Flight Dynamics Concept

A **stable** aircraft, if disturbed from its ‘trim’, either **through** unwanted forces and **moments** or through control inputs, has **an** initial tendency to either return to, or reach, another ‘trim’.

However, while doing so, it goes through a **state** of ‘disturbed’ equilibrium, **resulting** in time variation of **aircraft** states, as it transits from one ‘trim’ to another.

Flight dynamics is the formal study of the manner in **which** aircraft, under disturbed **condition**, transits from one trim **point** to another, as time progresses.

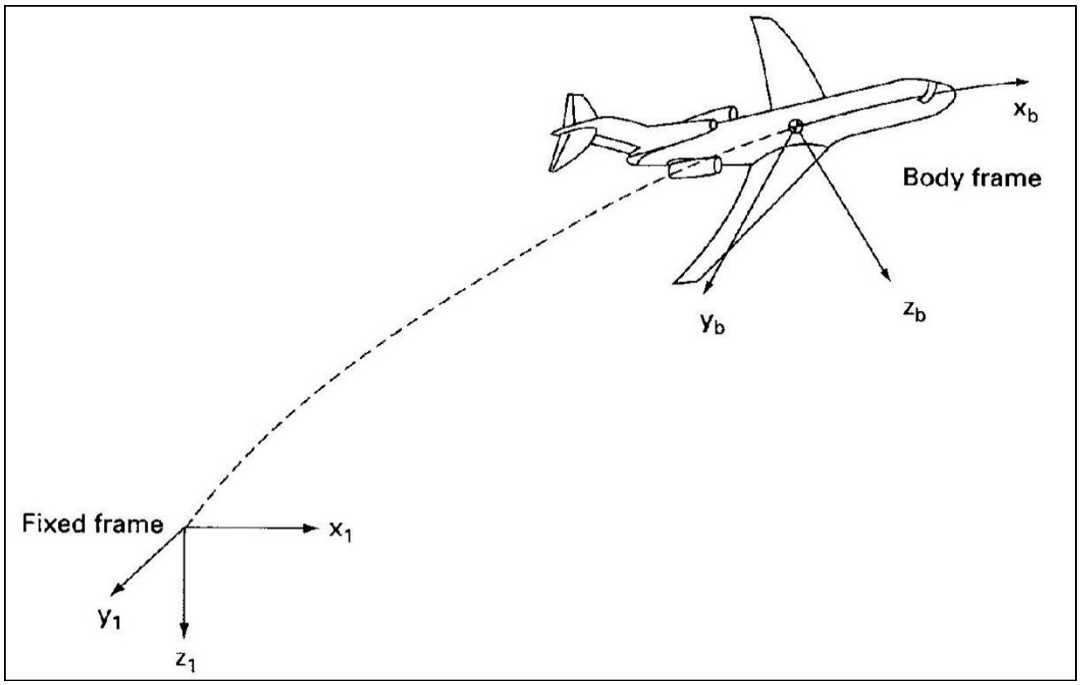
Why Study Flight Dynamics?

