AE 339 : High speed aerodynamics

(VIII) External aerodynamics: Hypersonic flow past bodies

Prof. Vineeth Nair

Dept. Aerospace Engg.

IIT Bombay

By definition $M_\infty = \dfrac{V_\infty}{a_\infty} \gg 1$

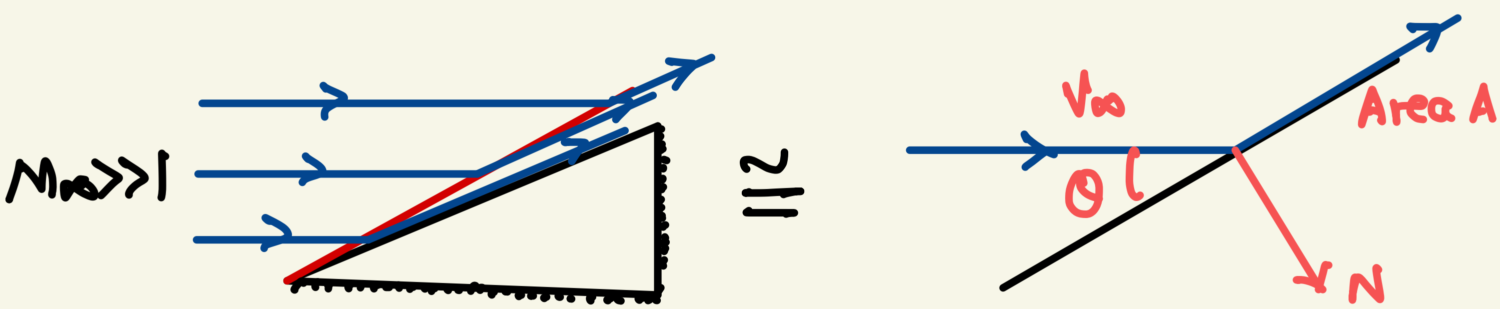
Newtonian theory

According to the Newtonian model, flow consists of a large number of individual particles that impact the surface and then move tangentially. (like pellets from a shotgun)

On collision the particles lose their component of momentum normal to the surface, but tangential component is preserved.

The time rate of change of normal momentum equals the force exerted on the surface by particle impacts.

The thin shock layers around the hypersonic bodies are the closest example in fluid mechanics to Newton's model.



Component of velocity normal to the surface

$$V_n = V_\infty \sin\theta$$

mass flow incident on the surface : $\rho_\infty V_\infty (A \sin\theta)$

Time rate of change of momentum:

$$= (\rho_\infty V_\infty A \sin\theta)(V_\infty \sin\theta)$$

$$= \rho_\infty V_\infty^2 A \sin^2\theta$$

$= N$ from Newton's second law

The normal force per unit area

$$\frac{N}{A} = \rho_\infty V_\infty^2 \sin^2\theta$$

This normal force must be construed as the pressure difference above p_∞ on the surface

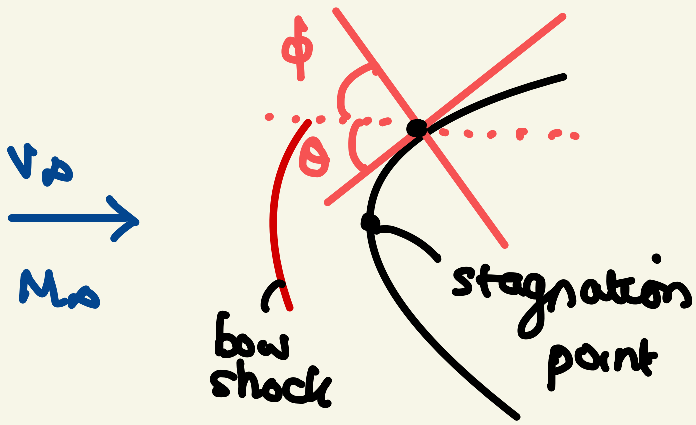
ie $\quad p - p_\infty = \rho_\infty V_\infty^2 \sin^2\theta$

$$\Rightarrow C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\sin^2\theta$$

$$\boxed{C_p = 2\sin^2\theta}$$

Called NEWTON'S SINE-SQUARED LAW

The law states that the pressure coefficient is proportional to the sine-square of the angle between a tangent to the surface and the direction of the free stream.

