AE 339 : High speed aerodynamics

(VII) External aerodynamics: Supersonic flow past bodies

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Key features

The leading and trailing edges of supersonic airfoils should be sharp (or only slightly rounded) and the section relatively thin to prevent formation of detached strong shock wave in front of the leading edge with consequent drag penalties.

Linearized supersonic flow

For such thin airfoils at small angles of attack, we can apply small perturbation theory to obtain theoretical approximations of aerodynamic characteristics. Flow is treated as isentropic without shock waves.

Linearized perturbation VPE is:

$$\underbrace{(M_\infty^2 - 1)}_{\lambda^2} \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0$$

↳ change from elliptic PDE to hyperbolic PDE

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0$$

which has solutions of the form

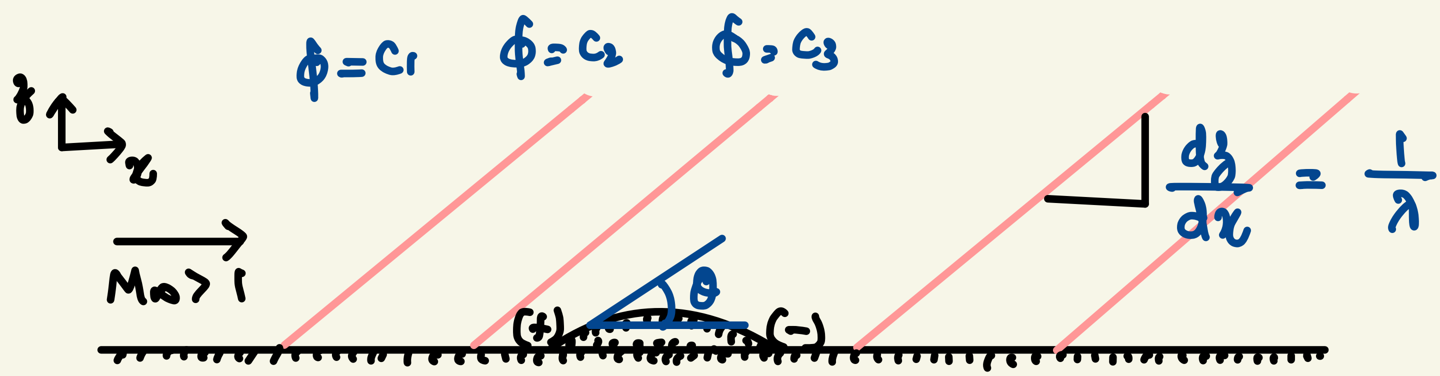
$$\boxed{\hat{\phi} = f(x - \lambda z)}$$ Verify by substitution!

The solution is invariant along the lines $x - \lambda z = \text{constant}$. The slope of these lines are obtained as

$$\boxed{\frac{dz}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \alpha}$$

α — Mach angle

The lines along which $\hat{\phi}$ is constant are Mach lines



Consider the supersonic flow over a small bump of shape $g(x)$ with θ the angle of surface relative to horizontal.

$$\hat{u} = \frac{\partial \hat{\phi}}{\partial x} = f'_\xi \qquad \text{where } \xi = x - \lambda z$$

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial z} = -\lambda f'_\xi$$

Eliminating f', we have

$$\hat{u} = -\frac{\hat{v}}{\lambda}$$

The boundary condition is given as

$$\hat{v} = V_\infty \frac{dy}{dx} = V_\infty \tan\theta \approx V_\infty \theta$$

$$\Rightarrow \boxed{\hat{u} = \frac{-V_\infty \theta}{\lambda}}$$

Hence, the linearized pressure coefficient is

$$\boxed{C_p = \frac{-2\hat{u}}{V_\infty} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$

Remember Prandtl-Meyer flows

Pressure coefficient is proportional to the local surface inclination.

