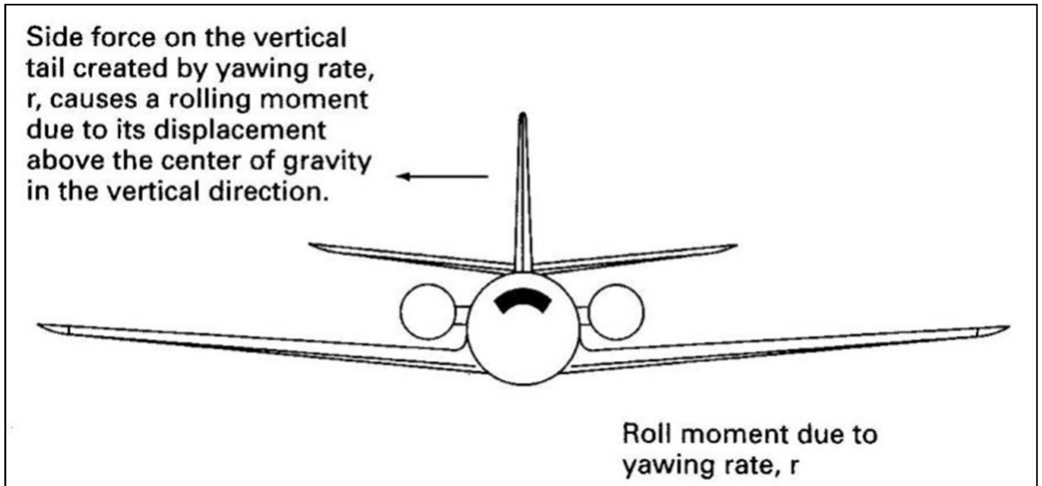
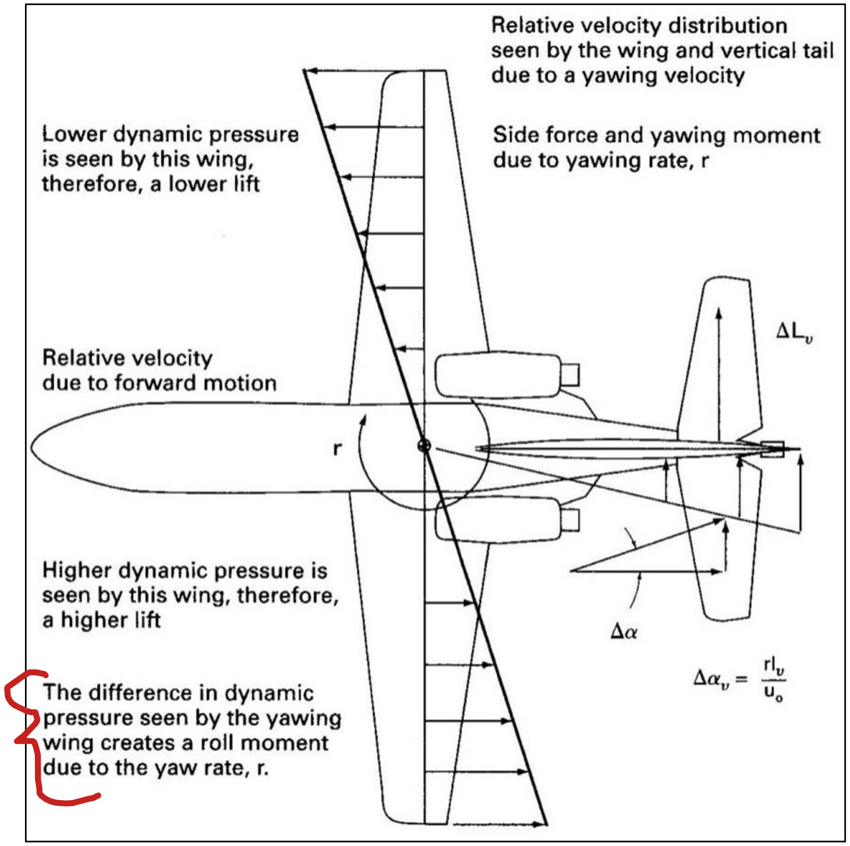
Yaw Rate (r) Effect

