AE 339 : High speed aerodynamics

(Module I : Isentropic Flows)

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What is a 'compressible flow'?

Density changes w.r.t. pressure are significant

$\rho = \rho(p, T)$ $\qquad \frac{1}{\rho}\frac{d\rho}{dp}$ compressibility

Temperature changes appreciably $\Rightarrow$ we need to worry about thermodynamics in addition to the flow dynamics

Primarily gases (gas dynamics)

High vs low speeds

Speed of sound is the reference chosen to compare velocities (a)

$$\text{Mach number } (M) = \frac{\text{gas velocity}}{\text{speed of sound}} = \frac{V}{a}$$

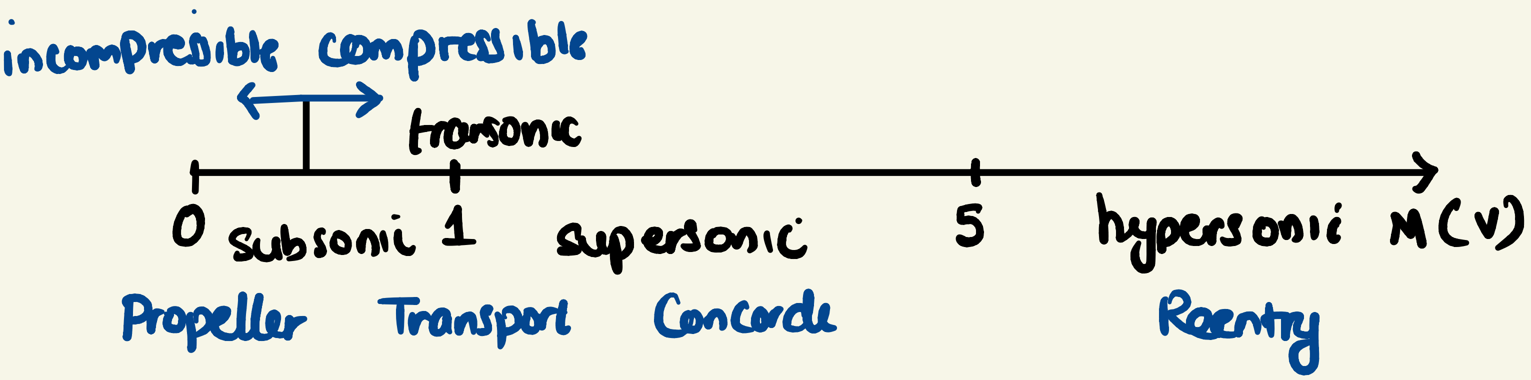
Regimes covered

(1) High speed subsonic flow

(2) Transonic flow

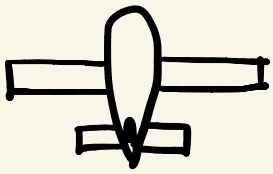
(3) Supersonic flow

(4) Hypersonic flow

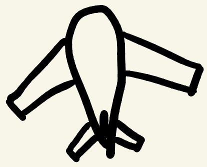


Aircrafts

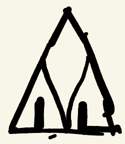
Propeller (low speed)
rounded nose
unswept wings

