Supersonic Nozzle Design

High Speed Aerodynamics

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Contents

Abstract

In this report, we design a supersonic rocket nozzle using the Method of Characteristics (MOC) to achieve an exit Mach number of 2.4. The process involves determining the wall contour that allows smooth and shock-free expansion from subsonic to supersonic flow within the nozzle. Results are presented to validate the effectiveness of the design and assess its minimum length configuration.

1 Introduction

A supersonic nozzle is crucial in applications requiring high-speed exhaust flows, such as rocket engines and jet engines. This report details the use of the Method of Characteristics (MOC) for designing a nozzle that minimizes length while achieving the target Mach number. Supersonic nozzles are designed to gradually expand the gas from subsonic to supersonic velocities without shock waves.

2 Theory of Method of Characteristics

The physical conditions of a two-dimensional, steady, isentropic, and irrotational flow are mathematically represented by a nonlinear differential equation for the velocity potential. The method of characteristics provides a mathematical approach to solve this velocity potential equation, allowing for the application of boundary conditions that transform the governing partial differential equations (PDEs) into ordinary differential equations (ODEs)

2.1 Governing Equations

Governing Equations for a two-dimensional compressible, irrotational flow can be written as

$$(u^2 - a^2)\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2 - a^2)\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

The above equation is a statement that the flow is irrotational. These governing equations form a non-linear set of Partial Differential Equations. The solutions can be classified as follows:

- **Elliptic**, if $(u^2 + v^2)/a^2 < 1$
- **Parabolic**, if $(u^2 + v^2)/a^2 = 1$
- **Hyperbolic**, if $(u^2 + v^2)/a^2 > 1$

Supersonic flows with $M > 1$, belong to the *Hyperbolic* class. One of the properties of Hyperbolic Equations is that there exist what are called the characteristic lines or directions. In a supersonic flow at every point there can exist small disturbance waves called Mach Waves.

These are, in fact, the characteristic lines. The direction of Mach Waves is the characteristic direction. Across a characteristic line, velocity derivatives may be discontinuous, but velocity itself will be discontinuous. Along the characteristic lines, the Compatibility Relations hold good.

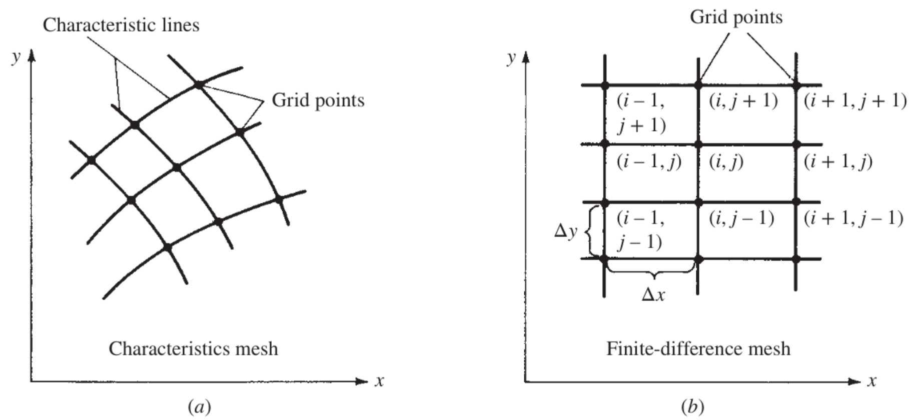


Figure 1: Characteristics lines

2.2 Compatibility Relations

- Since the velocity potential and its derivatives are functions of x and y, we have:

$$d\left(\frac{\partial \Phi}{\partial x}\right) = du = \frac{\partial^2 \Phi}{\partial x^2}dx + \frac{\partial^2 \Phi}{\partial x \partial y}dy$$

$$d\left(\frac{\partial \Phi}{\partial y}\right) = dv = \frac{\partial^2 \Phi}{\partial y^2}dy + \frac{\partial^2 \Phi}{\partial x \partial y}dx$$