$$\frac{dp}{p} = \frac{\gamma M^2}{\sqrt{M^2-1}} d\theta$$

$\boxed{\text{Sign of } \theta}$

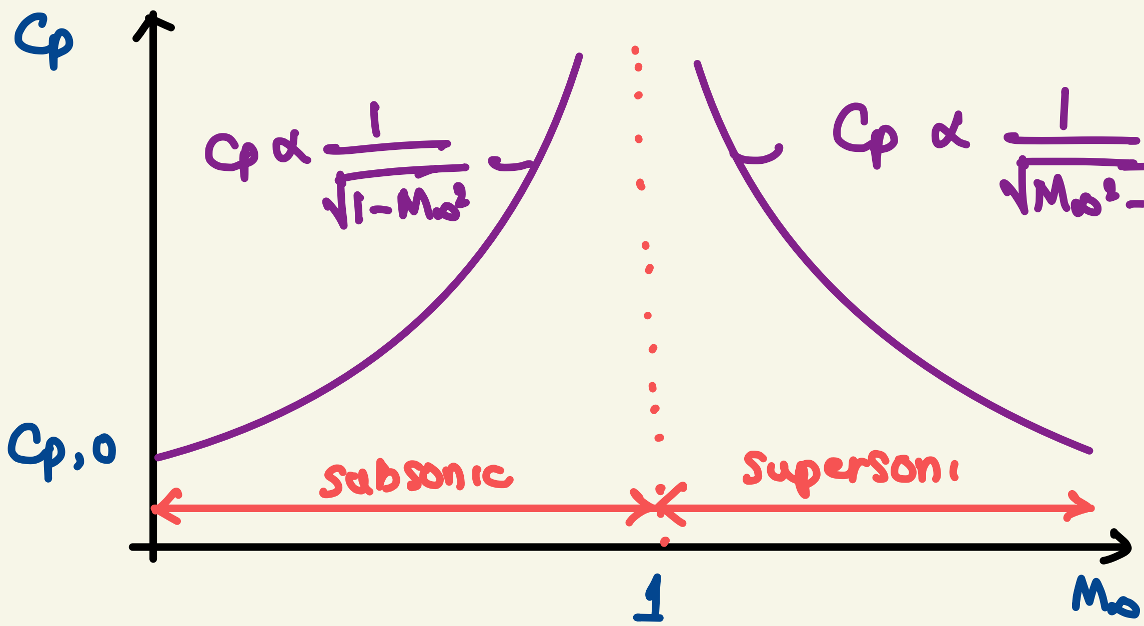
θ is positive for compressive changes and negative for expansive changes.

$\Rightarrow$ C_p positive on the forward portion of the hump and negative on the rear portion.

$\Rightarrow$ Drag force exists on the hump WAVE DRAG

Wave drag is a characteristic of supersonic flows

↳ Present even without shocks in isentropic flows with $M_\infty > 1$



Linearized theory predicts $C_p \to \infty$ as $M_\infty \to 1$ which we know is not valid in the transonic regime.

$\boxed{\text{Solving for } \hat{\phi}}$

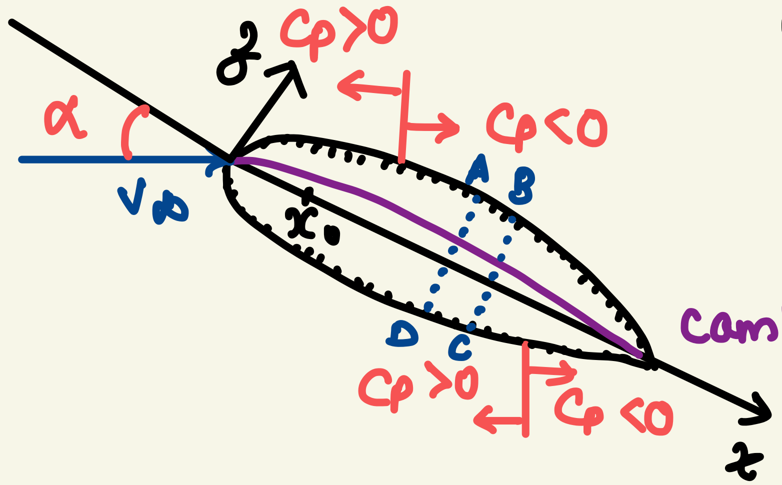
$$\hat{v} = -\lambda f'_\xi = V_\infty \left(\frac{dg}{dx}\right)_w$$