Consider the figures alongside and below **showing** the effect of yaw rate, **‘r’,** on the **directional** as well as lateral aerodynamics**.**





Yaw Rate (r) Derivatives for ΔY_A, ΔN_A

$$Y = -a_V \Delta\beta Q_V S_V; \quad \Delta\beta = -\frac{rl_V}{u_0}; \quad C_Y = \frac{Y}{Q_w S_w} = \frac{a_V r l_V Q_V S_V}{u_0 Q_w S_w}$$

$$C_Y = a_w r\left(\frac{l_V}{u_o} \times \frac{S_V}{S_w}\right)\eta_V; \quad \bar{r} = \frac{rb}{2u_0}; \quad C_{Y\bar{r}} = 2a_w \eta_V \frac{S_V l_V}{S_w b} \approx -2C_{Y\beta}\left(\frac{l_V}{b}\right)$$

$$N = a_V \Delta\beta Q_V S_V l_V = -a_V\left(\frac{rl_V}{u_0}\right)Q_V S_V l_V = -2a_V \bar{r} Q_V S_V l_V\left(\frac{l_V}{b}\right)$$

$$C_N = \frac{N}{QS_w b} = -2a_V \bar{r}\eta_V V_V\left(\frac{l_V}{b}\right); \quad C_{N\bar{r}} = -2a_V \eta_V V_V\left(\frac{l_V}{b}\right) \approx 2C_{Y\beta}\left(\frac{l_V}{b}\right)^2$$

$$C_{lr} = \frac{C_L}{4} - 2\left(\frac{l_V}{b}\right)\left(\frac{z_V}{b}\right)C_{y\beta V} \rightarrow \text{Due to both Wing and VT}$$

Roll Rate (p) Effect and Derivative

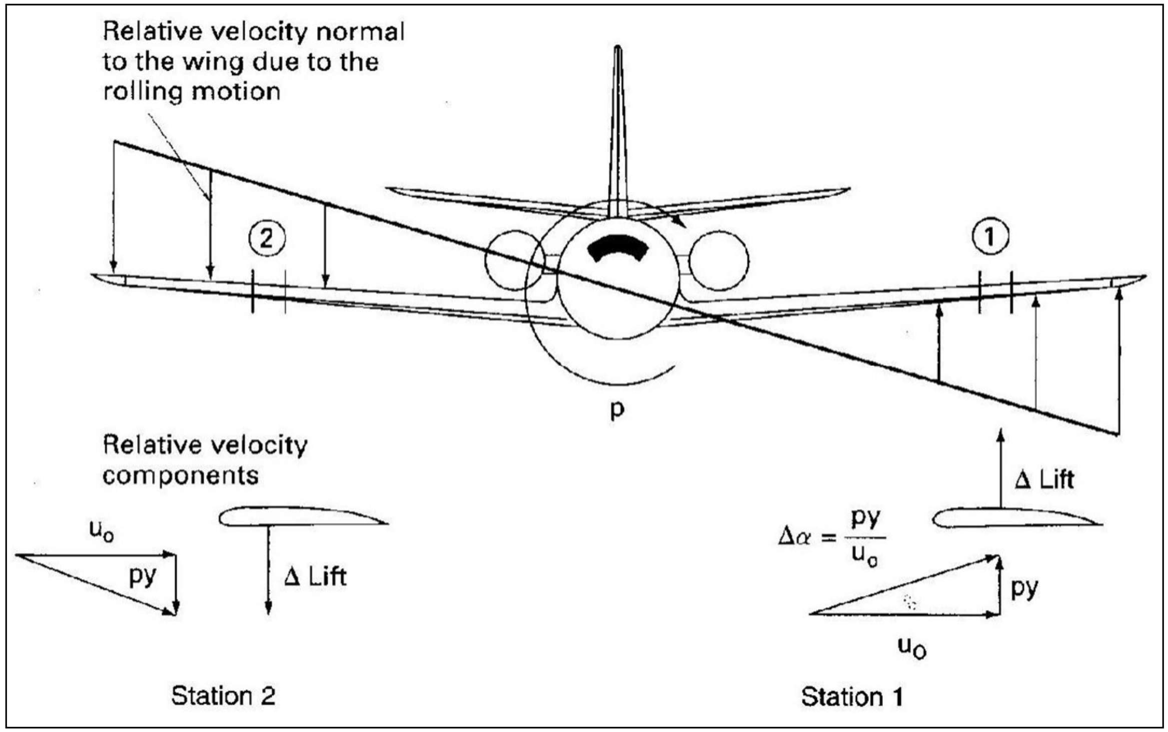
Roll rate effects are due to contributions **from** wings and the schematic along-side **brings** out the applicable physics.

