ALGORITHMS AND DATA STRUCTURES II

Lecture 9

Dynamic Programming,
Matrix chain multiplication,
The knapsack problem.

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DYNAMIC PROGRAMMING

- Dynamic Programming is a general algorithm design technique for solving problems defined by or formulated as recurrences with overlapping sub-instances.
- Invented by American mathematician Richard Bellman in the 1950s to solve optimization problems.
- "Programming" here means "planning".

DYNAMIC PROGRAMMING

- Main idea:
  - set up a recurrence relating a solution to a larger instance to solutions of some smaller instances.
  - solve smaller instances once and record solutions in a table.
  - extract solution to the initial instance from that table.
- Different from divide-and-conquer which partitions the problem into **independent** sub-problems and solves them recursively.

DYNAMIC PROGRAMMING

- Recall definition of Fibonacci numbers:
  - $F(n) = F(n-1) + F(n-2)$
  - $F(0) = 0$
  - $F(1) = 1$

- Computing the n^{th} Fibonacci number recursively (top-down):

$$F(n)$$

$$F(n-1) \quad + \quad F(n-2)$$

$$F(n-2) + F(n-3) \qquad F(n-3) + F(n-4)$$

$$\cdots$$

DYNAMIC PROGRAMMING

- Dynamic programming applications:
  - Computing a binomial coefficient.
  - Longest common subsequence.
  - **Warshall's** algorithm for transitive closure.
  - **Floyd's** algorithm for all-pairs shortest paths.
  - Constructing an optimal binary search tree.
  - Some instances of difficult discrete optimization problems:
    - - traveling salesman problem.
    - - **knapsack problem**.

MATRIX CHAIN MULTIPLICATION

- Given an $\boldsymbol{l} \times \boldsymbol{m}$ matrix $\boldsymbol{A}$ and an $\boldsymbol{m} \times \boldsymbol{q}$ matrix $\boldsymbol{B}$, the product

$$\underset{(\boldsymbol{l} \times \boldsymbol{q})}{\boldsymbol{C}} = \underset{(\boldsymbol{l} \times \boldsymbol{m})}{\boldsymbol{A}} \times \underset{(\boldsymbol{m} \times \boldsymbol{q})}{\boldsymbol{B}}$$

is an $\boldsymbol{l} \times \boldsymbol{q}$ matrix, where

$$\boldsymbol{C}[i,j] = \sum_{k=1}^{m} \boldsymbol{A}[i,k] \times \boldsymbol{B}[k,j]$$

and the number of multiplications used to compute $\boldsymbol{C}$ is:

$$l \times m \times q$$

MATRIX CHAIN MULTIPLICATION

- Assume $\boldsymbol{n}$ matrices are to be multiplied together:

$$\begin{matrix} (r_1 \times r_2) & (r_2 \times r_3) & (r_3 \times r_4) & \dots & (r_{n-1} \times r_n) & (r_n \times r_{n+1}) \\ \boldsymbol{M_1} & \boldsymbol{M_2} & \boldsymbol{M_3} & \dots & \boldsymbol{M_{n-1}} & \boldsymbol{M_n} \end{matrix}$$

where each matrix $\boldsymbol{M_i}$ has r_i rows and r_{i+1} columns for $1 \leq i \leq n$.

MATRIX CHAIN MULTIPLICATION

- The product can be computed in many different **orders**!
- Example 1: Let's compute the product of four matrices.

$$\underset{(4\times 2)}{A} \times \underset{(2\times 3)}{B} \times \underset{(3\times 1)}{C} \times \underset{(1\times 2)}{D}$$

there are 5 different ways:

1. $(((A\times B)\times C)\times D)$
2. $(A\times((B\times C)\times D))$
3. $((A\times B)\times(C\times D))$
4. $((A\times(B\times C))\times D)$
5. $(A\times(B\times(C\times D)))$

MATRIX CHAIN MULTIPLICATION

$$A \times B \times C \times D$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \end{pmatrix}$$

- Way 1: $\mathbf{44}$ multiplications

  $(\boldsymbol{A}\times\boldsymbol{B})$ $\quad$ $((\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C})$ $\quad$ $(((\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C})\times\boldsymbol{D})$

  $\mathbf{4\times2\times3 = 24}$ $\quad$ $\mathbf{4\times3\times1 = 12}$ $\quad$ $\mathbf{4\times1\times2 = 8}$

- Way 5: $\mathbf{34}$ multiplications

  $(\boldsymbol{C}\times\boldsymbol{D})$ $\quad$ $(\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D}))$ $\quad$ $(\boldsymbol{A}\times(\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D})))$

  $\mathbf{3\times1\times2 = 6}$ $\quad$ $\mathbf{2\times3\times2 = 12}$ $\quad$ $\mathbf{4\times2\times2 = 16}$

MATRIX CHAIN MULTIPLICATION

- Example 2: Given the matrix chain A_1, A_2, A_3, A_4 and matrix sizes:

  size(A_1) = 30×1, size(A_2) = 1×40,

  size(A_3) = 40×10, size(A_4) = 10×25.

