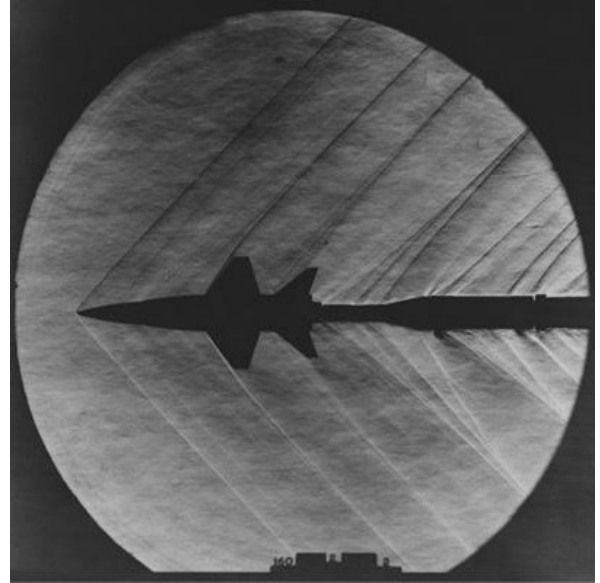# CHAPTER 2

Compressible Flow with Friction and Heat: *A Review*

Schlieren visualization of waves around an X-15 model in a supersonic wind tunnel. Source: Courtesy of NASA

2.1   Introduction

The study of propulsion is intimately linked to the understanding of internal fluid mechanics. This means that duct flows are of prime interest to propulsion. The physical phenomena in a real duct include the following effects:

- Friction
- Heat transfer through the walls
- Chemical reaction within the duct
- Area variation of the duct
- Compressibility effects, e.g. appearance of shocks

The laws of thermodynamics govern the relationship between the state variables of the gas, namely density ρ, pressure p, absolute temperature T, entropy s, internal energy e, and derived properties such as enthalpy h and specific heats at constant pressure and volume, c_p and c_v, respectively. In addition to the laws of thermodynamics, the fluid flow problems need to obey other conservation principles that were introduced in Newtonian mechanics. These are conservation of mass and momentum as described in classical mechanics. Since the study of gas turbine engines and propulsion in undergraduate curricula in mechanical and aerospace engineering follows the introductory courses in thermo-fluid dynamics, we shall review only the principles that have a direct impact on our study of jet engines.

The purpose of this chapter is thus to provide a review of the working principles in aerothermodynamics, which serve as the foundation of propulsion. The reader should consult textbooks on thermodynamics, such as the classical work of Sonntag, Borgnakke, and Van Wylen (2003), and modern fluid mechanics books, such as Munson, Young, and Okiishi (2006) or John Anderson's books on aerodynamics (2005). For detailed exposition of compressible flow, the two volumes written by Shapiro (1953) should be consulted.

2.2   A Brief Review of Thermodynamics

To get started, we need to characterize the medium that flows through the duct. In gas turbine engines the medium is a perfect gas, often air. The perfect gas law that relates the pressure, density, and the absolute temperature of the gas may be derived rigorously from the kinetic theory of gases. Prominent assumptions in its formulation are (1) intermolecular forces between the molecules are negligibly small and (2) the volume of molecules that occupy a space is negligibly small and may be ignored. These two assumptions lead us to

$$p = \rho R T \tag{2.1}$$

where R is known as the gas constant, which is inversely proportional to the molecular weight of the gas, i.e.,

$$R \equiv \frac{\overline{R}}{MW} \tag{2.2a}$$

where $\overline{R}$ is the universal gas constant expressed in two systems of units as

$$\overline{R} = 8314 \frac{\text{J}}{\text{kmol} \cdot \text{K}} \tag{2.2b}$$

$$\overline{R} = 4.97 \times 10^4 \frac{\text{ft} \cdot \text{lbf}}{\text{slug} \cdot \text{mol} \cdot {}^\circ\text{R}} \tag{2.2c}$$

The thermodynamic relations for a perfect gas in terms of specific heats at constant pressure and volume are

$$dh \equiv c_p dT \tag{2.3}$$

$$de \equiv c_v dT \tag{2.4}$$

In general, the specific heats at constant pressure and volume are functions of gas temperature,

$$c_p = c_p(T) \tag{2.5}$$

$$c_v = c_v(T) \tag{2.6}$$

The gas is then called a *thermally perfect* gas. There is often a simplifying assumption of constant specific heats, which is a valid approximation to gas behavior in a narrow

temperature range. In this case,

$$c_p = \text{Constant} \tag{2.5a}$$

$$c_v = \text{Constant} \tag{2.6a}$$

The gas is referred to as a *calorically perfect* gas.

EXAMPLE 2.1

Calculate the density of (dry) air where its static pressure and temperature are 3.0 MPa and 25°C, respectively.

First express the static temperature in absolute scale, $T = 25 + 273 = 298$ K.

The gas constant for air is calculated from Equation 2.2a based on its (average) molecular weight of 29 kg/kmol, i.e.,

$$R = \frac{\overline{R}}{MW} = \frac{8{,}314 \text{ J/kmol} \cdot \text{K}}{29 \text{ kg/kmol}}$$
$$= 286.69 \text{ J/kg} \cdot \text{K} \cong 287 \text{ J/kg} \cdot \text{K}$$

We use the perfect gas law (Equation 2.1) to get the density, i.e.,

$$\rho = \frac{p}{RT} = \frac{3 \times 10^6 \text{ N/m}^2}{287 \dfrac{\text{J}}{\text{kg} \cdot \text{K}} (298 \text{ K})}$$

$$\cong 35.08 \frac{\text{kg}}{\text{m}^3} \qquad \boxed{\rho \cong 35.08 \frac{\text{kg}}{\text{m}^3}}$$

The first law of thermodynamics is the statement of conservation of energy for a system of fixed mass m, namely,

$$\delta q = de + \delta w \tag{2.7}$$

where the element of heat transferred to the system from the surrounding is considered positive and on a per-unit-mass basis is δq with a unit of energy/mass, e.g., J/kg. The element of work done by the gas on the surrounding is considered positive and per-unit-mass of the gas is depicted by δw. The difference in the convention for positive heat and work interaction with the system explains the opposite sides of equation where the two energy exchange terms are located in Equation 2.7, otherwise they both represent energy exchange with the system. The net energy interaction with the system results in a change of energy of the system; where again on a per-unit-mass basis is referred to as de. The three terms of the first law of thermodynamics have dimensions of energy per mass. The elemental heat and work exchange are shown by a delta "δ" symbol instead of an exact differential "d" as in "de." This is in recognition of *path-dependent* nature of heat and work exchange, which differ from the thermodynamic property of the gas "e," which is independent of the path, i.e.,

$$\int_1^2 \delta q \neq q_2 - q_1 \tag{2.8}$$

rather

$$\int_1^2 \delta q = {_1}q_2 \tag{2.9}$$

Whereas in the case of internal energy (or any other thermodynamic property),

$$\int_1^2 de = e_2 - e_1 \tag{2.10}$$

Note that in the eyes of the first law of thermodynamics, there is no distinction between heat and mechanical work exchange with the system. It is their "net" interaction with the system that needs to be accounted for in the energy balance. The application of the first law to a closed cycle is of importance to engineering and represents a balance between the heat and work exchange in a cyclic process, i.e.,

$$\oint \delta q = \oint \delta w \tag{2.11}$$

Also, for an adiabatic process, i.e., $\delta q = 0$, with no mechanical exchange of work, i.e., $\delta w = 0$, the energy of a system remains constant, namely $e_1 = e_2 = $ constant. We are going to use this principle in conjunction with a control volume approach in the study of inlet and exhaust systems of an aircraft engine.

The second law of thermodynamics introduces the absolute temperature scale and a new thermodynamic variable s, the entropy. It is a statement of impossibility of a heat engine exchanging heat with a single reservoir and producing mechanical work continuously. It calls for a second reservoir at a lower temperature where heat is rejected to by the heat engine. In this sense, the second law of thermodynamics distinguishes between heat and work. It asserts that all mechanical work may be converted into system energy whereas not all heat transfer to a system may be converted into system energy continuously. A corollary to the second law incorporates the new thermodynamic variable s and the absolute temperature T into an inequality, known as the Clausius inequality,

$$Tds \geq \delta q \tag{2.12}$$

where the equal sign holds for a reversible process. The concept of irreversibility ties in closely with frictional losses, viscous dissipation, and the appearance of shock waves in supersonic flow. The pressure forces within the fluid perform reversible work, and the viscous stresses account for dissipated energy of the system (into heat). Hence the reversible work done by a system per unit mass is

$$\delta w_{\text{rev}} = pdv \tag{2.13}$$

where v is the specific volume, which is the inverse of fluid density ρ. A combined first and second law of thermodynamics is known as the Gibbs equation, which relates entropy to other thermodynamic properties, namely

$$Tds = de + pdv \tag{2.14}$$

Although it looks as if we have substituted the reversible forms of heat and work into the first law to obtain the Gibbs equation, it is applicable to irreversible processes as well. Note that in an irreversible process, all the frictional forces that contribute to *lost work*

are dissipated into heat. Now, we introduce a derived thermodynamic property known as enthalpy, h, as

$$h \equiv e + pv \tag{2.15}$$

This derived property, that is, h, combines two forms of fluid energy, namely internal energy (or thermal energy) and what is known as the flow work, pv, or the pressure energy. The other forms of energy such as kinetic energy and potential energy are still unaccounted by the enthalpy h. We shall account for the other forms of energy by a new variable called the *total enthalpy* later in this chapter.

