AE 339 : High speed aerodynamics

(Module III : Supersonic Flow Turning)

a. Oblique shock waves
b. Expansion waves

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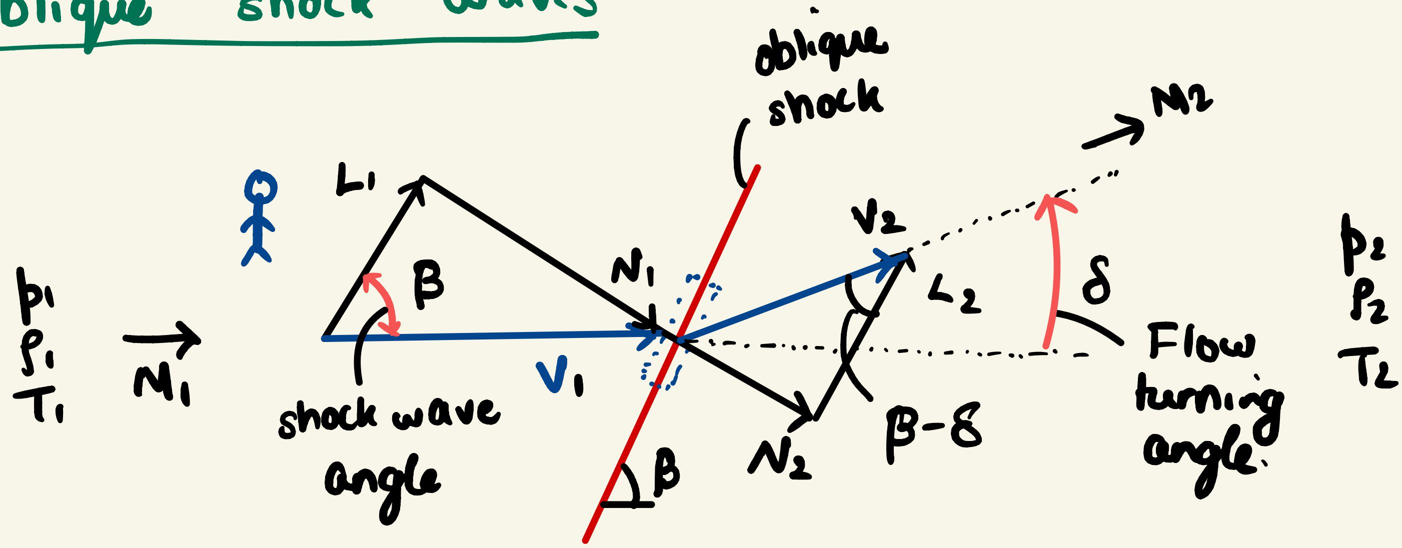
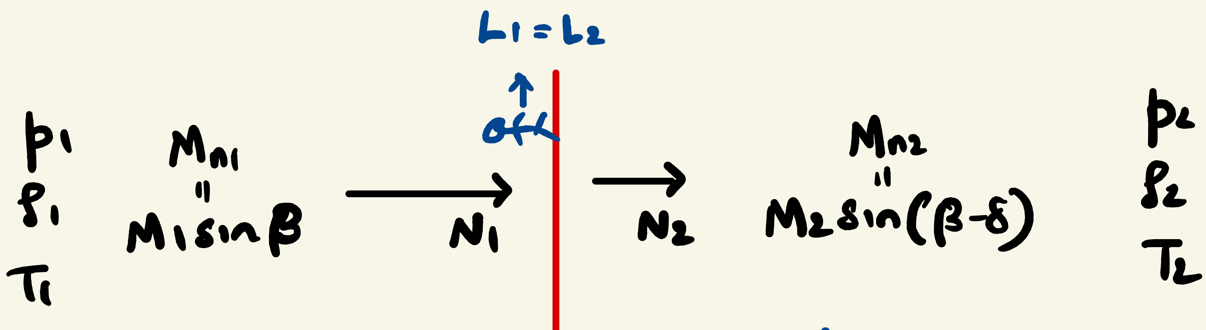
Oblique shock waves





Equivalent normal shock problem

$\boxed{\text{Continuity}}$

$$\rho_1 N_1 = \rho_2 N_2 \quad \text{--- (1)}$$

$\boxed{\text{Momentum}}$

$$p_1 - p_2 = \rho_2 N_2^2 - \rho_1 N_1^2 \quad \text{--- (2)}$$

$\boxed{\text{Energy}}$

$$\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1} + V_1^2 = \frac{2\gamma}{\gamma-1}\frac{p_2}{\rho_2} + V_2^2$$

$$\Rightarrow \left(\frac{2\gamma}{\gamma-1}\right)\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right] = N_1^2 - N_2^2 \quad \text{--- (3)}$$

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1} + 1}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_2}{p_1}} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1}}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_1}{p_2}}$$