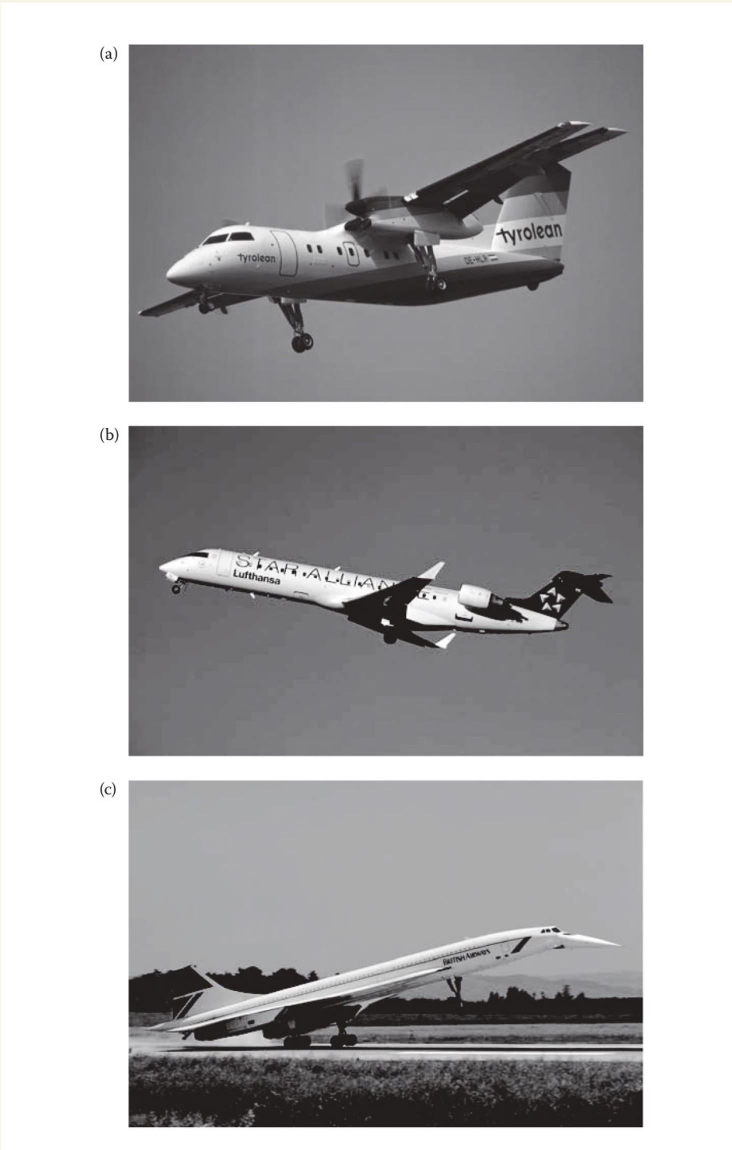


Transport (higher speeds)
rounded nose
swept wings



Concorde (even higher)
sharp nose
more sweep





low subsonic

transonic

M

supersonic

Compressible flow effects play a major role in aircraft design

Fundamental assumptions

1. Gas is continuous — Mean free path λ < flow dimensions

2. No chemical changes — No reactions, ionizations, dissociations

3. Gas is perfect

$$p = \rho R T$$ (perfect / ideal gas equation)

$$R = \frac{R_u}{M}$$

R_u = universal gas constant
M = molecular weight

$$= \frac{8314.3}{28.966} = 287.04\ J/kgK$$ (for air)

Calorically perfect — C_p, C_v are constants

$\gamma = C_p / C_v$, $C_p - C_v = R$

Thermally perfect — C_p, C_v are fn's of T

4. Gravitational effects are negligible
No gravitational potential

5. Magnetic & electric effects not considered

6. Effect of viscosity negligible
except close to bodies and shocks

$$F = \mu A \frac{dv}{dy}$$

μ → small; $\frac{dv}{dy}$: this is also small

Under these assumptions, the flow is completely described in terms of following variables

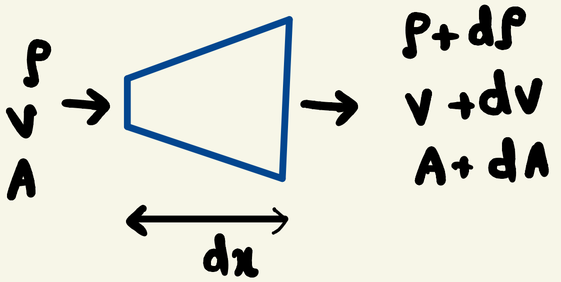
(a) V (b) p (c) ρ (d) T

Conservation laws

Additional assumptions:

1. Flow is steady
2. Not considering cases where gas does work

Differential control volume



We neglect $dV \cdot d\rho$ and other second and higher order terms

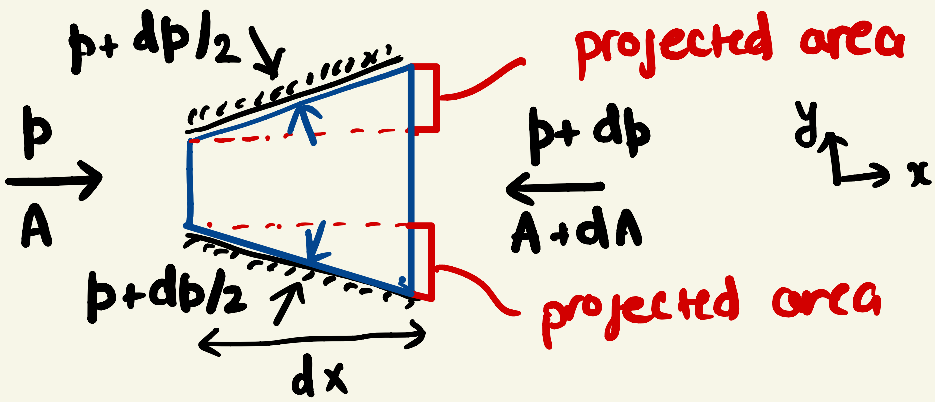
Continuity equation

$$\rho \cancel{V} A = (\rho + d\rho)(V + dV)(A + dA)$$

$$= \rho V dA + A V d\rho + \rho A dV + \rho \cancel{V} A$$

$$\boxed{\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0}$$

Momentum equation (Euler's equation)



Net forces

$$pA - (p+dp)(A+dA) + \left(p+\frac{dp}{2}\right)dA$$

$$= \cancel{pA} - \cancel{pA} - p\cancel{dA} - A\,dp + p\cancel{dA} = -A\,dp$$

Momentum flux

$$\rho v A[(v+dv)-v] = \rho v A\,dv$$

we have

$$\rho v \cancel{A}\,dv = -\cancel{A}\,dp$$

$$\boxed{-\frac{dp}{\rho} = v\,dv}$$

Euler's equation

If velocity increases pressure decreases and vice versa.

For incompressible flow, $\rho =$ const.

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{const.}$$

Bernoulli's equation

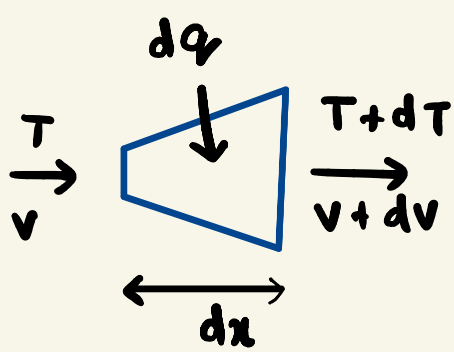
For general flows, we need to know $\rho = \rho(p)$ → Barotropic flows

$$\frac{v^2}{2} + \int \frac{dp}{\rho} = \text{const.}$$

Energy equation

$$h + dh + \frac{(v+dv)^2}{2} = h + \frac{v^2}{2} + dq$$

Now, $h = C_p T$



$$c_p(T+dT) + \frac{(v+dv)^2}{2} = c_p T + \frac{v^2}{2} + dq$$

$$\cancel{c_p T} + c_p\,dT + \cancel{\frac{v^2}{2}} + v\,dv = \cancel{c_p T} + \cancel{\frac{v^2}{2}} + dq$$

$$\boxed{c_p\,dT + v\,dv = dq}$$

Adiabatic flow : $dq = 0$

$$c_p\,dT + v\,dv = 0$$

$$\underbrace{c_p T}_{h} + \frac{v^2}{2} = \text{const}$$

If velocity increases temperature decreases and vice versa.

State equation

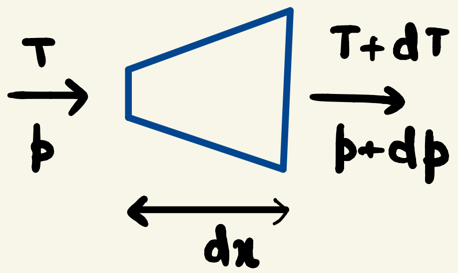
$$\frac{p}{\rho T} = R = \text{const.}$$

$$\cancel{\frac{p}{\rho T}} = \frac{p+dp}{(\rho+d\rho)(T+dT)}$$

$$= \cancel{\frac{p}{\rho T}}\left[\left(1+\frac{dp}{p}\right)\left(1-\frac{d\rho}{\rho}\right)\left(1-\frac{dT}{T}\right)\right]$$

$$\boxed{\frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0}$$

Entropy considerations



$$T\,ds = dh - dp/\rho$$

$$= C_p dT - dp/\rho$$

$$ds = C_p \frac{dT}{T} - R\frac{dp}{p}$$

$$s_2 - s_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$\frac{s_2 - s_1}{C_p} = \ln\frac{T_2}{T_1} - \frac{\gamma-1}{\gamma}\ln\frac{p_2}{p_1}$$