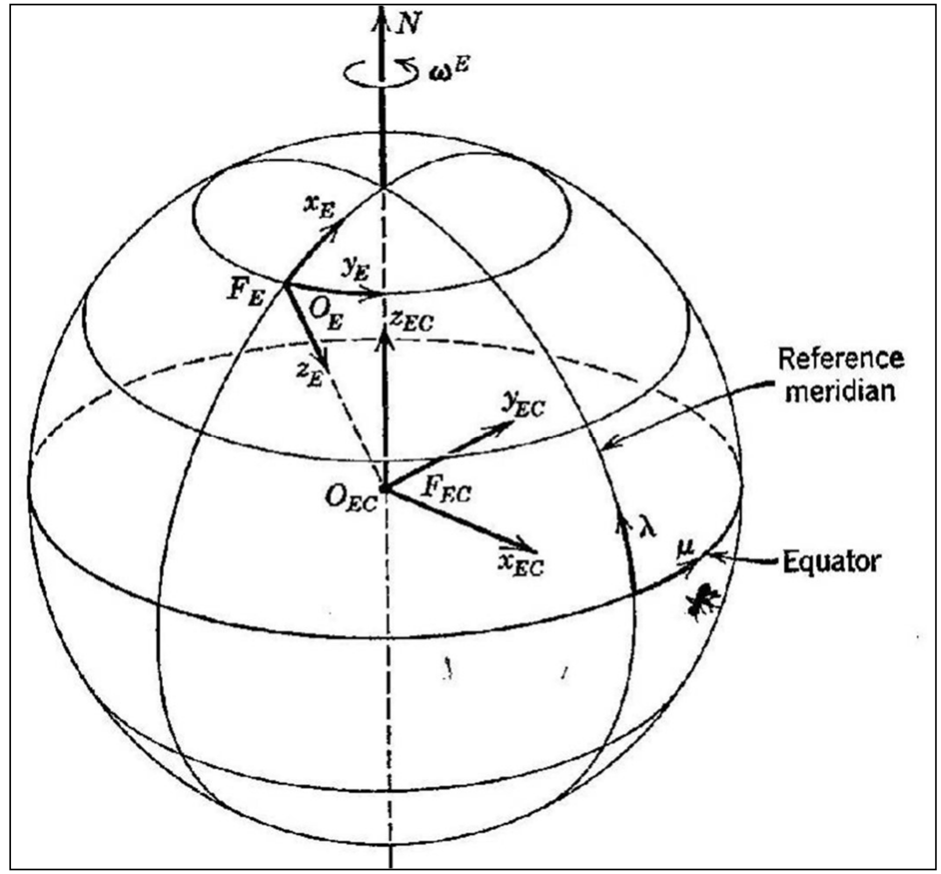
It is to be noted that ensuring of static stability **only** guarantees initial tendency to **return** to original trim and **there** is no information about either path or time duration.

In addition, we need to know if there are any **unstable**/ undesirable points along the **path** (trajectory).

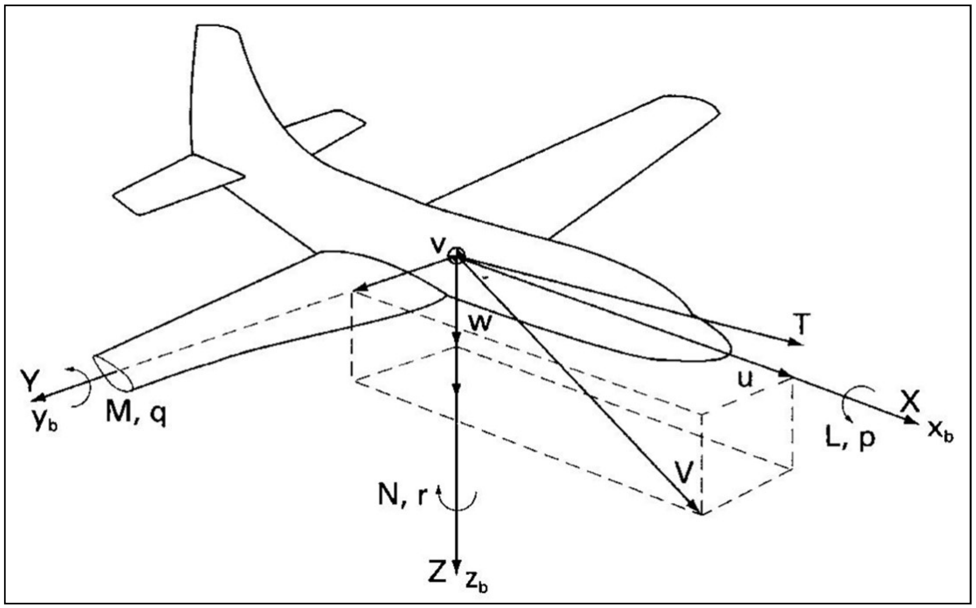
Above information helps us to either avoid such **points** or use a suitable control for **achieving** a stable and desirable **trajectory**, while going from one trim to another.