- As these equations represent three equations with three unknowns, using Cramer's rule we get:

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \begin{vmatrix} 1-\frac{u^2}{a^2} & 0 & 1-\frac{v^2}{a^2} \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix} \begin{vmatrix} 1-\frac{u^2}{a^2} & -\frac{2uv}{a} & 1-\frac{v^2}{a^2} \\ 0 & dy & dx \\ 0 & 0 & dy \end{vmatrix}^{-1} = \frac{N}{D}$$

- If the denominator D is chosen such that $D = 0$, then the numerator N must also be zero ($N = 0$), as we know that $\frac{\partial^2 \phi}{\partial x \partial y}$ has a specific defined value at every point in the flow.

- Thus, we can say that there is some direction at every point (A) along which $\frac{\partial^2 \phi}{\partial x \partial y}$ is **indeterminate**, which is the characteristic line. The precise direction of these lines can be calculated as follows:

- Consider the point A in the flow field and set the denominator D to zero. Expanding the determinant D and setting it to zero, we get:

$$\left[1-\frac{u^2}{a^2}\right]\left(\frac{dy}{dx}\right)^2_{\text{char}} + \left[1-\frac{v^2}{a^2}\right] + \frac{2uv}{a^2}\left(\frac{dy}{dx}\right)_{\text{char}} = 0$$

- Here $\left(\frac{dy}{dx}\right)_{\text{char}}$ represents the slope of the characteristic lines and is given by the roots of this quadratic equation:

$$\left(\frac{dy}{dx}\right)_{\text{char}} = \frac{-\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}}{1 - \frac{u^2}{a^2}}$$

- Since $u = V\cos\theta$ and $v = V\sin\theta$, and the local Mach angle μ is given by $\mu = \sin^{-1}\frac{1}{M}$, the slope becomes:

$$\left(\frac{dy}{dx}\right)_{\text{char}} = \left(\frac{-\cos\theta\sin\theta}{\sin^2\mu} \pm \sqrt{\frac{\cos^2\theta + \sin^2\theta}{\sin^2\mu} - 1}\right)\left(1 - \frac{\cos^2\theta}{\sin^2\mu}\right)^{-1}$$

- After some algebraic and trigonometric manipulation, we get:

$$\left(\frac{dy}{dx}\right)_{\text{char}} = \tan(\theta \mp \mu)$$

- This equation states that the two characteristic lines running through the point A have slopes equal to $\tan(\theta - \mu)$ shown by C_- and $\tan(\theta + \mu)$ depicted by C_+.
- The characteristic lines through the point A are simply the left and right running Mach waves through the point, i.e., the characteristic lines are Mach lines.
- Note that the characteristic lines are curved in space because the local Mach angle depends on the local Mach number, which is a function of both x and y. Moreover, the local streamline direction θ also varies throughout the flow.
- Along the characteristic lines, the governing partial differential equation describing the flow reduces to ordinary differential equations known as **compatibility equations**.
- These can be found by setting the numerator determinant to zero, $N = 0$, giving:

$$\frac{dv}{du} = -\frac{\left(1 - \frac{u^2}{a^2}\right)}{1 - \frac{v^2}{a^2}}\left(\frac{dy}{dx}\right)_{\text{char}}$$

- Substituting here the slope of the characteristic lines, $u = V\cos\theta$ and $v = V\sin\theta$, and after some algebraic manipulations, we get the following ODEs:

$$d\theta = -\sqrt{M^2 - 1}\,\frac{dV}{V} \quad \text{along } C_-$$

$$d\theta = \sqrt{M^2 - 1}\,\frac{dV}{V} \quad \text{along } C_+$$

- These equations can be integrated to obtain a result in terms of the Prandtl-Meyer function as shown:

$$\theta + \nu(M) = \text{const} = K_- \quad \text{along } C_-$$
$$\theta - \nu(M) = \text{const} = K_+ \quad \text{along } C_+$$