Now, let us differentiate Equation 2.15 and substitute it in the Gibbs equation, to get

$$Tds = dh - vdp \tag{2.16}$$

By expressing enthalpy in terms of specific heat at constant pressure, via Equation 2.3, and dividing both sides of Equation 2.16 by temperature T, we get

$$ds = c_p \frac{dT}{T} - \frac{v}{T}dp = c_p \frac{dT}{T} - R \frac{dp}{p} \tag{2.17}$$

We incorporated the perfect gas law in the last term of Equation 2.17. We may now integrate this equation between states 1 and 2 to arrive at

$$s_2 - s_1 \equiv \Delta s = \int_1^2 c_p \frac{dT}{T} - R \ln \frac{p_2}{p_1} \tag{2.18}$$

An assumption of a calorically perfect gas will enable us to integrate the first term on the right-hand side of Equation 2.18, i.e.,

$$\Delta s = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{2.19}$$

Otherwise, we need to refer to a tabulated thermodynamic function ϕ defined as

$$\int_1^2 c_p \frac{dT}{T} \equiv \phi_2 - \phi_1 \tag{2.20}$$

From the definition of enthalpy, let us replace the flow work, pv, term by its equivalent from the perfect gas law, i.e., RT, and then differentiate the equation as

$$dh \equiv c_p dT = de + RdT = c_v dT + RdT \tag{2.21}$$

Dividing through by the temperature differential dT, we get

$$c_p = c_v + R \tag{2.22a}$$

$$\frac{c_p}{R} = \frac{c_v}{R} + 1 \tag{2.22b}$$

This provides valuable relations among the gas constant and the specific heats at constant pressure and volume. The ratio of specific heats is given by a special symbol γ due to its frequency of appearance in compressible flow analysis, i.e.,

$$\gamma \equiv \frac{c_p}{c_v} = \frac{c_v + R}{c_v} = 1 + \frac{1}{c_v/R} \tag{2.23}$$

In terms of the ratio of specific heats γ and R, we express c_p and c_v as

$$c_p = \frac{\gamma}{\gamma - 1} R \tag{2.24a}$$

$$c_v = \frac{1}{\gamma - 1} R \tag{2.24b}$$

The ratio of specific heats is related to the degrees of freedom of the gas molecules, n, via

$$\gamma = \frac{n + 2}{n} \tag{2.25}$$

The degrees of freedom of a molecule are represented by the sum of the energy states that a molecule possesses. For example, atoms or molecules possess kinetic energy in three spatial directions. If they rotate as well, they have kinetic energy associated with their rotation. In a molecule, the atoms may vibrate with respect to each other, which then creates kinetic energy of vibration as well as the potential energy of intermolecular forces. Finally, the electrons in an atom or molecule are described by their own energy levels (both kinetic energy and potential) that depend on their position around the nucleus. As the temperature of the gas increases, the successively higher energy states are excited; thus the degrees of freedom increases. A monatomic gas, which may be modeled as a sphere, has at least three degrees of freedom, which represent translational motion in three spatial directions. Hence, for a monatomic gas, under "normal" temperatures the ratio of specific heats is
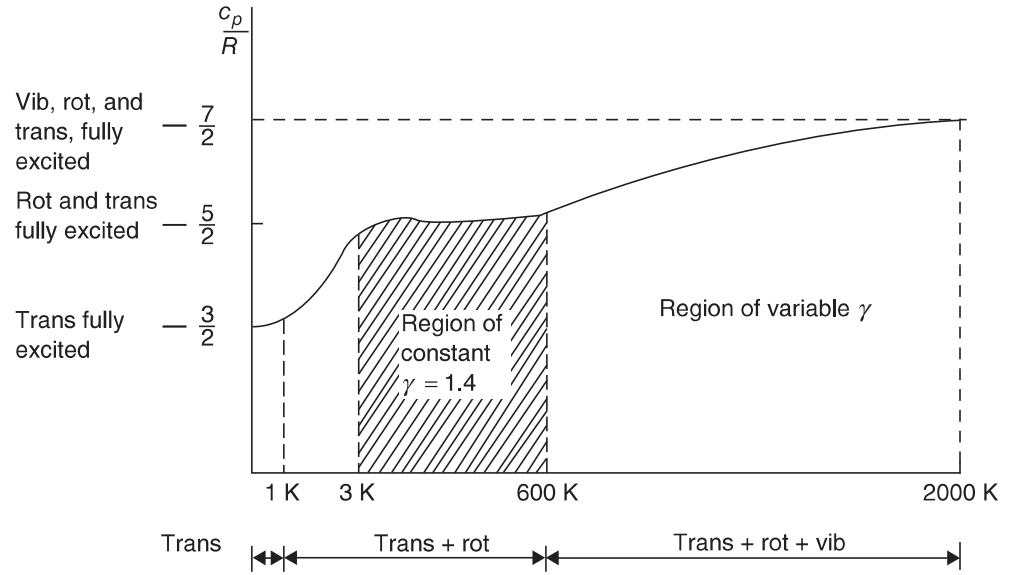
$$\gamma = \frac{5}{3} \cong 1.667 \quad \text{Monatomic gas at "normal" temperatures} \tag{2.26}$$

A monatomic gas has negligible rotational energy about the axes that pass through the atom due to its negligible moment of inertia. A monatomic gas will not experience a vibrational energy, as vibrational mode requires at least two atoms. At higher temperatures, the electronic energy state of the gas is affected, which eventually leads to ionization of the gas. For a diatomic gas, which may be modeled as a dumbbell, there are five degrees of freedom under "normal" temperature conditions, three of which are in translational motion and two of which are in rotational motion. The third rotational motion along the intermolecular axis of the dumbbell is negligibly small. Hence for a diatomic gas such as air (near room temperature), hydrogen, nitrogen, and so on, the ratio of specific heats is

$$\gamma = \frac{7}{5} = 1.4 \quad \text{Diatomic gas at "normal" temperatures} \tag{2.27a}$$

At high temperatures, molecular vibrational modes and the excitation of electrons add to the degrees of freedom and that lowers γ. For example, at ~600 K vibrational modes in air are excited, thus the degrees of freedom of diatomic gasses are initially increased

■ **FIGURE 2.1**
Temperature
dependence of specific
heat for a diatomic gas.
Source: Anderson
2003, Fig. 16.11, p. 613.
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McGraw Hill

by 1, that is, it becomes $5 + 1 = 6$, when the vibrational mode is excited. Therefore, the ratio of specific heats for diatomic gases at elevated temperatures becomes

$$\gamma = \frac{8}{6} \approx 1.33 \quad \text{Diatomic gas at elevated temperatures} \tag{2.27b}$$

The vibrational mode represents two energy states corresponding to the kinetic energy of vibration and the potential energy associated with the intermolecular forces. When fully excited, the vibrational mode in a diatomic gas, such as air, adds 2 to the degrees of freedom, that is, it becomes 7. Therefore, the ratio of specific heats becomes

$$\gamma = \frac{9}{7} \approx 1.29 \quad \text{Diatomic gas at higher temperatures} \tag{2.27c}$$

For example, air at 2000 K has its translational, rotational, and vibrational energy states fully excited. This temperature level describes the combustor or afterburner environment. Gases with a more complex structure than a diatomic gas have mote degrees of freedom, and thus their ratio of specific heats is less than 1.4. Figure 2.1 (from Anderson, 2003) shows the behavior of a diatomic gas from 0 to 2000 K. The nearly constant specific heat ratio between 3 and 600 K represents the calorically perfect gas behavior of a diatomic gas such as air with $\gamma = 1.4$. Note that near absolute zero (0 K), $c_v/R \to 3/2$; therefore, a diatomic gas ceases to rotate and thus behaves like a monatomic gas, that is, it exhibits the same degrees of freedom as a monatomic gas, that is, $n = 3$, $\gamma = 5/3$.

2.3   Isentropic Process and Isentropic Flow

For an isentropic process, where entropy remains constant, the Gibbs equation relates the pressure and temperature ratios by an isentropic exponent:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{c_p}{R}} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{(valid for a calorically perfect gas)} \tag{2.28}$$

Now, using the perfect gas law by replacing the temperature ratio by pressure and density ratios in Equation 2.28 and simplifying the exponents, we get

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \quad \text{(valid for a calorically perfect gas)} \tag{2.29}$$

EXAMPLE 2.2

Air is compressed from $p_1 = 100$ kPa, $T_1 = 15°C$ to $p_2 = 1.0$ MPa isentropically. Calculate the exit temperature and density of air. Assume $\gamma = 1.4$, $R = 287$ J/Kg · K, and the gas is calorically perfect.

First convert the temperature to absolute scale, i.e., $T_1 = 15 + 273 = 288$ K.

