Probability Midterm Solution

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1 Probability basics

1. Random variables A, B, C, and D satisfy following conditions:
   (a) $P(A) < 1$, $P(B) < 1$, $P(C) < 1$, $P(D) < 1$.
   (b) A and B are independent, and C and D are independent.
   (c) A and C are independent.

   Are B and D independent? Why?

   A: No. An counter example can be easily drawn for A, B, C, and D.

2. Random variables A, B, C, and D satisfy following conditions:
   (a) $P(A) < 1$, $P(B) < 1$, $P(C) < 1$, $P(D) < 1$.
   (b) A and B are mutual exclusive, and C and D are mutual exclusive.
   (c) A and C are independent.

   Are B and D independent? Why?

   A: Yes. $P(A) = P(A)(P(C) + (1 - P(C))) = P(A)P(C) + P(A)P(D)$. $P(C) + P(D) = 1$ since C and D are mutual exclusive. So A and D are independent since $P(A, D) = P(A)P(D)$. Using the similar approach, we can obtain B and D are independent.

3. In the communication network, link failures are independent, and each link has a probability of failure of P. Consider the physical situation before you write anything. A can communicate with B as long as they connected by at least one path which contains only in-service link.

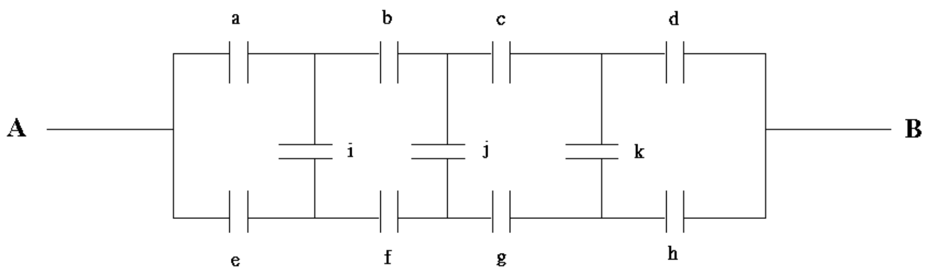

Figure 1: The communication network between A and B.

   (a) Given that exactly 7 links have failed, determine the probability that A can still communicate with B.

   A: Only $a.b, c, d$ or e, f, g, h are good can serve connection between A and B. The probability will be $2/C(11, 4) = 1/165$.

   (b) Given that a, h, and j have been failed without information about conditions of other links, Determine the probability that A can still communicate with B.

   A: $(1-P)(1-(1-(1-P)^3)^2)(1-P) = (1-P)^2(1-(1-(1-P)^3))(1+(1-(1-P)^3)) = (1-P)^5(2-(1-P)^3)$.

2 Random variables

1. We are given the following information about random variable x:
   $E[x]=24, E[x^2]=625,\ E[x|x>E[x]+\sigma_x]=32, Prob[x>E[x]+\sigma_x]=0.2.$
   Determine the numerical value of $E[x|x \leq E[x]+\sigma_x]$.
   A: $\sigma_x=\sqrt{E[x^2]-E^2[x]}=7.$
   $E[x|x\leq E[x]+\sigma_x]=\frac{E[x]-E[x|x>E[x]+\sigma_x]*Prob[x>E[x]+\sigma_x]}{1-Prob[x>E[x]+\sigma_x]}=22.$

2. Discrete random variable x is described by the PMF

$$p_x(x_0)=\begin{cases} K-\frac{x_0}{15} & \text{if } x_0=0,1,2 \\ 0 & \text{for all the other values of } x_0 \end{cases}$$

   Let $d_1, d_2, \cdots, d_N$ represent N successive independent experimental values of random variable x.

   (a) Determine the numerical value of K.
   A: $3K=1 \Rightarrow K=1/3$.

   (b) Determine the probability that $d_1>d_2$.
   A: $P(d_1=2\ \&\ d_2<2)+P(d_1=1\ \&\ d_2=0)=\frac{4}{15}\times\frac{11}{15}+\frac{5}{15}\times\frac{6}{15}=\frac{74}{225}$.

