Probability2013 Midterm1 (Prof. Shou-de Lin)

3/25/2013 14:30-17:30pm

Total Points: 110

You can answer in either Chinese or English

1. **[Probability defined on events]**

   Let *E* and *F* be mutually exclusive events of a random experiment. Suppose that the experiment is repeated until either event *E* or event *F* occurs. What does the sample space of this new super experiment look like? What is the probability that event *E* occurs before event *F*? (6pts)

2. **[set theory and events]**

   If $P(A) = 0.3$, $P(B) = 0.5$

   (a) Assume A and B are mutually exclusive, find $P(A \cap B)$ and $P(A \cup B)$. (4pts)

   (b) Assume A and B are independent, find $P(A' \cap B')$ (5pts)

3. **[Probability and Conditional Probability]**

   (a) In a modified Monty Hall Problem, assuming there are 4 doors and behind three of them there are goats, while the remaining one is a car. After a participant picks a door, the host (who knows where the car is) will intentionally open a door with goat. In this case, should the participant swap his current choice with one of the remaining door? (5pts)

   (b) If the host does not know where is the car, and he opens a door with a goat. Should the participant swap? (5pts)

   Please explain your answers using probability.

4. **[Probability and expectation]**

   You have two opponents with whom you alternate play. Whenever you play *A*, you win with probability p_A; whenever you play *B*, you win with probability p_B, where $p_B > p_A$. If your objective is to minimize *the number of games you need to play before winning two in a row*, should you start with *A* or with *B*? Prove it. [6pts]

5. **[Independency]:**

   (a) X, Y, and Z are three random variables. Can you proposal a real-world example of them that satisfy both of the following conditions (5 pts):

   (1) X and Y are independent

   (2) X and Y becomes dependent given Z

   (b) Given your grandparent's IQ, is your IQ independent of your father's? (2 pts)

   (c) Given your parent's IQ, is your IQ independent of your sister's IQ? (2 pts)

6. **[sampling]:**

   You are asked to estimate the circumference ratio π. The only function you can use is the random-value-generator random(). Please write a pseudo code (or java/C) that uses random() to obtain π. Note that you can use *whie, if*, and *+-*/* in the pseudo code. [8pts]

7. [Bayes rule]

Assume that there are only Salmons (60%) and Trout (40%) in the sea, and assume the weight and color of a fish are independent. Researches discover the weight (kg) of fishes, W , follows the following p.d.f,

$$f(w) = \frac{5}{11((w-\mu)^2+1)} \qquad \mu = \begin{cases} 4 & \text{if fish type="Trout"} \\ 5 & \text{if fish type="Salmon"} \end{cases}$$

$$w \in \{3,4,5,6\}$$

Further, 2/3 of the salmons are black, others are silver; 1/2 trouts are black, the remainders are silver. Today, a fisher catches a black fish with 5 kg, what is the probability that this fish is a Salmon? [8pts]

8. [Exponential Distribution]

The amount of delay time for a given flight is exponentially distributed with a mean of 0.5 hour, namely, the p.d.f. f(x)= $2e^{-x/0.5}$. Ten passengers on this flight need to take a subsequent connecting flight. 7 persons have connection time for 2 hours, but unfortunately the remaining 3 persons has only 1 hour between flights.

(a)Suppose John is one of the 10 passengers needing a connection. What is the probability that he will miss his connection? (5pts)

(b)Suppose he met Mike on the plane, who also needs to make a connection. However, Mike is going to another destination and thus has a different connection time from John's. What is the probability that both John and Mike will miss their connections? (5pts)

(c)A friend of John’s, named Mary, happens to live close to the airport where John makes his connection. She would like to take this opportunity to meet John at the airport. Suppose she has already waited for 30 minutes beyond John's scheduled arrival time. What is the probability that John will miss his connection so that they could have a leisurely dinner together? Assume John’s scheduled connection time is 1 hour. (5pts)

9. **[Prove Variance]** Prove that the variance of an exponential distribution is θ^2(8pts)

10. [pmf]

A pmf of X is defined as p=$\frac{1}{2}e^{-\lambda}\frac{\lambda^x}{x!}+\frac{1}{2}e^{-\mu}\frac{\mu^x}{x!}$, when x=0,1,2…., and p=0 otherwise.

