Scattering theory for time-dependant central potentials

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Introduction

The central aim of scattering theory is to describe how a wavefront is scattered off a potential in $\mathbb{R}^{n+1}_{x,t}$.

Mathematically, this means studying the asymptotics of solutions to partial differential equations (PDE's) as $t \to \pm\infty$.

In doing so, we get information about how an incoming wave φ^p (p: past) scatters to an outgoing wave φ^f (f: future), and encode this information in a 'scattering map' S.

Here, our PDE-of-choice is the Klein-Gordon equation (KGE), a model for particles travelling with relativistic energies.

This poster aims to understand understand how S depends on incoming waves when a perturbing potential V is introduced. We do this by studying the *phase shifts* of S.

Work of this kind is novel, and here we focus on how to extend the well-documented results for a time-*independent*, spherically symmetric (called *central*) potential [1], to the lesser known case of a time-*dependent* one.

Background

Klein-Gordon equation

The Klein-Gordon equation is

$$(D_t^2 - \Delta_x - m^2)u(x,t) = 0,$$

where $D_t = -i\partial_t$, $\Delta_x = \sum_{i=1}^n D_{x_i}^2$ is the 'positive' Laplacian on $\mathbb{R}^n_x$ and $m > 0$.

We perturb this equation by adding in a smooth potential $V \in C^\infty(\mathbb{R}^{n+1}_{x,t})$:

$$(\underbrace{D_t^2 - \Delta_x - V - m^2}_{=:P_V})u(x,t) = 0.$$

Here we consider the case of a time-dependent central potential $V(r,t)$, that has compact support in the region $\mathbb{R} \times [-\mathcal{T}, \mathcal{T}]$, for some $\mathcal{T}$.

The scattering map S

Given incoming waves $\varphi^p_\pm \in C_c^\infty(\mathbb{R}^{n+1})$, $\exists$ a unique u solving $P_V u = 0$, which encodes information about how these two waves scatter off V. This encoding comes from the asymptotics of u as $t \to \pm\infty$:

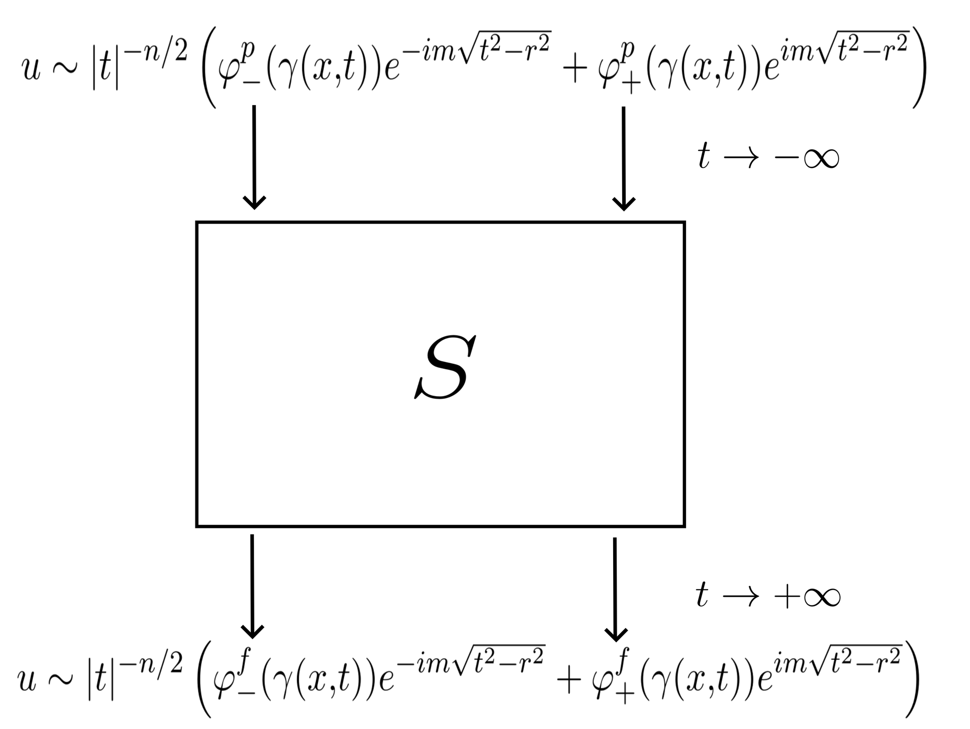

Figure 1. Link between the asymptotics of u as $t \to \pm\infty$, and S.