   (c) Determine the probability that $\sum_{i=1}^{N} d_i \leq 1.0$.
   A: $P(d_i=0,\forall i)+P(\text{only one} d_i=1)=(\frac{2}{5})^N+N\times(\frac{2}{5})^{N-1}\times\frac{1}{3}=(\frac{2}{5})^N\times(1+\frac{5N}{6})$.

3 Transform

Let k be a random variable with z-transform: $P_k^T(z)=0.25z+0.25z^4+0.5e^{-2(z-1)(z-2)}$
Let x be a random variable with s-transform: $f_x^T(s)=\frac{0.5}{s+0.5}$
Let w be the sum of k independent experimental values of random variable x.
Determine the numerical values of

1. $E(k^2)$
2. $E(w)$

A:

$$\begin{aligned} E(k^2) &= \left[\frac{d^2}{dz^2}P_k^T(z)\right]_{z=1}+\left[\frac{d}{dz}P_k^T(z)\right]_{z=1} \\ &= (3z^2-2e^{-2(z-1)(z-2)}+(3-2z)^2e^{-2(z-1)(z-2)})_{z=1}+ \\ &\quad (0.25+z^3+(3-2z)e^{-2(z-1)(z-2)})_{z=1} \\ &= (3-2+1)+(0.25+1+1)=6.25. \end{aligned}$$

$$\begin{aligned} E(w) &= E(k)E(x) \\ &= 2.25*\left[-\frac{d}{ds}f_x^T(s)\right]_{s=0} \\ &= 2.25*\left[-\frac{0.5\times(-1)}{(s+0.5)^2}\right]_{s=0} \\ &= 4.5. \end{aligned}$$

4 Poisson Process

A woman is seated beside a conveyer belt, and her jobis to remove certain items from the belt. She has a narrow line of vision and can get these items only when they are right in front of her.

She has noted that the probability that exactly k of her items will arrive in a minute is given by $p_k(k_0) = \frac{2^{k_0}e^{-2}}{k_0!} \qquad k_0 = 0, 1, 2, 3, \cdots$
and she assumes that the arrivals of her items constitute a Poisson process.

1. If she wishes to sneak out to have a beer but will not allow the expected value of the number of items she misses to be greater than 5, how much time can she take?

   A: Since the arrivals of her items constitute a Poisson process, arrival from different minute will be independent. So we only need to know expected number of her items per minute is enough for calculating the maximal time she can take.

$$\begin{aligned} E(k) &= \sum_{k_0} k_0 p_k(k_0) \\ &= \sum_{k_0 \geq 1} e^{-2} \frac{2^{k_0}}{(k_0 - 1)!} \\ &= 2e^{-2} \sum_{k_0 \geq 1} \frac{2^{k_0 - 1}}{(k_0 - 1)!} \\ &= 2e^{-2}e^2 = 2. \end{aligned}$$

   So the time she can take will be $5/E(k) = 2.5$ minute.

2. If she leaves for two minutes, what is the probability that she will miss a total of exactly 4 items?

   A: Performing z-transform on $p_k(k_0)$, we can obtain $p_k^T(z) = e^{2(z-1)}$. Since items from first minute and items from second minute will be independent, the z-transform for items from two minutes will be $(p_k^T(z))^2 = e^{4(z-1)} = e^{-4} \sum_{k_0} \frac{(4z)^{k_0}}{k_0!}$. Applying $z = 1$ and $k_0 = 4$, we can get the corresponding term for representing the probability of missing exactly 4 items in 2 minutes as $e^{-4} \times \frac{4^4}{4!} = \frac{32e^{-4}}{3}$.

3. The union has installed a bell which rings once a minute with precisely one-minute intervals between gongs. If, between two successive gongs, more than 3 items come along the belt, she will handle only 3 of them properly and will destroy the rest. Under this system, what is the probability that any particular item will be destroyed?

   A: The expected number of not-destroyed items will be $K' = 0 + \frac{2e^{-2}}{1!} + \frac{2\times 2^2 e^{-2}}{2!} + \sum_{k_0 \geq 3} \frac{3 \times 2^{k_0} e^{-2}}{k_0!} = 3 - 3e^{-2} - 2e^{-2} \times 2 - e^{-2} \times 2 = 3 - 9e^{-2}$. So the probability for items being destroyed will be $(2 - (3 - 9e^{-2}))/2 = (9e^{-2} - 1)/2$.