Rankine - Hugoniot relations for oblique shock waves

$N_1 = V_1 \sin\beta$ $\qquad$ $N_2 = V_2 \sin(\beta - \delta)$

$\Rightarrow$ Instead of M_1 and M_2, we can substitute $M_{n1} = M_1 \sin\beta$ and $M_{n2} = M_2 \sin(\beta - \delta)$ in the normal shock relationships

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\beta - (\gamma-1)}{\gamma+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2 \sin^2\beta}{2 + (\gamma-1) M_1^2 \sin^2\beta}$$

$$\frac{T_2}{T_1} = \frac{\left[2 + (\gamma-1) M_1^2 \sin^2\beta\right]\left[2\gamma M_1^2 \sin^2\beta - (\gamma-1)\right]}{(\gamma+1)^2 M_1^2 \sin^2\beta}$$

$$M_2^2 \sin^2(\beta - \delta) = \frac{M_1^2 \sin^2\beta + 2/(\gamma-1)}{2\gamma M_1^2 \sin^2\beta/(\gamma-1) - 1}$$

From consideration of normal shock waves

$$M_1 \sin\beta \geq 1$$

For a given M_1, $\beta_{max} = 90^\circ$ Normal shock

$\beta_{min} = \sin^{-1}\frac{1}{M_1}$ Mach wave

$$\boxed{\sin^{-1}\frac{1}{M_1} \leq \beta \leq 90^\circ}$$

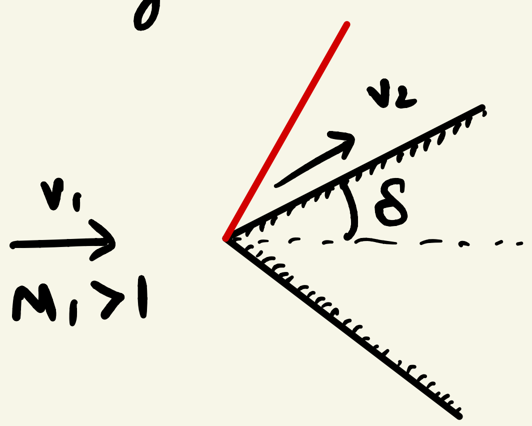
Also, from normal shock wave analysis

$$M_2 \sin(\beta - \delta) \leq 1$$

M_2 can be supersonic for oblique shock waves

Oblique shocks are one of two ways in which flow turning is achieved in supersonic flows.

For a given M_1 & δ, β is usually unknown



Flow has to remain parallel to the wall which is ensured by an oblique shock of appropriate strength/inclination

$\Rightarrow$ We need a relation connecting M_1, δ and β

We have

$$\tan\beta = \frac{N_1}{L_1}, \qquad \tan(\beta-\delta) = \frac{N_2}{L_2}$$

$$\Rightarrow \frac{\tan(\beta-\delta)}{\tan\beta} = \frac{N_2}{N_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta} = X$$

From l.h.s. and r.h.s., we get

$$\left(\frac{\tan\beta-\tan\delta}{1+\tan\beta\tan\delta}\right)\Big/\tan\beta = X$$

$$\Rightarrow \tan\delta = \frac{\tan\beta(1-X)}{1+X\tan^2\beta}$$

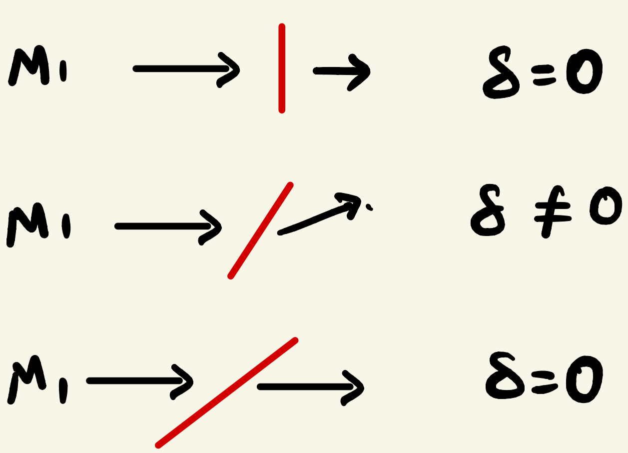
Substituting for X, we have

$$\boxed{\tan\delta = \frac{2\cot\beta\,(M_1^2\sin^2\beta-1)}{2+M_1^2(\gamma+\cos 2\beta)}} \quad - \circledast$$

