AE 339 : High speed aerodynamics

(Module II : Normal shock waves)

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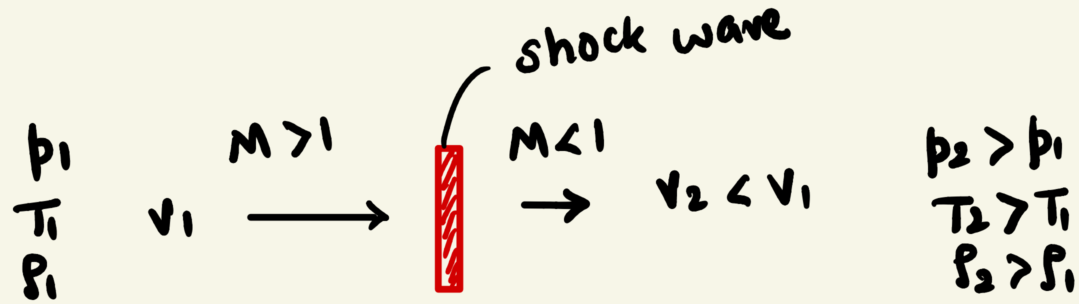
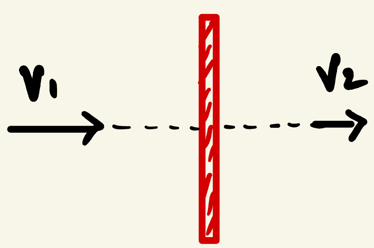
Shock waves

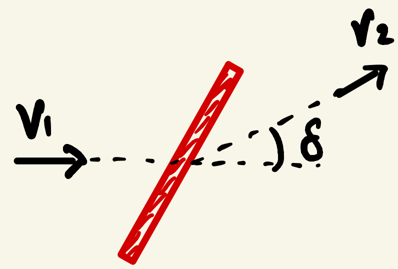
Extremely thin region in which the transition from the inital supersonic velocity, relatively low pressure state to the state that involves a relatively low velocity and high pressure is termed a shock wave



Thickness ~ few mean free paths



Normal shock wave



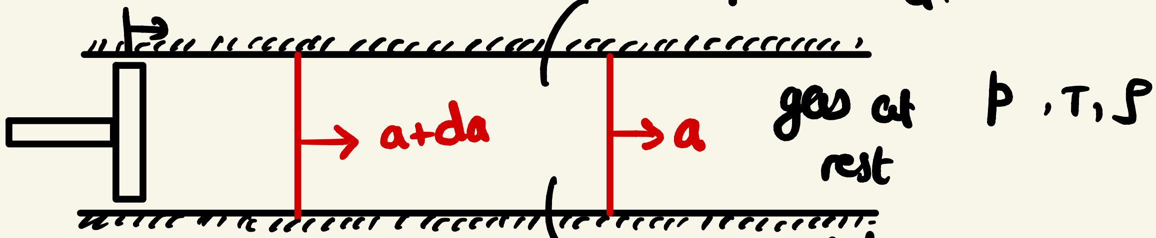
Oblique shock wave

A straight shock wave that is at right angles to the upstream flow is termed a normal shock wave

A straight shock wave that is at an angle to the upstream flow is termed an oblique shock wave

How are shocks formed?

$$dT = \frac{\gamma-1}{\gamma} \cdot T \, \frac{dV}{a}.$$



$$dp = \rho a \, dV$$

$$c_p \cancel{T} + \frac{a^2}{2} = c_p(\cancel{T} + dT) + \frac{(a - dV)^2}{2}$$

$$\Rightarrow \quad \boxed{dT = \frac{\gamma-1}{\gamma} T \cdot \frac{dV}{a}.}$$

Relative velocity of first wave $= a$

Relative velocity of second wave $= a + da + dV$

$> a$

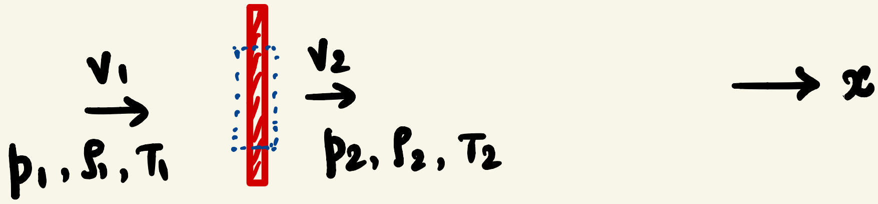
Waves following the first one are able to catch up to it, merge and form a single, stronger wave