$$= \ln\left[\left(\frac{T_2}{T_1}\right)\left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

$$C_p - C_v = R$$

$$C_p/C_v = \gamma$$

$$C_p = \frac{R\gamma}{\gamma-1}$$

$$C_v = \frac{R}{\gamma-1}$$

Isentropic flow : $ds = s_2 - s_1 = 0$

$$\boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}}$$

Relation b/n p & T

From the state relation, we have

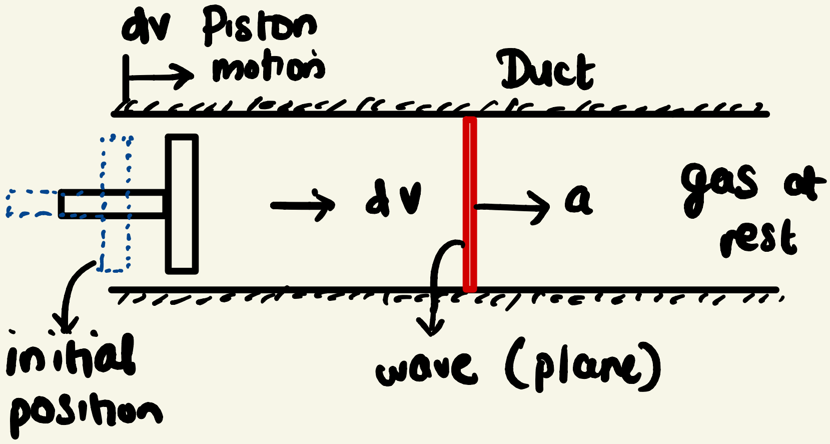
$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\cdot\frac{\rho_1}{\rho_2} \Rightarrow \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{p_2}{p_1}\cdot\frac{\rho_1}{\rho_2}$$

$$\boxed{\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}}$$

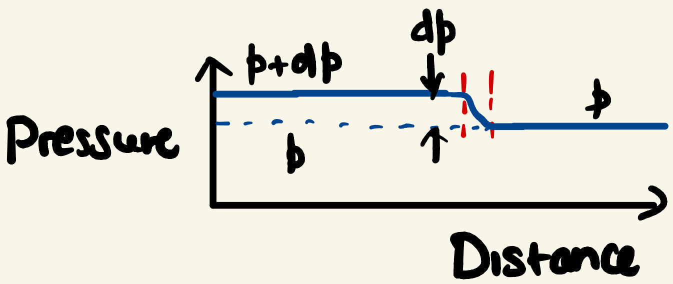
Relation b/n p & ρ

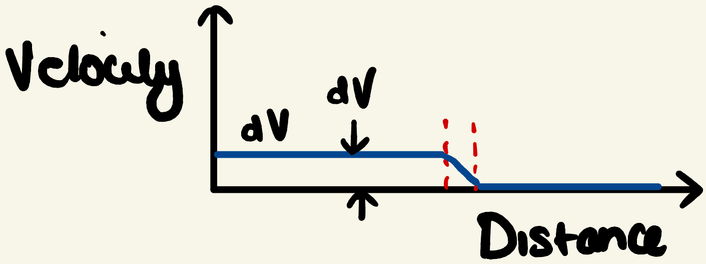
Speed of sound

Speed at which very weak pressure waves are transmitted through the gas.

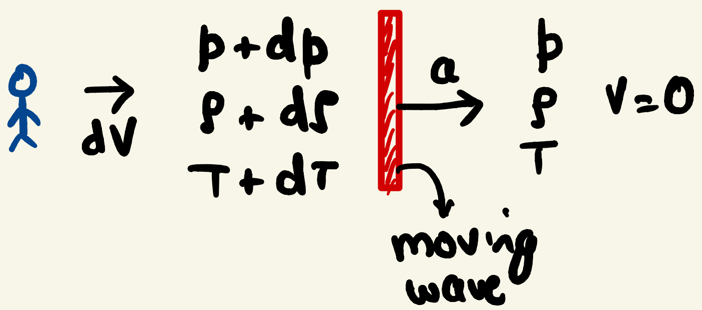


We have a wave propagating through the duct.

