Linear Systems Theory
JVN, Spring 2013

Time: 1:00pm - 4:00pm Thursdays
Location: Room 2, JVN Building
Notes: *Applied Numerical Computing* by Vandenberghe, UCLA.

Contents

Part 1. Linear Algebra

1. Vector Spaces

1.1. **Basic Objects.** We begin with the usual Euclidean spaces

$$\mathbb{R} = \{\text{all real numbers}\}$$

$$\dots$$

$$\mathbb{R} \times \dots \times \mathbb{R} = \mathbb{R}^n = \{\text{all } (x_1, \dots, x_n), x_i \in \mathbb{R}\}$$

Each element $(x_1, \dots, x_n)$ is called an n-tuple and each x_i is called its i^{th} component. Like real numbers, the elements in $\mathbb{R}^n$ can be added $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ and scaled by a real number $\alpha(x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$ for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n, \alpha \in \mathbb{R}$. These algebraic operations impose a structure on $\mathbb{R}^n$. We consider its abstract generalization.

Definition 1.1. A vector space over a field F is a set V together with two binary operations $+$ and $\cdot$ that satisfy the following axioms,

(1) (closure under addition) $u + v \in V$ for any $u, v \in V$.
(2) (associativity of addition) $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
(3) (commutativity of addition) $u + v = v + u$.
(4) (identity element under addition) there exists an element $0 \in V$ such that $0 + u = u + 0 = u$ for all $u \in V$.
(5) (inverse element under addition) there exists an element $-u$ such that $u + (-u) = -u + u = 0$ for all $u \in V$.
(6) (closure under scalar multiplication) $\alpha \cdot u \in V$ for any $\alpha \in F$ and $u \in V$.
(7) (distributivity of scalar multiplication with respect to vector addition) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.
(8) (distributivity of scalar multiplication with respect to field addition) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$.
(9) (compatibility of scalar multiplication with field multiplication) $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$.
(10) (identity element of scalar multiplication) $1 \cdot u = u$ for any $u \in V$ and 1 is the multiplicative identity in F.

For those who have followed some talks on abstract algebra, the first five axioms mean V is a group under $+$ and the next five axioms mean V is an F-module. The elements in V are called vectors while the elements in F are called scalars. We denote this vector space structure by $(V, +, \cdot_F)$ or $(V, +, \cdot)$ or V/F. Below are some examples.

Example 1.2. The set $V = \{*\}$ a single point over the field $\mathbb{Q}$ under the trivial $+$ and $\cdot$ is a vector space. It is called the zero vector space and denoted by $\{0\}$. The same holds when we replace $\mathbb{Q}$ with any field F.

Example 1.3. The plane $\mathbb{R}^2 = \{(x, y) \text{ with } x, y \in \mathbb{R}\}$ over the field $\mathbb{R}$ under the usual operations $+$ and $\cdot$ is a vector space. More generally, the space $\mathbb{R}^n = \{(x_1, \dots, x_n) \text{ with } x_i \in \mathbb{R}\}$ over the field $\mathbb{R}$ under the usual operations $+$ and $\cdot$ is a vector space.

Example 1.4. The rational plane $\mathbb{Q}^2 = \{(x, y) \text{ with } x, y \in \mathbb{Q}\}$ over the field $\mathbb{R}$ under the usual operations $+$ and $\cdot$ is not a vector space.

Example 1.5. The set $S = \{\sum_{i=1}^{n} a_i x^i\}$ of all finite sums of length n where $a_i \in \mathbb{C}$ and x is an indeterminate over the field $\mathbb{C}$ under the usual operations $+$ and $\cdot$ is a vector space.

Example 1.6. We consider different classes of maps from $\mathbb{R}^n$ to $\mathbb{R}$.

(1) A map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}, (x_1, ..., x_n) \mapsto f(x_1, ..., x_n)\}$ is called linear if $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for any $\alpha, \beta \in \mathbb{R}, u, v \in \mathbb{R}^n$. If we define addition $f + g$ as $(f + g)(x) = f(x) + g(x)$ and scalar multiplication as $(\alpha \cdot f)(x) = \alpha f(x)$ then the set $L = \{\text{all linear maps } \mathbb{R}^n \xrightarrow{f} \mathbb{R}\}$ over the field $\mathbb{R}$ under $+$ and $\cdot$ is a vector space.
(2) More generally, a map $\mathbb{R}^n \xrightarrow{g} \mathbb{R}, (x_1, ..., x_n) \mapsto g(x_1, ..., x_n)$ is called affine if $g(x) = f(x) + \alpha$ for some linear map f and some $\alpha \in \mathbb{R}$. One can verify that this condition is equivalent to the condition $g(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for any $\alpha + \beta = 1 \in \mathbb{R}, u, v \in \mathbb{R}^n$. The set $M = \{\text{all affine maps } \mathbb{R}^n \xrightarrow{g} \mathbb{R}\}$ over the field $\mathbb{R}$ under operations $+$ and $\cdot$ is a vector space.
(3) Most general is the set $N = \{\text{all maps } \mathbb{R}^n \xrightarrow{h} \mathbb{R}\}$ over the field $\mathbb{R}$ under same operations is a vector space.

One can see that $L \subset M \subset N$ and the operations on each subset agree with the operations from the larger set so in a sense L is a subspace of M which is a subspace of N. Abstractly, whenever we have objects with some structure, we also consider subobjects with the same structure.

Definition 1.7. A subset $U \subset V$ of a vector space $(V, +, \cdot)$ is called a subspace if U under $+$ and $\cdot$ is also a vector space over F.

Example 1.8. In example 1.6 $L \subset M \subset N$ as subspaces over $\mathbb{R}$. We will know more about them later.

Example 1.9. We can view $\mathbb{R}^2$ as a subspace $U = \{(x, y, 0), x, y \in \mathbb{R}\} \subset \mathbb{R}^3$. Why isn't $U' = \{(x, y, 1), x, y \in \mathbb{R}\} \subset \mathbb{R}^3$ a subspace?

We associate the first invariant to each vector space V over a field F, generalizing the notion of dimension that we often speak of for $\mathbb{R}^n$.

Definition 1.10. A finite sum $\sum_{i=1}^{n} \alpha_i u_i = \alpha_1 u_1 + ... + \alpha_n u_n$ with $\alpha_i \in F, u_i \in V$ is called a linear combination of $u_1, ..., u_n$. The set of all linear combinations of $u_1, ..., u_n$ is called their span and denoted by $\text{Span}(u_1, .., u_n)$. If there exist $\alpha_1, ..., \alpha_n$, not all zero, such that $\alpha_1 u_1 + ... + \alpha_n u_n = 0$ then we say the $u_1, ..., u_n$ are linearly dependent. Else we say they are linearly independent.

Example 1.11. It is easy to verify that $(1, 0), (0, 1)$ are linearly independent in $\mathbb{R}^2$ while $(2, 1, 1), (0, 1, 2), (4, 5, 8)$ are linearly dependent in $\mathbb{R}^3$.

Example 1.12. Any pair $x^i, x^j, i \neq j$ in example 1.5 are linearly independent.

Definition 1.13. A subset $B = \{u_i\}_{i \in I}, I$ an indexing set and $u_i \in V$, is called a basis for V if B is linearly independent and $\text{Span}(B) = V$.

By definition, every $v \in V$ can be written as a linear combination $v = \sum_{i \in I} \alpha_i v_i$ of members of basis $B = \{v_1, ..., v_n\}$ and such representation is unique by linear independence of B. Sometimes we write $v = (\alpha_i)$ in its coordinate form if the v_i are ordered. One can imagine that v has different representations and different coordinate forms in different bases.

Example 1.14. If we choose $B = \{(1,0),(0,1)\}$ as a basis for $\mathbb{R}^2$ then $v = (3,4)$ can be written as $3(1,0) + 4(0,1)$ with coordinate form $(3,4)$. If $B' = \{(0,1),(1,0)\}$ then still $v = 3(1,0) + 4(0,1) = 4(0,1) + 3(1,0)$ but its coordinate form is now $(4,3)$. If $B'' = \{(1,0),(0,2)\}$ is chosen then $v = 3(1,0) + 2(0,2) = (3,2)$ in its coordinate form.

Exercise 1.15. Find the representation and coordinate form of $(3,4)$ in basis $B = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$.

The previous examples show no basis is more special than the rest, only some bases are nicer for computation than others. Moreover, the order in each basis affects coordinate form representation. What is true is every vector space has a basis $B = \{v_i, i \in I\}$ and all bases of V have the same size, though that may be infinite. This is the first invariant we associate to each vector space V over a field F.

Definition 1.16. We define the dimension $\dim(V/F)$ of V/F as the size of any basis for V.

Example 1.17. The vectors $e_1 = (1,0,..,0), ..., e_n = (0,...,0,1)$ form a basis for $\mathbb{R}^n$ over $\mathbb{R}$ since they are clearly linearly independent and any $(x_1, ..., x_n)$ can be written as $x_1(1,0,...,0) + ... + x_n(0,...,1), x_i \in \mathbb{R}$. Therefore the dimension of $\mathbb{R}^n$ over $\mathbb{R}$ is n as conventionally known. They do not form a basis for $\mathbb{R}^n$ over $\mathbb{Q}$.

Example 1.18. The vectors $x, x^2, ..., x^n$ together form a basis for our vector space V over $\mathbb{C}$ in example 1.5. Its dimension is n. Viewed as a vector space over $\mathbb{R}$, however V has dimension $2n$ since one of its bases is $x, ix, x^2, ix^2, ..., x^n, ix^n$.

Example 1.19. The vector space N in example 1.6 has infinite dimension over $\mathbb{R}$. It fact, any of its bases must be uncountable.

1.2. **Inner Product.** This section focuses on vector spaces over $\mathbb{R}$. If they are to enjoy multiplication $V \times V \longrightarrow \mathbb{R}, (u,v) \mapsto u \cdot v$, we must expect the following.

Definition 1.20. An inner product on a vector space V over $\mathbb{R}$ is any map $V \times V \xrightarrow{\langle -,- \rangle} \mathbb{R}$ that satisfies the following axioms,

1. (symmetry) $\langle u, v \rangle = \langle v, u \rangle$.
2. (linearity) $\langle \alpha \cdot u + \beta \cdot v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$.
3. (positive definiteness) $\langle u, u \rangle \geq 0$ for all $u \in V$, with equality iff $u = 0$.

A vector space V over $\mathbb{R}$ equipped with an inner product is called an inner product space. Note that the product of two vectors is a scalar in $\mathbb{R}$. We can turn $\mathbb{R}^n$ into an inner product space as follows.

Definition 1.21. Given two vectors $u = (u_1, ..., u_n), v = (v_1, ..., v_n) \in \mathbb{R}^n$ we define their inner product to be $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + ... + u_n v_n$.

One can verify that this newly minted product satisfies the above three axioms. Warning: this inner product depends on basis. When we write $\langle v, w\rangle = \langle (u_1, ..., u_n), (v_1, ..., v_n)\rangle = u_1v_1 + ... + u_nv_n$ we assume the basis is $B = \{(1,0,...,0), ..., (0,...,0,1)\}$. Let us see some more examples.

Example 1.22. In example 1.14, $v = (3,4)$ has coordinate form $(3,4)$ and $w = (5,6)$ has coordinate form $(5,6)$ in $B = \{(1,0),(0,1)\}$. So $\langle v, w\rangle = 3\cdot 5 + 4\cdot 6 = 39$. In $B'' = \{(1,0),(0,2)\}$ v has coordinate form $(3,2)$ and w has coordinate form $(5,3)$ so $\langle v, w\rangle = 3\cdot 5 + 2\cdot 3 = 21$.

Example 1.23. The inner product of a vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ with the i^{th} unit vector e_i picks out its i^{th} coordinate x_i. This actually induces a linear map $\mathbb{R}^n \xrightarrow{f_{e_i}} \mathbb{R}, (x_1, ..., x_n) \mapsto \langle (x_1, ...x_n), e_i\rangle = x_i$.

Example 1.24. If $a_i, b_i \in \{0,1\}$ and $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{R}^n$ then $\langle a, b\rangle$ is the number of indices i where $a_i = b_i = 1$.

Following is a nice statement about inner product and linear maps as seen in example 1.6.