$dp + dp + \dots \; dp \rightarrow$ Large pressure change

$\Rightarrow$ Large temperature change (large gradients)

$\Rightarrow$ shock is not isentropic

Stationary Normal Shock wave Relations



Continuity

$$\dot{m} = \rho_1 V_1 \cancel{A} = \rho_2 V_2 \cancel{A}$$

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2$$

Momentum

$$p_1 A - p_2 A = \dot{m}(V_2 - V_1)$$

$$p_1 - p_2 = \left(\frac{\dot{m}}{A}\right)(V_2 - V_1)$$

$$= \rho_1 V_1 (V_2 - V_1) = \rho_2 V_2 (V_2 - V_1)$$

$$V_1 V_2 - V_1^2 = \frac{p_1 - p_2}{\rho_1} \quad \text{— (a)}$$

$$V_2^2 - V_1 V_2 = \frac{p_1 - p_2}{\rho_2} \quad \text{— (b)}$$

(a) + (b) gives

$$V_2^2 - V_1^2 = (p_1 - p_2)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] \quad \text{— (1)}$$

$\boxed{\text{Energy}}$

$$\frac{v_1^2}{2} + C_p T_1 = \frac{v_2^2}{2} + C_p T_2 = C_p T_0 = \text{const}.$$

where $C_p = \frac{R\gamma}{\gamma - 1}$ and $C_p T = \frac{\gamma R T}{\gamma - 1} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$.

$$\Rightarrow v_1^2 + \frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1} = v_2^2 + \frac{2\gamma}{\gamma-1}\frac{p_2}{\rho_2}$$

$$\text{or} \quad v_2^2 - v_1^2 = \frac{2\gamma}{\gamma-1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right] \quad — (2)$$

Using (1) & (2), we get

$$\frac{2\gamma}{\gamma-1}\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right] = (p_2 - p_1)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] \quad — (3)$$

Let us define p_2/p_1 as the strength of the shock

(3) . ρ_2/p_1 gives

$$\frac{2\gamma}{\gamma-1}\left[\frac{p_2}{p_1} - \frac{\rho_2}{\rho_1}\right] = \frac{\cancel{p_1}}{\cancel{\rho_2}}\left(\frac{p_2}{p_1} - 1\right)\left[\frac{\rho_2}{\rho_1} + 1\right]\cdot\frac{\cancel{\rho_2}}{\cancel{p_1}}$$

$$\Rightarrow \frac{2\gamma}{\gamma-1}\left[\frac{p_2}{p_1} - \frac{\rho_2}{\rho_1}\right] = \left(\frac{p_2}{p_1} - 1\right)\left(\frac{\rho_2}{\rho_1} + 1\right)$$

$$\Rightarrow \frac{\rho_2}{\rho_1}\left[\frac{p_2}{p_1} - 1 + \frac{2\gamma}{\gamma-1}\right] = 1 - \frac{p_2}{p_1} + \frac{2\gamma}{\gamma-1}\frac{p_2}{p_1}$$

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1} + 1}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_2}{p_1}}} \quad - \quad (\star)$$

$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} \quad \text{(continuity)}$$

$$\Rightarrow \boxed{\frac{V_2}{V_1} = \frac{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1} + 1}} \quad - \quad (\star\star)$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} \quad \text{(state equation)}$$

$$\Rightarrow \boxed{\frac{T_2}{T_1} = \frac{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1} + \frac{p_1}{p_2}}} \quad - \quad (\star\star\star)$$