Here K_- and K_+ are different constants along different C_- and C_+ characteristics and

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \arctan\left(\sqrt{M^2-1}\right)$$

- The above compatibility equations are now reduced to algebraic equations. In general inviscid supersonic steady flow, the compatibility equations are ODEs.
- In the case of a 2D irrotational flow, they further reduce to algebraic equations.
- The equations can be combined to obtain simple expressions to calculate θ and ν.

$$\theta = \frac{1}{2}\left[(K_-) + (K_+)\right]$$

$$\nu = \frac{1}{2}\left[(K_-) - (K_+)\right]$$

- Next, we discuss how we can use these results to calculate the supersonic flow inside a nozzle and determine a proper wall contour so that no shock waves appear inside the nozzle.

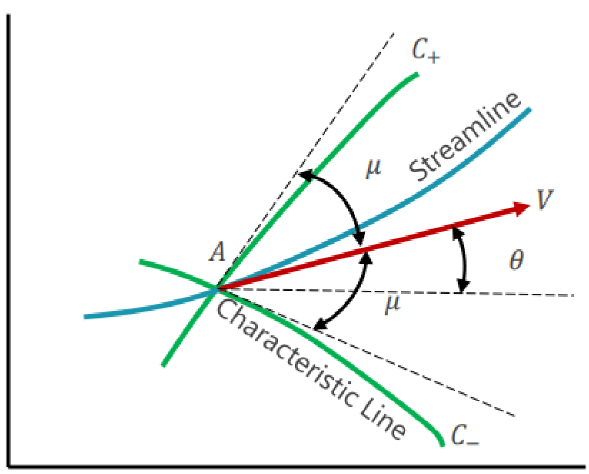


Figure 2: Left and Right Running Characteristics Lines

3 Nozzle Design

Supersonic nozzles are used in a variety of engineering applications to expand a flow to desired supersonic conditions. Supersonic nozzles can be divided into two different types: gradual-expansion nozzles and minimum-length nozzles.

Gradual-expansion nozzles are typically used in applications where maintaining a high-quality flow at the desired exit conditions is important (e.g., supersonic wind tunnels). For other types of applications (e.g., rocket nozzles), the large weight and length penalties associated with gradual-expansion nozzles make them unrealistic; therefore, minimum-length nozzles, which utilize a sharp corner to provide the initial expansion, are commonly used.

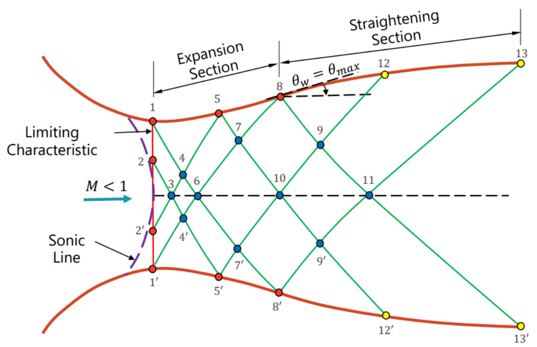


Figure 3: Gradual-Expansion Nozzle

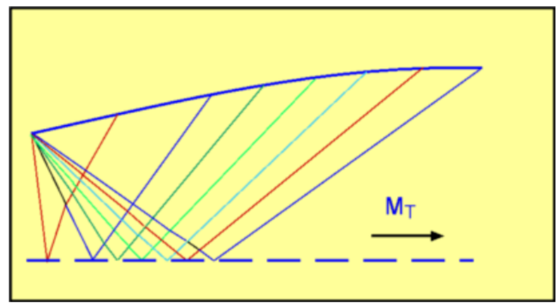


Figure 4: Minimum-Length Nozzle

Figure 5: Types of nozzles

For both gradual-expansion and minimum-length nozzles, the flow can be divided into simple and non-simple regions. A non-simple region is characterized by Mach wave reflections and intersections. In order to meet the requirement of uniform conditions at the nozzle exit, it is desirable to minimize the non-simple region as much as possible.

