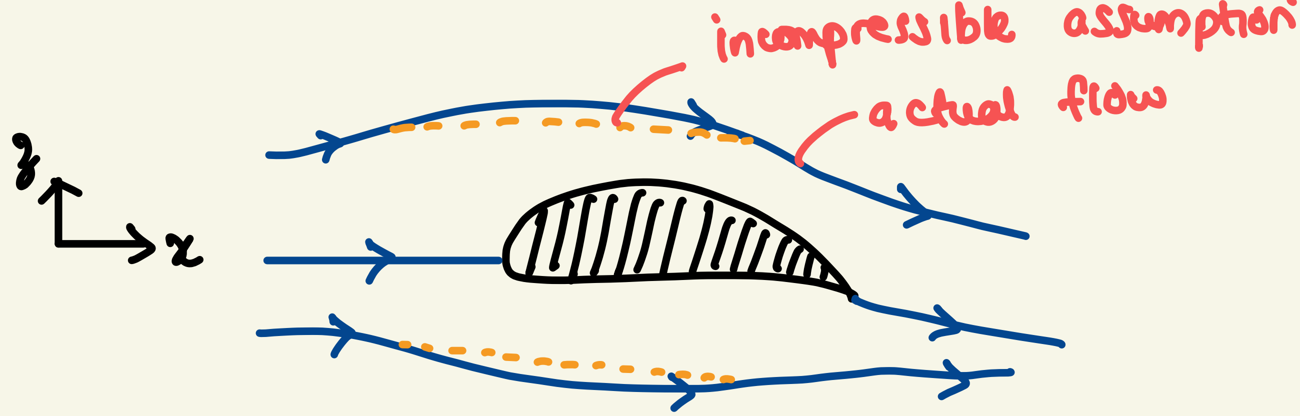
AE 339 : High speed aerodynamics

(V) External aerodynamics: Subsonic Compressible flow past bodies

Prof. Vineeth Nair
Dept. Aerospace Engg.
IIT Bombay

Effects of compressibility in flow



For compressible flows, density decreases as pressure decreases due to flow acceleration. This variable-density flow requires a relatively higher velocity and diverging streamlines in order to get the mass flow past the midsection of the airfoil. The expansion of the minimum cross-section of the streamtubes forces the streamlines outward so that they conform more nearly to the curvature of the airfoil surface. Thus, disturbances caused by the airfoil extends vertically to a greater distance.

Existence of potential motion

Taking curl of momentum equation,

$$\nabla \times \left[\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla p}{\rho} + \nu \nabla^2 u \right],$$

we get

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = (\omega \cdot \nabla) u - \omega (\nabla \cdot u) + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \omega$$

If flow is inviscid and incompressible, there are no sources of rotation in a fluid.

If flow is inviscid and compressible, no sources of rotation provided $p = p(\rho)$ → BAROTROPIC FLOWS

eg: isentropic flows.

The existence of potential motion for compressible flows depends on:

(1) absence of significant viscous forces
(2) existence of a unique relation between pressure and density.

Velocity potential equation (VPE) for 2D flows

Objective: To obtain an equation for ϕ which represents a combination of continuity, momentum and energy equations where

$$\vec{V} = \nabla \phi \quad \text{or}$$

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial z} \quad \text{— (1)}$$

Conservation of mass requires,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial}{\partial z}(\rho v) = 0$$

$$\rho \frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial z} + v\frac{\partial \rho}{\partial z} = 0 \quad \text{substitute } ①$$

$$\rho\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + \frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \rho}{\partial z} = 0 \quad — ②$$

From momentum (Euler's) equation

$$dp = -\rho V dV = -\frac{\rho}{2} d(V^2) = -\frac{\rho}{2} d(u^2 + v^2)$$

$$= -\frac{\rho}{2} d\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]$$

For isentropic flows, $dp = a^2 d\rho$

$$\Rightarrow d\rho = \frac{-\rho}{2a^2} d\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]$$

$$\left.\begin{aligned} \therefore \frac{\partial \rho}{\partial x} &= \frac{-\rho}{a^2}\left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x \partial z}\right] \\ \frac{\partial \rho}{\partial z} &= \frac{-\rho}{a^2}\left[\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial z^2}\right] \end{aligned}\right\} ③$$