These relations are called Rankine - Hugoniot normal shock wave relations

From equation ①

$$V_2^2 - V_1^2 = (p_1 - p_2)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right],$$

we see that

$$V_1^2 \underbrace{\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right]}_{f^n(p_2/p_1)} = \frac{p_1}{\rho_1} \underbrace{\left(1 - \frac{p_2}{p_1}\right)}_{f^n(p_2/p_1)} \underbrace{\left(1 + \frac{\rho_1}{\rho_2}\right)}_{f^n(p_2/p_1)}$$

⇒ For a particular value of p_2/p_1, there is an associated particular value of

$$\frac{V_1^2}{p_1/\rho_1} = \frac{V_1^2}{a_1^2/\gamma} = \gamma M_1^2$$

A particular shock strength is associated with a fixed M_1.

Entropy changes across a shock

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$= (R + C_v) \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

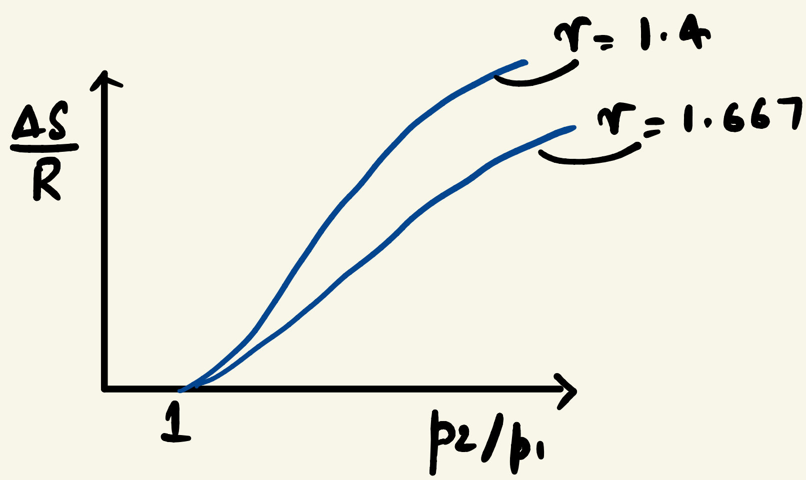
$$\frac{s_2 - s_1}{R} = \left[1 + \frac{1}{\gamma - 1}\right] \ln\left(\frac{T_2}{T_1}\right) - \ln\left(\frac{p_2}{p_1}\right)$$

$$= \left[1 + \frac{1}{\gamma - 1}\right] \ln\left[\frac{p_2}{p_1}\,\frac{\rho_1}{\rho_2}\right] - \ln\left(\frac{p_2}{p_1}\right)$$

$$\frac{s_2 - s_1}{R} = \ln\left[\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma-1}} \cdot \left(\frac{\rho_2}{\rho_1}\right)^{\frac{-\gamma}{\gamma-1}}\right]$$

$$\boxed{\frac{s_2 - s_1}{R} = \ln\left\{\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma-1}} \left[\frac{(\gamma+1)\frac{p_2}{p_1} + (\gamma-1)}{(\gamma+1) + (\gamma-1)\frac{p_2}{p_1}}\right]^{\frac{-\gamma}{\gamma-1}}\right\}}$$

From 2nd law of thermodynamics, $\Delta S \geq 0$



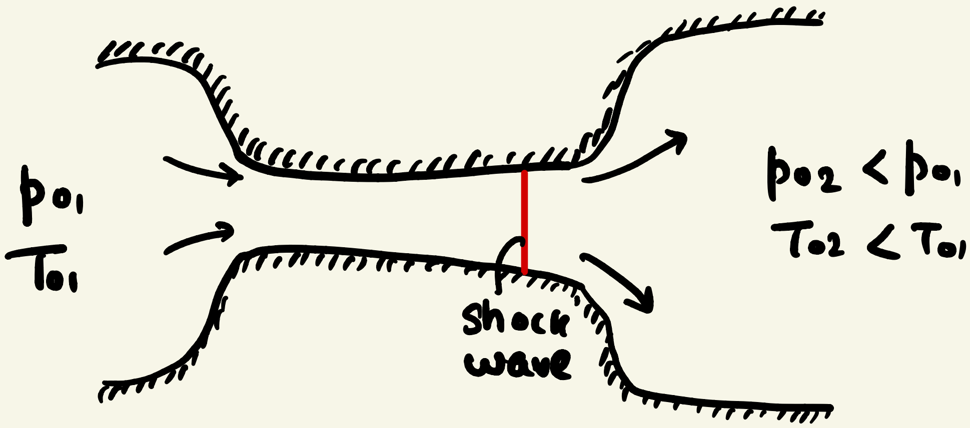
From the figure, $\Delta S \geq 0$ would mean