(a) Please confirm whether it is a legal pmf (4pts)

(b) What is E(X)? (4pts)

11. **[mgf]** Let R(t)=lnM(t), where M(t) is the mgf of a random variable. Show that

(a) $R'(0)=\mu$ (4pts)

(b) $R''(0)=\sigma^2$ (4pts)

12. [Terminology]

Please describe the following terms

(a) Theory of Total Probability (5pts)

(b) Draw an example pmf so that three random variables X,Y,Z are mutually independent (5pts)

(C) What is the relationship between Poisson, Binominal, Chi-Square, Gamma, and Exponential distribution (5pts)

Poisson Distribution

$$f(x)=\frac{\lambda^x e^{-\lambda}}{x!}$$

$\mu=\lambda$, $\sigma^2=\lambda$

Exponential Distribution

$f(x)=\lambda e^{-\lambda x}$, let $\theta=1/\lambda$

$\mu=\theta$, $\sigma^2=\theta^2$

Probability2014 Midterm (Prof. Shou-de Lin)

4/14/2013 14:30-17:30pm

Total Points: 120

You can answer in either Chinese or English

1. **[Independency]**

   X, Y, and Z are three random variables. Can you proposal a real-world example of them that satisfy the following conditions:

   (a) X and Y are independent, but becomes dependent given Z (5pts)

   (b) X and Y are dependent, but becomes independent given Z (5pts)

2. **[Application on Probability]**

   There is a group of n students who occupied the Legislative Yuan (LY). The observation is that: at each minute, there is 1/2 chance one student will leave LY (i.e. n=n-1), and 1/2 chance one student enters LY (i.e. n=n+1). The police make a deal with the leader: If they reach 2n students before everybody leaves LY (n=0), then the leader will not be arrested. However, if the number reaches 0 before it becomes 2n, then the leader will be arrested immediately.

   (a) Given t minutes passed, what is the chance that the leader has already been arrested? (12pts, if you cannot derive the close form solution, please describe how to generate this probability)

   (b) If t >> n, what is the answer for this question? (5pts)

3. **[Probability and Conditional Probability]**

   (a) In a modified Monty Hall Problem, assuming there are n doors and behind n-k of them there are goats, while the remaining k (k<<n) is a car. After a participant picks a door, the host (who knows where the cars are) will intentionally open a door with goat. In this case, should the participant swap his current choice with one of the remaining door? (5pts)

   (b) If the host does not know where are the car, and he opens a door with a goat. Should the participant swap? (5pts)

   Please explain your answers using probability.

4. **[Probability and expectation]**

   A grocery store has available *n* watermelons to sell and makes $1.00 on each sale. Say the number of consumers of these watermelons is a random variable that has a distribution that can be approximated by

   $$f(x) = \frac{1}{200} \quad , 0 < x < 200,$$

   a p.d.f. of the continuous type. If the grocer does not have enough water melons to sell to all consumers, she figures that she loses $5.00 for each unhappy customer. But if she has surplus watermelons, she loses 50 cents on each extra watermelon. What should *n* be to maximize "profit"? (10pts)

5. **[Random Experiment]:**

   You are asked to design a random experiment to estimate the circumference ratio π. The only function you can use is the random-value-generator random(), which returns a value between [0,1]. Please describe your experiment (you can use pseudo code or simply explain it in plain text).

   (10pts, Note that you can use *while, if*, and *+-*/* in the pseudo code)

6. **[Bayes rule]**

   Company1 announces a disease (occur rate= 20%) testing product T1. The performance looks like:

   P(T1= positive | Disease=true) = 0.7, P(T1= negative| Disease=false) = 0.7

   Company2 also announces a testing product T2 for the same disease. The performance looks like:

   P(T2= positive | Disease=true) = 0.9, P(T2= negative | Disease=false) = 0.6

   Q1: A careless doctor performed a test on a patient and found that the result is positive. However, this doctor forgot which testing product was chosen. Can you tell this doctor which product is more likely to be the one used given positive result? (5 pts)

   Q2: If a patient has been tested positive on both products, what is the probability that he/she really has the disease (assuming that the test results are conditionally independent given disease)? (5pts)

7. **[Exponential Distribution]**

   Let X be a random variable that represents the number of days that it takes a high-risk driver to have an accident. Assume that X has an exponential distribution. If P(X<50)=0.25, compute P(X>100|X>50). (5pt)

8. **[Poisson]** Given the following random experiments, please comment whether each of them is likely to produce a random variable that follows a Poisson distribution, and explain why:

   a) Observing the number of people entering CSIE R104 front door from 14:20-15:00 every Mon

   b) Observing the number of cars passing 長興街警衛亭 every Monday from 10-11am

   c) Observing the number of cars passing 新生南路忠孝東路交叉口 at 5-6pm every Mon

   (9pts)

9. Amy wants to buy the May-Day concern ticket online. Since it is really hard to get one, she used five computers (C1 ~C5) to in parallel to ensure she can get it. But there are different successful rate for each computer to get the ticket. The

successful rate of Ci computer is P(i+1)=1.05*P(i)-0.05. A 'round' is defined as one trail for every computer. A random variable X is defined as the number of rounds required to get one ticket. We know the variance of X is 2. So what's p(1)? (7pts)