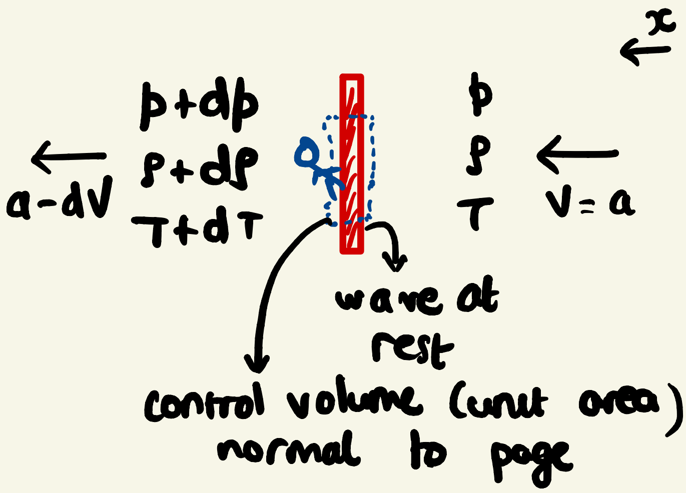




Lab frame



Wave frame



Continuity

$$\frac{\dot{m}}{A} = \rho a = (\rho + d\rho)(a - dV)$$

$$d\rho\, a - \rho\, dV = 0$$

$$d\rho = \frac{\rho}{a} dV \quad - \quad (1)$$

Momentum

$$pA - (p + dp)A = \dot{m}[(a - dV) - a]$$

$$-dp = \frac{\dot{m}}{A}(-dV)$$

$$dp = \rho a\, dV \quad - \quad (2)$$

② / ① $\Rightarrow \dfrac{dp}{d\rho} = \dfrac{\rho a}{\rho / a} = a^2$

$$\boxed{a = \sqrt{\frac{dp}{d\rho}}}$$

To evaluate this expression, we need to know the thermodynamic process the gas undergoes in passing through the wave

Wave is weak $\Rightarrow$ dv, dT are small
i.e. gradients of velocity and temperature are negligible $\Rightarrow$ heat transfer and viscous effects on the flow through the wave are negligible.

Gas undergoes an isentropic process.

For isentropic sound propogation,

$$\frac{p}{\rho^{\gamma}} = \text{const.} = c$$

$$\left.\frac{dp}{d\rho}\right)_s = \gamma c \rho^{\gamma-1} = \frac{\gamma p}{\rho} = \gamma R T$$

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}}$$

Speed of sound depends only on the absolute temperature for a given gas.

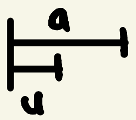
Mach waves

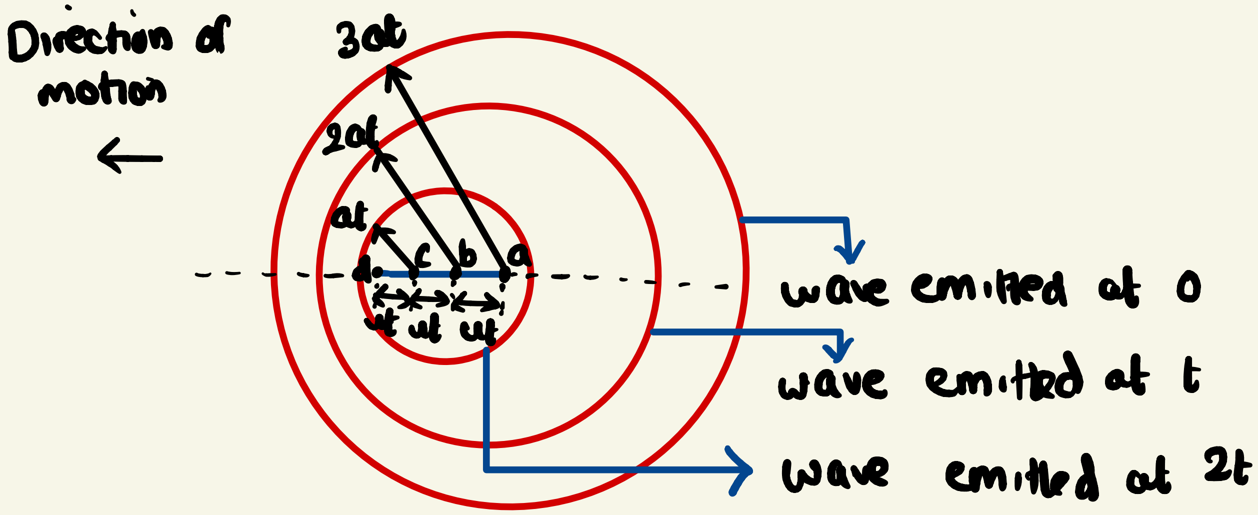
For the gas to move smoothly over a body, disturbances propagate ahead of the body to 'warn' the gas of the approach of the body.