Substituting ③ in ②, we get

$$\boxed{\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]\frac{\partial^2\phi}{\partial z^2} -\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial z}\right)\left(\frac{\partial^2\phi}{\partial x\partial z}\right)=0} \quad \text{— 4 (a)}$$

VELOCITY POTENTIAL EQUATION (non linear PDE)

where

$$a^2+\frac{\gamma-1}{2}V^2 = a^2+\frac{\gamma-1}{2}\left(u^2+v^2\right) = a_0^2$$

or

$$\boxed{a^2 = a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]} \quad \text{— 4 (b)}$$

known constant

In principle, 4 can be solved to obtain the flow field around any 2D shape subject to boundary conditions. (SOLVE & BECOME FAMOUS)

Once ϕ is known

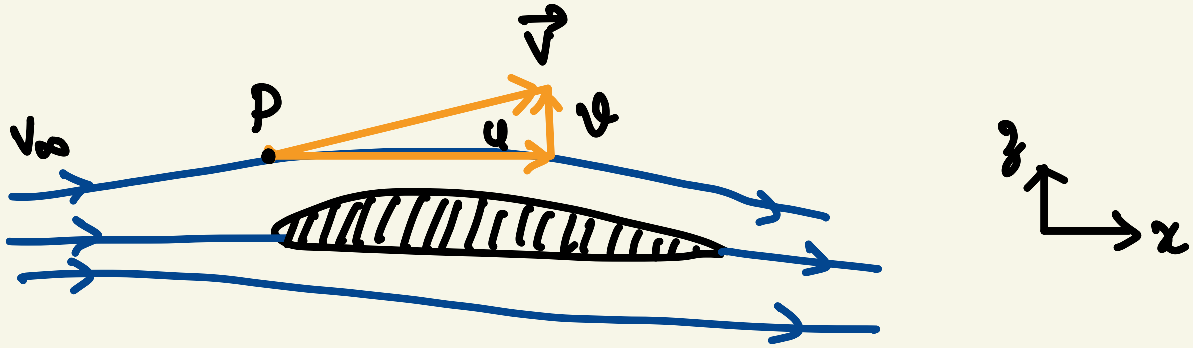
(1) $\vec{V} = \nabla\phi$

(2) $M = u/a$ where a is obtained from 4(b)

(3) T, p, ρ using isentropic relations.

Linearized VPE (small perturbation theory)

We can linearize eqn. 4 to apply it to thin airfoils at small angles of attack.



Body perturbs uniform flow with increments $\hat{u}$, $\hat{v}$ corresponding to potential $\hat{\phi}$

$$u = \frac{\partial \phi}{\partial x} = V_\infty + \hat{u} = V_\infty + \frac{\partial \hat{\phi}}{\partial x}$$

$$v = \frac{\partial \phi}{\partial z} = \hat{v} = \frac{\partial \hat{\phi}}{\partial z}$$

VPE becomes

$$\left[a^2 - (V_\infty + \hat{u})^2\right]\frac{\partial \hat{u}}{\partial x} + \left[a^2 - \hat{v}^2\right]\frac{\partial \hat{v}}{\partial z}$$

$$- 2(V_\infty + \hat{u})\hat{v}\frac{\partial \hat{u}}{\partial z} = 0 \quad — (5)$$

or $\partial \hat{v}/\partial x$

and

$$\frac{2a_\infty^2}{\gamma - 1} + V_\infty^2 = \frac{2a^2}{\gamma - 1} + (V_\infty + \hat{u})^2 + \hat{v}^2 \quad — (6)$$