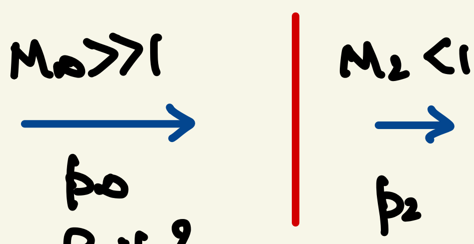


$$C_p = 2\sin^2\theta$$
$$= 2\cos^2\phi$$

$C_{p,max}$ occurs at the stagnation point where $\theta = \pi/2$, $\phi = 0$

$C_{p,max} = 2$ as per Newtonian theory.

Consider a normal shock at hypersonic speeds



large $\rho_\infty V_\infty^2$

$\rho_2 V_2^2 << \rho_\infty V_\infty^2$

$$p_\infty + \rho_\infty V_\infty^2 = p_2 + \rho_2 V_2^2$$
$$\approx p_2$$

$$C_p = \frac{p_2 - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2$$

For large but finite Mach numbers, the value of C_p at a stagnation point is less than 2 (say $C_{p,max}$)

Downstream of the stagnation point, C_p can be assumed to follow the sine-squared law predicted by Newtonian theory, i.e.

$$\boxed{C_p = C_{p,max} \sin^2\theta}$$

MODIFIED NEWTONIAN THEORY

↳ works better for 3D bodies

↳ comparison improves at larger values of M & θ.

↳ used in the preliminary design of hypersonic vehicles.

Estimates of $C_{p,max}$

For incompressible flow, $C_{p,max} = 1$

For $M_\infty = 1$,

$$C_p = \frac{2}{\gamma}\left[\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] = 1.28$$

$$C_{p\,max} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_{02}}{p_\infty} - 1\right)$$

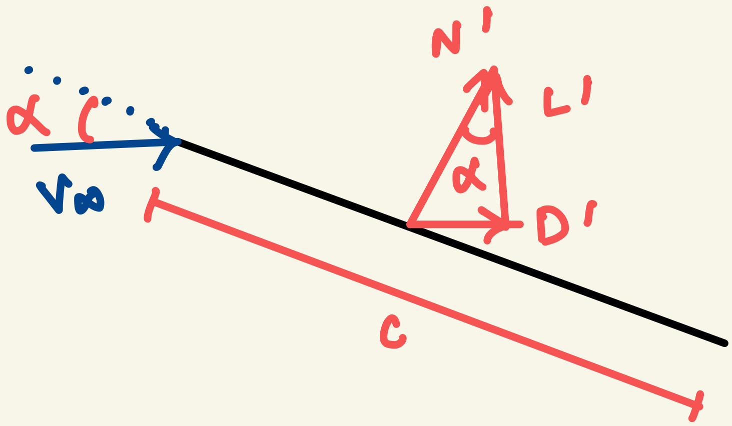
at stagnation point.

$$= \frac{2}{\gamma M_\infty^2}\left[\left(\frac{(\gamma+1)^2 M_\infty^2}{4\gamma M_\infty^2 - 2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} \frac{1-\gamma+2\gamma M_\infty^2}{\gamma+1} - 1\right]$$

$$C_{p_{max}, M\to\infty} = \frac{2}{\gamma}\left[\left(\frac{(\gamma+1)^2}{4\gamma}\right)^{\frac{\gamma}{\gamma-1}} \frac{2\gamma}{\gamma+1}\right] = 1.84$$

Lift and drag coefficients

In an approximate fashion, the section is modeled by a flat plate at an angle of attack.



Consider a 2D flat plate of chord length c and angle of attack α.