$$\boxed{p_2/p_1 \geq 1}$$

Shock waves are always compressive and there are no expansion shocks

$p_2 \to p_1$ corresponds to the Mach wave

Losses and stagnation pressure



$$S_{02} - S_{01} = C_p \ln \frac{T_{02}}{T_{01}} - R \ln \frac{p_{02}}{p_{01}}$$

Shock is adiabatic, i.e., $T_{02} = T_{01}$

$$\Rightarrow \quad s_{02} - s_{01} = -R \ln \frac{p_{02}}{p_{01}}$$

Now $s_{02} = s_2$ and $s_{01} = s_1$ since flow is isentropic upstream and downstream

$$\Rightarrow \quad \frac{s_2 - s_1}{R} = -\ln \frac{p_{02}}{p_{01}}$$

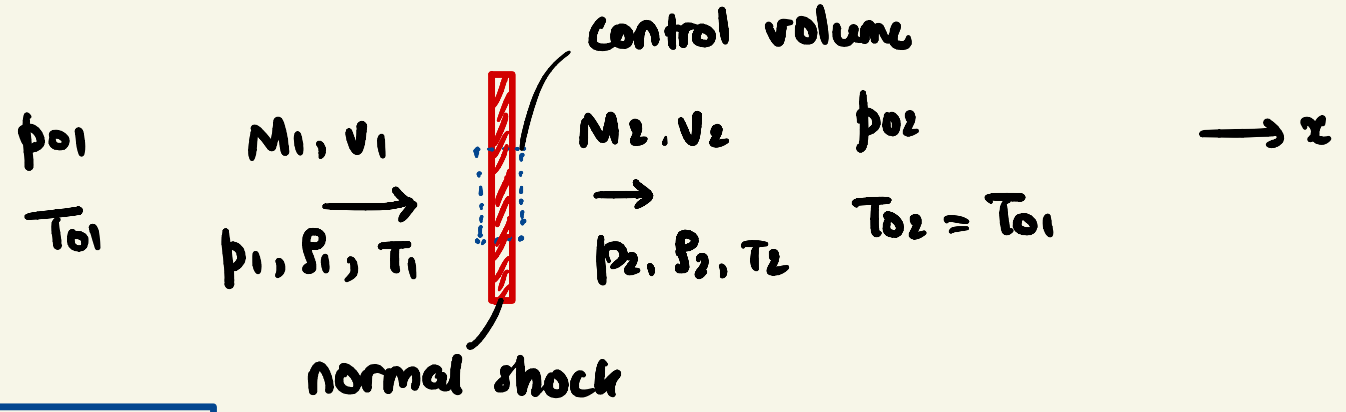
$$\text{or} \quad \frac{p_{02}}{p_{01}} = e^{-\Delta S/R}$$

Since $\Delta s \geq 0$, we have

$$\boxed{p_{02} \leq p_{01}}$$

There is always a loss of stagnation pressure across a shock wave (represents losses in the flow)

Normal shock relation in terms of M_1



Continuity

$$\rho_1 V_1 = \rho_2 V_2$$

Divide by a_1

$$\rho_1 M_1 = \rho_2 \frac{V_2}{a_2} \frac{a_2}{a_1}$$

$$\frac{\rho_2}{\rho_1} = \frac{a_1}{a_2} \cdot \frac{M_1}{M_2} \quad \text{— (1)}$$