Substituting (6) in (5), we get

$$\begin{aligned}
\overbrace{(1-M_\infty^2)}^{L_1}\frac{\partial \hat{u}}{\partial x} + \overbrace{\frac{\partial \hat{v}}{\partial z}}^{L_2} = \; & \underbrace{M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial \hat{u}}{\partial x}}_{R_1} \\
& + \underbrace{M_\infty^2\left[(\gamma-1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}\right]\frac{\partial \hat{v}}{\partial z}}_{R_2} \\
& + \underbrace{M_\infty^2\left[\frac{\hat{v}}{V_\infty}\left(1+\frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{v}}{\partial z}\right)\right]}_{R_3} \qquad \text{— (8)}
\end{aligned}$$

Now $\hat{u}/V_\infty, \; \hat{v}/V_\infty \ll 1$

★ For $0 \leq M_\infty \leq 0.8$ or $M_\infty \geq 1.2$ } not transonic

$R_1 \ll L_1$

For $M_\infty < 5$ (approximately) } not hypersonic

$R_2 \ll L_2$ and $R_3 \approx 0$

we have
$$\boxed{(1-M_\infty^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0}$$

Linearized VPE

$0 \leq M_\infty \leq 0.8$

$1.2 \leq M_\infty < 5$

Boundary conditions on linearized VPE

★ At infinity, $\hat{u}, \hat{v} = 0$ or $\hat{\phi}$ = constant

★ On the surface of the body, flow tangency condition holds.

$$\tan\theta = \frac{v}{u} = \frac{\hat{v}}{V_\infty + \hat{u}} \approx \frac{\hat{v}}{V_\infty}$$

or $\hat{v} = V_\infty \tan\theta$

$$\boxed{\frac{\partial\hat{\phi}}{\partial z} = V_\infty \tan\theta \approx V_\infty \theta}$$

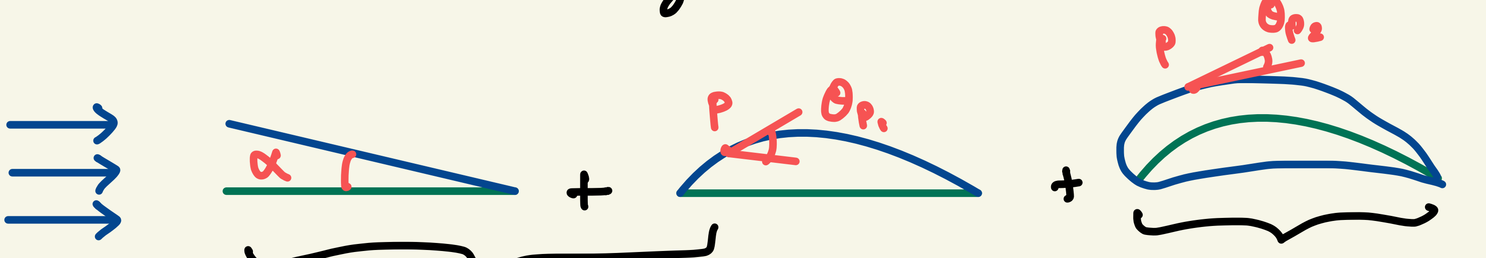
θ is the angle between tangent to surface and freestream.

For situations in which small perturbation theory is applicable, θ is small and to close approximations it may be taken as the sum of

① angle of attack,

② angle between chord line and tangent to the mean camberline, and

③ angle between tangent to the mean camber line and tangent to the surface,



$$\underbrace{\quad\quad\quad\quad}_{\text{Lifting}} \quad\quad \underbrace{\quad\quad\quad\quad}_{\text{Non-lifting}}$$

Lifting [same signs for angles at top & bottom]

Non-lifting [Opposite signs for angles at top & bottom]

SOLVED SEPARATELY & ADDED.