According to Newtonian theory, the pressure coefficient on the lower surface

$$C_{p,l} = 2\sin^2\alpha$$

The upper surface receives no 'direct impact' of freestream particles; the upper surface is said to be in the 'shadow' of the flow.

$$C_{p,u} = 0$$

Neglecting friction, the normal force coefficient is given as

$$C_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx$$

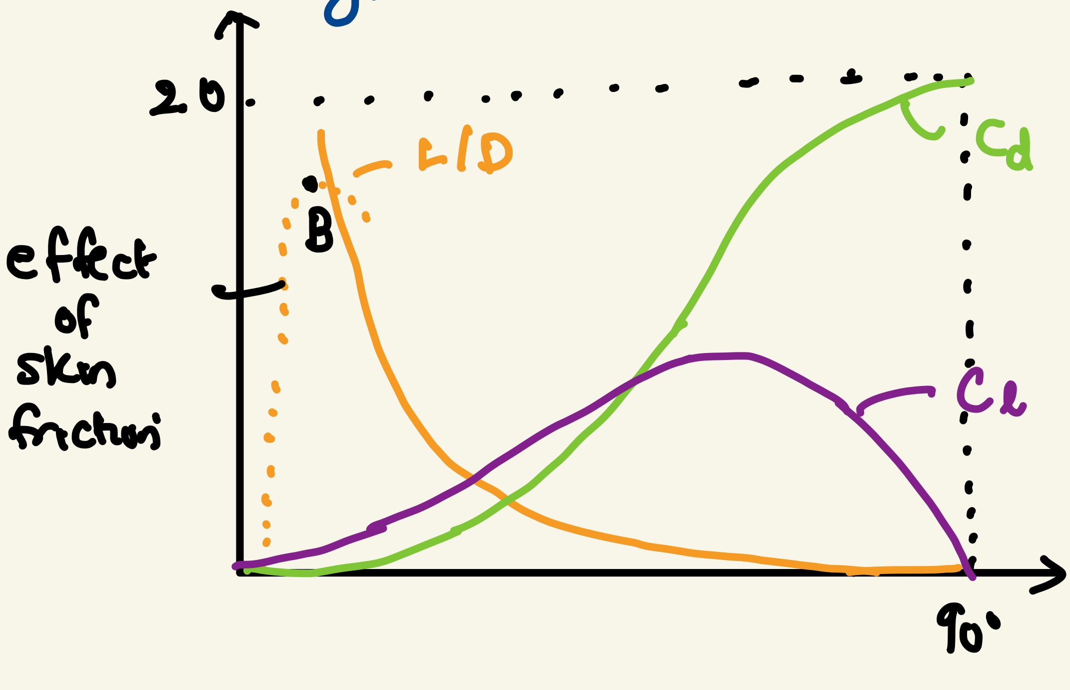
$$= \frac{1}{c}\, 2\sin^2\alpha\, c = 2\sin^2\alpha$$

$$C_l = C_n\cos\alpha = 2\sin^2\alpha\cos\alpha$$

$$C_d = C_n\sin\alpha = 2\sin^3\alpha$$

$$\boxed{\frac{L}{D} = \frac{C_l}{C_d} = \cot\alpha}$$

$L/D = \cot\alpha$ is a general result for inviscid supersonic or hypersonic flow over a flat plate.



Remarks

1. C_l varies in a nonlinear manner; i.e. $\frac{dC_l}{d\alpha}$ is not constant

↳ consistent since VPE is nonlinear

↳ $C_{l_{max}}$ at α such that

$$\frac{dC_l}{d\alpha} = 4\sin\alpha\cos^2\alpha - 2\sin^2\alpha\sin\alpha = 0$$

$$\Rightarrow \sin^2\alpha = 2/3 \Rightarrow \alpha = 54.7^\circ$$

$$C_{l,max} = 0.77$$

C_l peaks at $\alpha \approx 55^\circ$, then decreases and reaches 0 at 90° (practically seen in lots of hypersonic configurations)

Although C_l increases over a large range of α $(0, 54.7^\circ)$, its rate of increase is small; i.e. $dC_l/d\alpha$ is small

↳ The attainment of $C_{l,max}$ is purely a geometric effect and not due to any viscous. separated flow phenomenon (as in subsonic flow)

Reason: Although N increases with α, component of N along vertical reaches a maximum and gradually decreases.