Theorem 1.25. *For any $a = (a_1, ..., a_n) \in \mathbb{R}^n$ the map $\mathbb{R}^n \xrightarrow{f_a} \mathbb{R}, (x_1, ..., x_n) \mapsto \langle x, a\rangle$ is linear. Conversely, any linear map f equals f_a for some $a \in \mathbb{R}^n$.*

Proof. That $f_a = \langle -, a\rangle$ is linear follows from the linearity of inner product. Conversely, for any linear map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ and any $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we have $f(x) = f(x_1e_1 + ... + x_ne_n) = x_1f(e_1) + ... + x_nf(e_n) = \langle x, a\rangle = f_a(x)$ where $a = (f(e_1), ..., f(e_n))$. □

It is worth noting here that what f does to $\{e_1, ..., e_n\}$ completely determines its whole behavior. The same holds for any map $V \xrightarrow{f} W$ between vector spaces over F. Here are two nice results.

Corollary 1.26. *The space L of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}$ has dimension n.*

Proof. It follows from the theorem that any linear map $f = f_a = \langle -, a\rangle = \langle -, (f(e_1), ..., f(e_n))\rangle = f(e_1)f_{e_1} + ... + f(e_n)f_{e_n}$ where f_{e_i} was defined in example 1.23. Furthermore, these $\{f_{e_1}, ..., f_{e_n}\}$ are linearly independent, hence they form a basis for L. Summarily, every basis $\{b_1, ..., b_n\}$ for $\mathbb{R}^n$ corresponds to a basis $f_{b_1}, ..., f_{b_n}$ for L. Hence $\mathbb{R}^n$ and L share the same dimension n. □

We now look at some examples.

Example 1.27. Taking average of the coordinates of a vector $x, f(x) = (x_1 + ... + x_n)/n$ is linear. Surely $f = f_a$ where $a = (1/n, ..., 1/n)$.

Example 1.28. Taking maximum of the coordinates of a vector $x, f(x) = \max\{x_1, ..., x_n\}$ is not linear. To see this, pick $n = 2, x = (1,-1), y = (-1,1)$. Then $f(x+y) \neq f(x) + f(y)$. Hence it can not be represented by any inner product.

Another nice result from theorem 1.25 is the following statement about the space M of all affine maps from $\mathbb{R}^n$ to $\mathbb{R}$.

Corollary 1.29. *The space M of all affine maps from $\mathbb{R}^n$ to $\mathbb{R}$ has dimension $n+1$.*

Proof. This follows from definition of affine maps in example 1.6 and previous corollary. $\square$

While M is much smaller than N, every continuously differentiable map $f \in N$ has a good affine approximation in M. Recall that if $\mathbb{R}^n \xrightarrow{f} \mathbb{R}, x = (x_1, ..., x_n) \mapsto f(x)$ is a continuously differentiable then we can take the continuous partial derivatives $\frac{\partial f(x)}{\partial x_i}, i = 1, ..., n$ and form its gradient $\nabla f(x) = (\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_n})$. The first-order Taylor approximation of f near x is defined as $f_{\text{aff}}(x') = (\frac{\partial f(x)}{\partial x_1}, ..., \frac{\partial f(x)}{\partial x_n})(x' - x)^t + f(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}(x'_i - x_i) + f(x) = \langle \nabla f(x), (x' - x) \rangle + f(x)$. This map f_{aff} is certainly affine and gives a good approximation of f when x' is near x.

Example 1.30. When $n = 1$ this is none other than the usual Taylor approximation $f_{\text{aff}}(x') = f'(x)(x' - x) + f(x)$ we often see.

Example 1.31. Consider $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}, (x_1, x_2) \mapsto e^{x_1+x_2-1} + e^{x_1-x_2-1} + e^{-x_1-1}$. Then $\nabla f(x) = (e^{x_1+x_2-1} + e^{x_1-x_2-1} - e^{-x_1-1}, e^{x_1+x_2-1} - e^{x_1-x_2-1})$. At $(0,0), \nabla f((0,0)) = (1/e, 0)$. Hence the first-order Taylor approximation of f near $(0,0)$ is $f_{\text{aff}}(x) = \langle \nabla f((0,0)), x - (0,0) \rangle + f((0,0)) = x_1/e + 3/e$.

1.3. **Norm.** Given a vector space V over $\mathbb{R}$ we want to make precise how large each vector $v \in V$ is. Of course, there are different ways to do so but they all must meet certain expectations.

Definition 1.32. A norm on V over $\mathbb{R}$ is a map $V \xrightarrow{\|-\|} \mathbb{R}$ that satisfies,

1. (positive definiteness) $\|v\| \geq 0$, with equality iff $v = 0$.
2. (homogeneity) $\|\alpha \cdot v\| = |\alpha| \|v\|$.
3. (triangle inequality) $\|u + v\| \leq \|u\| + \|v\|$.

Any vector space V over $\mathbb{R}$ with norm is called a normed space.

Example 1.33. One obvious way to define a norm on $\mathbb{R}^n$ is $\|(u_1, ..., u_n)\| = |u_1| + ... + |u_n|$. This is called the 1-norm and denoted by $\| - \|_1$.

Example 1.34. Another norm we have on $\mathbb{R}^n$ is the usual Euclidean norm $\|(u_1, ..., u_n)\| = \sqrt{u_1^2 + ... + u_n^2} = \sqrt{\langle u, u \rangle}$. It is also known as the 2-norm and denoted by $\| - \|_2$. The Euclidean norm is related to the root mean square (RMS, instead of mean or mean square) value of a vector u, defined as $\text{RMS}(u) = \sqrt{\frac{1}{n}(u_1^2 + ... + u_n^2)} = \frac{1}{\sqrt{n}}\|u\|$. This quantity roughly tells us about the typical value of the coordinates u_i with respect to n.

Example 1.35. More generally we define the p-norm on $\mathbb{R}^n$ as $\|(u_1, .., u_n)\|_p = \sqrt[p]{|u_1|^p + ... + |u_n|^p}$ for any $1 \leq p < \infty$.

Example 1.36. We define the ∞-norm on $\mathbb{R}^n$ as $\|(u_1, ..., u_n)\| = \max \{|u_1|, ..., |u_n|\}$. It is denoted by $\| - \|_\infty$.

Exercise 1.37. Draw all the vectors of norm 1 in $\mathbb{R}^2$, where norm here is the 1-norm, the p-norm for $1 < p < 2$, the 2-norm, the p-norm for $2 < p < \infty$, and the ∞-norm.

Example 1.38. One more norm we can give $\mathbb{R}^n$ is $||u||_w = \sqrt{(u_1/w_1)^2 + ... + (u_n/w_n)^2}$ for some $w = (w_1, ..., w_n)$. What this norm does is assigning different weights $w_1, ..., w_n$ to the coordinates $u_1, ..., u_n$ of u. Euclidean norm is now a special case of weighted norm, where each coordinate is given the same weight 1. Moreover, when each coordinate of u is a physical quantity with unit then the weights are often of the same units, so that u has unitless norm.

We just saw that $||u||_2 = \sqrt{\langle u, u\rangle}$ on $\mathbb{R}^n$. In general, any inner product $\langle -, -\rangle$ on a vector space V over $\mathbb{R}$ induces a map $V \xrightarrow{||-||} \mathbb{R}, u \mapsto ||u|| = \sqrt{\langle u, u\rangle}$. Such definition easily clears the first two axioms of being a norm due to the properties of inner product. We establish a nice theorem that would imply the triangle inequality axiom for such norm.

Theorem 1.39. *(Cauchy-Schwarz Inequality) For all u, v in an inner product space V with induced norm $|| - ||$ we have $|\langle u, v\rangle| \le ||u||||v||$. Moreover, equality holds iff u, v are linearly dependent.*

Proof. If either $u = 0$ or $v = 0$ then inequality holds. Else consider the quadratic polynomial $p(t) = ||tu + v||^2 = \langle tu + v, tu + v\rangle = \langle u, u\rangle t^2 + 2\langle u, v\rangle t + \langle v, v\rangle$. Being a square, $p(t) \ge 0$, hence its discriminant $4\langle u, v\rangle^2 - 4\langle u, u\rangle\langle v, v\rangle \le 0$, or $\langle u, v\rangle^2 \le \langle u, u\rangle\langle v, v\rangle$. Equality holds iff the discriminant is 0 iff $p(t) = 0$ iff $ut + v = 0$ iff v is a multiple of u. $\square$

Do you see why we considered such polynomial $p(t)$ as to show us exactly what we needed?

Corollary 1.40. *(Triangle Inequality) The induced norm $|| - ||$ from an inner product on V satisfies triangle inequality $||u + v|| \le ||u|| + ||v||$, all $u, v \in V$.*

Proof. Clearly $||u + v||^2 = \langle u + v, u + v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle = ||u||^2 + 2\langle u, v\rangle + ||v||^2 \le ||u||^2 + 2||u||||v|| + ||v||^2 = (||u|| + ||v||)^2$, from which our claim follows. $\square$

Thus an inner product on V induces a norm.

1.4. **Distance between Vectors.** A norm in turn induces distance.

Definition 1.41. For any vectors u, v in a vector space V equipped with norm we define the distance between them to be $dist(u, v) = ||u - v||$.

Example 1.42. All the above norms on $\mathbb{R}^n$ induce different distances between vectors in $\mathbb{R}^n$, though some of them are rather antiintuitive.

1.5. **Angle between Vectors.** As an inner product $\langle -, -\rangle$ induces a norm for V, we notice there is a relationship between $\langle u, v\rangle$ and $||u||||v||$ for any pair $u, v \in V$.

Example 1.43. When u, v are in the same direction, $v = \alpha u \in V$ for some positive α and $\langle u, v\rangle = \alpha\langle u, u\rangle = \alpha||u||^2 = |\alpha|||u||||u|| = ||u||||v||$. When u, v are in opposite direction, $v = \alpha u$ for some negative α and $\langle u, v\rangle = -||u||||v||$

Hence we suspect that the ratio $\frac{\langle u,v\rangle}{||u||||v||}$ bears some correlation between u and v, perhaps how u lines up against v..

Definition 1.44. For nonzero $u, v \in V$ inner product $\langle -, - \rangle$ and induced norm $\|-\|$ we define their correlation coefficient $\rho(u,v) = \frac{\langle u,v \rangle}{\|u\|\|v\|}$.

This correlation coefficient, viewed as a map $V \times V \xrightarrow{\rho} \mathbb{R}$ is surely symmetric. Its range is $[-1,1]$ by Cauchy-Schwarz inequality. Two vectors u, v are said to be highly correlated if $\rho(u,v)$ is close to 1, uncorrelated if $\rho(u,v) = 0$, and highly anticorrelated if $\rho(u,v)$ is close to -1.

Example 1.45. Consider $u = (0.1, -0.3, 1.3, -0.3, -3.3), v = (0.2, -0.4, 3.2, -0.8, -5.2), w = (1.8, -1.0, -0.6, 1.4, -0.2)$. Then $\|u\| = 3.57, \|v\| = 6.17, \|w\| = 2.57, \langle u, v \rangle = 21.7, \langle u, w \rangle = -0.06, \rho(u,v) = 0.98, \rho(u,w) = -0.007$. Therefore u and v are more correlated than u and w are (v is roughly twice u).

Example 1.46. (Throw back to probability theory) The set $RV((\Omega, \mathcal{F}, P), (\mathbb{R}, \mathcal{B}(\mathbb{R}))) = \{(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$ of all real-valued random variables form a vector space over $\mathbb{R}$. If we define an inner product for this space as $\langle X, Y \rangle = E((X - \mu_X)(Y - \mu_Y))$ then $\langle X, Y \rangle$ is none other $cov(X,Y)$, $\langle X, X \rangle = \|X\|^2$ is none other than $var(X)$, $\|X\|$ is none other than σ_X, and correlation coefficient $\rho(X,Y)$ is none other than the usual correlation coefficient $corr(X,Y)$ as known in probability theory.

From correlation coefficient we define angle between two vectors.

Definition 1.47. In an inner product space V we define the angle between two vectors u and v to be $\angle(u,v) = arccos(\rho(u,v)) = arccos\left(\frac{\langle u,v \rangle}{\|u\|\|v\|}\right)$.

Example 1.48. We can now speak of "angle" between two random variables X, Y.