$$L_{roll} = -2\int_0^{b/2} C_{l\alpha}\left(\frac{py}{u_0}\right)Qc(y)y\,dy$$

$$C_l = -2\frac{a_W p}{Sbu_0}\int_0^{b/2} c(y)y^2 dy; \quad \lambda = \frac{c_t}{c_r}$$

$$\bar{p} = \frac{pb}{2u_0}; \quad C_{l\bar{p}} = -4\frac{a_W}{Sb^2}\int_0^{b/2} c(y)y^2 dy$$

$$c(y) = c_r - (c_r - c_t)\frac{2y}{b}; \quad C_{l\bar{p}} = -\frac{C_{L\alpha}(1+3\lambda)}{12(1+\lambda)}$$



Applicable x –z Plane Dimensional Derivatives

Imp

$$X_u = \frac{-(C_{D_u} + 2C_{D_0})QS}{mu_0}(\mathrm{s}^{-1}) \qquad X_w = \frac{-(C_{D_\alpha} - C_{L_0})QS}{mu_0}(\mathrm{s}^{-1})$$

$$Z_u = \frac{-(C_{L_u} + 2C_{L_0})QS}{mu_0}(\mathrm{s}^{-1})$$

$$Z_w = \frac{-(C_{L_\alpha} + C_{D_0})QS}{mu_0}(\mathrm{s}^{-1}) \qquad Z_{\dot{w}} = -C_{z_{\dot{\alpha}}}\frac{c}{2u_0}QS/(u_0 m)$$

$$Z_\alpha = u_0 Z_{\dot{w}} (\mathrm{ft/s^2}) \text{ or } (\mathrm{m/s^2}) \qquad Z_{\dot{\alpha}} = u_0 Z_{\dot{w}} \ (\mathrm{ft/s}) \text{ or } (\mathrm{m/s})$$

$$Z_q = -C_{Z_q}\frac{c}{2u_0}QS/m \ (\mathrm{ft/s}) \text{ or } (\mathrm{m/s}) \qquad Z_{\delta_e} = -C_{Z_{\delta_e}}QS/m \ (\mathrm{ft/s^2})$$

$$\mathrm{M}_u = C_{m_u}\frac{(QSc)}{u_0 I_y}\left(\frac{1}{\mathrm{ft \cdot s}}\right) \text{ or } \left(\frac{1}{\mathrm{m \cdot s}}\right)$$

$$\mathrm{M}_w = C_{m_\alpha}\frac{(QS\bar{c})}{u_0 I_y}\left(\frac{1}{\mathrm{ft \cdot s}}\right) \text{ or } \left(\frac{1}{\mathrm{m \cdot s}}\right) \qquad M_{\dot{w}} = C_{m_{\dot{\alpha}}}\frac{\bar{c}}{2u_0}\frac{QS\bar{c}}{u_0 I_y}(\mathrm{ft}^{-1})$$

$$\mathrm{M}_\alpha = u_0 \mathrm{M}_w \ (\mathrm{s}^{-2}) \qquad \mathrm{M}_{\dot{\alpha}} = u_0 \mathrm{M}_{\dot{w}} \ (\mathrm{s}^{-1})$$

$$\mathrm{M}_q = C_{m_q}\frac{\bar{c}}{2u_0}(QS\bar{c})/I_y \ (\mathrm{s}^{-1}) \qquad \mathrm{M}_{\delta_e} = C_{m_{\delta_e}}(QS\bar{c})/I_y \ (\mathrm{s}^{-2})$$