This can be achieved by designing the nozzle surface such that Mach waves (e.g., characteristics) are not produced or reflected while the flow is straightened. The Method of Characteristics is therefore applied to allow the design of a supersonic nozzle that meets these requirements.

4 Design of Gradual Expansion Supersonic Nozzle

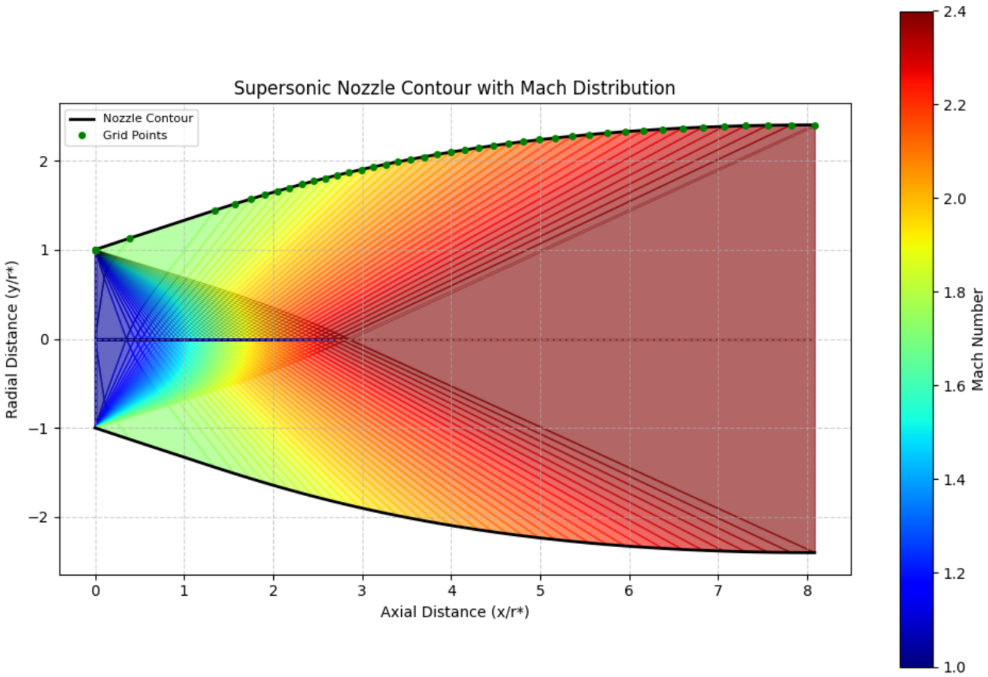
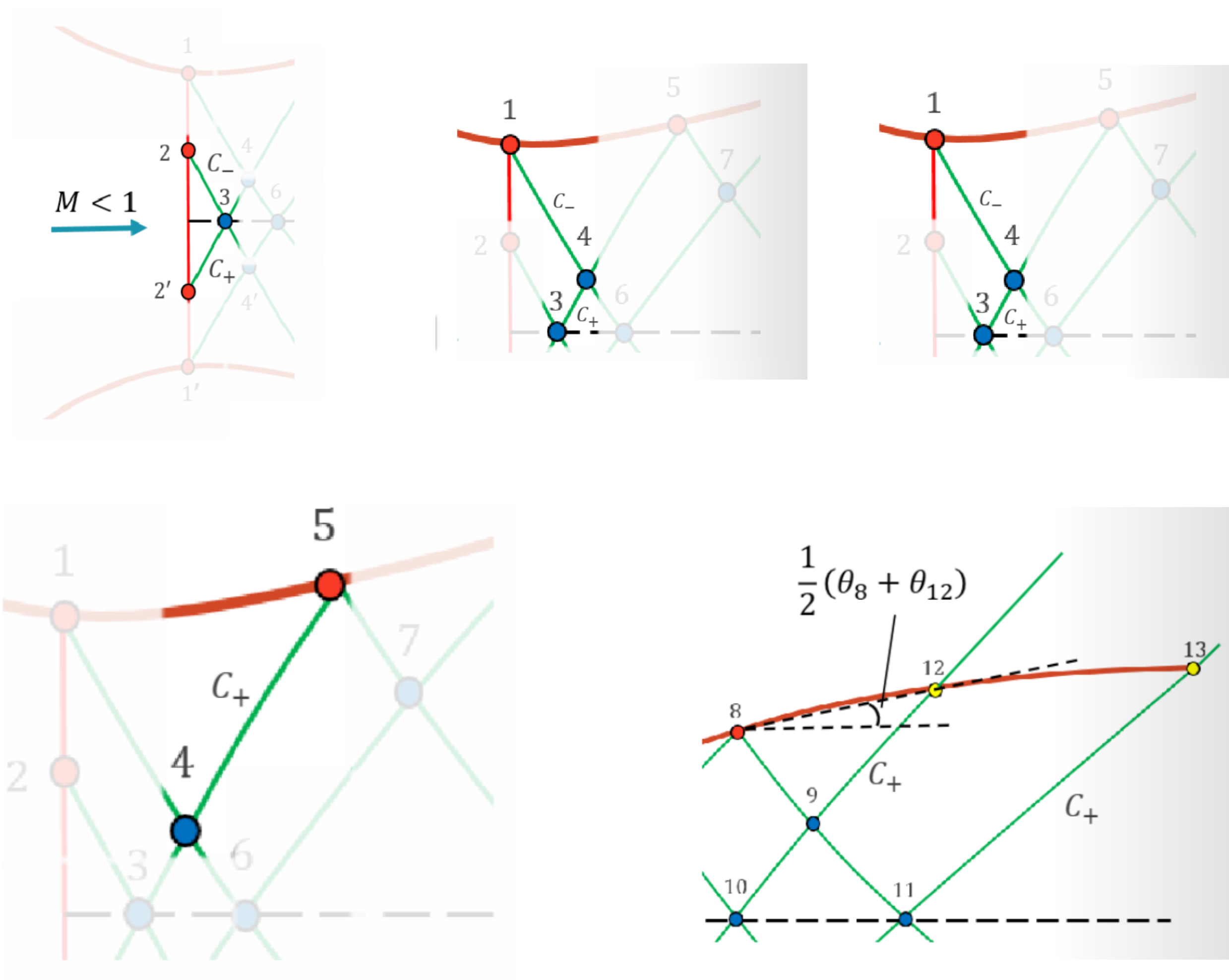