Pressure coefficient

$$C_p \equiv \frac{p - p_\infty}{q_\infty}$$

$$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2 \frac{\gamma p_\infty}{\gamma p_\infty} = \frac{\gamma}{2} p_\infty M_\infty^2$$

$$\Rightarrow \boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)}$$

Linearized pressure coefficient

Let us start with energy equation

$$T_\infty + \frac{V_\infty^2}{2C_p} = T + \frac{V^2}{2C_p}$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\underbrace{\frac{1}{\gamma R T_\infty}}_{a_\infty^2}\Big(V_\infty^2 - \overbrace{V^2}^{(V_\infty+\hat{u})^2 + \hat{v}^2}\Big)$$

$$\Rightarrow \frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2a_\infty^2}\left(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2\right)$$

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

small except for slender bodies

$$\approx 1 - \frac{\cancel{\gamma-1}}{2a_\infty^2}\cdot\frac{\gamma}{\cancel{\gamma-1}}\Big(2\hat{u}V_\infty + \underbrace{\hat{u}^2 + \hat{v}^2}_{\text{small}}\Big)$$

$$\approx 1 - \gamma M_\infty^2\left(\frac{\hat{u}}{V_\infty}\right) \quad \text{for small } \hat{u}, \hat{v}$$

$$\Rightarrow \boxed{C_p = -\frac{2\hat{u}}{V_\infty}}$$ Consistent with linearized VPE

Pressure coefficient depends only on $\hat{u}$

Prandtl-Glauert (PG) compressibility correction (2D)

Objective: Obtain simple corrections to existing incompressible flow results that take into account effects of compressibility
i.e., solve the lifting problem

* for thin airfoils at small angles of attack
* purely subsonic theory (inappropriate results for $M_\infty \geq 0.7$)

For a thin airfoil of shape $z = f(x)$, VPE is

$$\beta^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0$$

with $\beta \equiv \sqrt{1 - M_\infty^2}$

$(x, z) \xrightarrow{\text{TRANSFORM}} (\xi, \eta)$, with $\xi = x$, $\eta = \beta z$

In (ξ, η) coordinates, consider a new velocity potential such that $\phi_0(\xi, \eta) = m\,\hat{\phi}(x, z)$

$$\frac{\partial \hat{\phi}}{\partial x} = \frac{\partial \hat{\phi}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{\phi}}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \hat{\phi}}{\partial \xi} = \frac{1}{m}\frac{\partial \phi_0}{\partial \xi}$$

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{\beta}{m}\frac{\partial \phi_0}{\partial \eta}$$

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} = \frac{1}{m}\frac{\partial^2 \phi_0}{\partial \xi^2} \; ; \quad \frac{\partial^2 \hat{\phi}}{\partial z^2} = \frac{\beta^2}{m}\frac{\partial^2 \phi_0}{\partial \eta^2}$$

Linearized VPE becomes

$$\boxed{\frac{\partial^2 \phi_0}{\partial \xi^2} + \frac{\partial^2 \phi_0}{\partial \eta^2} = 0}$$

Incompressible flow in (ξ, η) space. where $\phi_0 = m\hat{\phi}$

Shape of body in (ξ, η) space

Let $\eta = g(\xi)$ be the shape of the body in transformed space and $\theta_0 = dg/d\xi$

Using this airfoil approximation

$$V_\infty \theta = V_\infty \frac{df}{dx} = \left(\frac{\partial \hat{\phi}}{\partial z}\right)_{z=0} = \frac{1}{m}\left(\frac{\partial \phi_0}{\partial z}\right)_{z=0}$$

$$= \frac{\beta}{m}\left(\frac{\partial \phi_0}{\partial \eta}\right)_{\eta=0} = \frac{\beta V_\infty}{m}\frac{dg}{d\xi} = \frac{\beta V_\infty}{m}\theta_0$$

or $\boxed{\theta = \frac{\beta}{m}\theta_0}$

$\boxed{\text{Case 1: } m = \beta}$

we have $\theta = \theta_0$ Incompressible flow is over an identical airfoil.

$$\phi_0(\xi, \eta) = \beta \hat{\phi}(x, z)$$

$$\Rightarrow \hat{u} = \hat{\phi}_x = \frac{1}{\beta} \phi_{0x} = \frac{1}{\beta} \phi_{0\xi} = \frac{u_0}{\beta}$$