Applicable Axes Systems



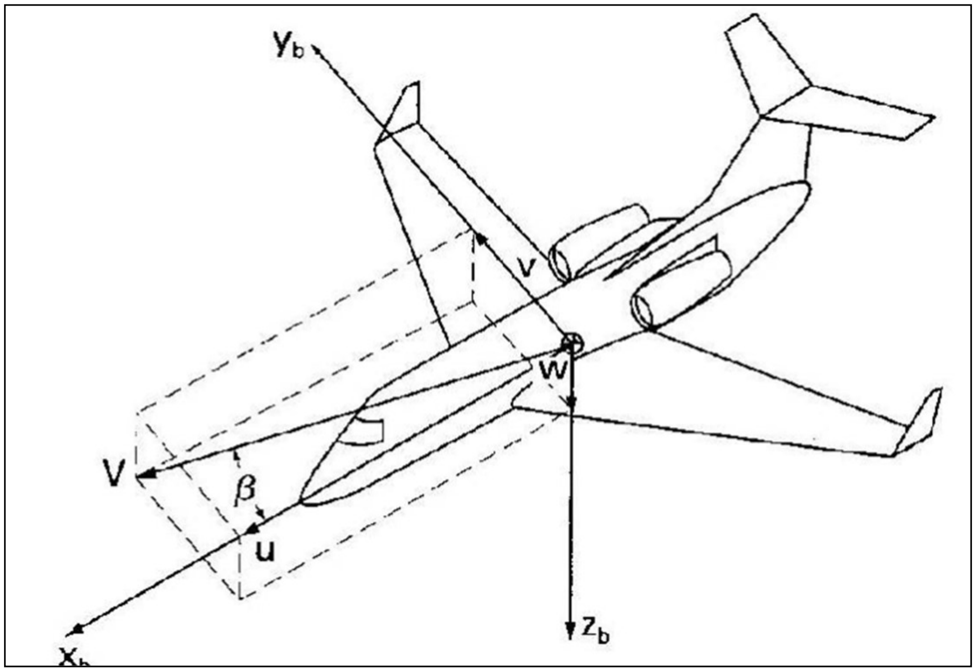


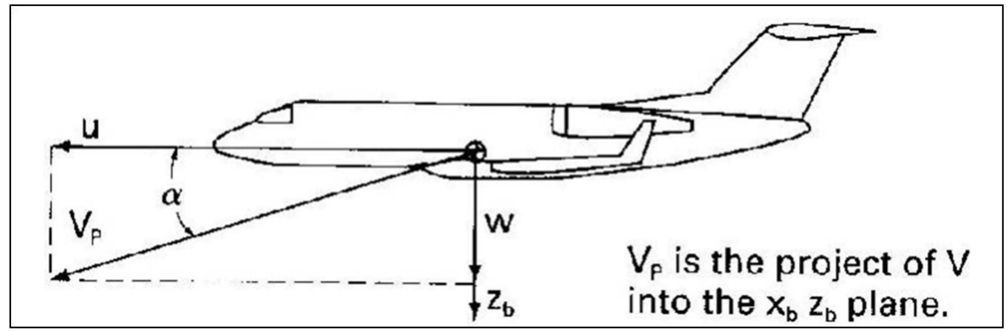
Body-fixed Frame & Motion Variables



	Roll Axis x_b	Pitch Axis y_b	Yaw Axis z_b
Angular rates	p	q	r
Velocity components	u	v	w
Aerodynamic force components	X	Y	Z
Aerodynamic moment components	L	M	N
Moment of inertia about each axis	I_x	I_y	I_z
Products of inertia	I_{yz}	I_{xz}	I_{xy}