Applicable x – y Plane Dimensional Derivatives

$$Y_\beta = \frac{QSC_{y\beta}}{m}\ (\text{ft/s}^2)\ \text{or}\ (\text{m/s}^2) \qquad N_\beta = \frac{QSbC_{n\beta}}{I_z}\ (\text{s}^{-2}) \qquad L_\beta = \frac{QSbC_{l\beta}}{I_x}\ (\text{s}^{-2})$$

$$Y_p = \frac{QSbC_{y_p}}{2mu_0}\ (\text{ft/s})\ (\text{m/s}) \qquad N_p = \frac{QSb^2C_{n_p}}{2I_x u_0}\ (\text{s}^{-1}) \qquad L_p = \frac{QSb^2C_{l_p}}{2I_x u_0}\ (\text{s}^{-1})$$

$$Y_r = \frac{QSbC_{y_r}}{2mu_0}\ (\text{ft/s})\ \text{or}\ (\text{m/s}) \qquad N_r = \frac{QSb^2C_{n_r}}{2I_x u_0}\ (\text{s}^{-1}) \qquad L_r = \frac{QSb^2C_{l_r}}{2I_x u_0}\ (\text{s}^{-1})$$

$$Y_{\delta a} = \frac{QSC_{y\delta_a}}{m}\ (\text{ft/s}^2)\ \text{or}\ (\text{m/s}^2) \qquad Y_{\delta_r} = \frac{QSC_{y\delta_r}}{m}\ (\text{ft/s}^2)\ \text{or}\ (\text{m/s}^2)$$

$$N_{\delta_a} = \frac{QSbC_{n\delta_a}}{I_z}\ (\text{s}^{-2}) \qquad N_{\delta_r} = \frac{QSbC_{n\delta_r}}{I_z}\ (\text{s}^{-2})$$

$$L_{\delta_a} = \frac{QSbC_{l\delta_a}}{I_x}\ (\text{s}^{-2}) \qquad L_{\delta_r} = \frac{QSbC_{l\delta_r}}{I_x}\ (\text{s}^{-2})$$

6-DOF Symmetric & Asymmetric Dynamics

Symmetric (Longitudinal) Dynamics:

$$\Delta\dot{u} = X_u\Delta u + X_w\Delta w - \left(g\cos\theta_0\right)\Delta\theta + X_{\delta e}\delta_e + X_{\delta T}\delta_T$$

$$\left(1 - Z_{\dot{w}}\right)\Delta\dot{w} = Z_u\Delta u + Z_w\Delta w + \left(u_0 + Z_q\right)\Delta\dot{\theta}$$
$$-\left(g\sin\theta_0\right)\Delta\theta + Z_{\delta e}\Delta\delta_e + Z_{\delta T}\Delta\delta_T$$

$$\Delta\dot{q} = M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta e}\Delta\delta_e + M_{\delta T}\Delta\delta_T$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{x}_I = \cos\theta_0\Delta u + \sin\theta_0\Delta w$$

$$\Delta\dot{z}_I = -\Delta\dot{h}_I = -\sin\theta_0\Delta u + \cos\theta_0\Delta w$$

Asymmetric (Lateral-Directional) Dynamics:

$$\Delta\dot{v} = Y_v\Delta v + Y_p\Delta p - \left(u_0 - Y_r\right)\Delta r + \left(g\cos\theta_0\right)\Delta\phi + Y_{\delta r}\delta_r$$

$$\Delta\dot{p} = L_v\Delta v + L_p\Delta p + L_r\Delta r + \left(\frac{I_{zx}}{I_{xx}}\right)\Delta\dot{r} + L_{\delta a}\Delta\delta_a + L_{\delta r}\Delta\delta_r$$

$$\Delta\dot{r} = N_v\Delta v + N_p\Delta p + N_r\Delta r + \left(\frac{I_{zx}}{I_{zz}}\right)\Delta\dot{p} + N_{\delta a}\Delta\delta_a + N_{\delta r}\Delta\delta_r$$

$$\Delta\dot{\phi} = \Delta p + \tan\theta_0\Delta r$$

$$\Delta\dot{\psi} = \left\{\frac{1}{\cos\left(\theta_0\right)}\right\}\Delta r$$

$$\Delta\dot{y}_I = \Delta v$$

Six equations for x – z **plane:** Δu, Δw, Δq, $\Delta\theta$, Δx_I, Δz_I, (along with θ_0).