From the isentropic relation (2.28), we get the exit temperature according to

$$T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = (288 \text{ K})(10)^{0.4/1.4} \cong 556 \text{ K}$$

There are two ways that we can calculate the exit density. The first method calls for the application of the perfect gas law at the exit, that is,

$$\rho_2 = \frac{p_2}{RT_2} = \frac{10^6 \text{ Pa}}{(287 \text{ J/kg} \cdot \text{K})(556 \text{ K})} \cong 6.267 \text{ kg/m}^3$$

The second method uses the isentropic relation between the inlet and exit density, i.e., Equation 2.29. We need to calculate the inlet density first,

$$\rho_1 = \frac{p_1}{RT_1} = \frac{100 \text{ kPa}}{(287 \text{ J/kg} \cdot \text{K})(288 \text{ K})} \cong 1.2098 \text{ kg/m}^3$$

Now, we may use the isentropic relation 2.29 to get the exit density ρ_2

$$\rho_2 = \rho_1 \left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma}} = (1.2098 \text{ kg/m}^3)(10)^{1/1.4} \cong 6.267 \text{ kg/m}^3$$

2.4   Conservation Principles for Systems and Control Volumes

A system is a collection of matter of fixed identity, hence fixed mass, whereas a control volume is a fixed region in space with fluid crossing its boundaries. A control volume approach seems to be a more practical method of treating the fluid flow problems in gas turbine engines. However, all the classical laws of Newtonian physics are written for a matter of fixed mass, i.e., the system approach. For example, the mass of an object (in our case a collection of matter described by a system) does not change with time, is a Newtonian mechanics principle. The counterpart of that expressed for a control volume is known as the continuity equation and is derived as follows. The starting point is the system expression, namely

$$\left.\frac{dm}{dt}\right|_{\text{system}} = 0 \tag{2.30}$$

We now express mass as the integral of density over volume

$$\frac{d}{dt} \iiint\limits_{V(t)} \rho dV = 0 \tag{2.31}$$

where $V(t)$ is the volume of the system at any time t. Note that to encompass the entire original mass of the gas in the system, the boundaries of the system become a function of time. Expression 2.31 is a derivative with respect to time of an integral with time-dependent limits. The Leibnitz rule of differentiating such integrals is the bridge between the system and the control volume approach, namely

$$\frac{d}{dt} \iiint\limits_{V(t)} \rho dV = \iiint\limits_{V(t_0)} \frac{\partial \rho}{\partial t} dV + \oiint\limits_{S(t_0)} \rho \vec{V} \cdot \hat{n} dS \tag{2.32}$$

The limits of the integrals on the right-hand side (RHS) of Equation 2.32 describe a volume $V(t_0)$ and a closed surface $S(t_0)$, which represents the boundaries of the system at time t_0. We choose our control volume to coincide with the boundaries of the system at time t_0 Since the time t_0 is arbitrary, the above equation represents the control volume formulation for the medium, namely our continuity equation or law of conservation of mass for a control volume is written as

$$\iiint\limits_{C.V.} \frac{\partial \rho}{\partial t} dV + \oiint\limits_{C.S.} \rho \vec{V} \cdot \hat{n} dS = 0 \tag{2.33}$$

The fluid velocity vector is $\vec{V}$ and $\hat{n}$ represents a unit vector normal to the control surface and pointing outward. The first integral in Equation 2.33 accounts for unsteady accumulation (or depletion) of mass within the control volume and, of course, vanishes for steady flows. We may demonstrate this by switching the order of integration and differentiation for the first integral, that is,

$$\iiint\limits_{C.V.} \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \iiint\limits_{C.V.} \rho dV = \frac{\partial}{\partial t}(m)_{C.V.} \tag{2.34}$$

A positive value depicts mass accumulation in the control volume and a negative value shows mass depletion. The second integral takes the scalar product of the fluid velocity vector and the unit normal to the control surface. Therefore, it represents the net crossing of the mass through the control surface per unit time. Flow enters a control volume through one or more inlets and exits the control volume through one or more exits. There is no mass crossing the boundaries of a control volume at any other sections besides the "inlets" and "outlets." Since the dot product of the velocity vector and the unit normal vector vanishes on all surfaces of the control volume where mass does not cross the boundaries, the second integral makes a contribution to the continuity equation through its inlets and outlets, namely

$$\oiint\limits_{C.S.} \rho \vec{V} \cdot \hat{n} dS = \iint\limits_{Outlets} \rho \vec{V} \cdot \hat{n} dS + \iint\limits_{Inlets} \rho \vec{V} \cdot \hat{n} dS \tag{2.35}$$

We also note that a unit normal at an inlet points in the opposite direction to the incoming velocity vector, therefore the mass flux at an inlet contributes a negative value to the

mass balance across the control volume (through a negative dot product). An outlet has a unit normal pointing in the same direction as the velocity vector, hence contributes positively to the mass balance in the continuity equation. In general, the inlet and exit faces of a control surface are not normal to the flow, but still the angle between the normal and the velocity vector is obtuse for inlets and acute for exits, hence a negative and a positive dot product over the inlets and outlets, respectively. Therefore, the net flux of mass crossing the boundaries of a control volume, in a steady flow, is zero. We may assume a uniform flow over the inlet and exits of the control volume, to simplify the integrals of Equation 2.35, to

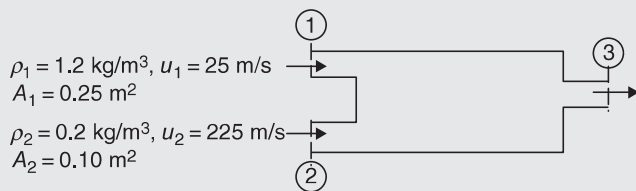$$\sum (\rho VA)_{\text{out}} = \sum (\rho VA)_{\text{in}} \qquad (2.36)$$

The product *VA* in the above equation is more accurately written as $V_n A$ or VA_n where a normal component of velocity through an area *A* contributes to the mass flow through the area, or equivalently, the product of velocity and a normal projection of the area to the flow contributes to the mass flow through the boundary. Often the subscript "*n*" is omitted in the continuity equation for convenience; however, it is always implied in writing the mass flow through a boundary. To write the conservation of mass for a control volume, let us combine the unsteady term and the net flux terms as

$$\frac{d}{dt}(m)_{C.V.} + \dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 0 \qquad (2.37)$$

where the mass flow rate is now given the symbol $\dot{m}$ (with units of kg/s, lbm/s, or slugs/s). We note that the difference between a positive outgoing mass and a negative incoming mass appears as a mass accumulation or depletion in the control volume. If the exit mass flow rate is higher than the inlet mass flow rate, then mass depletes within the control volume and vice versa.

EXAMPLE 2.3

Consider the control volume shown. There are two inlets boundaries and one exit boundary where flow crosses the control volume. Assuming the flow is steady and uniform, calculate the exit flow rate from the known inlet conditions.

$\rho_1 = 1.2$ kg/m³, $u_1 = 25$ m/s
$A_1 = 0.25$ m²

$\rho_2 = 0.2$ kg/m³, $u_2 = 225$ m/s
$A_2 = 0.10$ m²

The mass flow rate entering boundary 1, is

$$\dot{m}_1 = \rho_1 A_1 u_1 = (1.2 \text{ kg/m}^3)(0.25 \text{ m}^2)(25 \text{ m/s}) = 7.5 \text{ kg/s}$$

The mass flow rate entering boundary 2, is

$$\dot{m}_2 = \rho_2 A_2 u_2 = (0.2 \text{ kg/m}^3)(0.10 \text{ m}^2)(225 \text{ m/s}) = 4.5 \text{ kg/s}$$

Therefore, the mass flow rate out of boundary 3 is the sum of the mass flow rate into the box, i.e.,

$$\dot{m}_3 = \dot{m}_1 + \dot{m}_2 = (7.5 + 4.5) \text{kg/s} = 12 \text{ kg/s}$$

The momentum equation, according the Newtonian mechanics, relates the time rate of change of linear momentum of an object of fixed mass to the net external forces that act

on the object. We may write this law as

$$\frac{d}{dt}(m\vec{V}) = \vec{F}_{net} \tag{2.38}$$

Again, we propose to write the mass as the volume integral of the density within the system as

$$\frac{d}{dt}\iiint\limits_{V(t)} \rho\vec{V}dV = \vec{F}_{net} \tag{2.39}$$

Equation 2.39 is suitable for a system. We apply Leibnitz's rule to the momentum equation to arrive at the control volume formulation, namely

$$\iiint\limits_{C.V.} \frac{\partial}{\partial t}(\rho\vec{V})dV + \iint\limits_{C.S.} \rho\vec{V}(\vec{V}\cdot\hat{n})dS = \vec{F}_{net} \tag{2.40}$$