Wind Angles in Body Frame

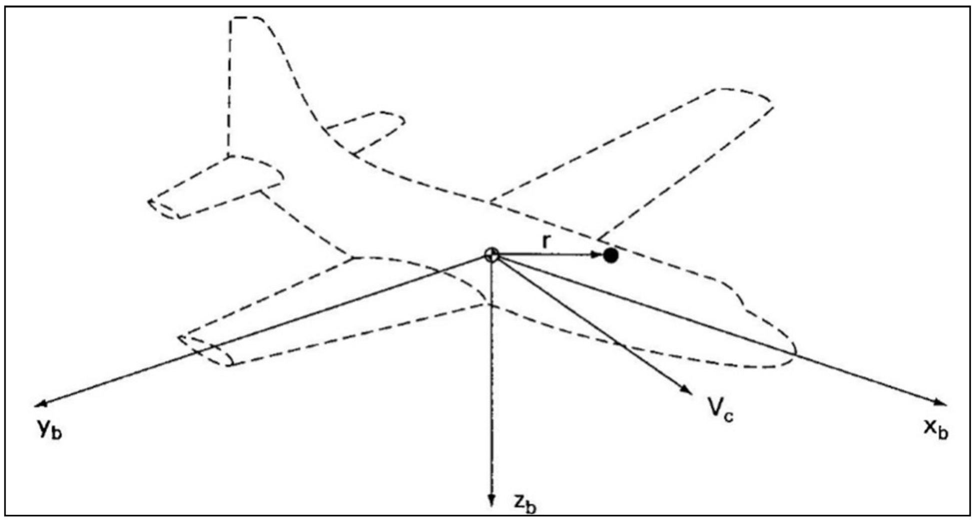




$$\alpha = \tan^{-1}\frac{w}{u}, \quad \beta = \sin^{-1}\frac{v}{V}$$

$$V = \sqrt{u^2 + v^2 + w^2}$$

Aircraft Dynamics Formulation



$$\delta\vec{F} = \delta m \frac{d\vec{v}}{dt}, \quad \vec{v} = \vec{v}_c + \frac{d\vec{r}}{dt}, \quad \sum \delta\vec{F} = \vec{F}$$

$$\vec{F} = \frac{d}{dt}\left(\sum \vec{v}\delta m\right) = m\frac{d\vec{v}_c}{dt} + \frac{d^2}{dt^2}\left(\sum \vec{r}\delta m\right) = m\frac{d\vec{v}_c}{dt}$$

$$\delta\vec{M} = \frac{d}{dt}\delta\vec{H} = \frac{d}{dt}(\vec{r}\times\vec{v})\delta m; \quad \vec{v} = \vec{v}_c + \vec{\omega}\times\vec{r}$$

$$\vec{H} = \sum \delta\vec{H} = \sum(\vec{r}\times\vec{v}_c)\delta m + \sum\left[\vec{r}\times(\vec{\omega}\times\vec{r})\right]\delta m$$

$$= \sum \vec{r}\times(\vec{\omega}\times\vec{r})\delta m; \quad \vec{\omega} = p\hat{i} + q\hat{j} + r\hat{k}, \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Scalar Expressions for Angular Momentum

$$\vec{H} = \sum \vec{r} \times (\vec{\omega} \times \vec{r})\,\delta m = \left(p\hat{i} + q\hat{j} + r\hat{k}\right)\sum\left(x^2 + y^2 + z^2\right)\delta m$$

$$-\sum\left(x\hat{i} + y\hat{j} + z\hat{k}\right)(px + qy + rz)\,\delta m \text{ (Triple Product Rule)}$$

$$\left[\text{Hint: } \vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\left(\vec{a}\cdot\vec{b}\right)\right]$$

$$H_x = p\sum\left(y^2 + z^2\right)\delta m - q\sum xy\,\delta m - r\sum zx\,\delta m$$

$$= pI_{xx} - qI_{xy} - rI_{zx}$$

$$H_y = -p\sum xy\,\delta m + q\sum\left(z^2 + x^2\right)\delta m - r\sum yz\,\delta m$$

$$= -pI_{xy} + qI_{yy} - rI_{yz}$$

$$H_z = -p\sum zx\,\delta m - q\sum yz\,\delta m + r\sum\left(x^2 + y^2\right)\delta m$$

$$= -pI_{zx} - qI_{yz} + rI_{zz}$$

$$\delta m = \rho\, dx\,dy\,dz$$

$$I_{xx} = \sum\left(y^2 + z^2\right)\rho\, dx\,dy\,dz = \iiint\left(y^2 + z^2\right)\rho\, dx\,dy\,dz$$