Figure 6: Wall contours of nozzle at exit Mach Number, $M_e = 2.4$

5 Method of Characteristics: Flow Calculation Procedure

- The calculation procedure involves analyzing the flow at grid points, which are the intersections of characteristic lines.
- There are two types of grid points:
  1. Internal Grid Points ($\bullet$)
  2. Wall Grid Points ($\circ$)
- Let's first analyze the internal grid points. Assume we know the location and flow properties at points 1 and 2.
- Point 3 lies on the centerline. The C_- and C_+ characteristics originate from points 2 and $2'$, respectively, and are symmetric.
- The constants along the given C_- and C_+ characteristics are the same and opposite, i.e.,

$$(K_-)_3 = -(K_+)_3.$$



The flow angle at point 3, θ_3, is zero since it is on the centerline, so v_3 can be calculated as:

$$\theta_2 + v_2 = \text{const} = (K_-)_2 = (K_-)_3 \quad \text{along } C_-. \tag{1}$$

$$\theta_2' - v_2' = \text{const} = (K_+)_{2'} = (K_+)_3 \quad \text{along } C_+. \tag{2}$$

$$\theta_3 + v_3 = \text{const} = (K_-)_3 \quad \text{along } C_-. \tag{3}$$

$$v_3 = (K_-)_3 = -(K_+)_3. \tag{4}$$

- Use the flow angle v_3 at point 3 to compute the local Mach number, M_3, which is essential for further calculations as it characterizes the flow speed relative to the speed of sound.
- With M_3 known, compute the static pressure (p_3) and temperature (T_3) at point 3 using stagnation conditions p_0 and T_0, derived from isentropic relations.
- Using T_3, calculate the local speed of sound c_3 as:

$$c_3 = \sqrt{\gamma R T_3},$$

where γ is the specific heat ratio and R is the specific gas constant. Finally, compute the flow velocity V_3 at point 3:

$$V_3 = M_3 c_3.$$

- Point 4 is located at the intersection of the C_- and C_+ characteristics through points 1 and 3. Therefore:

$$\theta_1 + \nu_1 = \text{const} = (K_-)_4 = (K_-)_1 \quad \text{along } C_-,$$

$$\theta_3 - \nu_3 = \text{const} = (K_+)_4 = (K_+)_3 \quad \text{along } C_+.$$

- At point 4, we can express the characteristics as:

$$\theta_4 + \nu_4 = (K_-)_4 \quad \text{along } C_-,$$

$$\theta_4 - \nu_4 = (K_+)_4 \quad \text{along } C_+.$$

  Solving these equations yields:

$$\theta_4 = \frac{1}{2}\left[(K_-)_1 + (K_+)_3\right],$$

$$\nu_4 = \frac{1}{2}\left[(K_-)_1 - (K_+)_3\right].$$

- Using θ_4 and ν_4, calculate the remaining flow properties at point 4 as done for point 3.

- Now consider an internal point (4) close to the wall. The C_+ characteristic through point 4 intersects the wall at point 5. Assume the slope of the wall at point 5, θ_5, is known. Using the known properties at point 4, we can find the flow properties at point 5, as K_+ is constant along the characteristic C_+:

$$\theta_5 + \nu_5 = \text{const} = (K_+)_4 = (K_+)_5 \quad \text{along } C_+.$$

- Given θ_5 and $(K_+)_5$, we can compute ν_5 and the remaining flow properties as outlined earlier.

- For both internal and wall points, the analysis begins with the known properties at specific grid points, followed by marching downstream along the grid defined by the intersection of the characteristic lines.

- Consider wall points in the straightening section of the nozzle, where the slope of the wall at points 12 and 13 is unknown.

- The straightening section is designed such that expansion wave cancellations occur at the wall, meaning no characteristic lines are generated.

- Thus, the flow properties are constant along the characteristic lines C_+ from points 9 and 11:

$$\theta_9 = \theta_{12} \quad \text{and} \quad \nu_9 = \nu_{12},$$

$$\theta_{11} = \theta_{13} \quad \text{and} \quad \nu_{11} = \nu_{13}.$$

- To draw the nozzle profile, start from point 8 and draw a straight line at an angle of $\frac{1}{2}(\theta_8 + \theta_{12})$, which intersects the C_+ line from point 9 at point 12. Then, using $\frac{1}{2}(\theta_{12} + \theta_{13})$, repeat the process from point 12 to point 13 along the C_+ line from point 11.

6 Results and Discussion

1. Ratio of exit Area to Throat Area $\left(\frac{A_e}{A_t}\right) = 2.4031$
2. Exit Mach number, $M_e = 2.4$

As we can see, the area ratio is satisfied by graph as well as output given by code. Also, since there is no abrupt change in Mach number (i.e., smooth variation in Mach number) implies that no shock is present on the nozzle and thereby shock free expansion through nozzle for exit Mach number, $M_e = 2.4$.

7 References

- Aerodynamics for Students
- Method of Characteristics-Ansys
- Google Colab
- Prof. Vineeth Nair Slides