10. In the movie theater there are 300 seats. 300 people including Jack lined up to enter the theater. Jack is the 200th person to enter the theater, but when it's his term, he found that his seat number faded so he has no idea where to seat. Jack decides to randomly pick the seat because everyone was so nice that if their seat is taken, they will simly find other seat randomly. What's the probability of the last person to seat his/her own seat correctly (10pts)
11. One day Takasi found that around him there were four zombies. He picked up the gun to kill them. Since it becomes easier to kill a zombie if there are more around, the probability for Takasi to kill the zombie is n/5 (n is the number of zombie) and he can only kill one zombie in a time. When the zombie was killed, the zombie will have 30% chance to be revived by their friend. That says, if there is only one zombie left, it won't rebirth.
What's the expected time Takasi needed to kill all the zombies? (10pts)
12. In the MMORPG world, Shiroe and his partners want to meet with the king. They started at "A" and is allowed to choose only three directions (right, up, down), with probabilities P(right)=1/2 ,P(up)=P(down)=1/4. However, if he can't go up or down, the probability becomes P(right)=1/2 , P(up or down)=1/2. In the graph there are many traps in each area, denoted as 'X'. Entering 'X' means game over. One special place is the rest area, there they can choose to start at " B " ,"C "and "D". Assuming Shiroe took SD's probability course previously and can make the best decision, what's the probability that Shiroe meets the king safely? (12pts)

<table>
<tr><td>O</td><td>O</td><td>O</td><td rowspan="3">Rest</td><td>B</td><td>O</td><td>O</td><td rowspan="3">King</td></tr>
<tr><td>A</td><td>O</td><td>X</td><td>C</td><td>O</td><td>X</td></tr>
<tr><td>O</td><td>O</td><td>O</td><td>D</td><td>O</td><td>O</td></tr>
</table>

Poisson Distribution

$$f(x)=\frac{\lambda^x e^{-\lambda}}{x!}$$

$\mu=\lambda$, $\sigma^2=\lambda$

The variance of geometry distribution is $(1-p)/p^2$

Exponential Distribution

$f(x)=\lambda e^{-\lambda x}$, let $\theta=1/\lambda$

$\mu=\theta$, $\sigma^2=\theta^2$

Probability2015 Midterm (Prof. Shou-de Lin)

4/23/2014 14:30-17:20pm

Total Points: 120

You can answer in either Chinese or English

1. **[MGF] 7pts**

   Find $P(|X| \leq 1)$, given that X has moment-generating function:

   $$M(t) = \frac{111}{345}e^{-2t} + \frac{1}{3}e^{-1t} + \frac{1}{5}e^{t} + \frac{41}{403}e^{2t} + \frac{1201}{27807}e^{3t}$$

2. **[Probability and Conditional Probability] 10pts**

   (a) In a modified Monty Hall Problem, assuming there are n doors and behind n-k of them there are goats, while the remaining k (k<<n) are cars. After a participant picks a door, the host (who knows where the cars are) will intentionally open a door with goat. In this case, should the participant swap his current choice with one of the remaining door? Please explain your answers using probability.

   (b) If the host does not know where the cars are, and he opens a door with a goat. Should the participant swap? Please explain your answers using probability.

3. **[Random Experiment] 10pts**

   The city transportation authority is claiming that they schedule 4 buses per hour on the average (according to certain unknown distribution) i.e. about 1 bus every **15 minutes** on average. However, major Ko doesn't believe it, since he thinks he usually waits for longer amount of time. So, he asked many people at the bus stop how long they have been waiting till the next bus arrives. He found out that the average waiting time is larger than 15min. Explain why the average waiting time for the passenger is larger than the average duration of a bus.

4. **[Events] 7pts**

   Let E, F. and G are mutually exclusive events of a random experiment (RE) with probability P(E),P(F),P(G). Suppose that a new random experiment is designed as repeated the original RE until either event E or F or G occurs. What is the probability that event E occurs before event F?

5. **[permutation] 10pts**

   S1, S2,…,Sn are playing a game. In the beginning they line up from S1 (first) to Sn (last). Then they in turn (starting from S1) throw a 6 side fair dice. When the dice obtain 1 or 2, the person go to the beginning of the line, when the dice obtain 3 or 4, the person stay the current position in the line, when the dice obtain 5 or 6, the person go to the end of the line. When a person go to the beginning or end of the line, the person behind him fill the left position.

   After all person throw the dice, what is the probability that the order of the line is not changed (the same as the initial line)?