This definition agrees with the usual notion of angle between vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, while generalizing to $\mathbb{R}^n, n > 3$. If $\angle(u,v) = 0°$, i.e. they have correlation coefficient 1 then each vector is a positive multiple of the other and we say u, v are aligned. If $0° < \angle(u,v) < 90°$, i.e. they have positive correlation coefficient then we say u, v make an acute angle. If $\angle(u,v) = 90°$, i.e. they have correlation coefficient 0 then we say u, v are orthogonal and write $u \perp v$. If $90° < \angle(u,v) < 180°$, i.e. they have negative correlation coefficient then we say u, v make an obtuse angle. Lastly, if $\angle(u,v) = 180°$, i.e. they have correlation coefficient -1 then each vector is a negative multiple of the other and we say u, v are antialigned. We are more interested in orthogonal vectors because they make excellent bases.

Definition 1.49. A vector v in a normed space V is called a unit vector if $\|v\| = 1$.

Any vector $v \in V$ can be easily normalized and replaced by an aligned unit vector $\frac{v}{\|v\|}$.

Definition 1.50. A collection of vectors $\{v_1, ..., v_n\}$ in an inner product space V with induced norm are said to be orthonormal if each v_i is a unit vector and $v_i \perp v_j$ for any $i \neq j$.

Example 1.51. Both collections $A = \{(1,0,0), (0,1,0)\}$ and
$B = \{(0,0,-1), (1/\sqrt{2}, 1/\sqrt{2}, 0), (1/\sqrt{2}, -1/\sqrt{2}, 0)\}$ are orthonormal while
$C = \{(1,-1,0), (1,1,1)\}$ can be normalized into an orthonormal one.

If $v = \alpha_1 v_1 + ... + \alpha_n v_n$ is a linear combination of an orthonormal collection $\{v_1, ..., v_n\}$ then surely $\langle v_i, v\rangle = \langle v_i, \alpha_1 v_1 + ... + \alpha_n v_n\rangle = \sum_{j=1}^{n} \langle v_i, \alpha_j v_j\rangle = \alpha_i \cdot 1 = \alpha_i$. So taking inner product with v_i picks out the i^{th} coefficient in the linear combination for v. It implies any orthonormal collection $\{v_1, ..., v_n\}$ are linearly independent. This is most useful when $n = \dim(V)$ and the $\{v_1, ..., v_n\}$ form a basis for V.

Exercise 1.52. page 18: 1.2, 1.7, 1.9, 1.10, 1.11, 1.12, 1.15, 1.16, 1.17, 1.21

2. Linear Maps between Vector Spaces

2.1. **Linear Maps.** We step back from vectors within a vector space V/F and consider the relationships between finite dimensional vector spaces over the same field F. Given two vector spaces V/F and W/F we would expect any map between them to respect their linear structures.

Definition 2.1. A map $V \xrightarrow{f} W$ between two vector spaces V, W over F is called linear if $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for all $u, v \in V$ and $\alpha, \beta \in F$.

Example 2.2. The trivial map $V \xrightarrow{f} W, v \mapsto 0$ between any two vector spaces is certainly linear.

Example 2.3. The map $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3, (x, y) \mapsto (x, y, 0)$ is linear. The image of f is the subspace we described in example 1.9 and this is how we often embed $\mathbb{R}^2$ into $\mathbb{R}^3$. In similar fashion can $\mathbb{R}^m$ be embedded into $\mathbb{R}^n, m < n$.

Example 2.4. With the usual basis $\{e_1, ..., e_n\}$ for $\mathbb{R}^n$ and inner product $\langle -, -\rangle$, each map $\mathbb{R}^n \xrightarrow{f_{e_i}} \mathbb{R}, (x_1, ..., x_n) \mapsto \langle (x_1, ..., x_n), e_i\rangle = x_i$ in example 1.23 is linear. For example, $f_{e_i}(e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$. People often denote f_{e_i} by e_i^* and call it the dual of e_i.

Example 2.5. More generally, if V is equipped with an inner product $\langle -, -\rangle$ then for each $v \in V$ we can define a linear map $V \xrightarrow{v^*} \mathbb{R}, w \mapsto \langle w, v\rangle$. It is called the dual of v.

The set $V^* = \{\text{all linear maps } V \xrightarrow{f} F\}$ is called the dual of V. Given two linear maps $V \underset{g}{\overset{f}{\rightrightarrows}} W$ we can define $V \xrightarrow{f+g} W, v \mapsto f(v) + g(v)$ and $V \xrightarrow{\alpha f} W, v \mapsto \alpha f(v)$. One can verify that they are both linear. Hence the space $L(V, W)$ of all linear maps between V and W is itself a vector space over F. The most notable thing about a linear map $V \xrightarrow{f} W$ is that it is completely determined by what it does to a basis $B = \{v_1, ..., v_n\} \subset V$. More precisely, if we write $V \ni v = a_1 v_1 + ... + a_n v_n$ then $f(v) = a_1 f(v_1) + ... + a_n f(v_n)$ by linearity of f. If we also choose a basis $C = \{w_1, ..., w_m\}$ for W then we can write $f(v_i) = b_{1i} w_1 + ... + b_{mi} w_m = (b_{1i}, ..., b_{mi})^t$. Hence $f(v) = a_1 f(v_1) + ... + a_n f(v_n) = a_1 (b_{11}, ..., b_{m1})^t + ... + a_n (b_{1n}, ..., b_{mn})^t = (a_1 b_{11} + ... + a_n b_{1n}, ..., a_1 b_{m1} + ... + a_m b_{mn})^t = \begin{pmatrix} b_{11} & ... & b_{1n} \\ . & . & . \\ b_{m1} & ... & b_{mn} \end{pmatrix} (a_1, ..., a_n)^t$ if we define multiplication between an $m \times n$ matrix and an $n \times 1$ matrix as such (we have used the transpose notation t here without having introduced it yet.) We state this in a theorem.

Theorem 2.6. *Any linear map $V \xrightarrow{f} W$ can be represented by a matrix A_f once V, W are given bases. Conversely any $m \times n$ matrix A is a linear map between a vector space V/F of dimension n and a vector space W/F of dimension m.*

Proof. It remains to show that as a map, $A(\alpha v + \alpha' v') = \alpha A(v) + \alpha' A(v)$, but this follows from definition of matrix multiplication above. □

Domain V and codomain W for A are often understood to be F^n with canonical basis $\{e_1, ..., e_n\}$ and F^m with $\{e_1, ..., e_m\}$. Let us look at some examples.

Example 2.7. The trivial map $V \xrightarrow{0} W, v \mapsto 0$ is represented by the 0 matrix with respect to any bases for V and W.

Example 2.8. The identity map $V \xrightarrow{id} V, v \mapsto v$ is linear and is represented by the identity matrix $I_n \in M(n, F)$ with respect to any basis for V. More generally, scaling $V \xrightarrow{\lambda} V, v \mapsto \lambda v$ is linear and represented by the matrix $(\lambda) \in M(n, F)$ with respect to any basis for V.

Example 2.9. Reflection across any line in the plane $\mathbb{R}^2$ is linear. If we choose an orthonormal basis $\{u_1, u_2\}$ such that $r(u_1) = u_2$ and $r(u_2) = u_1$ then clearly it is represented as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Similarly, reflection across u_1 is linear and represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ while projection onto u_1 is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 2.10. For any linear map $V \xrightarrow{A} W$ with respect to basis $\{v_1, ..., v_n\}$ for V and $\{w_1, ..., w_m\}$ for W, $A(v_i)$ equals the i^{th} column of A as seen below,

$$\begin{pmatrix} c_{11} & \dots & c_{1i} & \dots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & \dots & c_{ni} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{1i} \\ \vdots \\ \vdots \\ \vdots \\ c_{ni} \end{pmatrix}.$$

Example 2.11. This example shows that matrix representation for linear map depends on bases for domain and codomain. Consider the linear map

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2, (1,0) \mapsto (1,0), (0,1) \mapsto (0,2)$$

If we choose $\{e_1 = (1,0), e_2 = (0,1)\}$ as basis for both domain and codomain then $(1,0) = 1e_1 + 0e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $(0,1) = 0e_1 + 1e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $(0,2) = 0e_1 + 2e_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ in coordinate forms. Hence the matrix representation for f with respect to this basis is

$$\mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{R}^2, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

However, if we choose basis $\{e_1 = (1,0), e_2 = (0,1)\}$ for domain $\mathbb{R}^2$ and basis $\{u_1 = (1,0), u_2 = (0,2)\}$ for codomain $\mathbb{R}^2$ then $(1,0) = 1u_1 + 0u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (0,2) = 0u_1 + 1u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in coordinate forms. Hence the matrix representation for f with respect to this basis is

$$\mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{R}^2, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.2. **Kernel, Nullity, Image, Rank.** The first thing we look at in each linear map $V \xrightarrow{f} W$ is what it annihilates in V and what it covers in W.

Definition 2.12. For any linear map $V \xrightarrow{f} W$ we define $\ker(f) = \{v \in V \text{ such that } f(v) = 0\}$ and $\operatorname{im}(f) = \{w \in W \text{ such that } w = f(v) \text{ for some } v \in V\}$.

One can verify that $\ker(f)$ is a subspace of V and $\operatorname{im}(f)$ is a subspace of W. We say f is injective if $\ker(f) = \{0\}$, f is surjective if $\operatorname{im}(f) = W$, and f is bijective if it is both injective and surjective.

Definition 2.13. For any linear map $V \xrightarrow{f} W$ we define $\operatorname{nullity}(f) = \dim(\ker(f))$, called the nullity of f. We define $\operatorname{rank}(f) = \dim(\operatorname{im}(f))$, called the rank of f.

Clearly f is injective iff $\operatorname{nullity}(f) = 0$ and f is surjective if $\operatorname{rank}(f) = \dim(W)$. Furthermore, if $\{v_1, ..., v_k\}$ is a basis for $\ker(f)$ then it can be completed to a basis $\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ for V. Write any $V \ni v = \alpha_1 v_1 + ...\alpha_k v_k + \alpha_{k+1} v_{k+1} + ... + \alpha_n v_n$ then $f(v) = \alpha_1 f(v_1) + ...\alpha_k f(v_k) + \alpha_{k+1} f(v_{k+1}) + ... + \alpha_n f(v_n) = \alpha_{k+1} f(v_{k+1}) + ... + \alpha_n f(v_n)$. Therefore $\operatorname{im}(f) = \operatorname{Span}(f(v_{k+1}), ..., f(v_n))$. While $\dim(\ker(f)) = k$, we see $\dim(\operatorname{im}(f)) \le n - k$. Equality follows from the following theorem.

Theorem 2.14. *(Rank-Nullity) If $V \xrightarrow{f} W$ is a linear map then $\dim(V) = \operatorname{nullity}(f) + \operatorname{rank}(f)$.*

Proof. It remains to show $\{f(v_{k+1}), ..., f(v_n)\}$ are linearly independent and thus form a basis for $\operatorname{im}(f)$. Suppose $\beta_{k+1} f(v_{k+1}) + ... + \beta_n f(v_n) = 0$ for some $\beta_{k+1}, ..., \beta_n \in F$, then $f(\beta_{k+1} v_{k+1} + ... + \beta_n v_n) = 0$. So $\beta_{k+1} v_{k+1} + ... + \beta_n v_n \in \ker(f)$ and we can write $\beta_{k+1} v_{k+1} + ... + \beta_n v_n = \beta_1 v_1 + ... + \beta_k v_k$ for some $\beta_1, ..., \beta_k \in F$. Since the v_i form a basis for V, the β_i must all be 0. In particular, $\beta_{k+1} = ... = \beta_n = 0$. □

Example 2.15. The zero map $V \xrightarrow{0} W$ has $\operatorname{nullity}(0) = \dim(V)$ and $\operatorname{rank}(0) = 0$ while for $\lambda \neq 0$ scaling $V \xrightarrow{\lambda} V$ has $\operatorname{nullity}(\lambda) = 0$ and $\operatorname{rank}(\lambda) = \dim(V)$.

Example 2.16. Reflections in example 2.9 has nullity 0 and rank 2. On the other hand, projection onto u_1 has nullity 1 and rank 1. In general, if $\{u_1, ..., u_k\}$ are linearly independent in V then projection onto $\operatorname{Span}(u_1, ..., u_k)$ has rank $= k$ and nullity $= \dim(V) - k$.