Order of multiplications	Number of scalar multiplications
$(A_1 \times (A_2 \times (A_3 \times A_4)))$	40x10x25+1x40x25+30x1x25 =1,750
$(A_1 \times ((A_2 \times A_3) \times A_4))$	1x40x10+1x10x25+30x1x25 = 1,400
$((A_1 \times A_2) \times (A_3 \times A_4))$	30x1x40+40x10x25+30x40x25 = 41,200
$((A_1 \times (A_2 \times A_3)) \times A_4)$	1x40x10+30x1x10+30x10x25 = 8,200
$(((A_1 \times A_2) \times A_3) \times A_4)$	30x1x40+30x40x10+30x10x25 = 20,700

MATRIX CHAIN MULTIPLICATION

- The **matrix chain product problem** is to find the order of multiplying the matrices that **minimizes** the total number of multiplications used.

- We will use dynamic programming approach to find a solution for the matrix chain product problem.

MATRIX CHAIN MULTIPLICATION

- The **matrix chain product** algorithm.

$$(r_1 \times r_2) \quad (r_2 \times r_3) \quad (r_3 \times r_4) \quad \dots \quad (r_{n-1} \times r_n) \quad (r_n \times r_{n+1})$$

$$\boldsymbol{M_1 \quad M_2 \quad M_3 \quad \dots \quad M_{n-1} \; M_n}$$

  - First: there is only one way to compute $\boldsymbol{M_1 M_2}$ which takes $\boldsymbol{r_1 \times r_2 \times r_3}$ multiplications, $\boldsymbol{M_2 M_3}$ which takes $\boldsymbol{r_2 \times r_3 \times r_4}$ multiplications, … , $\boldsymbol{M_{n-1} M_n}$ and which takes $\boldsymbol{r_{n-1} \times r_n \times r_{n+1}}$ multiplications.
  - Next: We store these costs in a **TABLE**.

MATRIX CHAIN MULTIPLICATION

- Next: We find the best way to multiply successive triples:

$(M_1M_2M_3), (M_2M_3M_4), \dots, (M_{n-2}M_{n-1}M_n)$

The minimum cost of $(M_1M_2M_3)$ is the smaller of the cost of

1. $\left((M_1M_2)M_3\right) =$ the cost of (M_1M_2) plus $\boldsymbol{r_1} \times \boldsymbol{r_3} \times \boldsymbol{r_4}$
2. $\left(M_1(M_2M_3)\right) =$ the cost of (M_2M_3) plus $\boldsymbol{r_1} \times \boldsymbol{r_2} \times \boldsymbol{r_4}$

MATRIX CHAIN MULTIPLICATION

- When finding the costs of $((M_1M_2)M_3)$ and $(M_1(M_2M_3))$, we do not re-compute the costs of (M_1M_2) and (M_2M_3) but simply find them from the **TABLE**.

- The minimum costs of $(M_1M_2M_3)$, $(M_2M_3M_4), \ldots, (M_{n-2}M_{n-1}M_n)$ are also kept in the **TABLE**.

MATRIX CHAIN MULTIPLICATION

- Minimum Costs **TABLE.**

	M_1	M_2	M_3	$\dots$	M_{n-1}	M_n
M_1		M_1M_2	$M_1M_2M_3$		$M_1\dots M_{n-1}$	$M_1\dots M_n$
M_2			M_2M_3		$M_2\dots M_{n-1}$	$M_2\dots M_n$
M_3					$M_3\dots M_{n-1}$	$M_3\dots M_n$
$\dots$						
M_{n-1}						$M_{n-1}M_n$
M_n						

MATRIX CHAIN MULTIPLICATION

- In general, we find the best way to compute $(M_i M_{i+1} \ldots M_{i+j})$ by finding the minimum cost of computing $(M_i M_{i+1} \ldots M_{k-1})(M_k \ldots M_{i+j})$ for $i < k < j$

- The cost of $(M_i M_{i+1} \ldots M_{k-1})(M_k \ldots M_{i+j})$ is the sum of the cost of $(M_i M_{i+1} \ldots M_{k-1})$, the cost of $(M_k \ldots M_{i+j})$, and $r_i \times r_k \times r_{i+j+1}$.