Since hump is small, surface can be treated as $z = 0$. We have

$$\left[f'_\xi\right]_{z=0} = -\frac{V_\infty}{\lambda}\left[\frac{dg}{d\xi}\right]_{z=0}$$

$$\Rightarrow \boxed{f = \hat{\phi} = \frac{-V_\infty}{\sqrt{M_\infty^2-1}}\, g\left(x - \sqrt{M_\infty^2-1}\, z\right)}$$

Ackeret's theory



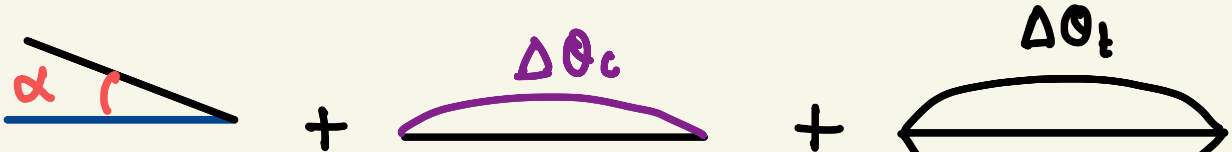
upper surface : $z_u(x)$

lower surface : $z_l(x)$

$$C_{p_u} = C_p(x, 0^+) = \frac{2\theta_u}{\lambda}$$

$$C_{p_l} = C_p(x, 0^-) = \frac{2\theta_l}{\lambda}$$

Decomposition



$$\frac{dz_u}{dx} = \Delta\theta_c + \Delta\theta_t$$

$$\frac{dz_l}{dx} = \Delta\theta_c - \Delta\theta_t$$

$$\boxed{\begin{aligned} \theta_u &= \Delta\theta_t + \Delta\theta_c - \alpha = \frac{dz_u}{dx} - \alpha \\ \theta_l &= \Delta\theta_t - \Delta\theta_c + \alpha = -\frac{dz_l}{dx} + \alpha \end{aligned}}$$

Lift

Lift per unit span acting over an infinitesimal segment ABCD

$$dL' \approx (p_l - p_u)\,dx$$

$$dC_l = (C_{p_l} - C_{p_u})\, d\frac{x}{c}$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}}(\theta_l - \theta_u)$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}}\left[2\alpha - 2\Delta\theta_c\right] d\left(\frac{x}{c}\right)$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}}\left[2\alpha - \frac{dz_l}{dx} - \frac{dz_u}{dx}\right] d\left(\frac{x}{c}\right)$$

Since $z_u = z_l = 0$ at both leading and trailing edges

$$\int_0^1 \frac{dz_l}{dx}\, d\left(\frac{x}{c}\right) = \int_0^1 \frac{dz_u}{dx}\, d\left(\frac{x}{c}\right) = 0$$

$$\Rightarrow \quad \boxed{C_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}} \quad \text{on integration}$$

In the linear approximation, the lift coefficient is independent independent of the camber and thickness distribution.

$$\frac{dC_l}{d\alpha} = a = \frac{4}{\sqrt{M_\infty^2 - 1}} \; ; \; \alpha_{L=0} = 0$$

For $M_\infty \gtrsim 1.185$, the lift curve slope is less than the theoretical value for incompressible flow past an airfoil.