If we view each matrix $A \in M(m, n, F)$ as a linear map $F^n \xrightarrow{A} F^m$ then from example 2.10 and theorem 2.14, $n - k$ of its columns will form a basis for its image while the other k columns are linearly dependent upon those. Looking closely at columns of a matrix reveals information about that matrix.

Example 2.17. The matrix $\begin{pmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ has rank 2 and nullity 1. It is neither injective nor surjective.

A linear map $V \xrightarrow{f} W$ is bijective iff $\text{nullity}(f) = 0$ and $\dim(V) = \text{rank}(f) = \dim(W)$ iff its associated matrix A_f is invertible iff its determinant is nonzero. The inverse map $W \xrightarrow{f^{-1}} V$ is also linear with associated matrix $A_{f^{-1}} = A_f^{-1}$.

3. Matrices

3.1. **Basic Concepts.** As any linear map between vector spaces over F with fixed bases is represented by a matrix, we study matrices more closely. First goes a formal definition of matrix.

Definition 3.1. A matrix M is a rectangular array of $(a_{ij})_{m\times n}$ of m rows and n columns, where each entry a_{ij} in the i^{th} row and j^{th} column is an element in F. We denote the set of all $m \times n$ matrices over F by $M(m, n, F)$.

Example 3.2. We have

(1) $\begin{pmatrix} 1/2 \\ 2 \\ 2/3 \end{pmatrix} \in M(3, 1, \mathbb{Q}), a_{3,1} = 2/3$

(2) $M = \begin{pmatrix} sin(\pi/10) & cos(\pi/10) \end{pmatrix} \in M(1, 2, \mathbb{R}), a_{1,2} = cos(\pi/10)$

(3) $M = \begin{pmatrix} i & e & 1 \\ \pi & 0 & 1/2 \\ ln5 & 1 & 3 \end{pmatrix} \in M(3, \mathbb{C}), a_{2,2} = 0$

Below are what we can do with matrices.

3.1.1. *Partition a matrix into submatrices.* such as $A_{4\times5} = \begin{pmatrix} A_{3\times3} & A_{3\times2} \\ A_{1\times3} & A_{1\times2} \end{pmatrix}$. This will be useful later when we multiply matrices by blocks without fretting too much about entries.

Example 3.3. If $A \in M(4, 5, F)$ and $B \in M(r + s, 4, F)$ then they can be partitioned into blocks for multiplication as follows,

$$\begin{pmatrix} B_{r\times3} & B_{r\times1} \\ B_{s\times3} & B_{s\times1} \end{pmatrix} \begin{pmatrix} A_{3\times3} & A_{3\times2} \\ A_{1\times3} & A_{1\times2} \end{pmatrix} = \begin{pmatrix} B_{r\times3}A_{3\times3} + B_{r\times1}A_{1\times3} & B_{r\times3}A_{3\times2} + B_{r\times1}A_{1\times2} \\ B_{s\times3}A_{3\times3} + B_{s\times1}A_{1\times3} & B_{s\times3}A_{3\times2} + B_{s\times1}A_{1\times2} \end{pmatrix}$$

.

3.1.2. *Addition of matrices of same size.* We equip the set $M(m, n, F)$ with addition.

Definition 3.4. For $A = (a_{ij}), B = (b_{ij}) \in M(m, n, F)$ we define $A+B = (c_{ij}) \in M(m, n, F)$ where $c_{ij} = a_{ij} + b_{ij}$.

Example 3.5. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & -3 \end{pmatrix}$.

.

Proposition 3.6. *Matrix addition enjoys the following properties,*

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$

Proof. Both statements follow from commutativity and associativity of addition in F. $\square$

Importantly, addition of matrices $A+B$ corresponds to addition of linear maps $V \xrightarrow{f_A+f_B} W$ once bases are chosen for V and W. This equips $M(m,n,F)$ with addition.

3.1.3. *Multiplication of matrices.* Next question is how to define scalar multiplication λA_f to represent the linear map λf. If we represent scaling by λ by the matrix $(\lambda,...,\lambda)$ then we are looking at how to define the product $(\lambda,...,\lambda)A$ to represent the composition $U \xrightarrow{f} V \xrightarrow{\lambda} V$. More generally, if $U \xrightarrow{f} V \xrightarrow{g} W$ are represented by A, B with respect to some chosen basses for U, V, W then we must define the right product BA to represent gf. Here is the picture.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "gf"'] & V \arrow[d, "g"] \\ & W \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "A"] \arrow[dr, "C"'] & V \arrow[d, "B"] \\ & W \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "A"] \arrow[dr, "BA"'] & V \arrow[d, "B"] \\ & W \end{tikzcd}$$

Definition 3.7. For $A = (a_{ij}) \in M(m,n,F)$ and $B = (b_{ij}) \in M(l,m,F)$ we define $BA = (c_{ij}) \in M(l,n,F)$ where $c_{ij} = a_{1j}b_{i1} + a_{2j}b_{i2} + ... + a_{mj}b_{im} = \sum_{k=1}^{m} a_{kj}b_{ik}$ is the inner product of i^{th} row of B and j^{th} column of A.

Proposition 3.8. *Consider U, V, W over F with dimensions n, m, l and bases $\{u_1,...,u_n\}$, $\{v_1,...,v_m\}$, and $\{w_1,...,w_l\}$. If $U \xrightarrow{f} V$ is represented by $A = (a_{ij}) \in M(m,n,F)$ and $V \xrightarrow{g} W$ is represented by $B = (b_{ij}) \in M(l,m,F)$ then $U \xrightarrow{gf} W$ is represented by BA.*

Proof. We write out $A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & ... & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & ... & b_{1m} \\ \vdots & & \vdots \\ b_{l1} & ... & b_{lm} \end{pmatrix}$. Then

$f(u_1) = A(1,0,...,0)^t = (a_{11},...,a_{m1})^t = a_{11}v_1 + ... + a_{m1}v_m$
...
...
$f(u_n) = A(0,...,0,1)^t = (a_{1n},...,a_{mn})^t = a_{1n}v_1 + ... + a_{mn}v_m$
$gf(u_1) = a_{11}B(v_1) + ... + a_{m1}B(v_m) = a_{11}(b_{11},...,b_{l1})^t + ... + a_{m1}(b_{1m},...,b_{lm})^t$
...
...
$gf(u_n) = a_{1n}B(v_1) + ... + a_{mn}B(v_m) = a_{1n}(b_{11},...,b_{l1})^t + ... + a_{mn}(b_{1m},...,b_{lm})^t$

Hence gf is represented by $\begin{pmatrix} a_{11}b_{11} + ... + a_{m1}b_{1m} & ... & a_{1n}b_{11} + ... + a_{mn}b_{1m} \\ \vdots & & \vdots \\ a_{11}b_{l1} + ... + a_{m1}b_{lm} & ... & a_{1n}b_{l1} + ... + a_{mn}b_{lm} \end{pmatrix}$, which is precisely BA. $\square$

Note that the number of columns of A must equal the number of rows of B. This definition agrees with our earlier one in 2.1.

Example 3.9. $\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 1 & 5 & 6 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 4 & 0 & 5 \\ 1 & 2 & 0 \end{pmatrix} = C_{2\times 3}$ where $c_{21} = 2\cdot 2 + 1\cdot 3 + 5\cdot 4 + 6\cdot 1 = 33$

Example 3.10. $5\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 4 & 0 & 5 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 4 & 0 & 5 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 15 \\ 15 & 0 & 5 \\ 20 & 0 & 25 \\ 5 & 10 & 0 \end{pmatrix}$

As a special case we define scalar multiplication as $\alpha \cdot A = \begin{pmatrix} \alpha & \dots & 0 \\ . & \alpha & . \\ 0 & \dots & \alpha \end{pmatrix} A$. This scalar multiplication together with matrix addition turn $M(m,n,F)$ into a vector space over F, just as $L(F^n, F^m)$ is a vector space over F. More is true.

Theorem 3.11. *For F^n and F^m with standard bases $\{e_1, ..., e_n\}$ and $\{e_1, ..., e_m\}$, the linear maps $(L(F^n, F^m), +, \cdot) \underset{\varphi}{\overset{\phi}{\rightleftarrows}} (M(m,n,F), +, \cdot), f \mapsto A_f, A \leftarrow\!\shortmid A$ are inverses of each other.*

Proof. obvious. ☐

Corollary 3.12. *The vector space $L(F^n, F^m)$ has dimension mn.*

Proof. The collection $\{E_{ij} = (1_{ij}), 1 \le i \le m, 1 \le j \le n\}$ form a basis for $M(m,n,F)$. ☐

Next, we explore some primary results about matrix multiplication.

Proposition 3.13. *Matrix multiplication enjoys the following properties,*

(1) (associativity) $(AB)C = A(BC)$
(2) (distributivity) $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$
(3) (identity element) $I_{m\times m}A_{m\times n} = A_{m\times n}I_{n\times n} = A$ for all $A_{m\times n}$ (best when $m = n$)
(4) (commutativity in scalar multiplication) $\alpha A = A\alpha$

Here are a few more examples to illuminate matrix multiplication.

Example 3.14. One can see $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ne \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ through direct multiplication or through geometry in $\mathbb{R}^2$. Matrix multiplication is not commutative in general.

Example 3.15. Matrix multiplication fails to cancel, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ but clearly $\begin{pmatrix} 3 \\ 1 \end{pmatrix} \ne \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Example 3.16. Matrix multiplication has zero divisors, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 3.17. Matrix multiplication has idempotents, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 3.18. Given $\alpha_1, ..., \alpha_k \in F$ and $A_1, ..., A_k \in M(m,n,F)$ we can form a linear combination $\alpha_1 A_1 + ... + \alpha_k A_k \in M(m,n,F)$.

Example 3.19. Given a polynomial such as $p(x) = x^2 + 2x + 3 \in \mathbb{R}[x]$, we can view it as $\mathbb{R} \xrightarrow{p(x)} \mathbb{R}, 4 \mapsto 4^2 + 2 \cdot 4 + 3 = 27$. Now we can also view it as $M(n,\mathbb{R}) \xrightarrow{p(x)} M(n,\mathbb{R}), A \mapsto A^2 + 2 \cdot A + 3$.

3.1.4. *Trace of a matrix.* We assign to each square matrix $A \in M(n,F)$ its first invariant.

Definition 3.20. For $A = (a_{ij}) \in M(n,F)$ we define its trace as sum along its diagonal $tr(A) = \sum_{i=1}^{n} a_{ii}$.

The following results are immediate from definition.

Proposition 3.21. *For any $A, B \in M(n,F)$ and $\alpha \in F$,*

(1) $tr(\alpha A) = \alpha tr(A)$.
(2) $tr(A+B) = tr(B+A)$
(3) $tr(AB) = tr(BA)$.

Proof. Straightforward. □

Hence trace is a linear map from $M(n,F)$ to F. Actually, these three properties characterize trace completely, any linear map $M(n,F) \xrightarrow{f} F$ satisfying the above three properties must be a multiple of trace. If A has eigenvalues $\lambda_1, ..., \lambda_n$ with corresponding nonzero eigenvectors $v_1, ..., v_n$ then $A(v_i) = \lambda_i v_i$, or equivalently $(\lambda_i I - A)(v_i) = 0$. This means $\det(\lambda_i I - A) = 0$ and $\lambda_1, ..., \lambda_n$ are roots of the polynomial $p(\lambda) = \lambda^n - (a_{11} + ... + a_{nn})\lambda^{n-1} + ... = \lambda^n - tr(A)\lambda^{n-1} + ...$

Example 3.22. $tr(\lambda) = n\lambda$ for any matrix $(\lambda) \in M(n,F)$.

Example 3.23. $tr\begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 1 \\ 5 & 3 & 1 \end{pmatrix} = 2 + 4 + 1 = 7.$

3.1.5. *Transpose of a matrix.* Next we attribute to each (not necessarily square) matrix $A \in M(m,n,F)$ a companion, it helps us express matrix theory.

Definition 3.24. For $A = (a_{ij}) \in M(m,n,F)$ we define its transpose $A^t = (b_{ij}) \in M(n,m,F)$ where $b_{ij} = a_{ji}$.