Six equations for **x – y** plane: Δv, **Δp,** Δr, $\Delta\phi$, $\Delta\psi$, Δy_I, (along with θ_0).

Longitudinal Dynamics

$$\Delta\dot{u} = X_u \Delta u + X_w \Delta w - \left(g\cos\theta_0\right)\Delta\theta + X_{\delta e}\delta_e + X_{\delta T}\delta_T$$

$$\left(1 - Z_{\dot{w}}\right)\Delta\dot{w} = Z_u \Delta u + Z_w \Delta w + \left(u_0 + Z_q\right)\Delta q - \left(g\sin\theta_0\right)\Delta\theta$$

$$+ Z_{\delta e}\Delta\delta_e + Z_{\delta T}\Delta\delta_T$$

$$\Delta\dot{q} = M_u \Delta u + M_w \Delta w + M_{\dot{w}}\Delta\dot{w} + M_q \Delta q + M_{\delta e}\Delta\delta_e + M_{\delta T}\Delta\delta_T$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{x}_I = \cos\theta_0 \Delta u + \sin\theta_0 \Delta w$$

$$\Delta\dot{z}_I = -\Delta\dot{h}_I = -\sin\theta_0 \Delta u + \cos\theta_0 \Delta w$$

$$\{\dot{x}\} = [A]\{x\} + [B]\{u\}$$

$$\{y\} = [C]\{x\} + [D]\{u\}$$

$$\{x\} = \begin{Bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \\ \Delta x_I \\ \Delta z_I \end{Bmatrix}; \quad \{u\} = \begin{Bmatrix} \Delta\delta_e \\ \Delta\delta_T \end{Bmatrix}$$

[A], [B], [C], [D] Matrices

$$[A] = \begin{bmatrix} X_u & X_w & 0 & -g\cos\theta_0 & 0 & 0 \\ Z_u & Z_w & Z_q + u_0 & -g\sin\theta_0 & 0 & 0 \\ M_u + M_{\dot{w}} Z_u & M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cos\theta_0 & \sin\theta_0 & 0 & 0 & 0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} X_{\delta e} & X_{\delta T} \\ Z_{\delta e} & Z_{\delta T} \\ M_{\delta e} + M_{\dot{w}} Z_{\delta e} & M_{\delta T} + M_{\dot{w}} Z_{\delta T} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[C] = [I]; \quad [D] = 0$$

Longitudinal Motion Features

It is seen that the position kinematics is decoupled so **that** we can solve the sixth order **dynamics** as two separate 4^{th} **order** and 2^{nd} order systems.

Further, pitch angle '$\Delta\theta$' is directly solvable from **'Δq'** solution so that in some **cases**, further simplification are possible.

Lastly, nature of the [A] matrix provides **additional** simplifications in the **context** of longitudinal dynamics.

4^{th} Order Longitudinal Model

$$\{x\} = \begin{Bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{Bmatrix}; \quad \{u\} = \begin{Bmatrix} \Delta \delta_e \\ \Delta \delta_T \end{Bmatrix}; \quad [B] = \begin{bmatrix} X_{\delta e} & X_{\delta T} \\ Z_{\delta e} & Z_{\delta T} \\ M_{\delta e} + M_{\dot{w}} Z_{\delta e} & M_{\delta T} + M_{\dot{w}} Z_{\delta T} \\ 0 & 0 \end{bmatrix}$$

$$[A] = \begin{bmatrix} X_u & X_w & 0 & -g\cos\theta_0 \\ Z_u & Z_w & Z_q + u_0 & -g\sin\theta_0 \\ M_u + M_{\dot{w}} Z_u & M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$