MATRIX CHAIN MULTIPLICATION

- The cost of $(M_i M_{i+1} \dots M_{k-1})$ and the cost of $(M_k \dots M_{i+j})$ are found from the **TABLE**.

- The minimum cost of $(M_i M_{i+1} \dots M_{i+j})$ is kept in the **TABLE**.

MATRIX CHAIN MULTIPLICATION

- The pseudo-code is:

```
def MATRIX-CHAIN-ORDER (r):
    // r- list of matrices dimensions.
    // m - table of costs, s – optimal cost index table.
    n = r.length - 1
    m = matrix (1…n,1…n), s = matrix (1…n-1, 2…n)
    for i = 1 to n: m[i,i] = 0
    for l = 2 to n:
        for i = 1 to n – l + 1:
            j = i + l – 1
            m[i,j] = ∞
            for k = i to j – 1:
                q = m[i,k] + m[k+1,j] + r_{i-1} r_k r_j
                if q < m[i,j]: m[i,j] = q; s[i,j] = k
    return m, s
```

MATRIX CHAIN MULTIPLICATION

- The pseudo-code for printing the optimal parenthesization is:

```
def PRINT-OPTIMAL-PARENS (s,i,j):
// Given s, prints the optimal parethesization.
    if i == j:
        print "A"_i
    else:
        print "("
        PRINT-OPTIMAL-PARENS (s,i,s[i,j])
        PRINT-OPTIMAL-PARENS (s,s[i,j]+1,j)
        print ")"
```

MATRIX CHAIN MULTIPLICATION

- Example - Cost **TABLE** with parenthesization:

[ABC][DEF] -> [A[BC]][DEF] -> [A[BC]][[DE]F]

	B	C	D	E	F
A	24 [A] [B]	14 [A] [BC]	22 [ABC] [D]	26 [ABC] [DE]	36 [ABC] [DEF]
B		6 [B] [C]	10 [BC] [D]	14 [BC] [DE]	22 [BC] [DEF]
C			6 [C] [D]	10 [C] [DE]	19 [C] [DEF]
D				4 [D] [E]	10 [DE] [F]
E					12 [E] [F]