Two cases where $\delta = 0$

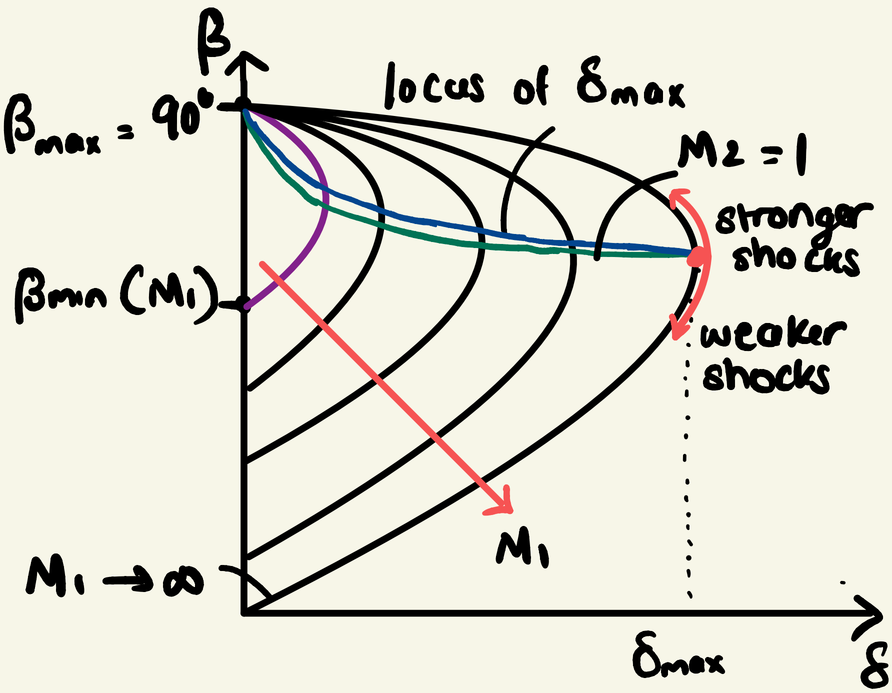
(i) $\cot\beta = 0 \Rightarrow \beta = 90^0$ Normal shock

(ii) $M_1^2\sin^2\beta = 1 \Rightarrow \beta = \sin^{-1}\frac{1}{M_1}$ Mach wave



No flow turning for normal shocks and Mach waves

β decreasing

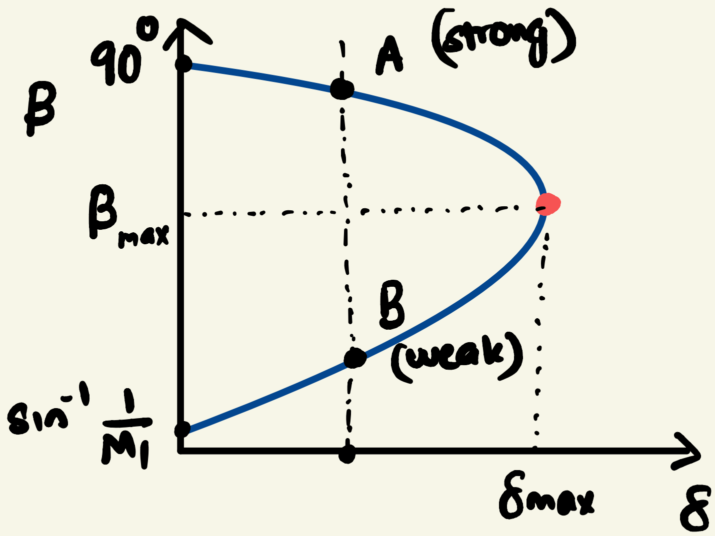


As β increases for fixed M_1, $M_{n1} = M_1 \sin\beta$ increases, p_2/p_1 increases and shocks become stronger

$\beta > \delta_{max}$ (strong shock solution) Always subsonic

$\beta < \delta_{max}$ (weak shock solution) Subsonic / supersonic

Given δ, M_1 two possible solutions for β



β_A – strong shock

β_B – weak shock

External flows – weak shock solutions as the required pressure gradients cannot be maintained

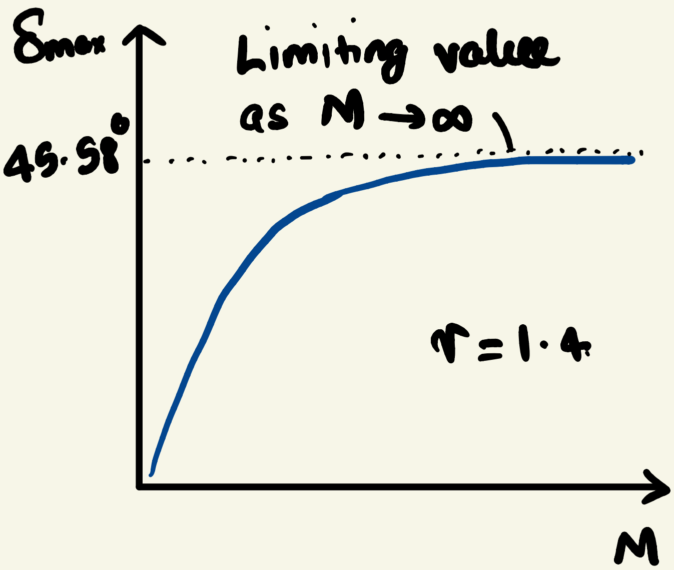
Internal flows – strong shock solution possible as required pressure gradients are easier to maintain

Finding δ_{max}

Set $\dfrac{d\delta}{d\beta} = 0$ in (*)