Pressure at the surface of the body $>$ Pressure of the gas nearby

$\Rightarrow$ The disturbances (pressure waves) propagate at the speed of sound.

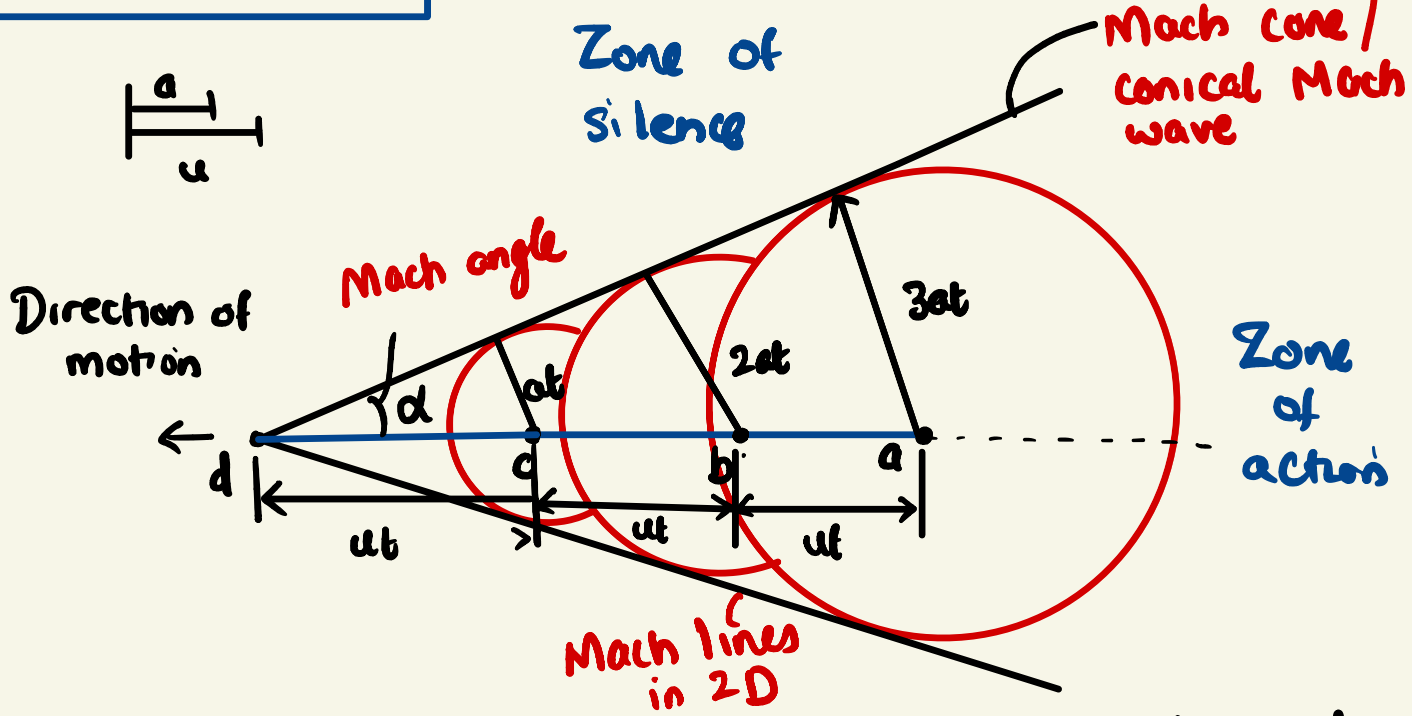
$\boxed{\text{Case 1: } M < 1}$





waves emitted at times 0, t, 2t as observed at time 3t when the body has moved from a-d.

Case 2 : $M > 1$



waves emitted at times 0, t, 2t as observed at time 3t when the body has moved from a-d

$$\sin\alpha = \frac{at}{ut} = \frac{1}{M}$$

$$\boxed{\alpha = \sin^{-1}\frac{1}{M}}$$ Mach angle

Used in the measurement of Mach numbers of a gas flow.