6. **[Poisson] 10 pts**

Please describe how to estimate the value of e (i.e 2.71828…) using only a random function r() that returns a real number between [0,1], and operation +,-,*,/. Please write a C or pseudo code to do so (hint: 'e' appears in the Poisson distribution)

7. **[Multivariate] 10 pts**

Suppose that A1 and A2 are independent uniform random variables on [0, 1]. Let X = max{A1, A2} and Y = min{A1, A2}. Compute the following:

(a) The probability density function for X and Y.

(b) The expectation E[X] and E[Y].

8. **[continuous RV] 10pts**

X follows a uniform distribution on $[0, 6]$.

Y follows exponential distribution with $\theta = 1/3$; Also X, Y are independent

(a) Find the mean and variance of *X+5Y*

(b) Find the probability such that $Y \geq X$.

9. **[Bayes Rule] 10pts**

Jennifer checks the weather forecast before deciding whether to carry an umbrella most of the mornings. If the forecast says rain, then the probability of actually having ''rain'' that day is 70%. On the other hand, if the forecast says ''no rain'', then the probability of actually having rain is 10%. During spring/summer the forecast is ''rain'' 15% of the time and during fall/winter it is 80%.

   (a) One day, Jennifer missed the forecast and it rained. What is the probability that the forecast was ''rain'' given it was during the spring?

10. The probability of Jennifer missing the morning forecast is 40% on any day for the whole year. If Jennifer misses the forecast, she will flip a fair coin to decide whether to carry an umbrella. On any day of a given season she sees the forecast, if it says ''rain'' she will always carry an umbrella, and if it says ''no rain'' she won't carry an umbrella. Now Jennifer is carrying an umbrella and it's not raining. What is the probability that she saw the forecast? Does it depend on the season?

11. **[correlation coefficient] 10pts**

In this question, we want to show that the correlation coefficient is a measure of the amount of linearity.

X and Y are discrete random variables that take values from $(-\infty, +\infty)$ and their outcome space is of the same size. They can form pairs from $(X_0,Y_0)(X_1,Y_1)$to(X_n,Y_n). Let us now consider all possible lines in 2D space passing through the point (μ_x, μ_y), that is the mean of X and Y. Such line can be expressed as the form $y-\mu_y=b(x-\mu_x)$ whereb is the slope. Thus, the vertical distance from any point (X,Y) to this line can be denoted as $|Y-\mu_y - b(X-\mu_x)|$

   (1) Find the $b=b_m$ that minimizes $E[|Y-\mu_y - b(X-\mu_x)|^2]$,which is the expectation of

the vertical distance to the line. Please express b_m using the correlation coefficient ρ,μ_x, μ_y (I do not sure E[|Z|]->E[Z^2] is monotonically transformed, origin question is E[Z^2])

(2) Let $E[|Y-\mu_y - b(X-\mu_x)|^2]=k(b)$, what is K(bm)? Use it to show that ρ lies in [-1,1] and why ρ is a measure of linearity

12. [discrete r.v] 10pts

B02 consists of X boys and Y girls, they have different taste of movies. Let X_1,X_2,X_3 be the number of boys who want to see Comic(X_1), Action(X_2) and both of them(X_3) ， Y_1,Y_2,Y_3 for girls respectively. We hold a blind date split B02 into pairs. Everybody has the same chance to group with others and there is no one left. Note that the conditional probability of a successful pair is defined in the below table. It says that if the pair is of same sex and wants to see the same movie, there is 50% chance it is a successful match, and so on.

	Same taste	Different taste
Same sex	0.5	0.1
Different sex	1	0.7

What is the expected number of successfully matched pairs (Hint: The method used in problem 2.7 will do you a favor)?

13. [conditional pmf]: 16pts

Suppose our 3 TAs have built a website containing lots of tools for preparing midterm. The TAs also need to collect data from the website (each TA have 0.5 chance to correct the data from website). Let random variable Y be the number of TAs who are correcting data on the website. However, the website's bandwidth can only allow at most α-2Y students downloading, where α is an integer larger than 6. The website crashes when there are more than α-2Y students downloading. We have known that 90% of the students are coming for surfing while 10% are for downloading (Once students in ,they do not leave). Let X denotes the total number of students the website serves before shutting down,

(1) What is P(Y)? (2pts)
(2) what is P(X|Y)? (6pts)
(3) what is the expected number of X given that there are more than one TA correcting data on the website? (8pts)

===

Cov (X1,X2)= $E[(X_1-\mu_1)(X_2-\mu_2)]$,

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

Poisson Distribution

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$,,$\mu=\lambda$, $\sigma^2=\lambda$

Exponential Distribution

$f(x)=\lambda e^{-\lambda x}$, let $\theta=1/\lambda$, $\mu=\theta$, $\sigma^2=\theta^2$