The first integral measures the unsteady momentum within the control volume and vanishes identically for a steady flow. The second integral is the net flux of momentum in and out of the control surface. Assuming uniform flow at the boundaries of the control surface inlets and outlets, we may simplify the momentum equation to a very useful engineering form, namely

$$(\dot{m}\vec{V})_{out} - (\dot{m}\vec{V})_{in} = \vec{F}_{net} \tag{2.40a}$$

Note that momentum equation (2.40a) is a vector equation and a shorthand notation for the momentum balance in three spatial directions. For example, in Cartesian coordinates, we have

$$(\dot{m}V_x)_{out} - (\dot{m}V_x)_{in} = F_{net,x} \tag{2.41a}$$

$$(\dot{m}V_y)_{out} - (\dot{m}V_y)_{in} = F_{net,y} \tag{2.41b}$$

$$(\dot{m}V_z)_{out} - (\dot{m}V_z)_{in} = F_{net,z} \tag{2.41c}$$

In cylindrical coordinates (r, θ, z), we may write the momentum equation as

$$(\dot{m}V_r)_{out} - (\dot{m}V_r)_{in} = F_{net,r} \tag{2.42a}$$

$$(\dot{m}V_\theta)_{out} - (\dot{m}V_\theta)_{in} = F_{net,\theta} \tag{2.42b}$$
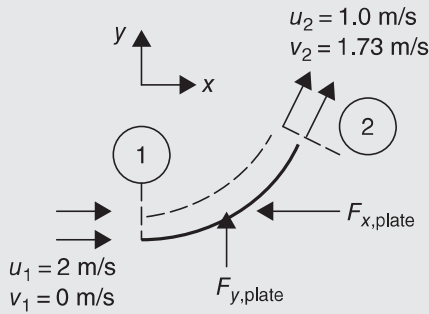
$$(\dot{m}V_z)_{out} - (\dot{m}V_z)_{in} = F_{net,z} \tag{2.42c}$$

We may write these equations in spherical coordinates (r, θ, ϕ) as well. The force terms in the momentum equation represent the net external forces exerted on the fluid at its boundaries and any volume forces, known as body forces, such as gravitational force. If the control volume contains/envelopes an object, then the force acting on the fluid is equal and opposite to the force experienced by the body. For example a body that experiences a drag force D imparts on the fluid a force equal to $-D$.

EXAMPLE 2.4

In placing a curved plate in front of a garden hose with a water flow rate of 0.1 kg/s, we have to exert an axial force on the plate, as shown, to hold it in place. Estimate the axial and lateral forces, F_x and F_y, respectively, that are needed to support the plate.

(Note: u and v are x- and y-component of velocity, respectively.)

$u_2 = 1.0$ m/s
$v_2 = 1.73$ m/s

$u_1 = 2$ m/s
$v_1 = 0$ m/s

$F_{x,plate}$

$F_{y,plate}$

For a quick estimation of the axial force, we assume the flow is uniform and steady. Then, we may apply Equation 2.41a

to get

$$F_{x,\text{fluid}} = \dot{m}_2 u_2 - \dot{m}_1 u_1 = \dot{m}(u_2 - u_1)$$
$$= 0.1 \text{ kg/s}(-1 \text{ m/s}) = -0.1 \text{ N}$$

The axial force acting on the curved plate is equal and opposite to $F_{x,\text{fluid}}$. Therefore, the external force needed to hold the plate horizontally is -0.1 N (that is in the negative x-direction, as shown). Also, the lateral force on the fluid is calculated from the lateral momentum balance, i.e.,

$$F_{y,\text{fluid}} = \dot{m}_2 v_2 - \dot{m}_1 v_1 = \dot{m}(v_2 - v_1)$$
$$= 0.1 \text{ kg/s}(1.73 \text{ m/s}) = 0.173 \text{ N}$$

The lateral force on the plate is in the opposite direction (i.e., in the $-y$ direction), and thus the external force to hold the plate laterally is $+0.173$ N, as shown.

The law of conservation of energy for a control volume starts with the first law of thermodynamics applied to a system. Let us divide the differential form of the first law by an element of time dt to get the rate of energy transfer, namely

$$\dot{Q} = \frac{dE}{dt} + \dot{W} \tag{2.43}$$

where the equation is written for the entire mass of the system. The energy E is now represented by the internal energy e times mass, as well as the kinetic energy of the gas in the system and the potential energy of the system. The contribution of changing potential energy in gas turbine engines or most other aerodynamic applications is negligibly small and often ignored. We write the energy as the mass integral of specific energy over the volume of the system, and apply Leibnitz's rule of integration to get the control volume version, namely

$$\frac{dE}{dt} = \frac{d}{dt} \iiint_{V(t)} \rho \left(e + \frac{V^2}{2} \right) dV = \iiint_{C.V.} \frac{\partial}{\partial t} \left[\rho \left(e + \frac{V^2}{2} \right) \right] dV + \oiint_{C.S.} \rho \left(e + \frac{V^2}{2} \right) \vec{V} \cdot \hat{n} dS$$

$$\tag{2.44}$$

The first term on the RHS of Equation 2.44 is the time rate of change of energy within the control volume, which identically vanishes for a steady flow. The second integral represents the net flux of fluid power (i.e., the *rate* of energy) crossing the boundaries of the control surface. The rates of heat transfer to the control volume and the rate of

mechanical energy transfer by the gases inside the control volume on the surrounding are represented by $\dot{Q}$ and $\dot{W}$ terms in the energy equation (2.43). Now, let us examine the forces at the boundary that contribute to the rate of energy transfer. These surface forces are due to pressure and shear acting on the boundary. The pressure forces act normal to the boundary and point inward, that is, opposite to $\hat{n}$,

$$- p\hat{n}dS \tag{2.45}$$

To calculate the rate of work done by a force, we take the scalar product of the force and the velocity vector, namely

$$- p\vec{V} \cdot \hat{n}dS \tag{2.46}$$

Now, we need to sum this elemental rate of energy transfer by pressure forces over the surface, via a surface integral, that is,

$$\oiint_{C.S.} -p\vec{V} \cdot \hat{n}dS \tag{2.47}$$

Since the convention on the rate of work done in the first law is positive when it is performed "on" the surroundings, and Equation 2.46 represents the rate of work done by the surroundings on the control volume, we need to incorporate an additional negative factor for this term in the energy equation. The rate of energy transfer by the shear forces is divided into a shaft power $\wp_s$ that crosses the control surface in the form of shaft shear and the viscous shear stresses on the boundary of the control volume. Hence,

$$\dot{W} = \oiint_{C.S.} p\vec{V} \cdot \hat{n}dS + \wp_s + \dot{W}_{\text{viscous-shear}} \tag{2.48}$$

Let us combine the expression 2.48 with Equations 2.44 and 2.43 to arrive at a useful form of the energy equation for a control volume, that is,

$$\oiiint_{C.V.} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right]dV + \oiint_{C.S.} \rho\left(e + \frac{V^2}{2}\right)\vec{V} \cdot \hat{n}dS + \oiint_{C.S.} p\vec{V} \cdot \hat{n}dS = \dot{Q} - \wp_s - \dot{W}_{\text{visc}}$$
$$\tag{2.49}$$

The closed surface integrals on the left-hand side (LHS) may be combined and simplified to

$$\oiiint_{C.V.} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right]dV + \oiint_{C.S.} \rho\left(e + \frac{V^2}{2} + \frac{p}{\rho}\right)\vec{V} \cdot \hat{n}dS = \dot{Q} - \wp_s - \dot{W}_{\text{visc}} \tag{2.50}$$

We may replace the internal energy and the flow work terms in Equation 2.50 by enthalpy h and define the sum of the enthalpy and the kinetic energy as the total or stagnation enthalpy, to get

$$\oiiint_{C.V.} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right]dV + \oiint_{C.S.} \rho h_t\vec{V} \cdot \hat{n}dS = \dot{Q} - \wp_s - \dot{W}_{\text{visc}}. \tag{2.51}$$

where the total enthalpy h_t is defined as

$$h_t \equiv h + \frac{V^2}{2} \tag{2.52}$$

In steady flows the volume integral that involves a time derivative vanishes. The second integral represents the net flux of fluid power across the control volume. The terms on the RHS are the external energy interaction terms, which serve as the *drivers* of the energy flow through the control volume. In adiabatic flows, the rate of heat transfer through the walls of the control volume vanishes. In the absence of shaft work, as in inlets and nozzles of a jet engine, the second term on the RHS vanishes. The rate of energy transfer via the viscous shear stresses is zero on solid boundaries (since velocity on solid walls obeys the no slip boundary condition) and nonzero at the inlet and exit planes. The contribution of this term over the inlet and exit planes is, however, small compared with the net energy flow in the fluid, hence neglected.

The integrated form of the energy equation for a control volume, assuming uniform flow over the inlets and outlets, yields a practical solution for quick engineering calculations

$$\sum (\dot{m}h_t)_{\text{out}} - \sum (\dot{m}h_t)_{\text{in}} = \dot{Q} - \wp_s \tag{2.53}$$

The summations in Equation 2.53 account for multiple inlets and outlets of a general control volume. In flows that are adiabatic and involve no shaft work, the energy equation simplifies to

$$\sum (\dot{m}h_t)_{\text{out}} = \sum (\dot{m}h_t)_{\text{in}} \tag{2.54}$$
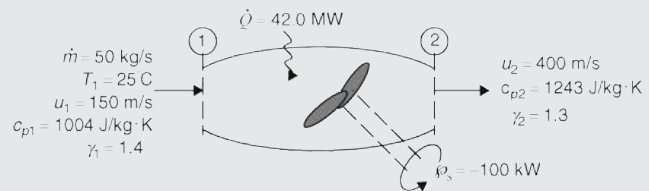
For a single inlet and a single outlet, the energy equation is even further simplified, as the mass flow rate also cancels out, to yield

$$h_{t-\text{exit}} = h_{t-\text{inlet}} \tag{2.55}$$

Total or stagnation enthalpy then remains constant for adiabatic flows with no shaft power, such as inlets and nozzles or across shock waves.