Solving 1D isentropic flow problems

we have

$$\frac{p}{\rho^\gamma} = \text{const.} \qquad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma$$

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1}\cdot\frac{\rho_1}{\rho_2} \ ; \quad a = \sqrt{\gamma R T}$$

$$\boxed{\frac{a_2}{a_1} = \left(\frac{T_2}{T_1}\right)^{1/2} = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{\gamma-1}{2}} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}}$$

From energy equation,

$$C_p T_1 + \frac{V_1^2}{2} = C_p T_2 + \frac{V_2^2}{2}$$

$$T_1\left(1 + \frac{V_1^2}{2C_p T_1}\right) = T_2\left(1 + \frac{V_2^2}{2C_p T_2}\right)$$

$$\frac{T_2}{T_1} = \frac{1 + \frac{V_1^2(\gamma-1)}{2\gamma R T_1}}{1 + \frac{V_2^2(\gamma-1)}{2\gamma R T_2}}$$

$$\boxed{\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}}$$

This equation applies in adiabatic flow — ①

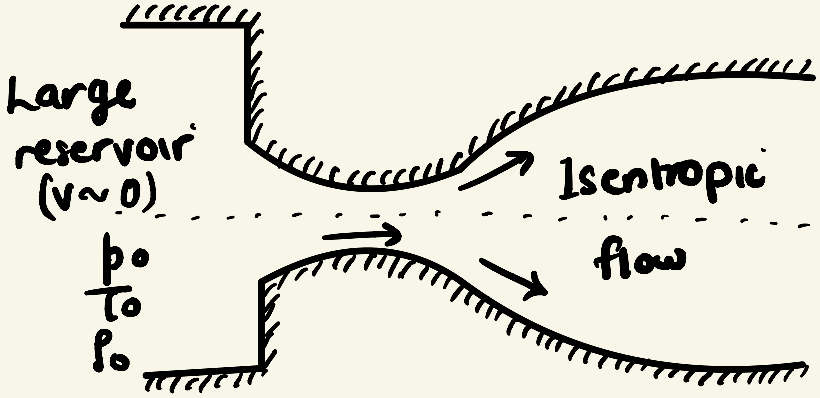
$$\boxed{\frac{p_2}{p_1} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{\gamma}{\gamma-1}}}$$ — ②

Only valid for isentropic flows

$$\boxed{\frac{\rho_2}{\rho_1} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{1}{\gamma-1}}}$$ — ③