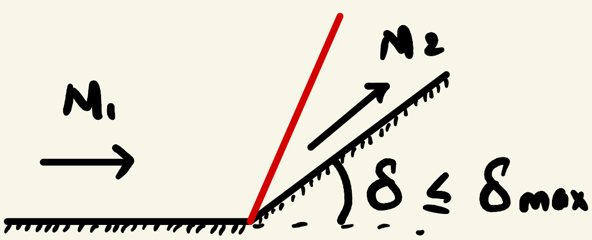
$$\Rightarrow \boxed{\sin^2\beta_{max} = \frac{\gamma+1}{4\gamma} - \frac{1}{\gamma M_1^2}\left[1 - \sqrt{(\gamma+1)\left(1 + \frac{\gamma-1}{2}M_1^2 + \frac{\gamma+1}{16}M_1^4\right)}\,\right]}$$

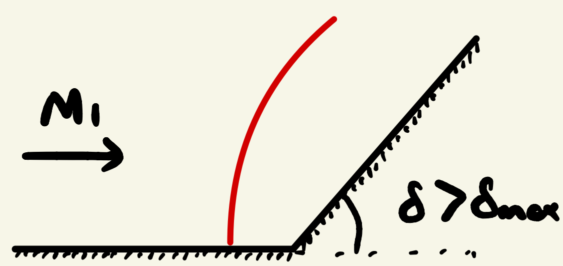
Substitute β_{max} in (*) to get δ_{max}



Tells us maximum achievable flow turning at a given M

Maximum possible turning by an oblique shock = 45.58°

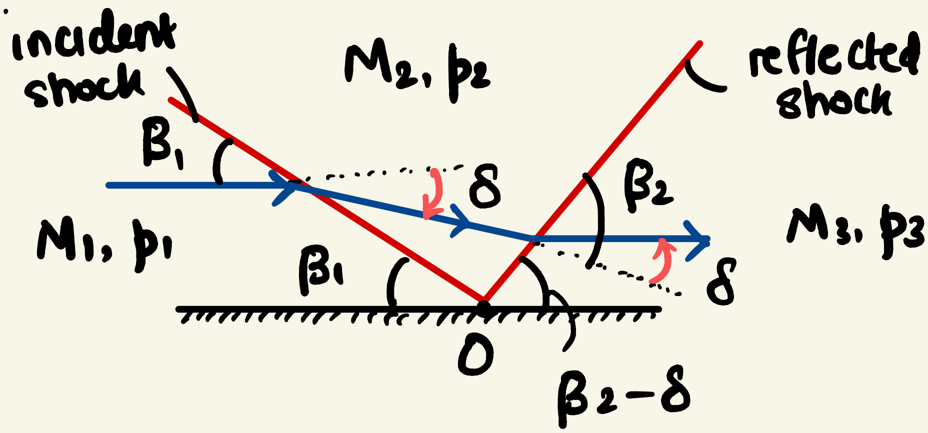




For $\delta > \delta_{max}$, we have a detached curved shock wave which consists of all possible solutions at the M_1 ($\sin^{-1}\frac{1}{M_1} \leq \beta \leq 90^\circ$)

Decreasing M_1 at a given δ shifts the solution from an oblique shock to a detached curved shock.

Reflections of oblique shocks



Normal reflection

(1) For the given M_1, δ, determine β_1 M_2, p_2/p_1

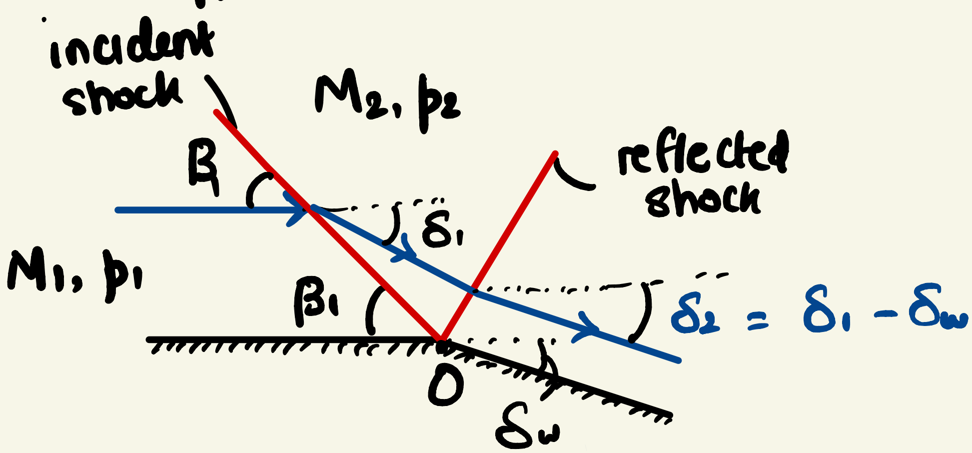
(2) For M_2 and known δ, find β_2, M_3, p_3/p_2

(3) Then overall pressure ratio

$$\frac{p_3}{p_1} = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1}$$

(4) Find $\beta_2 - \delta$ which gives angle of reflected shock

What happens if wall is inclined? (δ_w)



If $\delta_w = \delta_1 \Rightarrow \delta_2 = 0 \Rightarrow$ No reflected shock

What if $\delta_w > \delta_1$?