Momentum

$$p_1 - p_2 = \rho_2 V_2^2 - \rho_1 V_1^2$$

$$\boxed{p = \frac{a^2 \rho}{\gamma}}$$

Now $p = a^2 \rho / \gamma$

$$\Rightarrow \frac{a_1^2 \rho_1}{\gamma} - \frac{a_2^2 \rho_2}{\gamma} = \rho_2 V_2^2 - \rho_1 V_1^2$$

Simplifying

$$\frac{\rho_2}{\rho_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)\left(\frac{a_1}{a_2}\right)^2 \quad \text{— (2)}$$

$\boxed{\text{Energy}}$

$$V_1^2 + \frac{2}{\gamma-1} a_1^2 = V_2^2 + \frac{2}{\gamma-1} a_2^2$$

$$\Rightarrow \frac{\gamma-1}{2} M_1^2 + 1 = \left(\frac{V_2}{a_2}\right)^2 \cdot \left(\frac{a_2}{a_1}\right)^2 \cdot \frac{\gamma-1}{2} + \left(\frac{a_2}{a_1}\right)^2$$

Simplifying,

$$\left(\frac{a_2}{a_1}\right)^2 = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} \quad — \text{③}$$

Eliminate P_2/P_1 using ① & ②

$$\frac{a_1}{\cancel{a_2}} \cdot \frac{M_1}{M_2} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right) \cdot \left(\frac{a_1}{a_2}\right)^{\cancel{2}}$$

$$\frac{a_2}{a_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right) \frac{M_2}{M_1} \quad — \text{④}$$

Eliminate a_2/a_1 using ③ & ④

$$\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2$$

or

$$(\gamma-1)(M_2^4 - M_1^4) - 2\gamma M_2^2 M_1^2 (M_2^2 - M_1^2) + 2(M_2^2 - M_1^2) = 0$$

For nontrivial solution $M_2 \neq M_1$, we get

$$\boxed{M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)}} \quad — \text{⑤}$$

Substituting ⑤ in ③

$$\boxed{\left(\frac{a_2}{a_1}\right)^2 = \frac{T_2}{T_1} = \left\{ \frac{[2\gamma M_1^2 - (\gamma-1)][2+(\gamma-1)M_1^2]}{(\gamma+1)^2 M_1^2} \right\}}$$

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \cdot \frac{T_2}{T_1} = \frac{\rho_2}{\rho_1} \cdot \left(\frac{a_2}{a_1}\right)^2$$

$$= \frac{\cancel{\rho_2}}{\cancel{\rho_1}} \cdot \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \cdot \frac{\cancel{\rho_1}}{\cancel{\rho_2}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

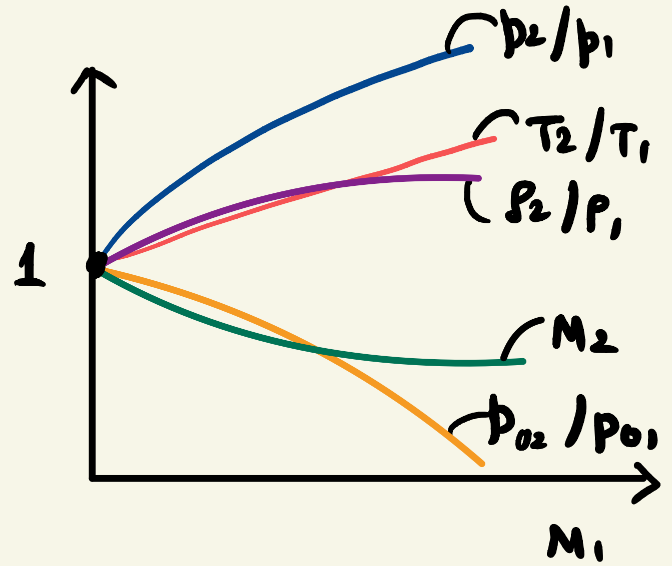
$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}}$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \cdot \frac{T_1}{T_2}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}}$$