$$I_{xy} = \sum xy\,\rho\, dx\,dy\,dz = \iiint xy\,\rho\, dx\,dy\,dz$$

$$I_{yy} = \sum\left(z^2 + x^2\right)\rho\, dx\,dy\,dz = \iiint\left(z^2 + x^2\right)\rho\, dx\,dy\,dz$$

$$I_{zx} = \sum zx\,\rho\, dx\,dy\,dz = \iiint zx\,\rho\, dx\,dy\,dz$$

$$I_{zz} = \sum\left(x^2 + y^2\right)\rho\, dx\,dy\,dz = \iiint\left(x^2 + y^2\right)\rho\, dx\,dy\,dz$$

$$I_{yz} = \sum yz\,\rho\, dx\,dy\,dz = \iiint yz\,\rho\, dx\,dy\,dz$$

In general, we have two options for the body **frame**.

1. **Aligned** with respect to Inertial frame.
2. Allowed to **rotate** with the body (Preferred Option).

Equations of Motion in Rotating Frame

$$\vec{F} = m\frac{d\vec{v}_c}{dt}\Big|_I = m\frac{d\vec{v}_c}{dt}\Big|_B + m\left(\vec{\omega}\times\vec{v}_c\right); \quad \vec{M} = \frac{d\vec{H}}{dt}\Big|_I = \frac{d\vec{H}}{dt}\Big|_B + \left(\vec{\omega}\times\vec{H}\right)$$

$$X = m\left(\dot{u} + qw - rv\right)$$

$$Y = m\left(\dot{v} + ru - pw\right)$$

$$Z = m\left(\dot{w} + pv - qu\right)$$

$$L = I_{xx}\dot{p} - I_{zx}\dot{r} - qr\left(I_{yy} - I_{zz}\right) - I_{zx}pq$$

$$M = I_{yy}\dot{q} - pr\left(I_{zz} - I_{xx}\right) + I_{zx}\left(p^2 - r^2\right)$$

$$N = I_{zz}\dot{r} - I_{zx}\dot{p} - pq\left(I_{xx} - I_{yy}\right) + I_{zx}qr$$

Note: General 6-DOF Aircraft Dynamic Model

Kinematic Model Concept

The 6-DOF model is a set of six coupled nonlinear **first** order ODEs that describe **completely**, {u, v, w} and {p, q, r} **in** the body frame.

However, in general, final outputs of interest are **position** and orientation of **aircraft** with respect to the 'I' frame, **requiring** a transformation between these two frames.

In this context, it is known that (u, v, w) and (p, q, r) are measured along instantaneous **directions** of (x, y, z), which are **not** aligned with respect to 'I' frame.

Concept of Euler Angles

Further, we find that these directions also keep **changing** due to the presence of **angular** rates and, thus, we need a **strategy** to convert B-frame quantities to I-frame.

In order to determine the inertial velocity, we need to **know** the body axes orientation **relative** to the inertial axes, which **can** be obtained by integrating the inertial attitude rates.

This has resulted in a procedure that first defines **inertial** attitude, through Euler angles, and **then** generates their rates, **which** are subsequently connected to body rates, (p, q, r).

Euler Angle Definition

Orientation of B-frame with respect to I-frame is **described** using 3 angles; ψ (yaw angle), θ (**pitch** angle) and ϕ (roll **angle**), which are about inertial 'z', 'y' and 'x' axes.

Here, basic premise is that if origin of the I-frame **coincides** with that of B-frame, there is a **unique** set of angles (ϕ, θ, ψ) for **which**, the I-frame will coincide with the B-frame**.**

Euler angles, thus, are an ordered set (or a **vector**) of sequential rotations from an **I-frame** to B-frame**,** and there are **six** possible orders that one can employ.

Euler Angle Ordering and Features

However, the most common ordering is **3-2-1**, in which sequence of rotation is; First **rotate** by ‘ψ’, then by ‘θ’ and **lastly** by ‘ϕ’.