$\boxed{\text{Drag}}$

The incremental drag force due to the inviscid flow acting on ABCD

$$dD' \approx p_l \theta_l \, dx + p_u \theta_u \, dx$$

$$dC_d = \left(C_{p_l} \theta_l + C_{p_u} \theta_u \right) d\left(\frac{x}{c}\right)$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}} \left[\theta_l^2 + \theta_u^2 \right] d\left(\frac{x}{c}\right)$$

$$= \frac{2}{\sqrt{M_\infty^2-1}}\left[2\alpha^2 + \left(\frac{dz_u}{dx}\right)^2 + \left(\frac{dz_l}{dx}\right)^2 - 2\left(\frac{dz_l}{dx} + \cancel{\frac{dz_u}{dx}}\right)\right] d\left(\frac{x}{c}\right)$$

$\to 0$ as earlier

Integrating,

$$\Rightarrow \quad C_d = \frac{4\alpha^2}{\sqrt{M_\infty^2-1}} + \frac{2}{\sqrt{M_\infty^2-1}} \int_0^1 \left[\left(\frac{dz_u}{dx}\right)^2 + \left(\frac{dz_l}{dx}\right)^2\right] d\left(\frac{x}{c}\right)$$

Let $\frac{1}{c}\int_0^1 \left(\frac{dz}{dx}\right)^2 dx = \bar{\sigma}$ mean-squared slope

$$\boxed{C_d = \frac{D'}{q_\infty c} = \frac{4\alpha^2}{\sqrt{M_\infty^2-1}} + \frac{2}{\sqrt{M_\infty^2-1}}\left(\bar{\sigma}_u^2 + \bar{\sigma}_l^2\right)}$$

Drag is non-zero even without viscosity and assuming infinite span (no finite-wing effects)

↳ WAVE DRAG

* $\frac{4\alpha}{\sqrt{M_\infty^2-1}}$ represents wave drag due to lift / induced wave drag

It is independent of airfoil shape

* $\frac{2}{\sqrt{M_\infty^2-1}}\left(\bar{\sigma}_u^2 + \bar{\sigma}_l^2\right)$ represents wave drag due to thickness

It depends only on airfoil shape

$$C_{d,\,lift} = \alpha C_l \qquad C_d = C_{d_0} + \alpha C_l$$

$$C_{d,\,thickness} = C_{d_0}$$

Pitching moment

The incremental moment about an arbitrary point x_0 on the chord (+ve for nose up moment) is

$$dM'_{x_0} = (p_u - p_l)(x - x_0)\,dx$$

$$dC_{m_{x_0}} = (C_{p_u} - C_{p_l})\left(\frac{x - x_0}{c}\right) d\left(\frac{x}{c}\right)$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}}(\theta_u - \theta_l)\left(\frac{x - x_0}{c}\right) d\left(\frac{x}{c}\right)$$

$$= \frac{2}{\sqrt{M_\infty^2 - 1}}\left[-2\alpha + \underbrace{\frac{dz_u}{dx} + \frac{dz_l}{dx}}_{2\Delta\theta_c \text{ or } 2\frac{dz_c}{dx}}\right]\left(\frac{x - x_0}{c}\right) d\left(\frac{x}{c}\right)$$

Integrating,

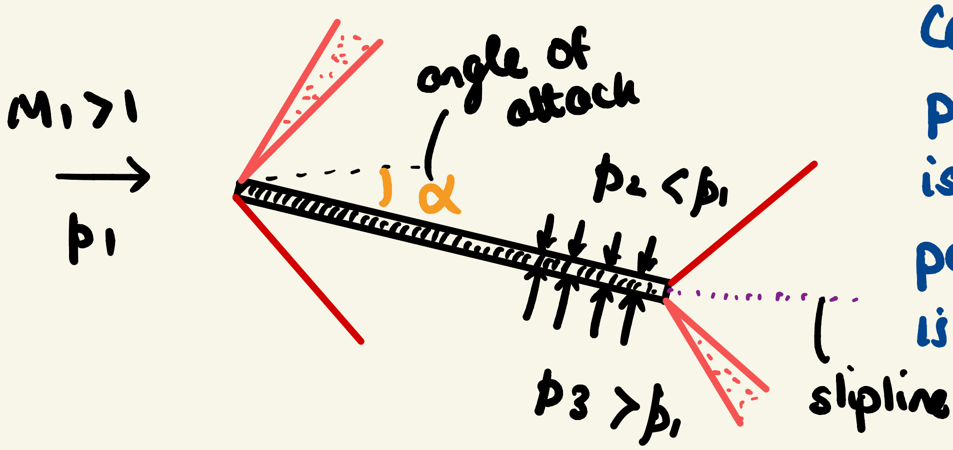
$$\boxed{C_{m_{x_0}} = \frac{-4\alpha}{\sqrt{M_\infty^2 - 1}}\left(\frac{1}{2} - \frac{x_0}{c}\right) + \frac{4}{\sqrt{M_\infty^2 - 1}}\int_0^1 \frac{dz_c}{dx}\left(\frac{x - x_0}{c}\right) d\left(\frac{x}{c}\right)}$$