Supersonic flow

↳ turning into itself — compression (oblique shock)

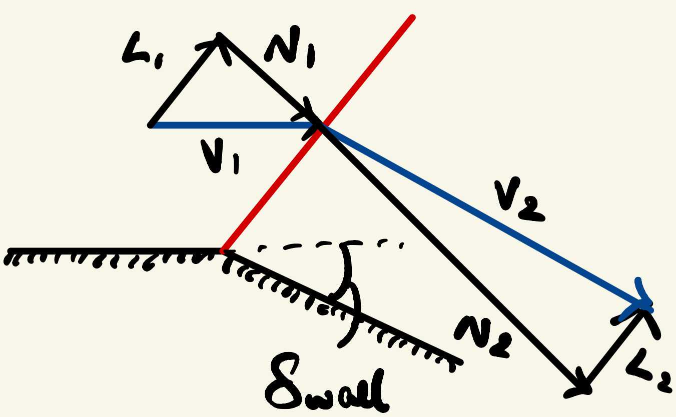
↳ turning away from itself — expansion ?

Happens when

(i) the wall is convex

(2) duct flows when pressure just at the exit is greater than outside pressure outside

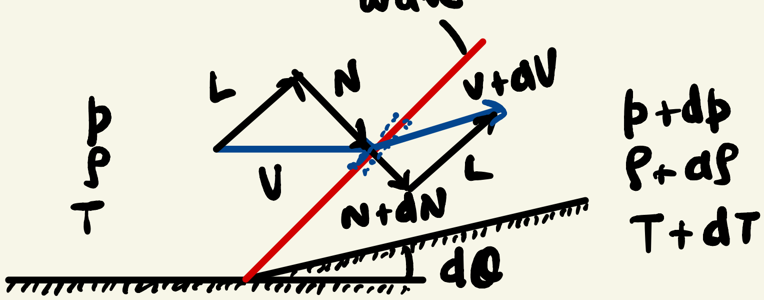
Can oblique shocks achieve this turn?



$N_2 > N_1$ which is unphysical because 2nd law of thermodynamics is violated (Expansion shocks)

Oblique shocks cannot make the flow turn this way

Solution procedure : consider an infinitesimal turn $d\theta$



$d\theta$ is infinitesimal. Also $d\rho$, dp, dT, dV small

$\Rightarrow$ flow $\approx$ isentropic

Since flow is isentropic $d\theta$ can be +ve or −ve

↳ +ve for 'compression', −ve for 'expansion'

Continuity

$$\rho N = (\rho + d\rho)(N + dN)$$

$$\Rightarrow \rho\, dN + N\, d\rho = 0 \quad — ①$$

Momentum

$$\cancel{p} - (\cancel{p} + dp) = \rho N(\cancel{N} + dN - \cancel{N})$$

$$\Rightarrow -dp = \rho N\, dN \quad — ②$$ Euler's equation

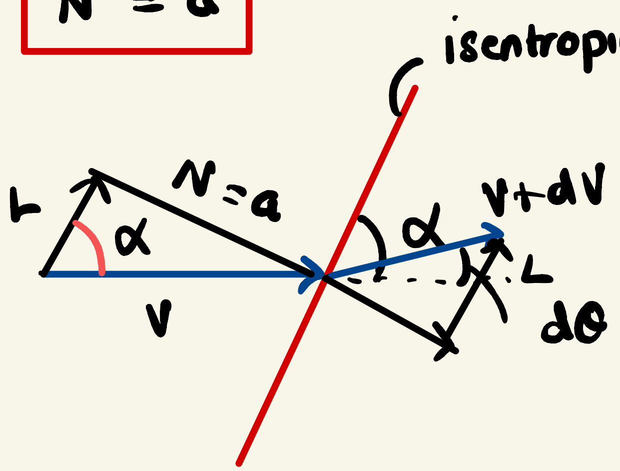
Substitute for dN from ① in ②

$$+dp = \cancel{\rho} N\left(+\frac{N\, d\rho}{\cancel{\rho}}\right)$$

$$\Rightarrow \frac{dp}{d\rho} = N^2$$

Process nearly isentropic $\left.\frac{dp}{d\rho}\right)_s = a^2 = \frac{\gamma p}{\rho}$

$$\Rightarrow \boxed{N = a}$$



α = Mach angle

$$L = V\cos\alpha = (V + dV)\cos(\alpha - d\theta)$$

$$\Rightarrow (V + dV)[\cos\alpha \underbrace{\cos d\theta}_{\approx 1} + \sin\alpha \underbrace{\sin d\theta}_{\approx d\theta}] = V\cos\alpha$$

$$\Rightarrow \cancel{V\cos\alpha} + V\sin\alpha\, d\theta + dV\cos\alpha = \cancel{V\cos\alpha}$$

$$\Rightarrow \frac{dV}{V} = -\tan\alpha \, d\theta \qquad \boxed{\tan\alpha = \frac{1}{\sqrt{M^2-1}}}$$

$$\Rightarrow \boxed{\frac{dV}{V} = \frac{-d\theta}{\sqrt{M^2-1}}} \quad — \quad (*)$$