EXAMPLE 2.5

Let us consider a control volume with heat transfer rate and mechanical power (i.e., shaft power) exchange specified at its boundaries. There is a single inlet and a single outlet where mass crosses the boundary in a steady and uniform flow, as shown. Calculate the exit total and static temperatures T_{t2} and T_2, respectively (note that the gas is not calorically perfect).

$\dot{Q} = 42.0$ MW
$\dot{m} = 50$ kg/s
$T_1 = 25$ C
$u_1 = 150$ m/s
$c_{p1} = 1004$ J/kg·K
$\gamma_1 = 1.4$
$u_2 = 400$ m/s
$c_{p2} = 1243$ J/kg·K
$\gamma_2 = 1.3$
$\wp_s = -100$ kW

First, we conclude that the exit mass flow rate is 50 kg/s, to satisfy the continuity equation for steady flow. Second, we cast the inlet static temperature in the absolute scale, i.e.,

$$T_1 = 25 + 273 = 298 \text{ K}$$

Third, we apply the conservation of energy equation (2.53) to this problem, which requires the knowledge of total enthalpy at the inlet. We use the definition of total enthalpy:

$$h_{t1} = h_1 + u_1^2/2 = c_{p_1}T_1 + u_1^2/2 = (1004 \text{ J/kg} \cdot \text{K})(298 \text{ K})$$
$$+ (150)^2/2 \text{ J/kg}$$
$$= 310,442 \text{ J/kg}$$

Now, Equation 2.53 gives the exit total enthalpy as

$$h_{t2} = h_{t1} + \left(\frac{\dot{Q} - \wp_s}{\dot{m}} \right) = 310.442 \text{ kJ/kg}$$

$$+ \left(\frac{(42,000 + 100) \times 1000 \text{ W}}{50 \text{ kg/s}} \right) \cong 1,152.4 \text{ kJ/kg}$$

$$T_{t2} = \frac{h_{t2}}{c_{p2}} = \frac{1,152.4 \text{ kJ/kg}}{1.243 \text{ kJ/kg} - \text{K}} \cong 927 \text{ K}$$

$$T_2 = T_{t2} - u_2^2/2c_{p2}$$

$$= 927 \text{ K} - (400)^2/(2 \times 1243 \text{ J/kg} \cdot \text{K}) \cong 863 \text{ K}$$

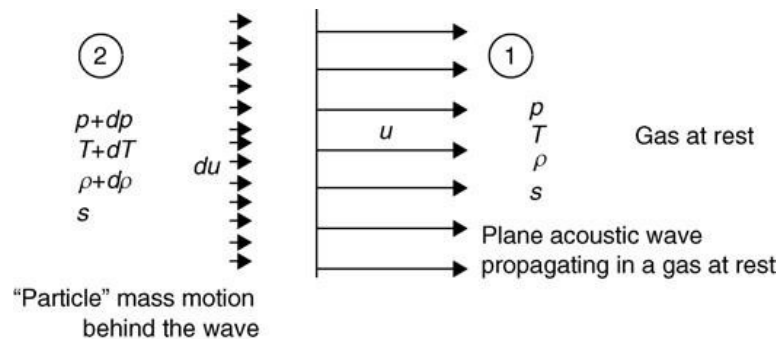## 2.5   Speed of Sound & Mach Number

Sound waves are infinitesimal pressure waves propagating in a medium. The propagation of sound waves, or acoustic waves, is reversible and adiabatic, hence isentropic. Since sound propagates through collision of fluid molecules, the speed of sound is higher in liquids than gas. The derivation of the speed of sound is very simple and instructive. Assume a plane sound wave propagates in a medium at rest with speed u. The fluid behind the wave is infinitesimally set in motion at the speed du with an infinitesimal change of pressure, temperature, and density. The fluid ahead of the wave is at rest and yet unaffected by the approaching wave. A schematic drawing of this wave and fluid properties are shown in Figure 2.2.

By switching observers from the gas at rest to an observer that moves with the wave at the wave speed u, we change the unsteady wave propagation problem into a steady one. This is known as the Lorentz transformation. The transformed problem is shown in Figure 2.3.
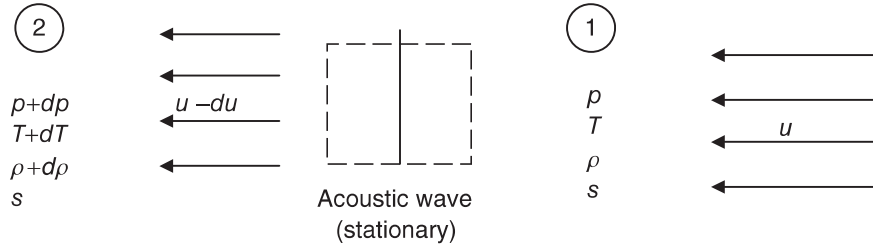
The fluid static properties, pressure, density, and temperature are independent of the motion of an observer; hence they remain unaffected by the observer transformation. We may now apply steady conservation principles to a control volume, as shown in Figure 2.3. The control volume is a box with its sides parallel to the flow, an inlet and an exit area normal to the flow. We may choose the entrance and exit areas of the box to be unity. The continuity demands

$$\rho u = (\rho + d\rho)(u + du) \qquad (2.56)$$

■ **FIGURE 2.2**
Propagation of a plane acoustic wave in a gas at rest (an unsteady but isentropic problem)

2

$p+dp$
$T+dT$
$\rho+d\rho$
s

du

"Particle" mass motion
behind the wave

u

1

p
T
ρ
s

Gas at rest

Plane acoustic wave
propagating in a gas at rest

which simplifies to

$$\rho du = -u d\rho \qquad (2.57)$$

The momentum equation for this one-dimensional wave problem is

$$(\rho u)(u + du) - (\rho u)u = p - (p + dp) \qquad (2.58)$$

The change of momentum from inlet to exit is shown on the LHS of Equation 2.58. The first parenthesis of each momentum term is the mass flow rate, which is multiplied by the respective flow speeds at the exit and inlet. The driving forces on the RHS of Equation 2.58 are the pressure-area terms in the direction and opposite to the fluid motion, acting on areas that were chosen to be unity. The momentum equation simplifies to

$$\rho u du = dp \qquad (2.59)$$

Now, let us substitute the continuity equation (2.57) in the momentum equation (2.59), to get an expression for the square of the acoustic wave propagation in terms of pressure and density changes that occur as a result of wave propagation in a medium at rest

$$u^2 d\rho = dp \qquad (2.60)$$

or

$$u^2 = \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s \qquad (2.61)$$

Since, for isentropic flow, $p = \text{const} \cdot \rho^\gamma$, Equation 2.61 reduces to

$$u^2 = \frac{\gamma p}{\rho} = \gamma RT \qquad (2.62)$$

We replaced the ratio of pressure to density by RT from the perfect gas law in Equation 2.62. The symbol we use in this book for the speed of sound is "a," hence, local speed of sound in a gas is

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} = \sqrt{(\gamma - 1) c_p T} \qquad (2.63)$$

The speed of sound is a *local* parameter, which depends on the local absolute temperature of the gas. Its value changes with gas temperature, hence it drops when fluid accelerates (or *expands*) and increases when gas decelerates (or *compresses*). The speed of sound in air at standard sea level conditions is ~340 m/s or ~1100 ft/s. The type of gas also affects the speed of propagation of sound through its molecular weight. We may observe this behavior by the following substitution:

$$a = \sqrt{\gamma RT} = \sqrt{\gamma \left(\frac{\overline{R}}{MW} \right) T} \qquad (2.64)$$

A light gas, like hydrogen (H_2) with a molecular weight of 2, causes an acoustic wave to propagate *faster* than a heavier gas, such as air with (a mean) molecular weight of 29. If we substitute these molecular weights in Equation 2.64, we note that sound propagates in gaseous hydrogen *nearly four times faster* than air. Since both hydrogen and air are diatomic gases, the ratio of specific heats remains (nearly) the same for both gases at the same temperature.