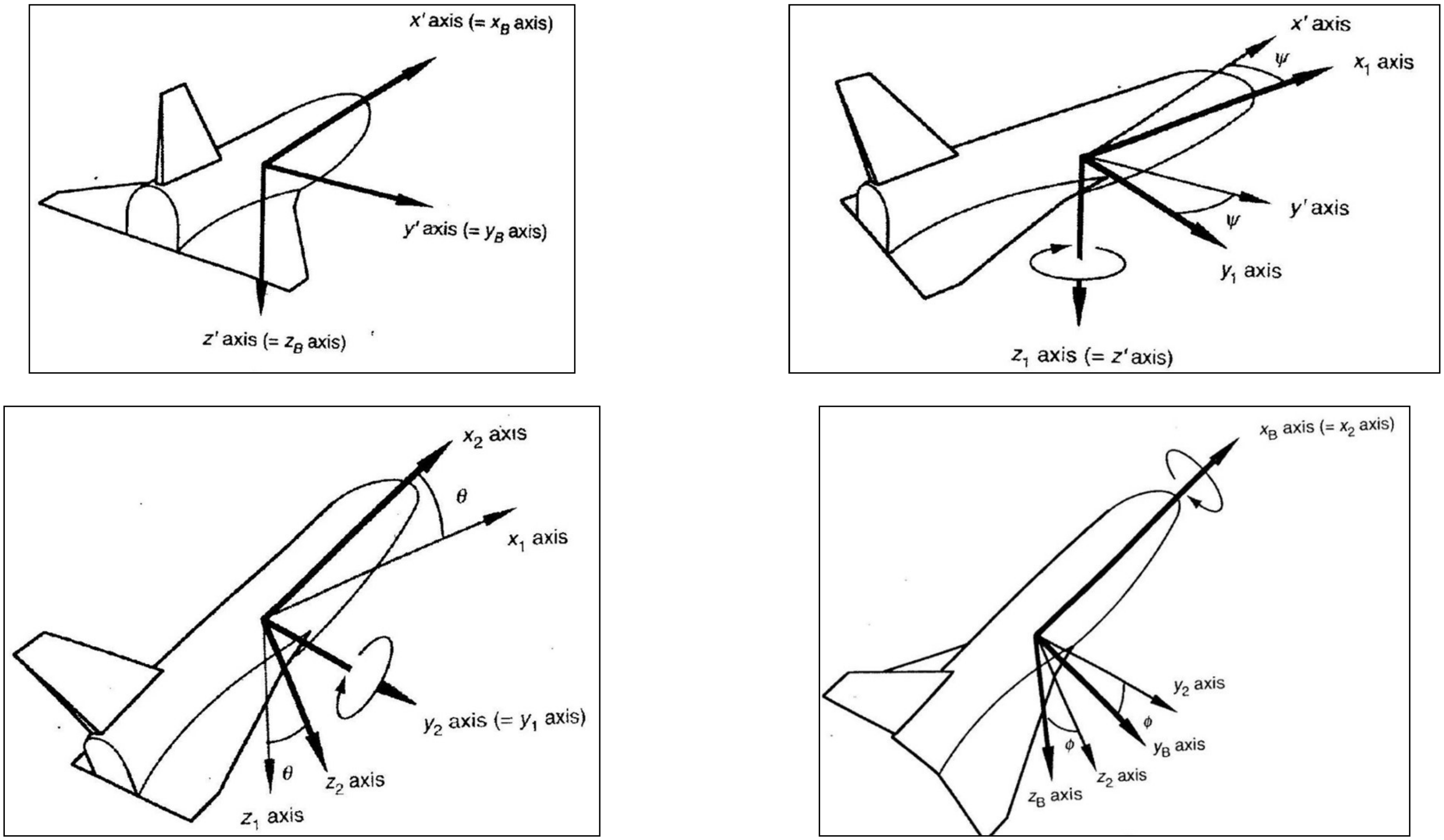
The process starts by initially aligning both **inertial** and body frames with each **other**, which is followed by;

(a) **Rotation** about inertial original z-axis, by angle ψ.