Energy

$$\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right) + V^2 = \left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p+dp}{\rho+d\rho}\right) + (V+dV)^2$$

$$\cancel{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right)} + \cancel{V^2} = \left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right)\left[\cancel{1} + \frac{dp}{p} - \frac{d\rho}{\rho}\right] + \cancel{V^2} + 2V\,dV$$

$$\Rightarrow \left(\frac{\cancel{2}\gamma}{\gamma-1}\right)\overbrace{\left(\frac{p}{\rho}\right)}^{a^2}\left[\frac{dp}{p} - \frac{1}{\gamma}\frac{dp}{p}\cdot\underbrace{\frac{\gamma p}{\rho}}_{a^2}\cdot\underbrace{\frac{d\rho}{dp}}_{1/a^2}\right] = -\cancel{2}V\,dV$$

$$\Rightarrow \left(\frac{a^2}{\gamma-1}\right)\frac{dp}{p}\left[1 - \frac{1}{\gamma}\right] = -V\,dV$$

$$\Rightarrow \frac{dp}{p} = -\frac{\gamma V\,dV}{a^2} = -\frac{\gamma V^2}{a^2}\frac{dV}{V} = -\gamma M^2\frac{dV}{V}$$

Substitute $(*)$

$$\boxed{\frac{dp}{p} = \frac{\gamma M^2}{\sqrt{M^2-1}}\, d\theta} \quad — \quad (**)$$

$$\frac{d\rho}{\rho} = \frac{d\rho}{dp}\cdot\frac{dp}{p}\cdot\frac{p}{\rho} = \frac{1}{a^2}\frac{dp}{p}\cdot\frac{a^2}{\gamma} = \frac{1}{\gamma}\frac{dp}{p}$$

$$\Rightarrow \boxed{\frac{d\rho}{\rho} = \frac{M^2}{\sqrt{M^2-1}}\, d\theta} \quad — \quad (***)$$

Entropy

$$\frac{ds}{R} = \underbrace{\frac{1}{\gamma-1}}_{C_v/R} \ln\underbrace{\left[1+\frac{dp}{p}\right]}_{p_2/p_1} - \underbrace{\frac{\gamma}{\gamma-1}}_{C_p/R} \ln\underbrace{\left[1+\frac{d\rho}{\rho}\right]}_{\rho_2/\rho_1}$$

$$\approx \frac{1}{\gamma-1}\frac{dp}{p} - \frac{\gamma}{\gamma-1}\frac{d\rho}{\rho} = 0 \quad \text{from } (**) \text{ and } (***)$$

Flow process is isentropic

what is the fractional change in Mach number?

$$M^2 = \frac{V^2}{a^2} = \frac{V^2\rho}{\gamma p}$$

$$\Rightarrow 2\frac{dM}{M} = 2\frac{dV}{V} + \frac{d\rho}{\rho} - \frac{dp}{p}$$

$$= \left[2 - M^2 + \gamma M^2\right]\frac{-d\theta}{\sqrt{M^2-1}}$$

$$\Rightarrow \boxed{\frac{dM}{M} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]\frac{(-d\theta)}{\sqrt{M^2-1}}} \quad - (****)$$

For a differential isentropic ($ds=0$) flow turn ($d\theta$)

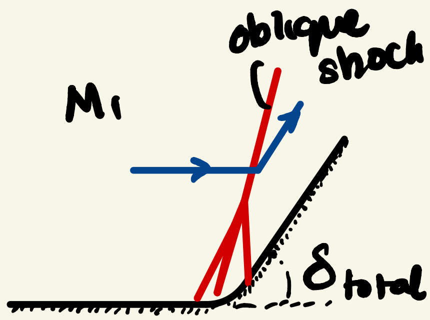
dV	$\propto$	$-d\theta$
dp	$\propto$	$d\theta$
$d\rho$	$\propto$	$d\theta$
dM	$\propto$	$-d\theta$

All property changes directly proportional to $d\theta$

If results seem counterintuitive to intuition, that is because our intuition is SUBSONIC