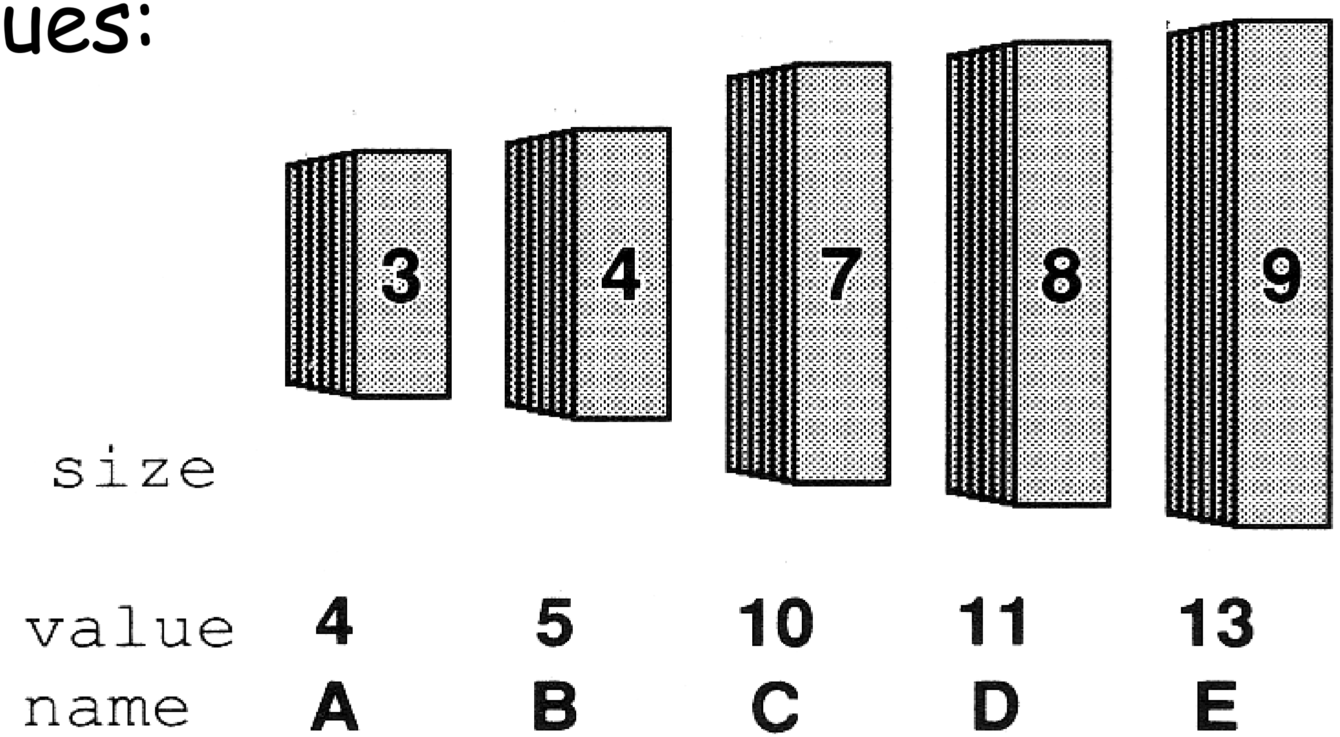
THE KNAPSACK PROBLEM

- A thief robbing a safe finds it filled with ***N*** types of items of varying size and value, but has only a small knapsack of capacity ***M*** to use to carry the goods.

- The knapsack problem is to find the combination of items which the thief should choose for his knapsack in order to **maximize the total value** of all the items it takes.

THE KNAPSACK PROBLEM

- For example, suppose that he has a knap-sack of capacity *17* and the safe contains many items of the following sizes and values:



THE KNAPSACK PROBLEM

- The thief can take:
  - 5 A's that make total value of 20 (space of 2 unused)
  - 1 D and 1 E that make total value of 24 (space of 3 unused)
- But how to **maximize** the total value?

THE KNAPSACK PROBLEM

- There are many commercial situations in which a solution to the knapsack problem could be important, for example a shipping company wishing to find the **best** way to load a truck or cargo plane, etc.

- In the dynamic-programming solution to the knapsack problem, we calculate the **best** combination for all knapsacks of sizes up to M.

THE KNAPSACK PROBLEM

- In this case, $cost[k]$ is the highest value that can be achieved with a knapsack of capacity k and $best[k]$ is the last item that was added to achieve that maximum.

- First, we calculate the best we can do for all knapsack sizes when only items of type A are taken, then we calculate the best that we can do when only A's and B's are taken, etc.

THE KNAPSACK PROBLEM

- Suppose an item $\boldsymbol{j}$ is chosen for the knapsack.
- Then the best value that could be achieved for the total would be:

$$\boldsymbol{val}[\boldsymbol{j}] + \boldsymbol{cost}[\boldsymbol{k} - \boldsymbol{size}[\boldsymbol{j}]].$$

- If this value exceeds the best value that can be achieved without an item $\boldsymbol{j}$, then we update $\boldsymbol{cost}[\boldsymbol{k}]$ and $\boldsymbol{best}[\boldsymbol{k}]$.

THE KNAPSACK PROBLEM

- Costs table for the previous example

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
j=1															
cost[k]	4	4	4	8	8	8	12	12	12	16	16	16	20	20	20
best[k]	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
j=2															
cost[k]	4	5	5	8	9	10	12	13	14	16	17	18	20	21	22
best[k]	A	B	B	A	B	B	A	B	B	A	B	B	A	B	B
j=3															
cost[k]	4	5	5	8	10	10	12	14	15	16	18	20	20	22	24
best[k]	A	B	B	A	C	B	A	C	C	A	C	C	A	C	C
j=4															
cost[k]	4	5	5	8	10	11	12	14	15	16	18	20	21	22	24
best[k]	A	B	B	A	C	D	A	C	C	A	C	C	D	C	C
j=5															
cost[k]	4	5	5	8	10	11	13	14	15	17	18	20	21	23	24
best[k]	A	B	B	A	C	D	E	C	C	E	C	C	D	E	C

THE KNAPSACK PROBLEM

- The first pair of lines shows the best that can be done with only A's.
- The second pair of lines shows the best that can be done with only A's and B's, etc.
- The highest value that can be achieved with a knapsack of size 17 is 24.

THE KNAPSACK PROBLEM

- The pseudo-code to calculate the $\boldsymbol{cost[k]}$ and $\boldsymbol{best[k]}$.

for $j = 1$ **to** n:
  for $i = 1$ **to** m:
    if $i >= size[j]$:
      if $cost[i] < cost[i\text{-}size[j]]+val[j]$
        $cost[\mathrm{i}] = cost[i\text{-}size[j]]+val[j])$
        $best[i] = j$

THE KNAPSACK PROBLEM

- The knapsack problem is easily solved for small size M, but the computing time may become unacceptable when the M is sufficiently large.

- This method does NOT work if M and the sizes or values are real numbers instead of integers.

That's All for Today!