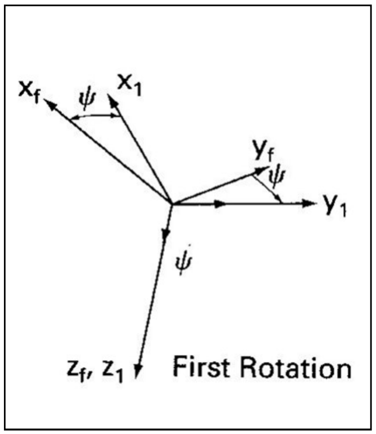
(b) Next, **rotation** about new y-axis by angle θ.

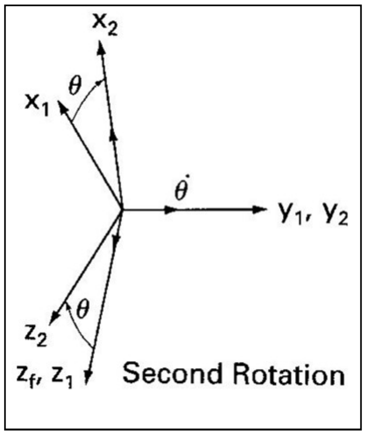
(c) Lastly, rotation **about** new x-axis by angle ϕ.

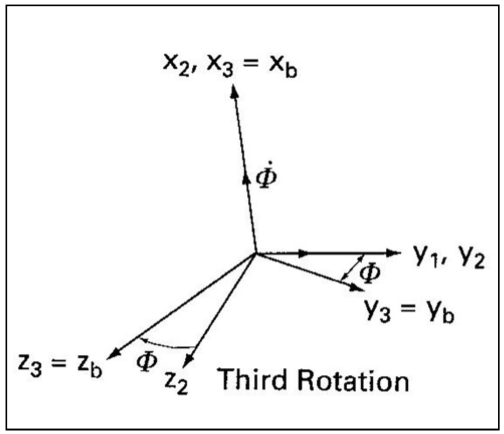
3-2-1 Sequence for an Aircraft



Applicable Unitary & Full Transformations







$$H_I^1(\psi) = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_1^2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

$$H_2^B(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

$$H_I^B(\phi,\theta,\psi) = H_2^B(\phi) \times H_1^2(\theta) \times H_I^1(\psi)$$

$$= \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ (-c\phi s\psi + s\phi s\theta c\psi) & (c\phi c\psi + s\phi s\theta s\psi) & s\phi c\theta \\ (s\phi s\psi + c\phi s\theta c\psi) & (-s\phi c\psi + c\phi s\theta s\psi) & c\phi c\theta \end{bmatrix}$$

$$H_B^I = \left[H_I^B\right]^{-1} = \left[H_I^B\right]^T \rightarrow \text{ Orthonormal Transformation}$$

Body Rate – Inertial Rate Scalar Relations

$$\{\dot{\vec{\Theta}}\} \equiv \left\{\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}\right\}^T \rightarrow \text{I-Frame Rates;} \quad \left\{\vec{\omega}\right\} = \{p \quad q \quad r\}^T \rightarrow \text{B-Frame Rates}$$

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \equiv I_n \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} + H_2^B \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} + H_2^B H_1^2 \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & s\phi tn\theta & c\phi tn\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi sc\theta & c\phi sc\theta \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

Inertial Rates from Body rates: $\dot{\phi} = (1)p + (\sin\varphi \tan\theta)q \; + (\cos\phi \tan\theta)r$

$$\dot{\theta} = (0)p - (\cos\phi)q \qquad - (\sin\phi)r$$

$$\dot{\psi} = (0)p + (\sin\phi \sec\theta)q \; + (\cos\phi \sec\theta)r$$

Body Rates from Inertial Rates: $p = (1)\dot{\phi} + (0)\dot{\theta} \qquad - (\sin\theta)\dot{\psi}$

$$q = (0)\dot{\phi} - (\cos\phi)\dot{\theta} \; - (\cos\theta \sin\phi)\dot{\psi}$$

$$r = (0)\dot{\phi} - (\sin\phi)\dot{\theta} \; + (\cos\theta \cos\phi)\dot{\psi}$$