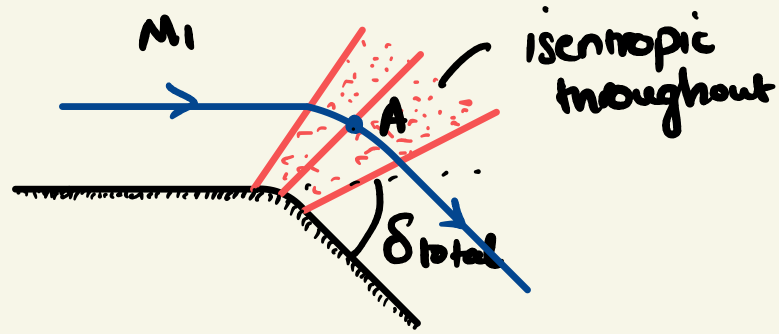
Prandtl - Meyer flows

$\boxed{d\theta > 0}$



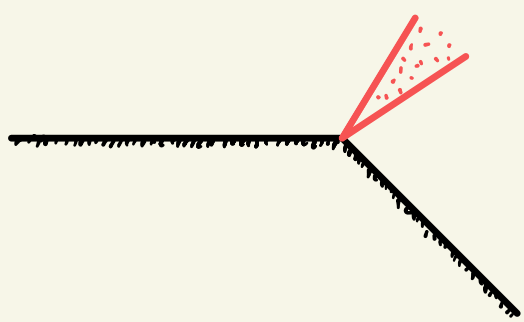
Mach waves converge

$\boxed{d\theta < 0}$

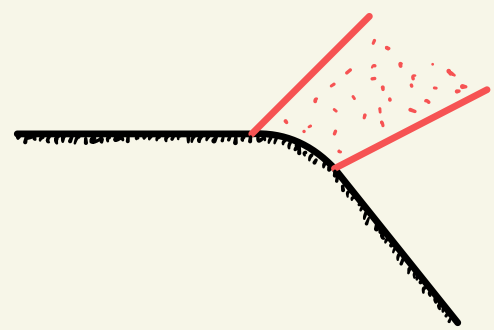


Mach waves diverge

$d\theta < 0$ produces Prandtl-Meyer expansion waves



centered expansion fan



noncentered expansion fan

$$\int_{\theta_1}^{\theta_2} -d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

At $M = 1$, $\theta = 0$, $V = a = a^*$ Reference condition

$$\int_{0}^{\theta_2} d\theta - \int_{0}^{\theta_1} d\theta = \int_{1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M} - \int_{1}^{M_1} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

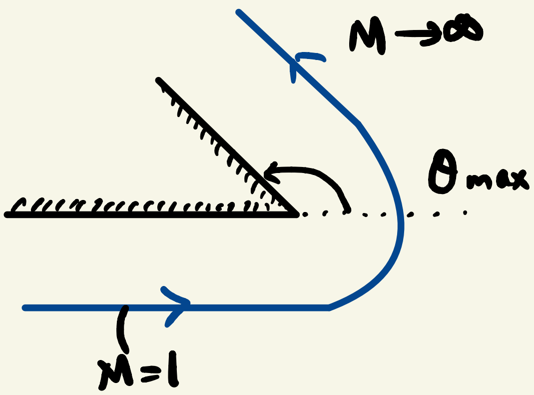
$$\Rightarrow \theta = \int_{1}^{M} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$ function only of M

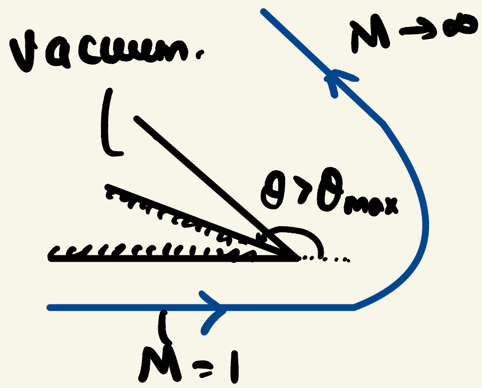
$$\boxed{\theta = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}}$$

tabulated in isentropic tables

As $M \to \infty$, $\theta_{max} = \sqrt{\frac{\gamma+1}{\gamma-1}}\,\frac{\pi}{2} - \frac{\pi}{2} = 130.5^\circ$ for $\gamma = 1.4$

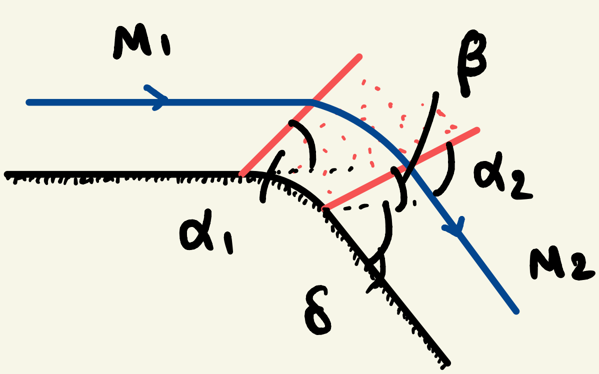
If we turn a $M=1$ flow by 130.5° we achieve $M \to \infty$ downstream.





$p, \rho, T \to 0$ downstream

Not realized physically as continuum violated, ideal gas assumption violated, liquefaction etc.