The effect of compressibility on the flow past an airfoil is to increase the horizontal perturbation velocities over the airfoil surface by the factor $1/\sqrt{1-M_\infty^2}$

$$\Rightarrow \boxed{C_p = \frac{C_{p0}}{\beta}}$$

If we know the incompressible pressure distribution over an airfoil, thus the compressible pressure distribution over the same airfoil is

$$C_p = \frac{C_{p0}}{\sqrt{1-M_\infty^2}} \leftarrow \text{COMPRESSIBILITY CORRECTION}$$

The position of resultant aerodynamic force for a compressible, subsonic flow is the same as that for an incompressible flow

Since, lift and pitching moment are essentially integrals of pressure distribution

$$\boxed{C_l = \frac{C_{l0}}{\sqrt{1-M_\infty^2}}}$$

$$\boxed{C_m = \frac{C_{m0}}{\sqrt{1-M_\infty^2}}}$$

Also, slope of $C_l - \alpha$ curve satisfies

$$\boxed{a = \frac{a_0}{\sqrt{1-M_\infty^2}}}$$

Assuming Prandtl's lifting line theory holds for compressible flows

$$a' = \frac{a}{1 + a/\pi e_1 AR}$$

$$\boxed{a' = \frac{a_0}{\sqrt{1+M_\infty^2} + a_0/\pi e_1 AR}}$$

$$e_1 = (1+\tau)^{-1}$$

$\boxed{\text{Case 2} : m \neq \beta}$

Incompressible flow is over a new airfoil shape.

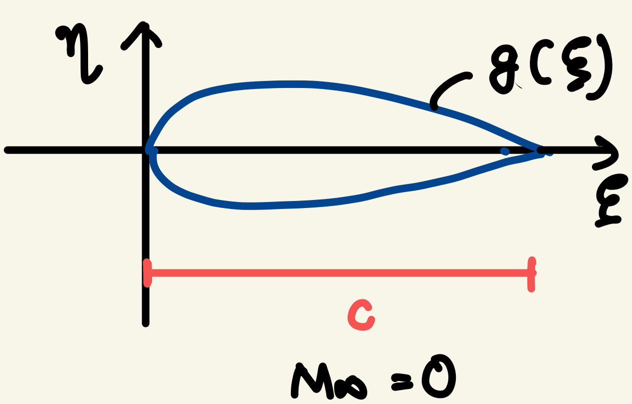
The pressure coefficients over the airfoils in original and transformed space are equal provided

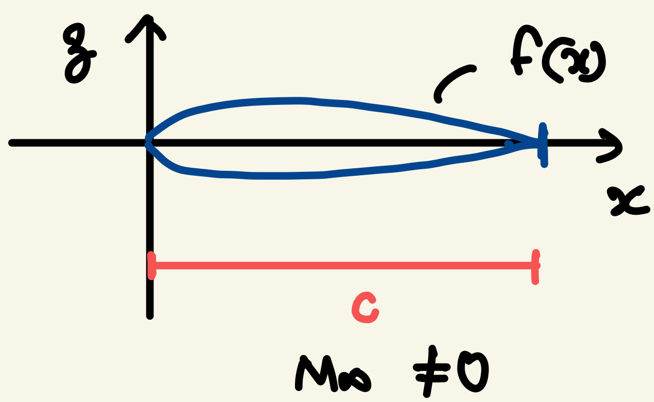
$$\hat{u} = u_0 \Rightarrow \hat{\phi}_x = \phi_{0\xi} \Rightarrow \hat{\phi}(x,z) = \phi_0(\xi,\eta)$$

or $m=1$ giving

$$\boxed{\theta = \theta_0 \beta}$$

C_l, C_m are also identical for the two airfoils





Maximum thickness related as

$$\boxed{\tau = \tau_0 \beta}$$

within the validity of linear theory, C_p will be identical only if thickness, camber and angle of attack are all reduced by the factor β from their values at $M_\infty = 0$