The equation for the speed of propagation of sound that we derived is for a gas at rest. Let us superimpose a uniform collective gas speed in a particular direction to the wave front, then the wave propagates as the vector sum of the two, namely, $\vec{V} + \vec{a}$. For waves propagating normal to a gas flow, we get either $(V + a)$ or $(V - a)$ as the propagation speed of sound. It is the $(V - a)$ behavior that is of interest here. In case the flow is sonic, then $(a - a = 0)$, which will not allow the sound to travel upstream and hence creates a zone of silence upstream of the disturbance. In case the flow speed is even faster than the local speed of sound, that is, known as supersonic flow, the acoustic wave will be confined to a cone. These two behaviors for small disturbances are shown in Figure 2.4.
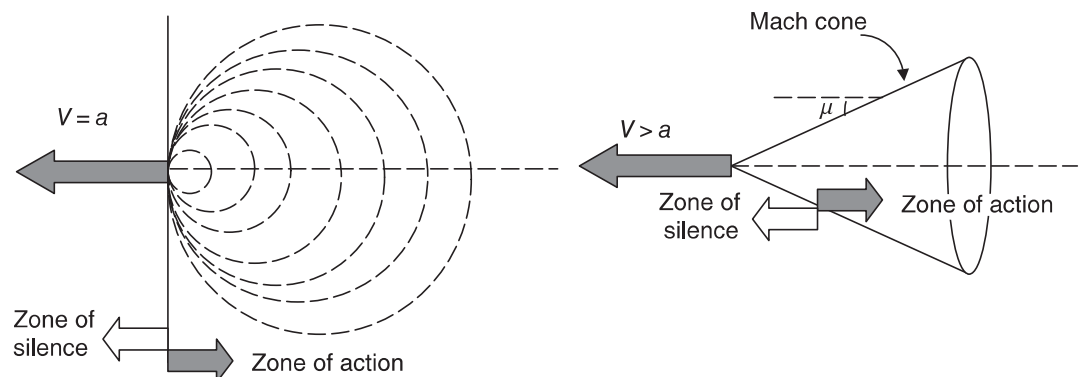
The ratio of local gas speed to the speed of sound is called Mach number, M
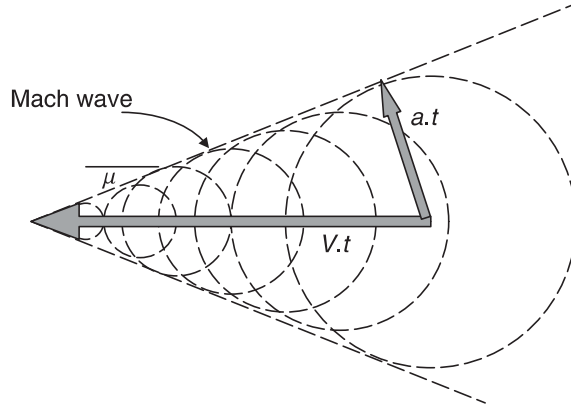
$$M \equiv \frac{V}{a} \qquad (2.65)$$

The envelope of the waves that create the zones of action and silence is the Mach wave. It makes a local wave angle with respect to the flow μ, which from the geometry of wave propagation, as shown in Figure 2.5, is

$$\mu = \sin^{-1} \left(\frac{a \cdot t}{V \cdot t} \right) = \sin^{-1} \left(\frac{1}{M} \right) \qquad (2.66)$$

■ FIGURE 2.4
Acoustic wave propagation in sonic and supersonic flows (or the case of a moving source)

■ **FIGURE 2.5**
**Wave front created by
a *small disturbance*
moving at a supersonic
speed**

2.6   Stagnation State

We define the stagnation state of a gas as the state reached in decelerating a flow to rest reversibly and adiabatically and without any external work. Thus, the stagnation state is reached isentropically. This state is also referred to as the *total* state of the gas. The symbols for the stagnation state in this book use a subscript "t" for total. The total pressure is p_t, the total temperature is T_t, and the total density is ρ_t. Since the stagnation state is reached isentropically, the static and total entropy of the gas are the same, that is, $s_t = s$. Based on the definition of stagnation state, the total energy of the gas does not change in the deceleration process, hence the stagnation enthalpy h_t takes on the form

$$h_t \equiv h + \frac{V^2}{2} \tag{2.52}$$

which we defined earlier in this chapter. Assuming a *calorically perfect gas*, we may simplify the total enthalpy relation 2.52 by dividing through by c_p to get an expression for total temperature according to

$$T_t = T + \frac{V^2}{2c_p} \quad \text{(valid for a calorically perfect gas)} \tag{2.67}$$

This equation is very useful in converting the local static temperature and gas speed into the local stagnation temperature. To nondimensionalize Equation 2.67, we divide both sides by the static temperature, that is,

$$\frac{T_t}{T} = 1 + \frac{V^2}{2c_p T} \tag{2.68}$$

The denominator of the kinetic energy term on the RHS is proportional to the square of the local speed of sound a^2 according to Eq 2.63, which simplifies to

$$\frac{T_t}{T} = 1 + \left(\frac{\gamma - 1}{2}\right)\frac{V^2}{a^2} = 1 + \left(\frac{\gamma - 1}{2}\right)M^2 \tag{2.69}$$

Therefore to arrive at the stagnation temperature of the gas, we need to have its static temperature as well as either the gas speed V or the local Mach number of the gas M to substitute in Equations 2.68 or 2.69, respectively. The ratio of stagnation to static temperature of a (calorically perfect) gas is a unique function of local Mach number, according to Equation 2.69. We may also use a spreadsheet tabulation of this function for later use for any gas characterized by its ratio of specific heats γ. From isentropic relations between the pressure and temperature ratio that we derived earlier based on Gibbs equation of thermodynamics, we may now relate the ratio of stagnation to static pressure and the local Mach number via
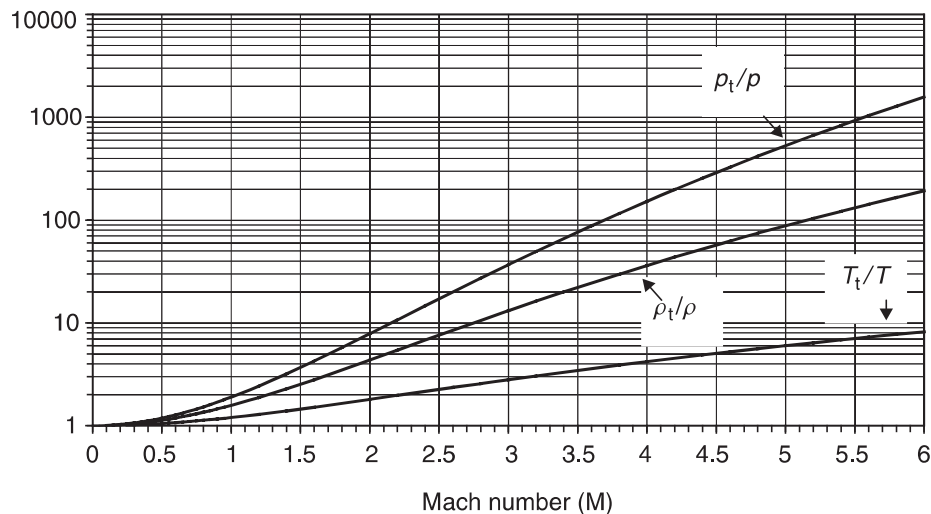
$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma-1}} \qquad \text{(valid for a calorically perfect gas)} \qquad \textbf{(2.70)}$$

We used the isentropic relation between the stagnation and static states based on the definition of the stagnation state that is reached isentropically. Also, the stagnation density is higher than the static density according to
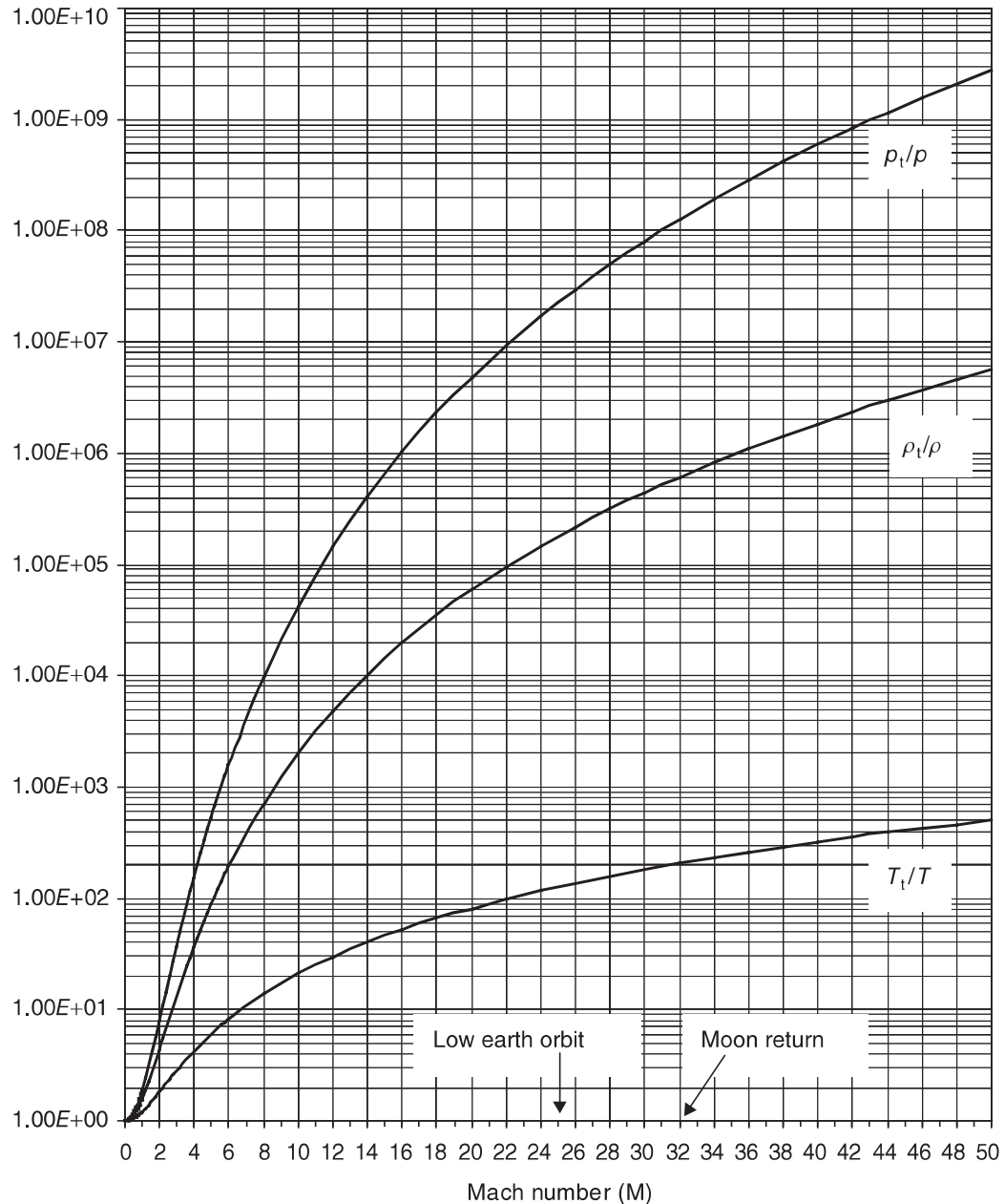
$$\frac{\rho_t}{\rho} = \left(\frac{p_t}{P}\right)^{\frac{1}{\gamma}} = \left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{1}{\gamma-1}} \qquad \text{(Valid for a calorically perfect gas)} \qquad \textbf{(2.71)}$$