For a thin airfoil in supersonic flow, aerodynamic center is located at $x_{ac} = \frac{c}{2}$

Shock-expansion theory

An exact method; however, it is essentially a numerical method which doesn't give a closed form solution to evaluate airfoil performance parameters such as C_l, C_d

Flat plate



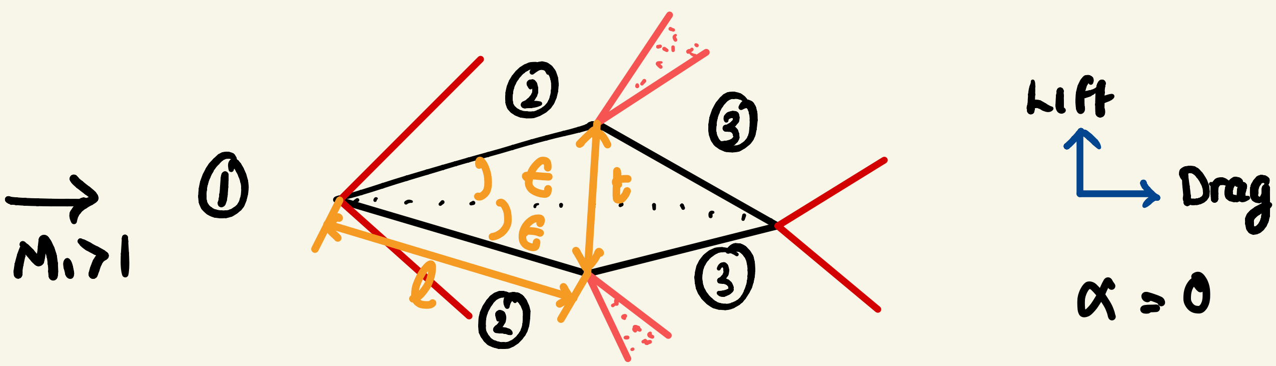
Component of force perpendicular to flow is LIFT and component parallel to the flow is DRAG (WAVE DRAG)

For unit span,

$$L' = (p_3 - p_2)\, c \cos\alpha$$

$$D' = (p_3 - p_2)\, c \sin\alpha$$

Diamond airfoil



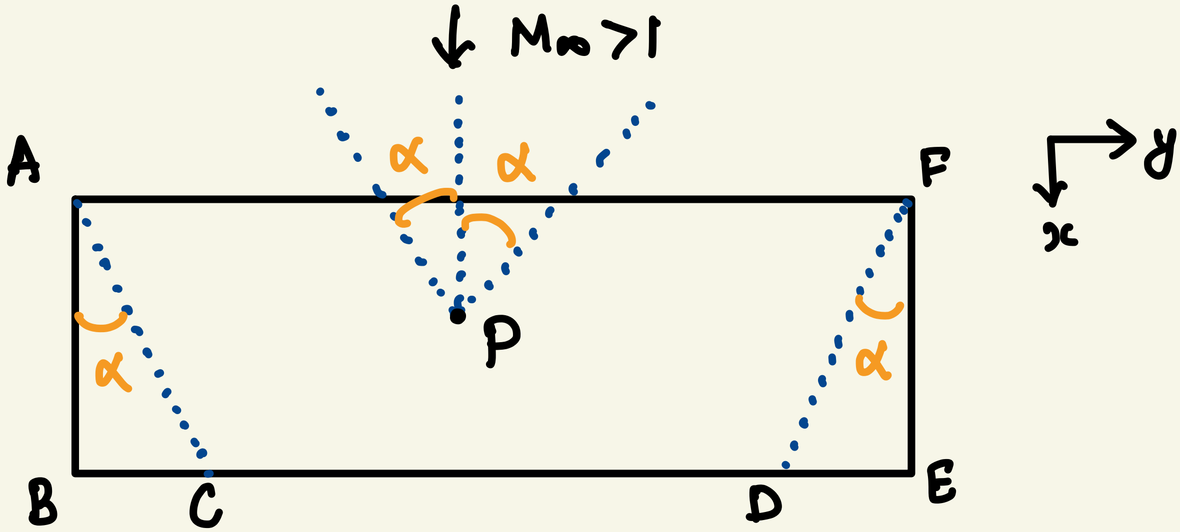
$L' = 0$ since pressure distributions are symmetrical on top and bottom surfaces for $\alpha = 0$