Proposition 3.25. *Transpose has the following properties,*

(1) $(A^t)^t = A$.
(2) $(A+B)^t = A^t + B^t$.
(3) $(\alpha A)^t = \alpha A^t$.
(4) $(AB)^t = B^t A^t$.

Proof. Straightforward. □

Example 3.26. $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 4 & 1 \\ 5 & 3 & 1 \end{pmatrix}^t = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 3 \\ 4 & 1 & 1 \end{pmatrix}$

Example 3.27. $\begin{pmatrix} 2 & 1 & 7 \end{pmatrix}^t = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$

Transpose can be used to describe many interesting classes of matrices. The first class is triangular matrices.

Definition 3.28. A matrix $A \in M(n,F)$ is called lower triangular if all entries above the diagonal are zero, i.e. $a_{ij} = 0$ for all $i < j$. A matrix $A \in M(n,F)$ is called upper triangular if $a_{ij} = 0$ for all $i > j$. Furthermore, if $a_{ii} = 1$ then we say A is unit lower triangular or unit upper triangular, respectively.

Example 3.29. We have $A = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 3 & 6 & 1 \end{pmatrix}$ is lower triangular and $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ is upper triangular.

Definition 3.30. A matrix $A \in M(n,F)$ is called diagonal if $a_{ij} = 0$ whenever $i \neq j$.

Example 3.31. These matrices are all diagonal,

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^n = \begin{pmatrix} 4^n & 0 & 0 \\ \vdots & 2^n & \vdots \\ 0 & \dots & 1^n \end{pmatrix}^n$$

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B^{-1} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3.32. One can verify that if $A = (a_{ii}) \in M(n,F)$ is a diagonal matrix and $M \in M(n,F)$ is any matrix then AM is the matrix whose row i^{th} is the i^{th} row or A multiplied by a_{ii}.

One sees from definition that A is lower triangular iff A^t is upper triangular and vice versa, while A is diagonal iff it is both lower triangular and upper triangular. Furthermore, each of $L(n,F), U(n,F), D(n,F)$ is closed under addition, multiplication and inverse. For example, if $A, B \in L(n,F)$ then $AB \in L(n,F)$ while $A^{-1} \in U(n,F)$ for any invertible $A \in U(n,F)$. Another class of matrices that transpose helps describe is symmetric matrices.

Definition 3.33. A square matrix $A \in M(n,F)$ is called symmetric if $A = A^t$, or equivalently if $a_{ij} = a_{ji}$ for all i,j.

Example 3.34. We have

(1) $A = \begin{pmatrix} 0 & 4 & 1 \\ 4 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ is symmetric.

(2) Any diagonal matrix is symmetric while other triangular matrices are not. Draw a Venn's diagram of $L(n,F), U(n,F), D(n,F), S(n,F)$.

Note that symmetric matrices are closed under addition and inverse, but not under multiplication. One sure source for symmetric matrices is taking inner product. If x, y are two vectors in an inner product space V and $\{v_1, ..., v_n\}$ are a basis for V then $x = \alpha_1 v_1 + ... + \alpha_n v_n,\ y = \beta_1 v_1 + ... + \beta_n v_n$ and $\langle x, y\rangle = \langle \alpha_1 v_1 + ... + \alpha_n v_n, \beta_1 v_1 + ... + \beta_n v_n\rangle = (\alpha_1, ..., \alpha_n) A (\beta_1, ..., \beta_n)^t$ where $A = (\langle v_i, v_j\rangle)$. Even if $\{v_1, ..., v_n\}$ are not a basis for V, we can still form $A = (\langle v_i, v_j\rangle)$.

Definition 3.35. A square matrix $A \in M(n,F)$ is named Gram if $A = (\langle v_i, v_j\rangle)$ for some vectors $v_1, ..., v_n \in V$.

This matrix is symmetric since inner product is symmetric. Moreover, if we choose a basis for V and write $v_i = (\alpha_{1i}, ..., \alpha_{ni})^t$ then $\langle v_i, v_j\rangle = (\alpha_{1i}, ..., \alpha_{ni})(\alpha_{1j}, ..., \alpha_{nj})^t$, so $A = B^t B$ where B has i^{th} column $(\alpha_{1i}, ..., \alpha_{ni})^t$.

Proposition 3.36. *The following statements for $v_1, ..., v_n \in V/\mathbb{R}$ are equivalent,*

(1) they are orthonormal.
(2) $B^t B = I$ where B has i^{th} column $(\alpha_{1i}, ..., \alpha_{ni})^t$.
(3) their Gram matrix is I.

Proof. follows from above discussion. □

Such matrices as B also have a name.

Definition 3.37. A matrix $A \in M(m,n,\mathbb{R})$ is called orthonormal (or orthogonal) if its columns form an orthonormal collection in V.

The number n of columns of A may not be m, so they need not form a basis and A need not be square. We will prove that it has as many or more rows than columns. Orthonormal matrices are special. For one, $A^t A = I$ by definition, so their transpose is their inverse from the left. Moreover, they preserve inner product, hence norm and angle when viewed as maps between inner product spaces.

Proposition 3.38. *If $V \xrightarrow{A} W$ is an orthonormal map between vector spaces over $\mathbb{R}$ then $\langle A(u), A(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$.*

Proof. If $(\alpha_1, ..., \alpha_n)^t$ and $(\beta_1, ..., \beta_n)^t$ are the coordinate forms for u and v then $\langle A(u), A(v)\rangle = (A(\alpha_1, ..., \alpha_n)^t)^t A(\beta_1, ..., \beta_n)^t = (\alpha_1, ..., \alpha_n) A^t A (\beta_1, ..., \beta_n)^t = (\alpha_1, ..., \alpha_n)(\beta_1, ..., \beta_n)^t = \langle v, v'\rangle$. □

The picture looks like this,

$$\begin{tikzcd} V \times V \arrow[r, "{(A,A)}"] \arrow[d, "{\langle -,- \rangle}"'] & W \times W \arrow[d, "{\langle -,- \rangle}"] \\ F \arrow[r, "id"] & F \end{tikzcd}$$

Definition 3.39. A square matrix $A \in M(n,\mathbb{R})$ is called positive semidefinite if it is symmetric and $x^tAx \geq 0$ for all $x \in \mathbb{R}^n$. It is called positive definite if in addition to being semidefinite, A satisfies $x^tAx = 0$ only if $x = 0$.

Example 3.40. For matrices of small sizes, we can verify the sign of x^tAx to see if A is positive semidefinite or positive definite. Concretely, $A = \begin{pmatrix} 9 & 6 \\ 6 & 5 \end{pmatrix}$ is positive definite as $(x_1,x_2)A(x_1,x_2)^t = (3x_1+2x_2)^2+x_2^2 \geq 0$ for all $(x_1,x_2) \in \mathbb{R}^2$ and $(x_1,x_2)A(x_1,x_2)^t = 0$ iff $(x_1,x_2) = (0,0)$. On other hand $B = \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$ is positive semidefinite as $(x_1,x_2)A(x_1,x_2)^t = (3x_1+2x_2)^2 \geq 0$ for all $(x_1,x_2) \in \mathbb{R}^2$. However it is not positive definite since $(2,-3)A(2,-3)^t = 0$. Lastly, $B = \begin{pmatrix} 9 & 6 \\ 6 & 3 \end{pmatrix}$ is not positive semidefinite as $(x_1,x_2)A(x_1,x_2)^t = (3x_1+2x_2)^2 - x_2^2$ for all $(x_1,x_2) \in \mathbb{R}^2$ and $(\frac{2}{3},1)A(\frac{2}{3},1)^t < 0$.

Example 3.41. Another source for positive semidefinite matrices are Gram matrices $A \in M(n,\mathbb{R})$, since $x^tAx = x^tB^tBx = (Bx)^t(Bx) = \langle Bx, Bx\rangle \geq 0$. Clearly A is positive definite iff $B(x) = 0$ implies $x = 0$ iff B has trivial kernel iff B^t has full image.

3.1.6. *Norm of a matrix.* Viewed as a linear map between normed spaces, a matrix $A \in M(m,n,F)$ will either stretch or shrink a vector. This behavior is measured by $\frac{||A(v)||}{||v||}, v \in V$.

Definition 3.42. For each linear map $V \xrightarrow{A} W$ between normed spaces we define its norm $||A|| = \max\{\frac{||A(v)||}{||v||}, v \in V\}$.

The following instances examplify this norm.

Example 3.43. The scalar matrix (λ) has norm $|\lambda|$.

Example 3.44. The $n \times 1$ matrix $(a_1,...,a_n)^t$ has norm $(a_1^2 + ... + a_n^2)^{1/2}$ as it would when viewed as a vector.

It is not easy to calculate norm of a matrix in general, although MATLAB and wolframalpha can approximate matrix norm by numerical methods. We list some facts about matrix norm.

Proposition 3.45. *Matrix norm enjoys the following properties,*

1. *(Homogeneity)* $||\alpha A|| = |\alpha|||A||$.
2. *(Triangle inequality)* $||A + B|| \leq ||A|| + ||B||$.
3. *(Definiteness)* $||A|| \geq 0$ *for all* A *and equality holds iff* $A = 0$.
4. $||A|| = \max\{||A(x)||, ||x|| = 1\}$.
5. $||A(x)|| \leq ||A||||x||$ *for all vectors* x.
6. $||AB|| \leq ||A||||B||$ *for all* A, B.
7. $||A^t|| = ||A||$.

Proof. straightforward. □

3.2. **Invertible Matrices and their Inverses.**

3.2.1. *Square matrices.* When $V \xrightarrow{f} W$ is a linear map between vector spaces of equal dimension n over F, its matrix representation A with respect to a basis is square and it is feasible to determine whether f is bijective by looking at A. Or abstractly, $M(n,F)$ has been equipped with $+$ and $\cdot$ and we want to consider those matrices that are invertible under $\cdot$.

Definition 3.46. A square matrix $A \in M(n,F)$ is called invertible (or nonsingular) if there exists $B \in M(n,F)$ such that $AB = BA = I$, in which case we denote B by A^{-1}. Else we say A is noninvertible (or singular).

Example 3.47. The inverse of $I \in M(n,F)$ is I itself. More generally, for $\alpha \neq 0 \in F$,

$$\begin{pmatrix} \alpha & \dots & 0 \\ \vdots & \alpha & \vdots \\ 0 & \dots & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} 1/\alpha & \dots & 0 \\ \vdots & 1/\alpha & \vdots \\ 0 & \dots & 1/\alpha \end{pmatrix}$$

Example 3.48. One can verify that $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} -5/12 & 3/12 & 4/12 \\ 7/12 & 3/12 & -8/12 \\ 1/12 & -3/12 & 4/12 \end{pmatrix}$ are inverses of each other. Or one can input $\{\{1,2,3\},\{3,2,1\},\{2,1,3\}\}$ into wolframalpha and let it do the work.

Example 3.49. Any matrix with a whole row of zeros or a whole column of zeros is singular. For example, if $M(3,\mathbb{R}) \ni A = \begin{pmatrix} c_1 & c_2 & 0 \end{pmatrix}$ then $BA = B\begin{pmatrix} c_1 & c_2 & 0 \end{pmatrix} = \begin{pmatrix} Bc_1 & Bc_2 & 0 \end{pmatrix} \neq I$ for any matrix B, so A is not invertible. Or one can look at what A does to the basis $\{e_1, e_2, e_3\}$ to see its kernel and range.

Theorem 3.50. *Any invertible matrices $A, B \in M(n,F)$ satisfy the following,*

(1) A has unique inverse A^{-1} and $(A^{-1})^{-1} = A$.
(2) $(AB)^{-1} = B^{-1}A^{-1}$
(3) $(A^n)^{-1} = (A^{-1})^n$.
(4) $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ for $k \neq 0 \in \mathbb{R}$.
(5) $(A^t)^{-1} = (A^{-1})^t$.

Proof. prove a few of these in class. □

Example 3.51. We reuse A and A^{-1} in example 3.48. One can verify that $A^2 = \begin{pmatrix} 13 & 9 & 14 \\ 11 & 11 & 14 \\ 11 & 9 & 16 \end{pmatrix}$ and $(A^{-1})^2 = \begin{pmatrix} 25/72 & -9/72 & -14/72 \\ -11/72 & 27/72 & -14/72 \\ -11/72 & -9/72 & 22/72 \end{pmatrix}$ are inverses of each other.