We note that the ratio of stagnation-to-static state of a gas is totally described by the local Mach number and the type of gas γ. The tabulation of these functions, that is, p_t/p, T_t/T, and ρ_t/ρ are made in isentropic tables for a gas described by its ratio of specific heats γ. Figure 2.6 shows the variation of the stagnation state with Mach number for a diatomic gas, such as air ($\gamma = 1.4$). The least variation is noted for the stagnation temperature, followed by the density and pressure. To graph these variations (up to Mach 6), we had to use a logarithmic scale, as the total pressure rise with Mach number reaches above one thousand (times the static pressure) while the stagnation temperature stays below ten (times the static temperature). We may examine higher Mach numbers in Figure 2.7. The assumption in arriving at the stagnation state properties relative to the static properties was *a calorically perfect gas*. The impact of high-speed flight on

■ **FIGURE 2.6** Variation of the stagnation state with Mach number for a diatomic gas, $\gamma = 1.4$ (For a calorically perfect gas, i.e., c_p, c_v, and γ are constants)
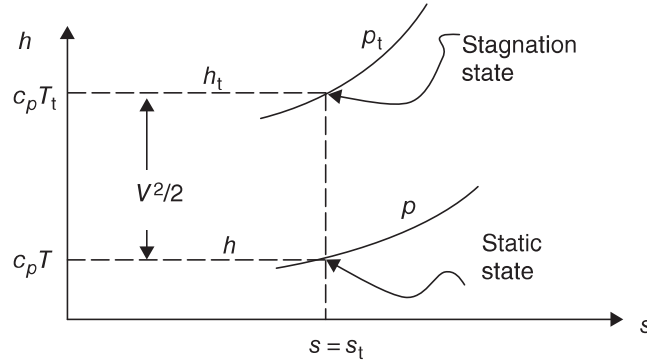
the atmosphere is to cause molecular dissociation, followed by ionization of the oxygen and nitrogen molecules. A host of other chemical reactions takes place, which result in a violation of our initial assumption. In reality, both the specific heats (c_p and c_v) are functions of temperature as well as the pressure at hypersonic Mach numbers, that is, $M > 5$. There are two lessons to be learned here. The first is to always examine the validity of your assumptions; the other is to be cautious about data extrapolations!
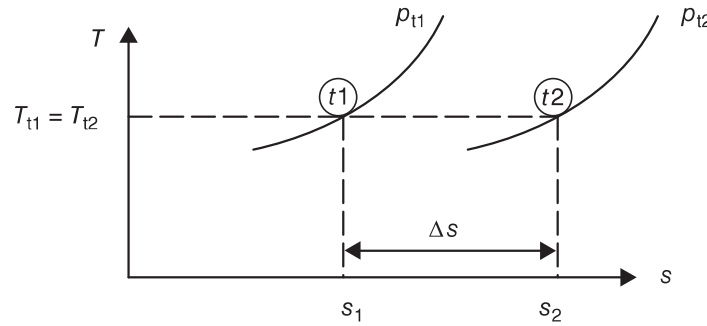
Although the pressure ratio (p_t/p) is relieved through real gas effects due to a high-speed flight, it still remains high. Consequently, the high-speed flight has to be scheduled at a high altitude where the static pressure is low. For example, the static pressure at 25 km is only ~2.5% and at 50 km altitude is 0.07% of the sea level pressure.

To identify the local static and stagnation states of a gas in a Mollier (h–s) diagram is important in studying propulsion. We use the stagnation enthalpy definition to identify

■ **FIGURE 2.8**
Stagnation and static states of a gas in motion shown on an *h–s* **diagram**

■ **FIGURE 2.9**
Adiabatic flow that shows total pressure loss Δp_t**, e.g., flows containing shocks**

the local stagnation properties of a gas, for example, we start with the static state and then build up a kinetic energy, as in Figure 2.8.

We may also apply the Gibbs equation to the stagnation state of the gas to get

$$\Delta s = c_p \ell n(T_{t2}/T_{t1}) - R\ell n(p_{t2}/p_{t1}) \tag{2.72}$$

For adiabatic flows the first term on the RHS identically vanishes, since the total temperature remains constant. Hence, the total pressure for adiabatic flows with losses, for example, due to frictional losses, always decreases, since the entropy change has to be positive. The exponential relationship between the total pressure ratio and the entropy rise in an adiabatic flow is very useful to propulsion studies and we write it in Equation 2.73
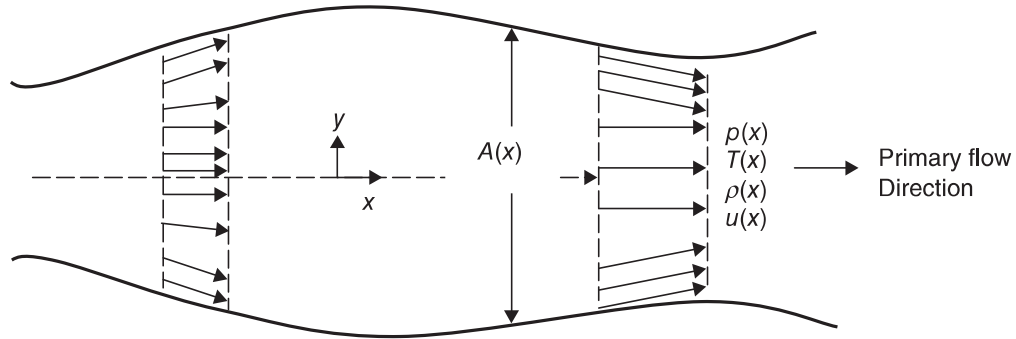
$$p_{t2}/p_{t1} = e^{-\frac{\Delta s}{R}} \tag{2.73}$$

All adiabatic flows with loss result in a drop of total pressure. For example, in a supersonic inlet with shocks and frictional losses in the boundary layer, we encounter a total pressure loss. Since, the total pressure is now cast as a measure of loss via Equation 2.73, it serves as a commodity that propulsion engineers try to preserve as much as possible. The *T–s* diagram of an adiabatic flow with loss is shown in Figure 2.9.

2.7   Quasi-One-Dimensional Flow

In duct flows with area variations, such as a diffuser or a nozzle, where the boundary layers are completely attached, the streamwise variation of flow variables dominate the lateral variations. In essence, the flow behaves as a uniform flow with pressure gradient

■ **FIGURE 2.10**
Schematic drawing of a
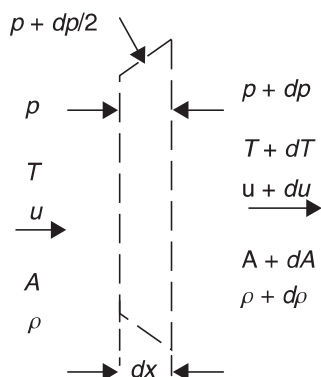variable-area duct with
attached boundary
layers

in a variable-area duct. A schematic drawing of a duct with area variation is shown in Figure 2.10. The primary flow direction is labeled as "x" and the lateral direction is "y." Although the flow next to the wall assumes the same slope as the wall, the cross-sectional flow properties are based on uniform parallel flow at the cross-section. The flow area at a cross-section enters the conservation laws, but the variation in the cross-section from the centerline to the wall is neglected. Such flows are called quasi-one-dimensional flows.