Aircraft Velocity in I-frame

$$
\begin{aligned}
&\dot{x}_I = u_1 \cos\psi - v_1 \sin\psi; \quad \dot{y}_I = u_1 \sin\psi + v_1 \cos\psi; \quad \dot{z}_I = w_1 \\
&u_1 = u_2 c\theta + w_2 s\theta; \quad v_1 = v_2; \quad w_1 = -u_2 s\theta + w_2 s\theta \\
&u_2 = u; \quad v_2 = vc\phi - ws\phi; \quad w_2 = vs\phi + wc\phi \\
&\dot{x}_I = (\mathrm{c}\theta\,\mathrm{c}\psi)u + (-\mathrm{c}\phi\,\mathrm{s}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi)v + (\mathrm{s}\phi\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi)w \\
&\dot{y}_I = (\mathrm{c}\theta\,\mathrm{s}\psi)u + (\mathrm{c}\phi\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi)v + (-\mathrm{s}\phi\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi)w \\
&\dot{z}_I = (-\mathrm{s}\theta)u + (\mathrm{s}\phi\,\mathrm{c}\theta)v + (\mathrm{c}\phi\,\mathrm{c}\theta)w
\end{aligned}
$$

Complete System of Dynamic Equations

Given alongside are the complete set of 12 **nonlinear** 1st order ODEs that are applicable for the **general** dynamics of any aircraft.

Thus, we see that solution of these equations is **possible** through either iterative or numerical **procedures** only.

However, useful results can be obtained by **considering** disturbances to be small departures from the **trim** condition.

$$X = m(\dot{u} + qw - rv); \quad Y = m(\dot{v} + ru - pw); \quad Z = m(\dot{w} + pv - qu)$$

$$\vec{X} = \vec{X}_g + \vec{X}_T + \vec{X}_A; \quad \vec{Y} = \vec{Y}_g + \vec{Y}_T + \vec{Y}_A; \quad \vec{Z} = \vec{Z}_g + \vec{Z}_T + \vec{Z}_A$$

$$\vec{L} = \vec{L}_T + \vec{L}_A; \quad \vec{M} = \vec{M}_T + \vec{M}_A; \quad \vec{N} = \vec{N}_T + \vec{N}_A$$

$$L = I_{xx}\dot{p} - I_{zx}\dot{r} - qr\left(I_{yy} - I_{zz}\right) - I_{zx}pq$$

$$M = I_{yy}\dot{q} - pr\left(I_{zz} - I_{xx}\right) + I_{zx}\left(p^2 - r^2\right)$$

$$N = I_{zz}\dot{r} - I_{zx}\dot{p} - pq\left(I_{xx} - I_{yy}\right) + I_{zx}qr$$

$$\dot{\phi} = p + (\mathrm{s}\varphi\,\mathrm{tn}\,\theta)q + (\mathrm{c}\phi\,\mathrm{tn}\,\theta)r; \quad \dot{\theta} = (\mathrm{c}\phi)q - (\mathrm{s}\phi)r$$

$$\dot{\psi} = (s\phi\,\mathrm{sc}\,\theta)q + (\mathrm{c}\phi\,\mathrm{sc}\,\theta)r$$

$$\dot{x}_I = \left(\mathrm{c}\theta\,\mathrm{c}\psi\right)u + \left(-\mathrm{c}\phi\,\mathrm{s}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi\right)v + \left(\mathrm{s}\phi\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi\right)w$$

$$\dot{y}_I = \left(\mathrm{c}\theta\,\mathrm{s}\psi\right)u + \left(\mathrm{c}\phi\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi\right)v + \left(-\mathrm{s}\phi\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi\right)w$$

$$\dot{z}_I = \left(-\mathrm{s}\,\theta\right)u + \left(\mathrm{s}\phi\,\mathrm{c}\theta\right)v + \left(\mathrm{c}\phi\,\mathrm{c}\theta\right)w$$