2. Wave drag coefficient C_d monotonically increases from 0 at $\alpha = 0$ to a maximum of 2 at $\alpha = 90^0$

The Newtonian drag is essentially wave drag at hypersonic speeds.

C_d has a cubic variation with α for low angles of attack as opposed to a quadratic variation for supersonic flows.

$$C_d = 2\alpha^3$$

3. $L/D \to \infty$ as $\alpha \to 0$ due to the neglect of skin friction drag. When skin friction is added L/D reaches a maximum value at a small, non-zero angle of attack and goes to 0 as $\alpha \to 0$ as $L=0, D\neq 0$ (due to skin friction)

at $\alpha = 0$

$$C_d = 2\alpha^3 + C_{d,0}$$

$$C_l = 2\alpha^2$$

$$\frac{C_l}{C_d} = \frac{2\alpha^2}{2\alpha^3 + C_{d,0}}$$

For $(L/D)_{max}$ $\frac{d}{d\alpha}(C_l/C_d) = 0$

$$\frac{(2\alpha^3 + C_{d,0})\,4\alpha - 2\alpha^2(6\alpha^2)}{(2\alpha^3 + C_{d,0})^2} = 0$$

$$\Rightarrow 8\alpha^4 + 4\alpha C_{d,0} - 12\alpha^4 = 0$$

or $$\boxed{\alpha_{(L/D)_{max}} = (C_{d,0})^{1/3}}$$

$$(L/D)_{max} = (C_L/C_d)_{max} = \frac{2(C_{d,0})^{2/3}}{2C_{d,0} + C_{d,0}} = \frac{2/3}{(C_{d,0})^{1/3}}$$

$$\boxed{(L/D)_{max} = (C_L/C_d)_{max} = \frac{0.67}{(C_{d,0})^{1/3}}}$$

The value of $(L/D)_{max}$ and $\alpha_{(L/D)_{max}}$ are strictly a function of the zero-lift drag coefficient $C_{d,0}$. At small angles of attack should be the skin friction exerted on the plate essentially that at $\alpha = 0$.

↳ $C_{d,0} \uparrow$, then $(L/D)_{max} \downarrow$ and $\alpha_{(L/D)_{max}} \uparrow$