Eqns. ① - ③ sufficient to determine all features of 1D isentropic flow

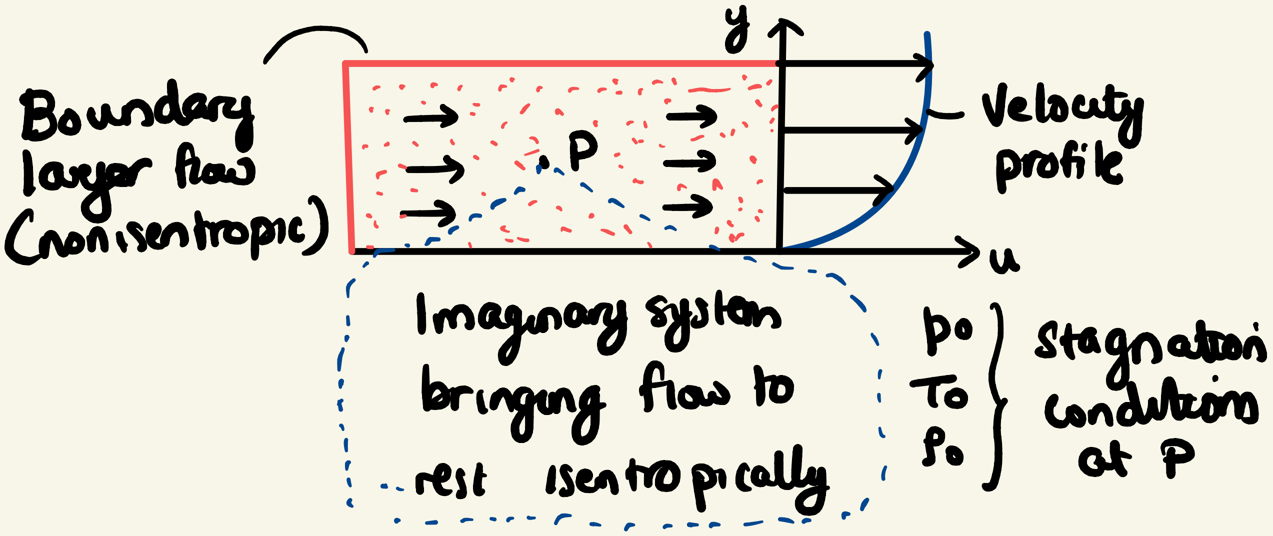
Stagnation conditions



Stagnation condition at all points is p_0, ρ_0, T_0

Stagnation conditions are those that would exist if the flow at any point is brought to rest **isentropically**

For an isentropic flow, the stagnation conditions at all points in the flow will be those existing at the zero velocity point



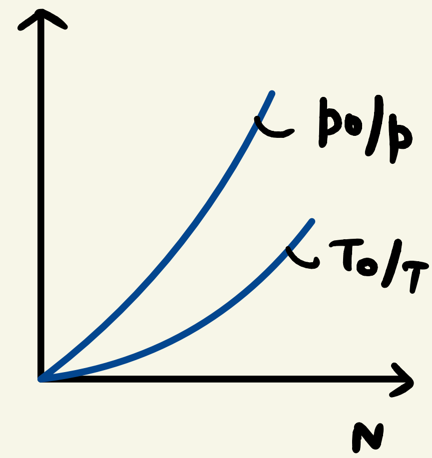
Even for nonisentropic flows, we can imagine stagnation conditions at any local point

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{p_0}{p} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

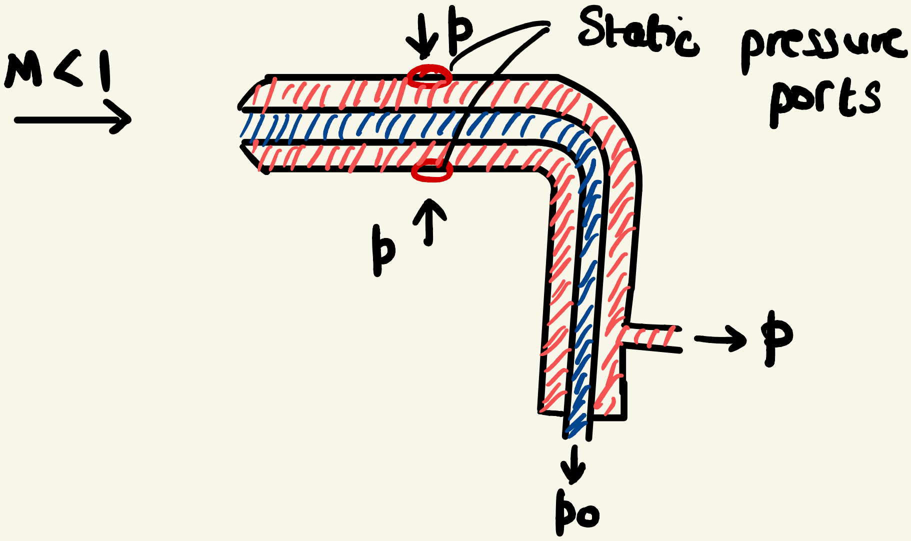
$$\frac{\rho_0}{\rho} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}}$$

For air, $\gamma = 1.4$



Pitot - static tube

Pitot - static can be used to measure Mach number in subsonic flow.



For incompressible flows,

$$p_0 = p + \frac{1}{2} \rho V_{inc}^2$$

$$V_{inc} = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

For compressible flows,

$$\frac{p_0}{p} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\Rightarrow M = \sqrt{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

$$V_{com} = M a = M\sqrt{\gamma R T}$$

To determine velocity in incompressible flow, only $\Delta p\,(p_0 - p)$ has to be measured

For compressible flows, to find the Mach number, p_0 & p have to be separately measured and for velocity, temperature also has to be measured (for the speed of sound)

Error using incompressible formula.

$$\epsilon = \left|\frac{V_{com} - V_{inc}}{V_{com}}\right| = \left|1 - \frac{V_{inc}}{V_{com}}\right|$$

For $M \sim 0.3$, $\epsilon < 1\%$

For $M \sim 0.6$, $\epsilon \sim 5\%$