AE 341: Flight mechanics of aircrafts and spacecrafts

Complete equations of motion

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Aircraft equations of motion

- Rigid body
- Earth is an inertial reference frame
- Body fixed coordinate system is rotating and accelerating
- C.G. moving with V_c
- Body rotating with ω_c
- R is the position vector of the CM
- r_c is the position vector of any point w.r.t. CM

Translation

- $V = V_c + \omega_c \times r_c$
- Easy to see that momentum of a/c: $p = mV_c$
- $F = m\frac{dV_c}{dt}$
- Most important: $\frac{dA}{dt}\Big|_I = \frac{dA}{dt}\Big|_B + \omega_c \times A$

$$\omega_c \times V = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Translation of body

$$F = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + m \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- Forces: gravity, aerodynamic, propulsive

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}_E + \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}_B + \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}_B$$

Aside: coordinate transformations

- Consider a coordinate system change
- Look at rotation of the axis by ψ
- Look at unit basis vectors
- Compare/contrast with components of the vector

Euler angles

- Given the above, we need to convert between frame E and B
- Euler angles: ϕ, θ, ψ
- Sequence: 321, rotate about $z : \psi$, $y : \theta$, $x : \phi$
- $X^E, Y^E, Z^E \xrightarrow{\psi} X^1 Y^1 Z^1$

$$\begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^1 \\ Y^1 \\ Z^1 \end{bmatrix} = R_\psi \begin{bmatrix} X^1 \\ Y^1 \\ Z^1 \end{bmatrix}$$

Euler angles ...

- $X^1, Y^1, Z^1 \xrightarrow{\theta} X^2 Y^2 Z^2$

$$\begin{bmatrix} X^1 \\ Y^1 \\ Z^1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X^2 \\ Y^2 \\ Z^2 \end{bmatrix} = R_\theta \begin{bmatrix} X^2 \\ Y^2 \\ Z^2 \end{bmatrix}$$

Euler angles ...

- $X^2, Y^2, Z^2 \xrightarrow{\phi} X^B Y^B Z^B$

$$\begin{bmatrix} X^2 \\ Y^2 \\ Z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} = R_\phi \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix}$$

Euler angles ...

$$\begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} = R_\psi R_\theta R_\phi \begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix}$$

$$\begin{bmatrix} X^B \\ Y^B \\ Z^B \end{bmatrix} = R_\phi^T R_\theta^T R_\psi^T \begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix}$$

Exercise

- Write the components of weight in body axis

Wind axis to inertial frame

$$\begin{bmatrix} X^E \\ Y^E \\ Z^E \end{bmatrix} = R_\chi R_\gamma R_\mu \begin{bmatrix} X^W \\ Y^W \\ Z^W \end{bmatrix}$$

$$\begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & -\sin\mu \\ 0 & \sin\mu & \cos\mu \end{bmatrix}$$

p, q, r and $\dot{\phi}, \dot{\theta}, \dot{\psi}$

- Can see that $\omega_c = \dot{\psi}\hat{k}_E + \dot{\theta}\hat{j}_1 + \dot{\phi}\hat{i}_2$
- So,

$$\omega_c = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = R_\phi^T R_\theta^T R_\psi^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + R_\phi^T R_\theta^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_\phi^T \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Other transformations

- Wind axis rates w.r.t. body axis rates p_w, q_w, r_w and $\dot{\mu}, \dot{\gamma}, \dot{\chi}$
- Inertial velocity to body axis, $\dot{x}_E, \dot{y}_E, \dot{z}_E$ and u, v, w
- Wind fixed coordinates and body-fixed coordinate axes
- Body-axis and wind-axis Euler angles
- Body axis and wind axis angular rates

Linear motion

- Recall:

$$F = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}_E + \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}_B + \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}_B ; F = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + m \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- So,

$$
\begin{aligned}
\dot{u} &= rv - qw + \frac{X^A}{m} + \frac{T}{m} - g\sin\theta \\
\dot{v} &= pw - ru + \frac{Y^A}{m} + g\sin\phi\cos\theta \\
\dot{w} &= qu - pv + \frac{Z^A}{m} + g\cos\phi\cos\theta
\end{aligned}
$$

Linear motion

- With the transformation matrices
  - Can convert this to wind axis
  - Can represent in any convenient fashion

Rotational motion

- Write angular momentum vector as:

$$
h = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}
$$

- Recall

$$
\left.\frac{dh}{dt}\right|_I = \left.\frac{dh}{dt}\right|_B + \omega_c \times h
$$

Rotational motion ...

- Aircraft (usually) has symmetry and choose body axes as principal axes
- Cross moments are zero

$$
h = \begin{bmatrix} I_{xx}p \\ I_{yy}q \\ I_{zz}r \end{bmatrix}
$$

$$\begin{bmatrix} \mathcal{L} \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_{xx}p \\ I_{yy}q \\ I_{zz}r \end{bmatrix}$$

- Write this in terms of $\dot{p}, \dot{q}, \dot{r}$

Full set of equations

- 12 equations along body axes:
  - 3 translational dynamics: $\dot{u}, \dot{v}, \dot{w}$
  - 3 translational kinematics: $\dot{x}_E, \dot{y}_E, \dot{z}_E$
  - 3 rotational dynamics: $\dot{p}, \dot{q}, \dot{r}$
  - 3 rotational kinematics: $\dot{\phi}, \dot{\theta}, \dot{\psi}$
- Can also have these along wind axes: important as we can use C_L, C_D here
- Non-linear system of equations
- Can solve these: usually numerically
  - Challenge is to find the aerodynamic forces