Example 3.52. Or $12A^{-1} = \begin{pmatrix} -5/12 & 3/12 & 4/12 \\ 7/12 & 3/12 & -8/12 \\ 1/12 & -3/12 & 4/12 \end{pmatrix} = \begin{pmatrix} -5 & 3 & 4 \\ 7 & 3 & -8 \\ 1 & -3 & 4 \end{pmatrix}$ is the inverse of $\frac{1}{12}A$.

Example 3.53. Or $(A^{-1})^t = \begin{pmatrix} -5/12 & 7/12 & 1/12 \\ 3/12 & 3/12 & -3/12 \\ 4/12 & -8/12 & 4/12 \end{pmatrix}$ is the inverse of $A^t = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{pmatrix}$

We relate invertibility of a matrix A with its behavior as a linear map.

Theorem 3.54. *A matrix $A \in M(n,F)$ is invertible iff $F^n \xrightarrow{A} F^n$ has trivial kernel iff it has full image.*

Proof. show in class. ☐

Corollary 3.55. *Positive definite matrices $A \in M(n,\mathbb{R})$ are invertible while positive semidefinite matrices $A \in M(n,\mathbb{R})$ that are not positive definite are noninvertible.*

Proof. If A is positive definite and $x \in \ker(A)$ then $x^tAx = x^t0 = 0$ implies $x = 0$. Hence A has trivial kernel and is invertible by the theorem. On the other hand, if A is positive semidefinite but not positive definite then there exists some $x \neq 0$ such that $x^tAx = 0$. We have the quadratic polynomial $p(t) = (x+tAx)^tA(x+tAx) = (x^t+tx^tA^t)(Ax+tA^2x) = x^tAx+tx^tA^2x+tx^tA^tAx+t^2x^tA^tA^2x = 0+2tx^tA^2x+t^2x^tA^3x = t(2x^tA^2x+tx^tA^3x) \geq 0$ with minimum 0 at $t = 0$. It follows that $p'(0) = 2x^tA^2x = 2x^tA^tAx = 2\langle Ax, Ax\rangle = 2||Ax||^2 = 0$. By positive definiteness of inner product, $Ax = 0$. So A has nontrivial kernel and is noninvertible by the theorem. ☐

Corollary 3.56. *For $A \in M(n,\mathbb{R})$, the product $A^tA \in M(n,\mathbb{R})$ is positive definite iff A^tA is invertible.*

Proof. Surely $x^tA^tAt = (Ax)^t(Ax) = \langle Ax, Ax\rangle = ||Ax||^2 \geq 0$ for all $x \in V$ so A^tA is positive semidefinite. The claim now follows from corollary 3.55. ☐

3.2.2. *General matrices.* When $V \xrightarrow{f} W$ is a linear map between vector spaces V and W of possibly different dimensions n and m then its matrix representation A has size $m \times n$ and we have more to keep track of. If $m > n$ then A has more rows than columns and is called a tall matrix. If $m < n$ then A has more columns than rows and is called a wide matrix.

Definition 3.57. A matrix $A \in M(m,n,F)$ is called left invertible if there exists a matrix $B \in M(n,m,F)$ such that $BA = I \in M(n,F)$. It is called right invertible if there exists a matrix $B \in M(n,m,F)$ such that $AB = I \in M(m,F)$.

Again we relate invertibility of A with its behavior as a linear map.

Theorem 3.58. *A matrix $A \in M(m,n,F)$ is left invertible iff A has trivial kernel.*

Proof. If A has a left inverse B then $A(x) = 0, x \in V$ implies $x = I(x) = BA(x) = B(0) = 0$, so A has trivial kernel. Conversely, suppose A has trivial kernel. By the proof of corollary 3.56 A^tA is positive definite, hence invertible with inverse $(A^tA)^{-1}$. But then $((A^tA)^{-1}A^t)A = (A^tA)^{-1}(A^tA) = I$, so A is left invertible. ☐

Corollary 3.59. *Any left invertible matrix A must be square or tall.*

Proof. By theorem 3.58, A has trivial kernel and V embeds into W, so $n \leq m$ (recall Nullity-Rank theorem). ☐

This left inverse $(A^tA)^{-1}A^t$ for A is called Moore-Penrose pseudoinverse, it is not unique. Recall that a matrix A is called orthogonal if $A^tA = I$, so it has left inverse and consequently as many or more rows than columns. The case A is right invertible and $m \leq n$ is mirrored.

Theorem 3.60. *A matrix $A \in M(m,n,F)$ is right invertible iff A has full image.*

Proof. We see A is right invertible iff A^t is left invertible iff A^t has trivial kernel by theorem 3.58 iff $(A^t)^t = A$ has full image. □

Corollary 3.61. *Any right invertible matrix A must be square or wide.*

Proof. By theorem 3.60, A has full image and $\mathrm{im}(V)$ covers W, so $n \geq m$. □

A right inverse for a right invertible matrix A is $A^t(AA^t)^{-1}$ as we can imagine. It is also called Moore-Penrose pseudoinverse. Note that a right inverse for A may not be its left inverse and vice versa.

3.3. **Exercises.**

Exercise 3.62. Show that $M(m,n,F)$ has dimension mn as a vector space over F.

Exercise 3.63. If $u_1, ..., u_n$ is an orthonormal basis for $\mathbb{R}^n$ with respect to the ususal inner product $\langle -,- \rangle$ and $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$ is linear such that $\langle v, f(w) \rangle = \langle f(v), w \rangle$ for all $v, w \in \mathbb{R}^n$ then show that its matrix representation A_f with respect to this basis is symmetric. Such a linear map f is called self-adjoint with respect to $\langle -,- \rangle$.

Exercise 3.64. page 41: 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.10, 2.12.

Exercise 3.65. page 63: 3.1, 3.4, 3.6, 3.17

Part 2. Matrix Algorithms

3.4. **Complexity of Matrix Algorithms.** As engineers, you care about the amount of effort and time to do your work, even if that is matrix operations. For example you want to know how many steps an algorithm takes to multiply two matrices $A, B \in M(n,F)$ or to solve a linear system of equations $Ax = b$.

Definition 3.66. A flop is an execution of either $+$ or $\cdot$ in F.

To evaluate the complexity of an algorithm, we write the total number of flops in that algorithm as a function of the sizes of objects at hand, for example $f(m,n)$ for matrices $A \in M(m,n,F)$, and simplify the expression by discarding minor terms. Flop count played a larger role when flops were slow, so their count gave an accurate prediction of time cost. Nowadays that is no longer the case and flop count gives a good rough estimate of time cost, especially when we consider its order.

Example 3.67. Taking inner product of $(x_1, ..., x_n), (y_1, ..., y_n) \in F^n$ takes n multiplications and $n-1$ additions for $2n-1$ flops. We only keep $p(n) = 2n$ as a polynomial of the size n of our vectors.

Example 3.68. Adding $A, B \in M(m,n,F)$ takes mn additions for mn flops. We keep $p(m,n) = mn$ as a polynomial of the size of our matrices.

Example 3.69. If $V \xrightarrow{A} W$ is a linear map between vector spaces of dimensions n and m then computing $A(x)$ takes m inner products, hence $m(2n-1)$ flops. We keep $p(m,n) = 2mn$. In the special case A is diagonal of size $n \times n$, computing $A(x)$ takes only

n flops. In another case $A = BC$ where $B \in M(m,p,F), C \in M(p,n,F)$ then computing $A(x) = B(C(x))$ takes $2pn + 2mp = 2p(m+n)$ flops, much few than $2mn$ when $p \ll \min\{m,n\}$.

Example 3.70. Multiplying two matrices $A \in M(m,n,F), B \in M(n,p,F)$ takes mn inner products of vectors of size n for $mp(2n-1)$ flops, or $2mnp$ if $n \gg 1$. In the special case $m = p$ and AB is symmetric, this number decreases to $(m^2/2+m)(2n-1) = m(m+1)(2n-1)/2$, or m^2n as the leading term.

Flop count in multiplying more than two matrices depends on how we parse the product. For $A \in M(m,n,F), B \in M(n,p,F), C \in M(p,q,F)$, computing $(AB)C$ takes $2mp(n+q)$ flops while computing $A(BC)$ takes $2nq(m+p)$ flops. The first computation is better when $2mp(n+q) < 2nq(m+p)$, i.e. when $1/n + 1/q < 1/m + 1/p$.

3.5. **Triangular Matrices.** In this section we show that solving the system $Ax = b$ where A is a triangular square matrix of size n takes n^2 flops. Assume A is lower triangular.

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Surely $x_1 = b_1/a_{11}, x_2 = (b_2 - a_{21}x_1)/a_{22}, x_3 = (b_3 - a_{31}x_1 - a_{32}x_2)/a_{33}, \dots, x_n = (b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n(n-1)}x_{n-1})/a_n n$, each step taking $1, 3, 5, \dots, (2n-1)$ flops for a total of n^2 flops. This algorithm is called forward substitution. If A is diagonal then there are only n divisions, so the algorithm take n flops. Or if A is sparse with at most k nonzeros in each row then each substitution step takes $2k-1$ flops and the algorithm takes $2(k-1)n \approx 2kn$ flops. If A is upper triangular, the algorithm is called backward substitution and works in a transposed fashion.

Whether we get through all the substitution steps, i.e. whether the system is solvable depends on the innate properties of A. We study the linear algebra of triangular matrices.

Theorem 3.71. *A triangular matrix $A \in M(n,F)$ is invertible iff its diagonal has no zero iff* $\det(A) \neq 0$.

Proof. Recall that the columns $c_1, \dots, c_n$ span $im(A)$. So A is invertible iff $im(A) = \mathbb{R}^n$ iff $\{c_1, \dots, c_n\}$ are linearly independent iff $\alpha_1 c_1 + \dots + \alpha_n c_n = 0$ implies $\alpha_j = 0$, all j iff $a_{ii} \neq 0$ for all i iff $\det(A) \neq 0$. ☐

Corollary 3.72. *If a triangular matrix $A \in M(n,F)$ has diagonal without zero then the system $Ax = b$ has unique solution for any b.*

The above proof does not reveal A^{-1}. If $A^{-1} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}$ then $AA^{-1} = I$ sets up a system of linear equations $Ac_1 = e_1, \dots, Ac_n = e_n$. We perform either algorithm above to solve each equation for c_i and get A^{-1}.

3.6. **Cholesky Factorization.** We next examine the complexity of working with positive definite matrices. Assume that every positive definite matrix A can be factored as LL^t where L is a lower triangular matrix with positive diagonal entries. This is called the Cholesky factorization of A and L is called the Cholesky factor.

Example 3.73. $A = (\alpha) \in M(1, F)$ is positive definite then $\alpha > 0$ and $A = (\sqrt{\alpha})(\sqrt{\alpha})$ is its Cholesky factorization.

Example 3.74. $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ is its Cholesky factorization.

We will return to how Cholesky factorization works and how it takes $n^3/3$ flops. Meanwhile, let's see what it does for us. Suppose we had to solve a system of linear equations $Ax = b$ where A is positive definite. If we write $A(x) = L(L^t(x)) = L(y) = b$ then we see,

- Cholesky factorization takes $n^3/3$ flops.
- Solving $L(y) = b, L$ lower triangular takes n^2 flops.
- Solving $L^t(x) = y, L$ upper triangular takes n^2 flops.
- Hence the total cost is $n^3/3 + 2n^2 \approx n^3/3$ flops.

Example 3.75. Suppose we had to solve $4x_1 + 2x_2 = 3, 2x_1 + 10x_2 = 5$. Factoring $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix}$ into $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ takes $2^3/3$ flops. Solving $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ to get $(y_1, y_2) = (3/2, 7/6)$ takes 2^2 flops. Lastly solving $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 7/6 \end{pmatrix}$ to get $(x_1, x_2) = (5/9, 7/18)$ takes another 2^2 flops. The total cost is $8/3 + 2 \cdot 2^2$ flops.