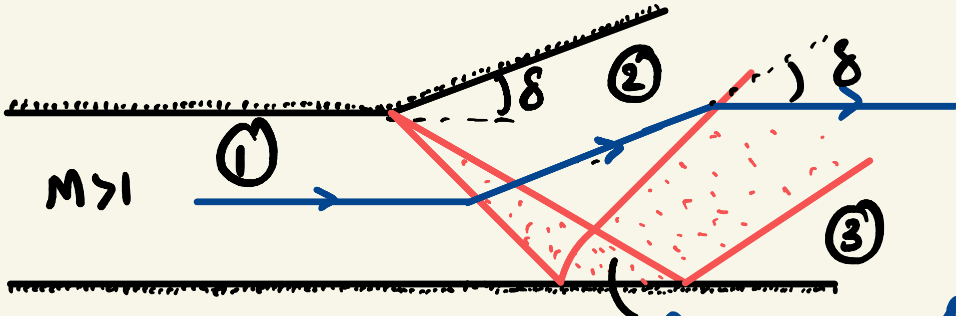


$\alpha_1 = \sin^{-1}\frac{1}{M_1}$

$\alpha_2 = \sin^{-1}\frac{1}{M_2}$

$\beta = \alpha_2 - \delta$

Reflection of expansion waves

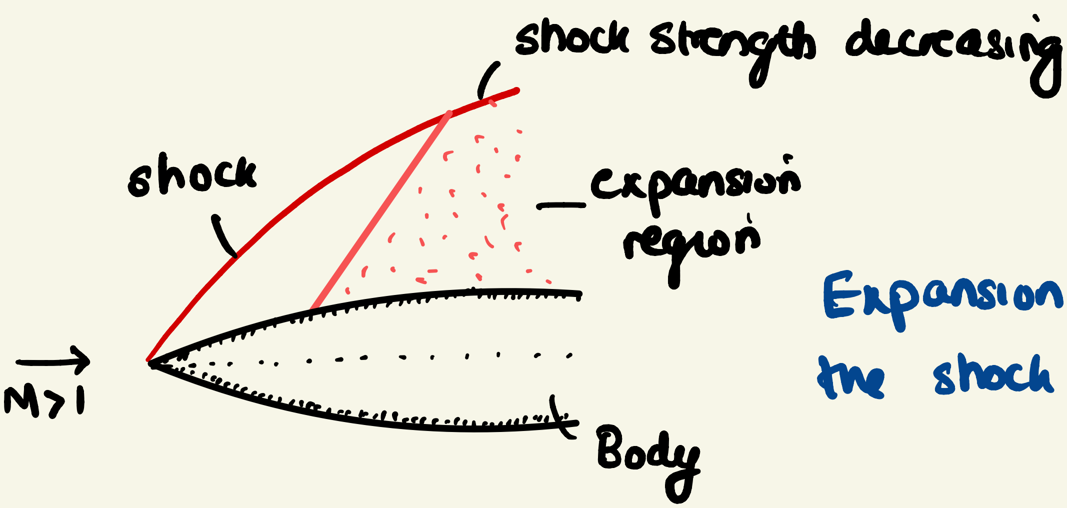


cannot use θ equation here
(NON SIMPLE REGION)

$$\theta_2 = \theta_1 + \delta$$

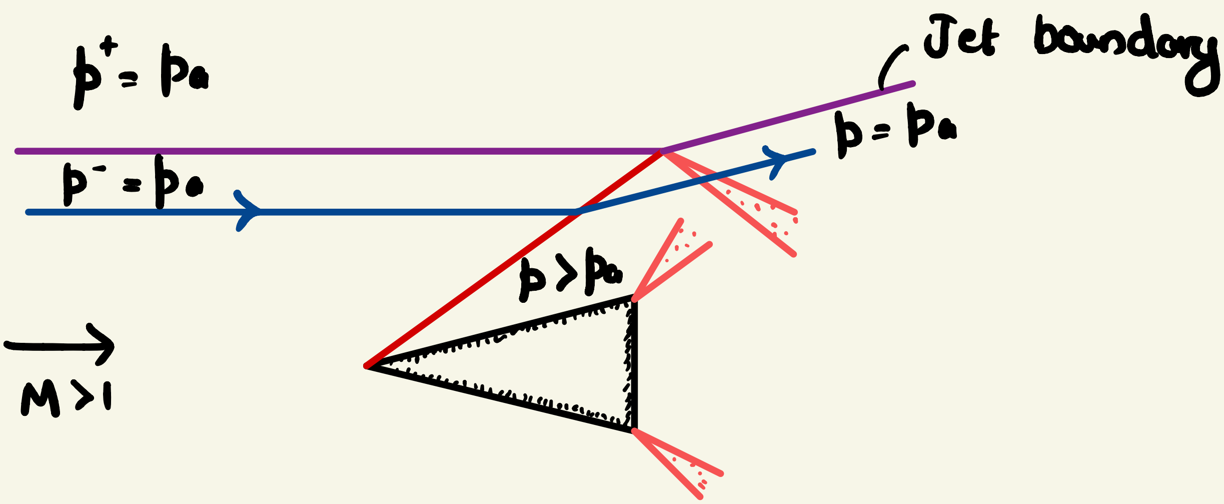
$$\theta_3 = \theta_2 + \delta = \theta_1 + 2\delta$$

Interaction of expansion waves with shock



Expansion fans weaken the shock on interaction

Interaction of an oblique shock with a jet boundary



Oblique shock interacting with a jet boundary produces an expansion wave and vice versa.