$$D' = 2(p_2 l \sin\epsilon - p_3 l \sin\epsilon)$$

or $$\boxed{D' = (p_2 - p_3)\, t}$$

Thus airfoils have least wave drag.

General remarks about supersonic wings



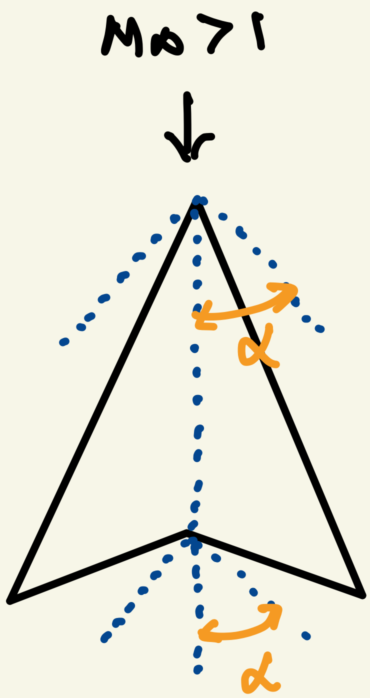
The pressure at $P(x, y)$ is only influenced by disturbances generated at points within the upstream Mach cone.

Tips affect the flow only in regions ABC and DEF as shown.

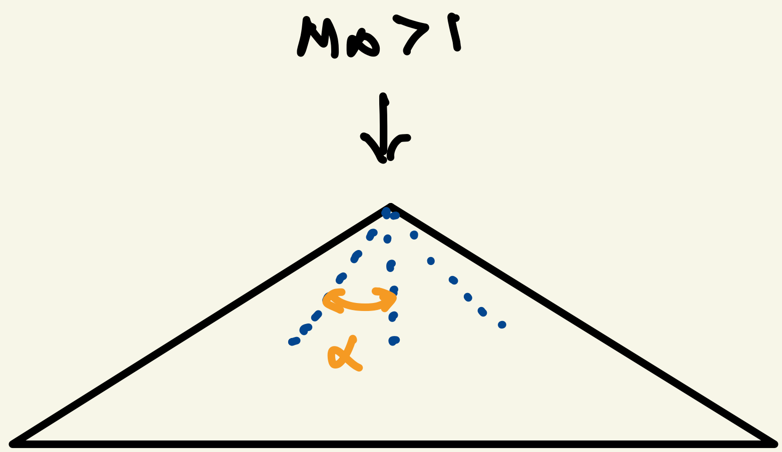
The 2D theory developed can be applied on ACDF as wing tip effects are not felt here.

Supersonic and subsonic edges

A supersonic edge is that portion of the wing where the normal component of freestream velocity is supersonic



ARROW WING



DELTA WING

* subsonic leading edge & supersonic trailing edge
* lower area of induced drag

* supersonic leading and trailing edges
* higher area of induced drag

A subsonic leading edge allows wings to have relatively blunt noses without substantial zero-lift wave drag penalty.

↳ high L/D for subsonic operations

↳ lower wave drag at supersonic cruise