Suppose we had to solve multiple systems $Ab_1 = c_1, ..., Ab_m = c_m$, which is equivalent to solving the matrix equation $A_{n\times n}B_{n\times m} = C_{n\times m}$ where B has i^{th} column b_i and C has j^{th} column c_j. Of course we don't repeat Cholesky factorization, so the total flop count is $n^3/3 + 2mn^2$. The effort of solving a small number of linear systems with the same A is roughly the effort of solving one system. This applies to finding the inverse of A, i.e. finding B such that $AB = I$ since we know any positive definite matrix $A = LL^t$ is invertible by corollary 3.56 with inverse $A^{-1} = (L^t)^{-1}L^{-1} = (L^{-1})^t L^{-1}$. The cost for A^{-1} is then $n^3/3 + 2n \cdot n^2 = 7n^3/3$. Is this cheaper than finding L^{-1} and multiplying with $(L^{-1})^t$?

We return to the existence of Cholesky factorization $A = LL^t$ as well as how to do it recursively, starting with the partition

$$\begin{pmatrix} a_{11} & A_{21}^t \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} l_{11} & L_{21}^t \\ 0 & L_{22}^t \end{pmatrix} = \begin{pmatrix} l_{11}^2 & l_{11}L_{21}^t \\ l_{11}L_{21} & L_{21}L_{21}^t + L_{22}L_{22}^t \end{pmatrix}$$

Hence $l_{11} = \sqrt{a_{11}}, L_{21} = \frac{1}{l_{11}}A_{21}$, and $L_{22}L_{22}^t = A_{22} - L_{21}L_{21}^t$. This means we now must find L_{22} in a Cholesky factorization of the positive definite matrix $A_{22} - L_{21}L_{21}^t = A_{22} - \frac{1}{a_{11}}A_{21}A_{21}^t$ of smaller sizer $(n-1) \times (n-1)$. Eventually we arrive at a Cholesky factorization of a 1×1 matrix, which is just taking square root. As a process,

(1) Compute the first column of $l_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{1}{l_{11}}A_{21}$ of L.
(2) Compute Cholesky factorization $A_{22} - L_{21}L_{21}^t = L_{22}L_{22}^t$.

The whole algorithm takes $n^3/3$ flops. It also verifies whether a matrix A is positive definite, as it will stall at some recursive step with $a_{11} = 0$ if A is not.

3.7. **Permutation Matrices.** Reordering the rows in the system of linear equations $Ax = b$ sometimes makes a difference in the cost of solving it. Let $\sigma \in S_n$ be a permutation of $\{1, ..., n\}$ then we can view it as a map $V \xrightarrow{\sigma} V, (x_1, ..., x_n) \mapsto (x_{\sigma(1)}, ..., x_{\sigma(n)})$. This map is linear with matrix representation $P_\sigma = (a_{ij})$ where $a_{ij} = 1$ if $\sigma(i) = j$ and $a_{ij} = 0$ otherwise.

Example 3.76. If $\sigma = (1324)$ then it maps (x_1, x_2, x_3, x_4) to (x_4, x_3, x_1, x_2) and $P_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

If σ^{-1} is the inverse of σ as permutations then $\sigma^{-1}(j) = i$. As linear maps, σ_{-1} is the inverse of σ and $P_{\sigma^{-1}} = (a_{ji}) = P_\sigma^t = P_\sigma^{-1}$. That means solving a system $P(x) = b$ where P is a permutation matrix costs 0 flop, with $x = P^t(b)$.

3.8. **LU Factorization.** We now tackle the general case of solving $A(x) = b$ where $A \in M(n, \mathbb{R})$ is invertible by using the same factor-solve approach as in previous subsections. First we look to factor A as $A_1 ... A_k$ where A_i is special, k is small, and the flop count is f. Then we solve $A(z_1) = b, A(z_2) = z_1, ..., A_k(x) = z_{k-1}$ sequentially, here the flop count is s. The total flop count is then $f + s \approx f$ when f is dominant.

Example 3.77. Solving $Ax = b$ where $A \in M(n, \mathbb{R})$ is positive definite costs $n^2/3 + 2n^2$ flops.

Again this approach is effective in solving m systems of equations sharing the same coefficient matrix A, as the cost is then $f + ms \approx f$ when f is dominant. We look at the existence of such factorization.

Lemma 3.78. *(LU factorization) Every invertible matrix $A \in M(n, \mathbb{R})$ can be written as PLU where P is permutation, L is unit lower triangular and U is invertible upper triangular.*

Proof. We will furnish an explicit algorithm to do this. □

Example 3.79. $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1/2 & -7/2 \\ 0 & 0 & -11 \end{pmatrix}$. Wolframalpha will do this with command “LU decomposition”.

Here is the explicit algorithm to solve a system $Ax = b$ with invertible $A \in M(n, \mathbb{R})$:

(1) Factoring $A = PLU$ in $2n^3/3$ flops.
(2) Computing $v = P^t(b)$ in 0 flop.
(3) Forward Substituting $L(w) = v$ in n^2 flops.
(4) Backward Substituting $U(x) = w$ in n^2 flops.

Total cost is $2n^3/3 + 2n^2 \approx 2n^3/3$ flops. This is the standard method in the industry.

Example 3.80. Given a system $2x_1 + x_2 + 3x_3 = 4, 0x_1 + 2x_2 + 3x_3 = 2, 3x_1 + x_2 + x_3 = 1$ we form A as in example 3.79 and get its factorization. Next we calculate

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -18 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1/2 & -7/2 \\ 0 & 0 & -11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -18 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -24/11 \\ -16/11 \\ 18/11 \end{pmatrix}$$

What step 2 actually does is follow $x = A^{-1}(b) = U^{-1}(L^{-1}(P^t(b)))$. Moreover, solving multiple systems of m equations with the same invertible coefficient matrix $A \in M(n, \mathbb{R})$ would cost $2n^3/3 + 2mn^2 \approx 2n^3/3$ if $n \gg m$. Lastly, solving for A^{-1} is the same as solving n systems $Ac_1 = e_1, ..., Ac_n = e_n$, which would take $2n^3/3$ flops for LU factorization plus $n \cdot 2n^2$ flops for n forward substitutions and n backward substitutions.

3.8.1. *LU factorization without pivoting.* We now return to the problem of LU factorization. Up first is the special case $P = I$ and $A = LU$, which sets up an matrix equation in blocks,

$$\begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

where $L_{21} \in M(n-1, 1, \mathbb{R}), U_{12} \in M(1, n-1, \mathbb{R})$, $L_{22} \in M(n-1, n-1, \mathbb{R})$ is unit lower triangular and $U_{22} \in M(n-1, n-1, \mathbb{R})$ is invertible upper triangular. This reduces the problem to LU factorization for an $(n-1) \times (n-1)$ matrix $A_{22} - \frac{1}{a_{11}}A_{21}A_{12} = L_{22}U_{22}$. Repeat until we arrive at an 1×1 matrix, whose factors are square roots. This algorithm works only if $a_{11} \neq 0$ at each recursive step and no permutation is needed to permute a nonzero with a_{11}, hence the name no pivoting (or no permutation). Step by step,

(1) Calculate the first row of $u_{11} = a_{11}, U_{12} = A_{12}$ of U.
(2) Calculate the first column $L_{21} = \frac{1}{a_{11}}A_{21}$ of L.
(3) Factor $A_{22} - \frac{1}{a_{11}}A_{21}A_{12}$.

Example 3.81. Consider the invertible matrix $\begin{pmatrix} 4 & 3 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}$. In step 1, $u_{11} = a_{11} = 4, U_{12} = A_{12} = \begin{pmatrix} 3 & 2 \end{pmatrix}$. In step 2, $L_{21} = \frac{1}{4}A_{12} = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$. In step 3, $A_{22} - \frac{1}{4}A_{12}A_{21} = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{3}{2} & 3 \end{pmatrix}$ is invertible so we start over recursively. Again $u_{11} = a_{11} = -\frac{3}{4}, U_{12} = A_{12} = \frac{1}{2}$. Next $L_{21} = \frac{1}{a_{11}}A_{21} = -\frac{4}{3}\frac{3}{2} = -2$. Thirdly $A_{22} - \frac{1}{a_{11}}A_{21}A_{12} = 3 + \frac{4}{3}\frac{3}{2}\frac{1}{2} = 4 = 2 \cdot 2$. Now we put everything together,

$$\begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{3}{2} & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ 0 & 2 \end{pmatrix} = L_{22}U_{22}$$

$$\begin{pmatrix} 4 & 3 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -2 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ 0 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix} = LU$$

.

Example 3.82. The matrix $\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$ can not be factored as $\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$ since the algorithm halts at first step with $a_{11} = 0$.

3.8.2. *LU factorization with pivoting.* When $a_{11} = 0$ in the previous algorithm, we use a permutation matrix P to permute a_{11} with a nonzero. Such permutations allow us to continue the steps and prove the existence of LU factorization $A = PLU$ for the general case. Surely any invertible $A = (a_{11}) \in M(1, \mathbb{R})$ can be written as $(1)(1)(a_{11})$. Suppose any invertible matrix of size $(n-1) \times (n-1)$ can be factored. Consider invertible $A \in M(n, \mathbb{R})$. Since its first column is nonzero, there exists a permutation matrix P_1^t such that $A' = P_1^t A = \begin{pmatrix} a'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}$ with $a'_{11} \neq 0$. The difference $A'_{22} - \frac{1}{a'_{11}} A'_{21} A'_{12}$ is called the Schur complement of a'_{11} in A'. It is invertible of size $(n-1) \times (n-1)$, hence has a LU factorization $P_2 L_{22} U_{22}$. But then $A = P_1 A' = P_1 \begin{pmatrix} a'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix} = P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} a'_{11} & A'_{12} \\ P_2^t A'_{21} & P_2^t A'_{22} \end{pmatrix} =$
$P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} a'_{11} & A'_{12} \\ P_2^t A'_{21} & L_{22} U_{22} + \frac{1}{a'_{11}} P_2^t A'_{21} A'_{12} \end{pmatrix} =$
$P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{a'_{11}} P_2^t A'_{21} & L_{22} \end{pmatrix} \begin{pmatrix} a'_{11} & A'_{12} \\ 0 & U_{22} \end{pmatrix}$. Thus we are done with induction. This proof also reveals to us how to factor A. Summarily, the algorithm for LU factorization is,

(1) Permute A to A' by a permutation matrix P_1^t so that $a'_{11} \neq 0$.
(2) Compute LU factorization for $A'_{22} - \frac{1}{a'_{11}} A'_{21} A'_{12} = P_2 L_{22} U_{22}$.
(3) Put together $P = P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ \frac{1}{a'_{11}} P_2^t A'_{21} & L_{22} \end{pmatrix}$, $U = \begin{pmatrix} a'_{11} & A'_{12} \\ 0 & U_{22} \end{pmatrix}$.

This recursive algorithm also takes $2n^3/3$ flops.

Example 3.83. Consider the invertible matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{pmatrix}$. In step 1, we permute it to $A' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix}$ with $a'_{11} = 1$ by $P_1^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In step 2, $A'_{22} - \frac{1}{a'_{11}} A'_{21} A'_{12} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is invertible so we start over recursively. This time $a'_{11} = 1$ is already nonzero so $P_2^t = I$. Next $A'_{22} - \frac{1}{a'_{11}} A'_{21} A'_{12} = -1 - \frac{1}{1} \cdot 1 \cdot 1 = (1)(1)(-2)$. So in step 3, $P_3 = 1, L_3 = 1, U_3 = -2$. Again in step 3, $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, L_{22} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $U_{22} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$. Lastly $P = P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$L = \begin{pmatrix} 1 & 0 \\ \frac{1}{a'_{11}} P_2^t A'_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, and $U = \begin{pmatrix} a'_{11} & A'_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$.

3.9. **Effect of Rounding in Using LU Factorization.** We discuss the effect of rounding errors on the accuracy of the LU factorization method for solving a system of linear equations. Consider the system $10^{-5}x_1 + x_2 = 1, x_1 + x_2 = 0$. Its exact solution is $(\frac{-1}{1-10^{-5}}, x_2 = \frac{1}{1-10^{-5}})$. On the other hand, $A = \begin{pmatrix} 10^{-5} & 1 \\ 1 & 1 \end{pmatrix}$ has two LU factorization

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 10^5 & 1 \end{pmatrix}\begin{pmatrix} 10^{-5} & 1 \\ 0 & 1-10^{-5} \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 10^{-5} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1-10^{-5} \end{pmatrix}$$

If we introduce some errors by rounding $a_{22} = 1-10^5 = -0.9999\cdot 10^5$ in the first factorization to four significant digits $-1.0000\cdot 10^5 = -10^5$ and proceed to solve the system then we get solution $(0,1)$. The difference in result is huge. This phenomenon is called numerical instability and an algorithm is called numerically unstable if small errors along the way can cause significant error in the end. Solving linear equations using LU factorization is numerically unstable.