↳ At $(L/D)_{max}$

$$C_d = 2C_{d,0} + C_{d,0} = 3C_{d,0}$$
$$= C_{d,w} + C_{d,0}$$

or $$\boxed{C_{d,w} = 2C_{d,0}}$$

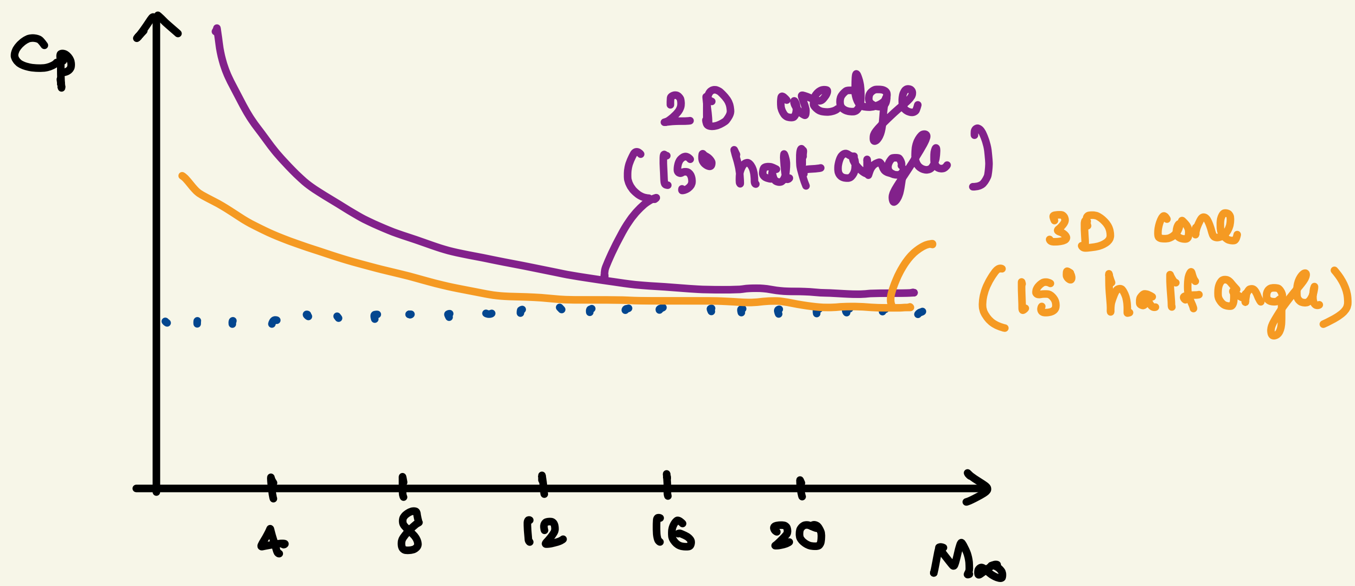
For hypersonic flat plates, wave drag is twice skin friction drag at $(L/D)_{max}$

Mach number independence

At high M_∞, certain aerodynamic quantities such as C_p, C_L, $C_{D,w}$ and flow field structure (such as shock wave shapes and Mach wave patterns) become essentially independent of M_∞

↳ consistent with Newtonian theory that gives

$$C_p = 2\sin^2\theta$$



↳ Accuracy of Newtonian theory increases as M_∞ increases

↳ Accuracy is better for 3D than 2D bodies

$\boxed{\text{Proof}}$

For flow through an oblique shock

$$\frac{p}{p_\infty} = \frac{2\gamma}{\gamma+1}\underbrace{\left(M_\infty^2 \sin^2\beta\right)}_{M_{n_1}} - \frac{\gamma-1}{\gamma+1} \quad \text{(Exact)}$$

As $M_\infty \to \infty$

$$\boxed{\frac{p}{p_\infty} = \frac{2\gamma}{\gamma+1} M_\infty^2 \sin^2\beta}$$

$$C_p = \frac{2}{\gamma M_\infty^2}\left[\frac{p}{p_\infty} - 1\right]$$

$$= \frac{4}{\gamma+1}\left[\sin^2\beta - \frac{1}{M_\infty^2}\right]$$ (Exact)

As $M_\infty \to \infty$

$$\boxed{C_{p_{M_\infty\to\infty}} = \frac{4}{\gamma+1}\sin^2\beta}$$ Independent of M_∞

If we additionally take the limit $\gamma \to 1$

$$C_{p_{\substack{M_\infty\to\infty \\ \gamma\to 1}}} = 2\sin^2\beta$$

$\delta - \beta - M_\infty$ relationship gives

$$\tan\theta = 2\cot\beta\left[\frac{M_\infty^2\sin^2\beta - 1}{M_\infty^2(\gamma + \cos 2\beta) + 2}\right]$$ (Exact)

For small θ, β is small

$\tan\theta \approx \theta$, $\sin\beta \approx \beta$, $\cos 2\beta \approx 1$

$$\theta = \frac{2}{\beta}\left[\frac{M_\infty^2\beta^2 - 1}{M_\infty^2(\gamma+1) + 2}\right]$$

As $M_\infty \to \infty$

$$\theta = \frac{2}{\beta}\left[\frac{M_\infty^2 \beta^2}{M_\infty^2 (\gamma+1)}\right] = \frac{1}{\beta}\,\frac{2}{\gamma+1}$$

or

$$\boxed{\frac{\beta}{\theta} = \frac{\gamma+1}{2}}$$

For $\gamma = 1.4$, $\beta = 1.2\,\theta$ (only 20% larger)

In the limit $\gamma \to 1$ $\beta = \theta$

$$\therefore \boxed{\underset{\substack{M_\infty \to 0 \\ \gamma \to 1 \\ \beta, \theta \text{ small}}}{C_p} = 2\sin^2\theta}$$

result from exact oblique shock theory.

