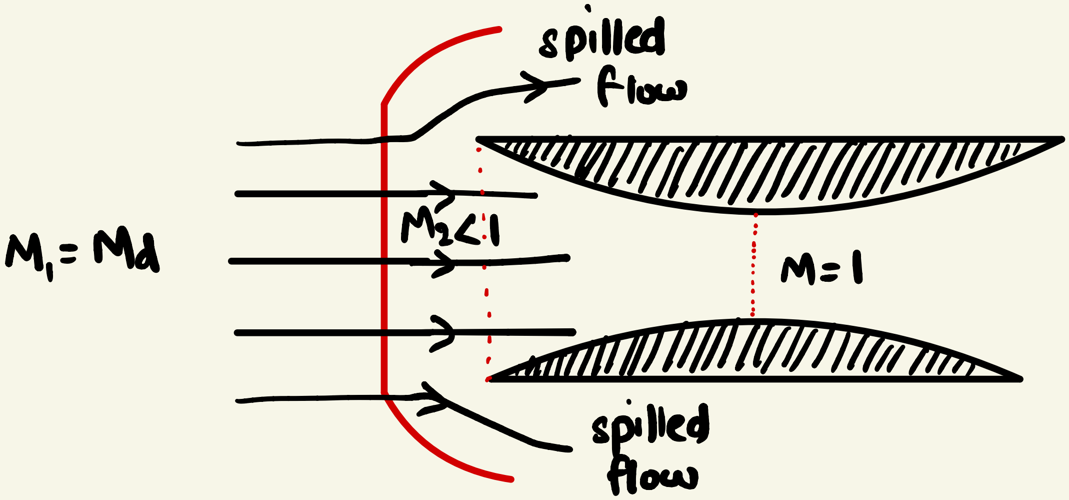


Ideally, we want shockless diffusion to $M<1$ before flow reaches engine

Q. What happens at off-design conditions?

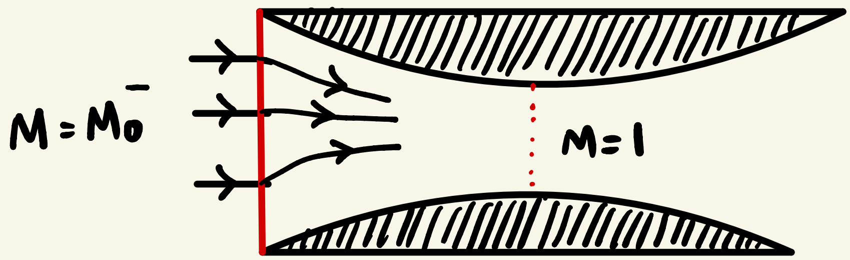
Fixed diffusers : A_i, A_t are fixed $\Rightarrow \frac{A^*}{A}$ fixed.

A^*/A is larger at $M < M_d$ which for a fixed diffuser results in a larger mass flow rate trying to enter it. As a result, normal shock is present before the inlet which persists even at $M = M_d$ although the shock is now closer to the intake



Q. How to get rid of the shock in a fixed diffuser?

Overspeeding: Increase freestream M until shock reaches the inlet and is swallowed. Shock swallowing happens when M_2 has the design A/A^* value. ($M_1 = M_0$). All of the mass flow is ingested