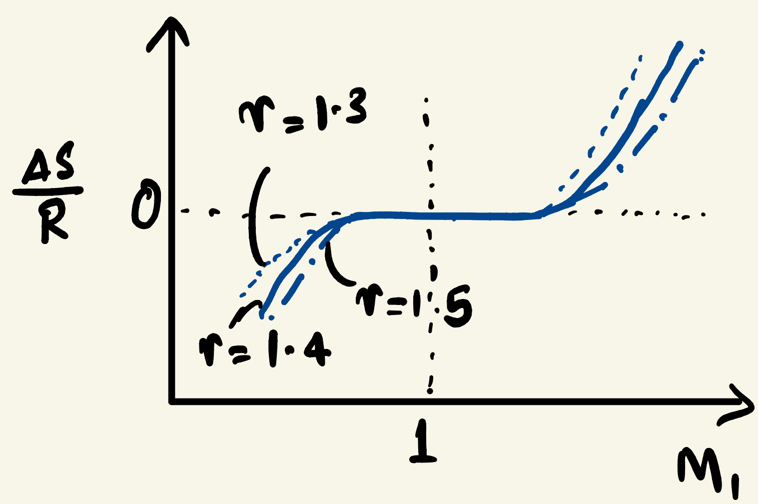
$$\frac{p_{02}}{p_{01}} = \frac{p_{02}/p_2}{p_{01}/p_1} \cdot \frac{p_2}{p_1}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \left\{ \frac{(\gamma+1)}{2} \frac{M_1^2}{\left(1+\frac{\gamma-1}{2}M_1^2\right)} \right\}^{\frac{\gamma}{\gamma-1}} \left\{ \left(\frac{2\gamma}{\gamma+1}\right) M_1^2 - \left(\frac{\gamma-1}{\gamma+1}\right) \right\}^{\frac{-1}{\gamma-1}}}$$



ρ_2/ρ_1 and M_2 become invariant ; p_2/p_1 and T_2/T_1 tend to ∞ ; p_{02}/p_{01} decreases as $M_1 \to \infty$

$$\boxed{\frac{s_2 - s_1}{R} = \ln\left\{\left[\frac{2\gamma}{\gamma+1}(M_1^2 - 1) + 1\right]^{\frac{1}{\gamma-1}}\left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{\frac{-\gamma}{\gamma-1}}\right\}}$$



From 2^{nd} law of thermodynamics $\Delta s \geq 0$

$$\boxed{M_1 \geq 1}$$

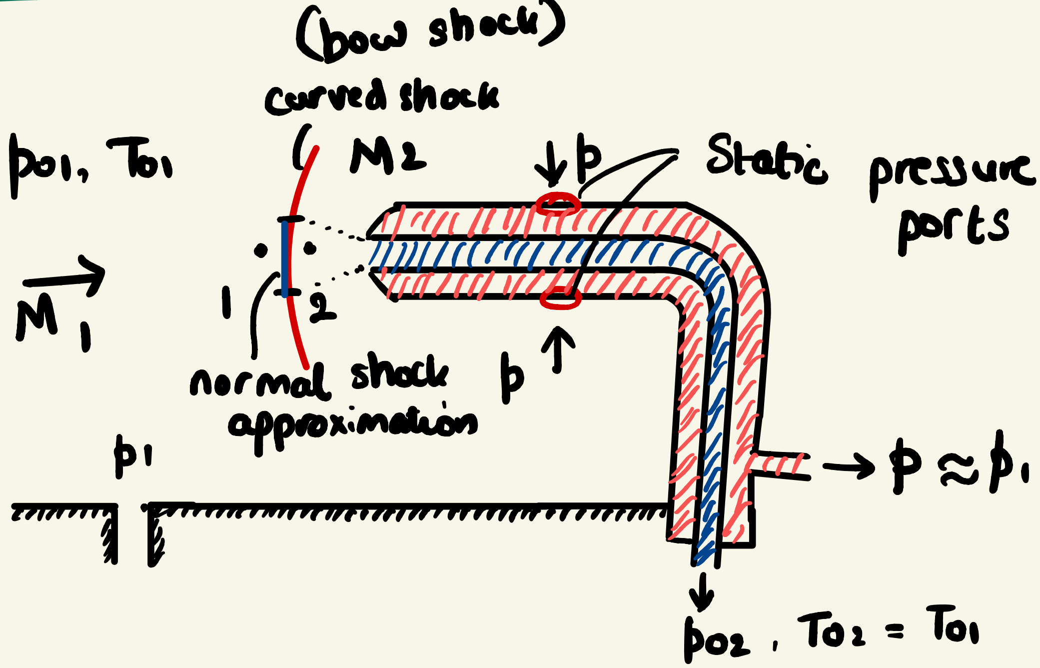
Normal shocks can only occur in a supersonic flow