Similarly, if we round $a_{22} = 1 - 10^{-5} = 0.99999$ in the second factorization to 1 and solve the system then we get solution $(-1,1)$. In this case, the difference is tiny, 10^{-5} of the same order as the error we manufactured. This means choice of factorization, i.e. choice of permutation matrix P, matters. Careful error analysis and extensive practical experience tell us to go with P that permutes the element of largest absolute value in the first column of A to a'_{11}. Our example agrees with this.

3.10. **Exercises.**

Exercise 3.84. Given a permutation $\sigma = (1\,3\,4\,2) \in S_4$,

(a) Compute the inverse permutation σ^{-1}.
(b) Compute the associated permutation matrices P_σ and $P_{\sigma^{-1}}$.
(c) Understand why $P_\sigma P_{\sigma^{-1}} = I$ and $P_{\sigma^{-1}} = P_\sigma^t$.
(d) Compute how P_σ permutes the rows of $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$

Exercise 3.85. Perform Cholesky factorization for $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Exercise 3.86. Perform LU factorization for $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Exercise 3.87. page 77: 4.2.

Exercise 3.88. page 83: 5.2, 5.3.

Exercise 3.89. page 92: 6.2b, 6.5, 6.6.

3.11. **Least Square Problem.** Consider a linear system of equations

$$Ax = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & ; \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

where $m > n$. Viewed as a linear map $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$, one sees that $im(A) \subsetneq \mathbb{R}^m$ and there exists $b \in \mathbb{R}^m$ such that $A(x) \neq b$ for all $x \in \mathbb{R}^n$. This holds even when $im(A)$ is largest possible, i.e. when $ker(A)$ is trivial and A is left invertible. In that case, we can only hope to find an $\hat{x}$ such that $A\hat{x}$ is closest to b, or equivalently $||A\hat{x} - b||$ is minimal. In other words, the problem becomes minimization of

$$||Ax - b||^2 = \sum_{i=1}^{m} (a_{i1}x_1 + \dots + a_{in}x_n - b_i)^2 = \sum_{i=1}^{m} r_i(x)^2$$

This explains the name least square problem. The word "linear" is sometimes included to emphasize that the r_i is linear in the $x_1, \dots, x_n$ plus a constant.

Example 3.90. The system $\begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has no solution. The corresponding least square problem is to minimize $f(x_1, x_2) = (2x_1 - 1)^2 + (-x_1 + x_2)^2 + (2x_2 + 1)^2$. Taking partial derivatives, we get

$$\frac{\partial f}{\partial x_1} = 4(2x_1 - 1) - 2(-x_1 + x_2) = 0$$
$$\frac{\partial f}{\partial x_2} = 2(-x_1 + x_2) + 4(2x_2 + 1) = 0$$

It yields solution $\hat{x} = (\frac{1}{3}, -\frac{1}{3})$.

The following theorem tells us when we can expect existence and uniqueness of such solution.

Theorem 3.91. *If A is a left invertible matrix then the least square problem for $Ax = b$ has unique solution given by $\hat{x} = (A^tA)^{-1}A^tb$.*

Proof. For any $x \in \mathbb{R}^n$, we have $||Ax - b||^2 = ||(Ax - A\hat{x}) + (A\hat{x} - b)||^2 = ||Ax - A\hat{x}||^2 + ||A\hat{x} - b||^2 + 2(Ax - A\hat{x})^t(A\hat{x} - b) = ||A(x - \hat{x})||^2 + ||A\hat{x} - b||^2$. It follows $||Ax - b||^2 \geq ||A\hat{x} - b||^2$. Moreover, equality holds iff $||A(x - \hat{x})||^2 = 0$ iff $x = \hat{x}$. □

By above theorem, $A^tA\hat{x} = A^tb$ and so we are reduced to solving the linear system $A^tAx = A^tb$. These are called the system of normal equations associated to the least square problem for $Ax = b$.

3.12. **Cholesky Factorization to Solve Least Square Problem.** Since A^tA is positive definite, it has Cholesky factorization and we have an algorithm for least square problem.

(1) Form $C = A^tA$ and $d = A^tb$.
(2) Compute the Cholesky factorization $C = LL^t$.
(3) Solve $Lz = d$ by forward substitution.
(4) Solve $L^tx = z$ by backward substitution.

The cost to form $C = A^tA$ in step 1 is mn^2 flops, note that C is symmetric so we only need to compute $\frac{n(n+1)}{2} \simeq \frac{n^2}{2}$ elements in its lower triangle. The cost to calculate $d = A^tb$ is $2mn$. Step 3 and step 4 cost n^2 each. Hence the whole algorithm costs $mn^2 + \frac{n^3}{3} + 2mn + 2n^2$ or roughly $mn^2 + \frac{n^2}{3}$. Note that left invertibility of A implies A is square or tall $m \geq n$, so computation of A^tA costs the most.

Example 3.92. Consider the linear system

$$Ax = \begin{pmatrix} \frac{3}{5} & -\frac{6}{5} & \frac{26}{5} \\ \frac{4}{5} & -\frac{8}{5} & -\frac{7}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \\ 0 & -\frac{3}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$C = A^tA = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

in its Cholesky factorization and $d = (\frac{7}{5}, -\frac{13}{5}, 4)$. It follows from there that $(z_1, z_2, z_3) = (\frac{7}{5}, \frac{1}{5}, \frac{1}{5})$ by forward substitution and $(x_1, x_2, x_3) = (\frac{41}{25}, \frac{4}{25}, \frac{1}{25})$ by backward substitution.

3.13. **QR Factorization.** There is an alternative, more accurate way to solve the system of normal equations $A^tAx = A^tb$ associated to the least square problem for $Ax = b$ when A is left invertible.

Lemma 3.93. *(QR factorization) Every left invertible matrix $A \in M(m, n, \mathbb{R})$ can be written as $A = QR$ where Q is an $m \times n$ orthogonal matrix with $Q^tQ = I$ and R is an $n \times n$ upper triangular matrix with positive diagonal entries.*

Proof. see the explicit algorithm below. ☐

We begin by partitioning A, Q, R as

$$A = \begin{pmatrix} a_1 & A_2 \end{pmatrix} \qquad Q = \begin{pmatrix} q_1 & Q_2 \end{pmatrix} \qquad R = \begin{pmatrix} r_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_1 & A_2 \end{pmatrix} = \begin{pmatrix} q_1 & Q_2 \end{pmatrix} \begin{pmatrix} r_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix} = \begin{pmatrix} q_1 r_{11} & q_1 R_{12} + Q_2 R_{22} \end{pmatrix}$$

where $a_1 \in M(m, 1, \mathbb{R}), A_2 \in M(m, n-1, \mathbb{R}), q_1 \in M(m, 1, \mathbb{R}), Q_2 \in M(m, n-1, \mathbb{R}), r_{11} \in M(1, 1, \mathbb{R}), R_{12} \in M(1, n-1, \mathbb{R}), R_{22} \in M(n-1, n-1, \mathbb{R})$. For Q to be orthogonal

$$QQ^t = \begin{pmatrix} q_1^t \\ Q_2^t \end{pmatrix} \begin{pmatrix} q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} q_1^t q_1 & q_1^t Q_2 \\ Q_2^t q_1 & Q_2^t Q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we must have $q_1^t q_1 = 1, q_1^t Q_2 = 0, Q_2^t Q_2 = I$. For R to be upper triangular with positive diagonal entries, we must have $r_{11} > 0$ and R_{22} is upper triangular with positive diagonal entries.

Putting things together, we first see that $a_1 = q_1 r_{11}$ so we choose $r_{11} = \|a_1\|, q_1 = \frac{a_1}{r_{11}}$. Next we see $A_2 = q_1 R_{12} + Q_2 R_{22}$ which implies $q_1^t A_2 = q_1^t q_1 R_{12} + q_1^t Q_2 R_{22} = R_{12}$. So we compute $R_{12} = q_1^t A_2$. Lastly, we see $A_2 - q_1 R_{12} = Q_2 R_{22}$. This is QR factorization of an $m \times (n-1)$ matrix. Continuing recursively, we arrive at QR factorization of an $m \times 1$ matrix $A = QR$ where $Q = \frac{1}{\|A\|} A, R = \|A\|$. This algorithm is called the modified Gram-Schmidt method. In summary, it runs

(1) $r_{11} = \|a_1\|$.
(2) $q_1 = \frac{a_1}{r_{11}}$.
(3) $R_{12} = q_1^t A_2$.
(4) Compute QR factorization $A_2 - q_1 R_{12} = Q_2 R_{22}$.

The total cost is $2mn^2$ flops. Note that $\|a_1\| \neq 0$ in step 1 because A has trivial kernel. Also $A_2 - q_1 R_{12}$ is left invertible in step 4. Indeed, if $x \neq 0$ then $(A_2 - q_1 R_{12})x = A_2 x - \frac{a_1}{r_{11}} R_{12} x = (a_1 \;\; A_2)(-\frac{1}{r_{11}} R_{12} x \;\; x)^t = A(-\frac{1}{r_{11}} R_{12} x \;\; x)^t \neq 0$.

Example 3.94. The matrix $A = \begin{pmatrix} \frac{3}{5} & -\frac{6}{5} & \frac{26}{5} \\ \frac{4}{5} & -\frac{8}{5} & -\frac{7}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \\ 0 & -\frac{3}{5} & -\frac{3}{5} \end{pmatrix}$ has QR factorization

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} & 0 \\ 0 & -\frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise 3.95. Perform QR factorization for $\begin{pmatrix} 3 & 3 & 2 \\ 4 & 4 & 1 \\ 0 & 6 & 2 \\ 0 & 8 & 1 \end{pmatrix}$

3.14. **QR Factorization to Solve Least Square Problem.** Consider a linear system of equations $Ax = b$ where $A \in M(m, n, \mathbb{R})$ is left invertible. By lemma 3.93 $A = QR$ where Q is an $m \times n$ orthogonal matrix with $Q^t Q = I$ and R is an $n \times n$ upper triangular matrix with positive diagonal entries. Then $A^t A = (QR)^t (QR) = R^t Q^t Q R = R^t R$ and $R^t R x = R^t Q^t b$. Since R is invertible, this gives $Rx = Q^t b$. Packaged as an algorithm, it is

(1) Compute the QR factorization $A = QR$.
(2) Compute $d = Q^t b$.
(3) Solve $Rx = d$ by backward substitution.

Step 1 costs $2mn^2$ flops while step 2 and step 3 cost $2mn$ and n^2. Total cost is $2mn^2 + 2mn + n^2$ or roughly $2mn^2$, about twice as slow if $m \gg n$.

Example 3.96. Again consider the linear system

$$Ax = \begin{pmatrix} \frac{3}{5} & -\frac{6}{5} & \frac{26}{5} \\ \frac{4}{5} & -\frac{8}{5} & -\frac{7}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \\ 0 & -\frac{3}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

in example 3.92. By example 3.94

$$\begin{pmatrix} \frac{3}{5} & -\frac{6}{5} & \frac{26}{5} \\ \frac{4}{5} & -\frac{8}{5} & -\frac{7}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \\ 0 & -\frac{3}{5} & -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} & 0 \\ 0 & -\frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

So

$$d = Q^t b = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & \frac{4}{5} & -\frac{3}{5} \\ \frac{4}{5} & -\frac{3}{5} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}$$

It follows from there that $(x_1, x_2, x_3) = (\frac{41}{25}, \frac{4}{25}, \frac{1}{25})$.

3.15. **Exercises.**

Exercise 3.97. Consider the linear system

$$Ax = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

(a) Perform Cholesky factorization for A^tA.
(b) Use it to solve the least square problem for the above linear system.

Exercise 3.98. Consider the linear system

$$Ax = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

(a) Perform QR factorization for A.
(b) Use it to solve the least square problem for the above linear system.

Exercise 3.99. page 123: 8.1, 8.12, 8.13, 8.14.

Exercise 3.100. page 142: 9.3, 9.5.