Prandtl-Glauert-Göthert transformation (3D)

$$(1-M_\infty^2)\,\hat{\phi}_{xx} + \hat{\phi}_{yy} + \hat{\phi}_{zz} = 0$$

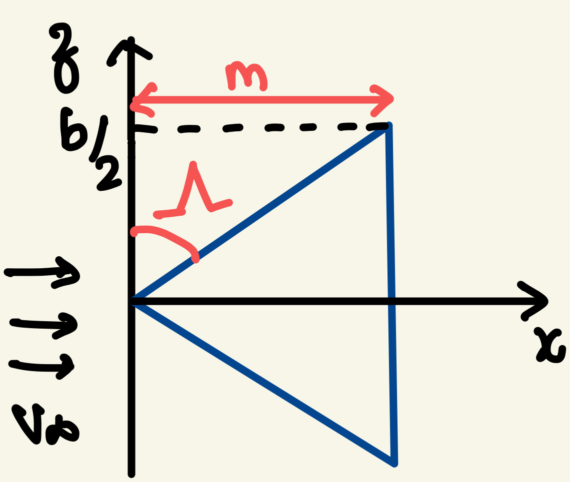
Using the affine transformation

$$x_0 = \frac{x}{\beta}, \quad y_0 = y, \quad z_0 = z, \text{ we get}$$

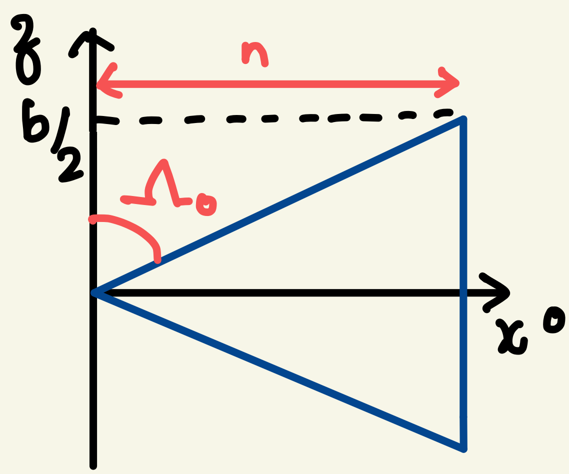
$$\hat{\phi}_{x_0x_0} + \hat{\phi}_{y_0y_0} + \hat{\phi}_{z_0z_0} = 0$$

$$\text{with} \quad \phi_0 = \hat{\phi} \text{ and } \Theta = \frac{(\hat{\phi}_z)_{z=0}}{V_\infty} = \frac{(\hat{\phi}_{z_0})_{z_0=0}}{V_\infty} = \Theta_0$$

The transformation stretches x-cordinate while retaining z (& y) coordinates



Original wing ($M_\infty \neq 0$)



Transformed wing ($M_\infty = 0$)

Aspect ratios are related as

$$\frac{AR_0}{AR} = \frac{b^2/S_0}{b^2/S} = \frac{S}{S_0} = \frac{c}{c_0} = \beta$$

$$\boxed{AR = \frac{AR_0}{\beta}}$$

Leading edge sweeps are related as

$$\frac{\tan \Lambda_0}{\tan \Lambda} = \frac{n / b/2}{m / b/2} = \frac{n}{m} = \frac{1}{\beta}$$

$$\boxed{\Lambda = \tan^{-1}(\beta \tan \Lambda_0)}$$

Pressure coefficients

$$C_p = \frac{-2\hat{\phi}_x}{V_\infty} \; ; \; C_{p_0} = \frac{-2\phi_{0 x_0}}{V_\infty}$$

$$\phi_x = \frac{\phi_{0 x_0}}{\beta} \Rightarrow \boxed{C_p = \frac{C_{p_0}}{\beta}}$$

The pressure coefficient on the subsonic wing is greater than the $M_\infty = 0$ wing.

However,

$$L' = \int_{LE}^{TE} (C_{p_l} - C_{p_u}) q_\infty \, dx$$

$$= \int_{LE}^{TE} \left(\frac{C_{p_{0l}} - C_{p_{0u}}}{\beta} \right) q_\infty \beta \, dx_0 = L_0'$$

The total lifts of the two wings are identical at given α.