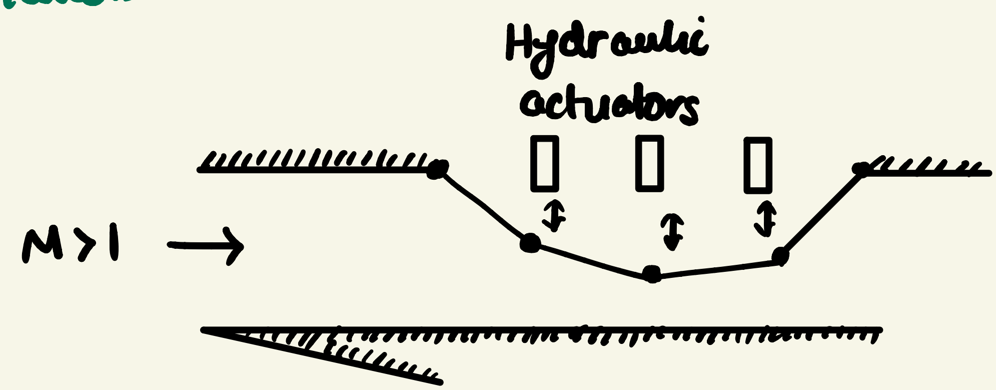


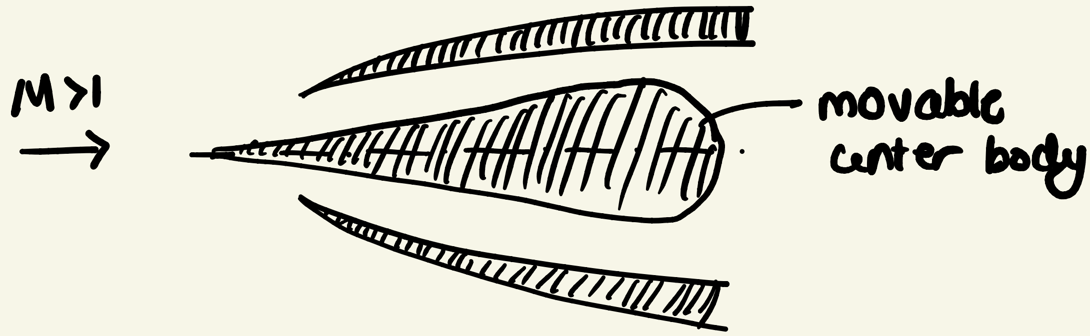
Once the shock is swallowed, the flight Mach number can be decreased which moves the shock upstream towards the throat until at $M = M_d$, the shock reachs throat and vanishes.

In reality, it would not be practical to operate under these design conditions because the slightest decrease in M would cause the shock to be disgorged, i.e., move out of the diffuser

Because of poor performance at off-design conditions and requirement of overspeeding, fixed c-d diffusers are seldom used.

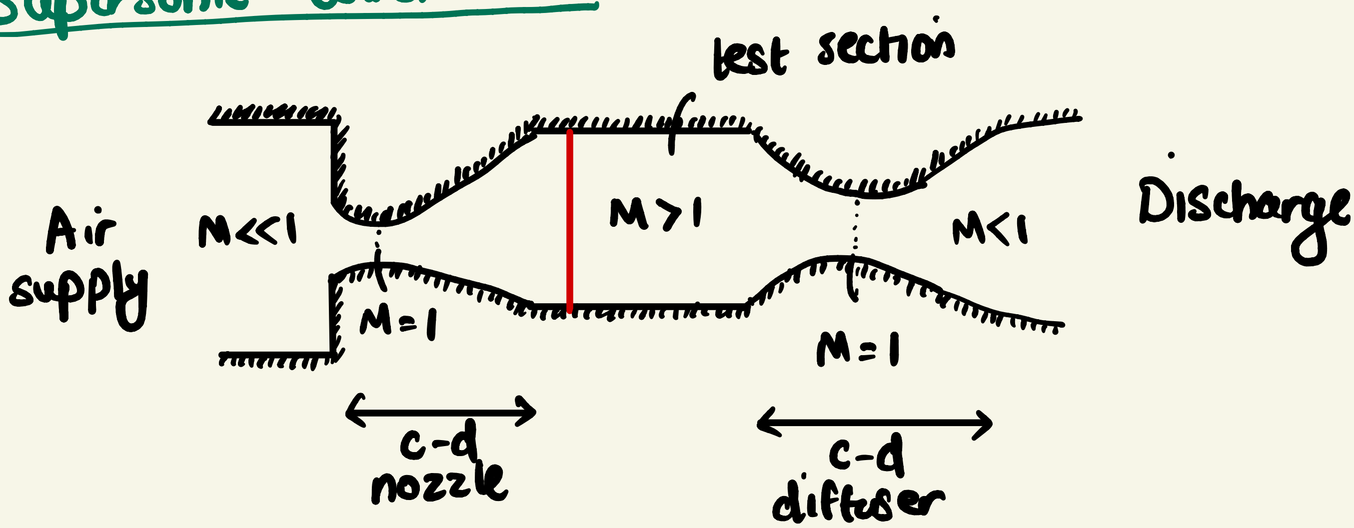
Solution: Diffuser with a variable area throat





With a variable A/A^*, the throat can be opened to allow the shock to be swallowed and then the throat area can be decreased until the shock disappears at all Mach numbers ⇒ no overspeeding required

Supersonic wind tunnel



The tunnel is started by either increasing the pressure ahead of the nozzle or decreasing the pressure behind the diffuser.

A normal shock wave exists in the nozzle during startup and moves downstream to the test section. Because of p_0 loss across the shock, A_t of diffuser > A_t of nozzle to swallow the starting shock.