Also, from equation ⑥ & $M_1 \geq 1$

$$\boxed{M_2 \leq 1}$$

Flow downstream of a normal shock is always subsonic

Pitot probes in supersonic flows



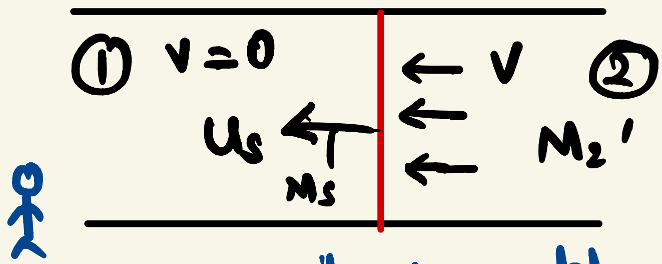
If we blindly use subsonic Pitot probe equation, we get M_2. We need M_1

$$\underset{\text{measured}}{\frac{p_{02}}{p_1}} = \underbrace{\frac{p_{02}}{p_2} \cdot \frac{p_2}{p_1}}_{f^{ns} \text{ of } M_1}$$

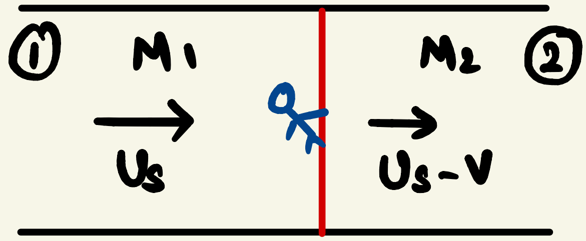
$$\boxed{\frac{p_{02}}{p_1} = \frac{\left[(\gamma+1) M_1^2/2\right]^{\frac{\gamma}{\gamma-1}}}{\left[\left(\frac{2\gamma M_1^2}{\gamma+1}\right) - \left(\frac{\gamma-1}{\gamma+1}\right)\right]^{\frac{1}{\gamma-1}}}}$$

Rayleigh supersonic Pitot probe equation.

Moving normal shock waves



moving shock problem



Equivalent stationary shock problem

We have $V_1 = U_s$ and $V_2 = U_s - V$

$$M_1 = \frac{U_s}{a_1} = M_s$$

$$M_2 = \frac{U_s - V}{a_2} \Rightarrow \boxed{V = M_1 a_1 - M_2 a_2}$$

$$\underbrace{M_2}_{f^n(M_1)} = \underbrace{\frac{U_s}{a_1}}_{M_1} \cdot \underbrace{\frac{a_1}{a_2}}_{f^n(M_1)} - M_2'$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_s^2}{2 + (\gamma-1) M_s^2}$$

$$\frac{T_2}{T_1} = \left(\frac{a_2}{a_1}\right)^2 = \frac{\left[2 + (\gamma-1) M_s^2\right]\left[2\gamma M_s^2 - (\gamma-1)\right]}{(\gamma+1)^2 M_s^2}$$

$M_2' \to 1.89$
$M_s \to \infty$

$$M_2' = \frac{2 (M_s^2 - 1)}{\left[2\gamma M_s^2 - (\gamma-1)\right]^{1/2} \left[2 + (\gamma-1) M_s^2\right]^{1/2}}$$