For high Reynolds number flows where the boundary layer is thin compared with lateral dimensions of a duct and assuming an attached boundary layer, the assumption of an *inviscid* fluid leads to a reasonable approximation of the duct flow. This means that an inviscid fluid model accurately estimates the pressure, temperature, and flow development along the duct axis. Attached boundary layers require a slow variation in the duct area, which is almost invariably the case in propulsion engineering. Inviscid flow analysis also produces a limit performance capability of a component when boundary layers are infinitely thin. In a practical sense, ducts with boundary layer suction approach an inviscid flow model. To develop the quasi-one-dimensional flow equations in a variable-area duct with inviscid fluid, we examine a slice, known as a *slab*, of the flow and apply the Taylor series approximation to the two sides of the slab. Figure 2.11 shows a thin slice of a duct flow with streamwise length dx separating the two sides of the slab.

We assume general parameters on one side and step in the x-direction using the Taylor series approximation of an analytical function expanded in the neighborhood of a point. Since the step size dx is small, we may truncate the infinite Taylor series after the first derivative. Note that the walls exert one-half of the incremental pressure $(dp/2)$ over the upstream pressure p, since we take only one-half the step size $(dx/2)$ to reach

■ **FIGURE 2.11**
Slab of inviscid fluid in
a variable-area duct
(walls exert pressure
on the fluid)

its center. Now, let us apply the conservation principles to this thin slice of the flow. Continuity demands

$$\rho u A = (\rho + d\rho)(u + du)(A + dA) \tag{2.74}$$

Ignoring higher order terms involving the products of two small parameters, such as $du \cdot dA$, we get

$$\rho u dA + uA d\rho + \rho A du = 0 \tag{2.75}$$

Now dividing both sides by a nonzero constant, namely puA, we get the continuity equation in differential form

$$\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0 \tag{2.76}$$

This is also known as the *logarithmic derivative* of the local mass flow rate, $puA =$ constant. Since we use the logarithmic derivative often in our analysis, let us examine it now. From the continuity equation

$$\rho u A = \text{Constant} \tag{2.77}$$

We take the natural logarithm of both sides, to get the sum

$$\ell n(\rho) + \ell n(u) + \ell n(A) = \text{Constant} \tag{2.78}$$

Now, if we take the differential of both sides we get Equation 2.76, which is called the logarithmic derivative of the continuity equation.

The momentum balance in the streamwise direction gives

$$(\rho u A)(u + du) - (\rho u A) u = pA - (p + dp)(A + dA) + (p + dp/2) dA \tag{2.79}$$

The first term on the LHS is the rate of momentum out of the slab, which is the mass flow rate times the velocity out of the box. The second term on the LHS is the rate of momentum into the box. The first term on the LHS is the pressure force pushing the fluid out. The second term on the LHS is the pressure force in the opposite direction to the flow, hence negative. The last term on the LHS is the pressure force contribution of the walls. First, we note that its direction is in the flow direction, hence positive. Second, we note that the projection of the sidewalls in the flow direction is dA, which serves as the *effective* area for the wall pressure to push the fluid out of the box. We may simplify this equation by canceling terms and neglecting higher order quantities to get

$$\rho u A du = -A dp \tag{2.80}$$

The flow area term is cancelled from Equation 2.80 (as expected) to yield

$$\rho u du = -dp \tag{2.81}$$

The energy equation for an adiabatic (non heat-conducting) flow with no shaft work is the statement of conservation of total enthalpy, which differentiates into

$$dh_{\text{t}} = dh + udu = 0 \tag{2.82}$$

The equation of state for a perfect gas may be written in logarithmic derivative form as

$$dp/p = d\rho/\rho + dT/T \qquad (2.83)$$

Also, as we stipulated an isentropic flow through the duct, that is, $ds = 0$, the pressure–density relationship follows the isentropic rule, namely $p/\rho^\gamma = $ constant, which has the following logarithmic derivative:

$$\frac{dp}{p} - \gamma \frac{d\rho}{\rho} = 0 \qquad (2.84)$$

The set of governing equations for an isentropic flow through a duct of variable area are summarized as:

$$d\rho/\rho + du/u + dA/A = 0$$
$$\rho u du = -dp$$
$$dh + u du = 0$$
$$dp/p = d\rho/\rho + dT/T$$

The unknowns are the pressure p, density ρ, temperature T, and velocity u. The enthalpy for a perfect gas follows $dh = c_p dT$, hence we need to know the specific heat at constant pressure of the gas as well. The entropy remains constant, hence the static pressure is related to the speed of sound and density according to:

$$p = \rho a^2/\gamma \qquad (2.85)$$

throughout the flow. We shall apply these governing equations in the following sections to fluid flow problems of interest to propulsion.

2.8   Area–Mach Number Relationship

Let us divide the momentum equation by $d\rho$, to get

$$\frac{\rho}{d\rho} u du = -\frac{dp}{d\rho} = -a^2 \qquad (2.86)$$

which yields

$$\frac{d\rho}{\rho} = -\frac{u du}{a^2} \qquad (2.87)$$

Now, let us substitute the density ratio in the continuity equation to relate the area variation to Mach number according to
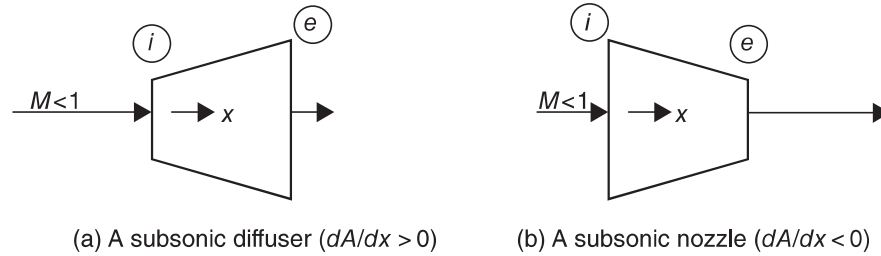
$$-\frac{u du}{a^2} + \frac{du}{u} + \frac{dA}{A} = 0 \qquad (2.88a)$$
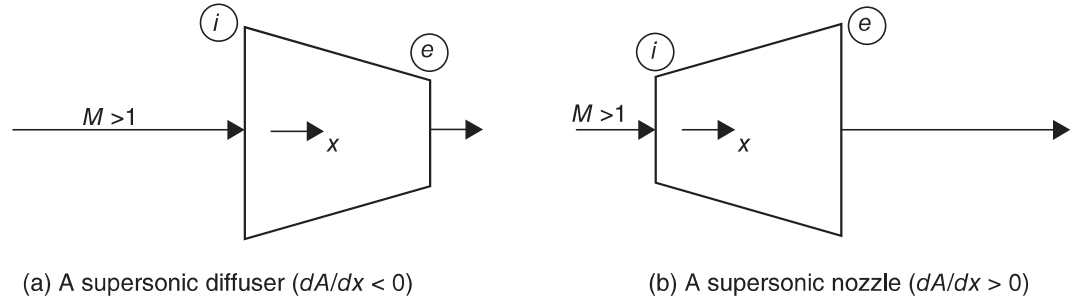
This simplifies to

$$(M^2 - 1)\frac{du}{u} = \frac{dA}{A} \qquad (2.88b)$$

■ FIGURE 2.12
Area variations for a subsonic diffuser and a nozzle

(a) A subsonic diffuser ($dA/dx > 0$) (b) A subsonic nozzle ($dA/dx < 0$)

■ FIGURE 2.13
Area variations for a supersonic diffuser and nozzle

(a) A supersonic diffuser ($dA/dx < 0$) (b) A supersonic nozzle ($dA/dx > 0$)

This equation relates area variation to speed variation for the subsonic and supersonic flows. Assuming a subsonic flow, the parenthesis involving Mach number becomes a negative quantity. Hence, an area increase in a duct, that is, $dA > 0$, results in $du < 0$, to make the signs of both sides of Equation 2.88b consistent. A negative du means flow deceleration. Hence, a subsonic diffuser requires a duct with $dA/dx > 0$. A subsonic nozzle with $du > 0$ demands $dA < 0$. These two duct geometries for a subsonic flow are shown in Figure 2.12.

For a supersonic flow, the parenthesis that involves Mach number in Equation 2.88b is positive. Hence, an area increase results in flow acceleration and vice versa. These geometric relationships between area variation and flow speed for supersonic flow are shown in Figure 2.13.

We note that the geometric requirements for a desired fluid acceleration or deceleration in a duct are opposite to each other in subsonic and supersonic flow regimes. This dual behavior is seen repeatedly in aerodynamics. We shall encounter several examples of this in this chapter.

2.9 Sonic Throat

A unique flow condition separates the subsonic from supersonic flow. It is the sonic flow that marks the boundary between the two flows. In a subsonic flow, we learned that area reduction accelerates the gas and, ultimately, could reach a sonic state at the exit of the subsonic nozzle. A sonic flow may accelerate to supersonic Mach numbers if the area of the duct increases in the streamwise direction. This initial contraction and later expansion of the duct is capable of accelerating a subsonic flow to supersonic exit flow. This is called a convergent–divergent duct, with an internal throat, that is, the sonic throat. Conversely, a convergent–divergent duct with a supersonic entrance condition has the capability of decelerating the flow to subsonic exit conditions through a sonic throat. These duct geometries are shown in Figure 2.14.