Remarks

1. $\beta = \theta$ implies that the shock wave lies on the body.

$$\frac{\rho}{\rho_\infty} = \frac{(\gamma+1) M_\infty^2 \sin^2\beta}{(\gamma-1) M_\infty^2 \sin^2\beta + 2}$$

as $M_\infty \to \infty$ $\dfrac{\rho}{\rho_\infty} = \dfrac{\gamma+1}{\gamma-1} \approx 6$ for $\gamma = 1.4$

$\dfrac{\rho}{\rho_\infty} \approx 20$ during reentry of Apollo Command Module (ACM)

as $\gamma \to 1$ $\dfrac{\rho_2}{\rho_\infty} \to \infty$

The density behind shock is infinitely large requiring shock to lie on the body to satisfy mass flow considerations.

2. The closer actual problem is to $M_\infty \to \infty$ and $\gamma \to 1$, the closer the problem is to Newtonian flow. The latter is not practically possible ($\gamma > 1$) (the agreements with experiments is fortuitous)

Hypersonic Expansion wave relations

$$\theta = \nu(M_2) - \nu(M_1) \quad \text{where}$$

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\left[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right] - \tan^{-1}\sqrt{M^2-1}$$

is the Prandtl-Meyer function.

For $M_\infty \gg 1$

$$\nu(M) \approx \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}}M - \tan^{-1}M$$

Now

$$\tan^{-1}x = \frac{\pi}{2} - \tan^{-1}\frac{1}{x}$$

and

$$\tan^{-1}\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} \cdots$$

$$\Rightarrow \quad \nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\left(\frac{\pi}{2} - \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{1}{M} + \cdots\right)$$
$$- \left(\frac{\pi}{2} - \frac{1}{M} + \cdots\right.$$

$$\Rightarrow \cdot \nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\ \frac{\pi}{2} - \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{M} - \frac{\pi}{2} + \frac{1}{M}$$

$$\therefore \quad \theta = \left(\frac{\gamma+1}{\gamma-1}\right)\left[\frac{1}{M_1} - \frac{1}{M_2}\right] + \left(\frac{1}{M_2} - \frac{1}{M_1}\right)$$

$$\boxed{\theta = \frac{2}{\gamma-1}\left(\frac{1}{M_1} - \frac{1}{M_2}\right)}$$

Alternately,

$$\frac{M_1}{M_2} = 1 - \frac{\gamma-1}{2} M_1 \theta$$

Since the flow through an expansion fan is isentropic

$$\frac{p_2}{p_1} = \frac{p_2/p_{0_2}}{p_1/p_{01}} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}} \approx \left(\frac{M_1}{M_2}\right)^{\frac{2\gamma}{\gamma-1}}$$

$$\Rightarrow \boxed{\frac{p_2}{p_1} = \left[1 - \frac{\gamma-1}{2} M_1 \theta\right]^{\frac{\gamma}{\gamma-1}}}$$

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p_2}{p_\infty} - 1\right)$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left\{\left(1 - \frac{\gamma-1}{2} M_\infty \theta\right)^{\frac{\gamma}{\gamma-1}} - 1\right\}}$$

Hypersonic similarity

Consider a slender body in hypersonic flow.

Flow tangency condition gives

$$v' \approx U_\infty \theta$$

$$\text{or} \quad \frac{v'}{a_\infty} = M_\infty \theta = K$$

$\Rightarrow$ $M_\infty \theta$ is a disturbance indicator in hypersonic flows; a similarity parameter K.

If two different flow problems have the same values of K, then they are similar flows and will have like solutions.