Where $\gamma(x,t)$ is a smooth function for $|x| < |t|$ (see 'The light cone').

As in the figure, the scattering map S is defined by

$$\begin{pmatrix}\varphi^p_+ \\ \varphi^p_-\end{pmatrix} \mapsto \begin{pmatrix}\varphi^f_+ \\ \varphi^f_-\end{pmatrix}.$$

Simplifying things: spherical harmonics

To take advantage of the spherical symmetry of V, we use *spherical harmonics*. For a given l, write

$$u(t,x) = v(r,t)Y_l^\kappa(\theta),$$

where Y_l^κ is a function on the sphere $\mathbb{S}^{n-1}$. This separation of variables helps us deal with the Laplacian in the KGE—the only x-dependant operator—since we can write the Laplacian in spherical coordinates as

$$\Delta_x = \text{function in } r \text{ of } \partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}.$$

The Y_l^κ are eigenvectors of the operator $\Delta_{\mathbb{S}^{n-1}}$, hence pull through $P_V u$, leaving us to solve the much simpler PDE

$$P_V v(r,t) = 0.$$

The free case

We show that in the free case ($V \equiv 0$), $S = I$.

Let $\varphi^p_\pm$ be incoming waves

$\downarrow$

write $u = v(r,t)Y_l^\kappa(\theta)$, substitute into $P_0 u = 0$ and take the Fourier transform in time

$\downarrow$

$$(\tau^2 + \partial_r^2 + \frac{n-2}{r} - \frac{l(l+n-2)}{r^2} - m^2)\hat{v}(r,\tau) = 0$$

$\downarrow$

write $w(r,\tau)r^{-(n-2)/2} = \hat{v}(r,\tau)$, $E = \tau^2 - m^2$ to get

$\downarrow$

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{(l+(n-2)/2)^2}{r^2} - E\right)w = 0.$$

This PDE has solutions which are the *Hankel functions*, $H^{(1)}, H^{(2)}$, defined in terms of Bessel functions [2]. We can then make the ansatz

$$w = \varphi^p_- H^{(1)}(r,\tau) + \varphi^p_+ H^{(2)}(r,\tau)$$

$\downarrow$

take inverse Fourier transform:

$$v = \int_{\mathbb{R}} e^{it\tau} r^{-(n-2)/2} w d\tau. \qquad (\dagger)$$

The asymptotics of the Hankel functions [2] and the *method of stationary phase* [3] gives us the asymptotics to this integral. Namely, as $t \to \pm\infty$

$$v \sim |t|^{-1/2} r^{-(n-1)/2} \times \left(\varphi^p_-(\gamma(r,|t|))e^{-im\sqrt{t^2-r^2}} + \varphi^p_+(\gamma(r,|t|))e^{im\sqrt{t^2-r^2}}\right).$$

Since these asymptotics are equal for $t \to \pm\infty$, $S = I$.

Interpretation: When no potential is present, the waves have nothing to scatter off.

Open question and motivation

Consider the following logical sequence for a time-*dependent* central potential.

it's known S is unitary and decomposes as $S = I + A$, hence A is normal ($A^*A = AA^*$)

$\downarrow$

Conjecture: A is compact, if true $\qquad (\star)$

$\downarrow$

A is diagonalisable, i.e., $\exists$ basis $\{\varphi_j, \psi_j\}_{j=1}^\infty$ s.t. $A(\varphi_j, \psi_j) = \mu_j(\varphi_j, \psi_j)$ that diagonalises A

$\downarrow$

$S(\varphi_j, \psi_j) = (1 + \mu_j)(\varphi_j, \psi_j)$ i.e., φ_j are eigenvectors of S

$\downarrow$

S unitary gives $|1 + \mu_j| = 1$, hence $1 + \mu_j = e^{i\theta_j}$.

These θ_j are called *phase-shifts*, and their name is derived from the following important corollary, implied by their existence:

Corollary: Given incoming data (basis vectors) $\varphi_j(x), \psi_j(x)$, $\exists u_{j,free}, u_j$ s.t. $P_0 u_{j,free} = 0$ and $P_V u_j = 0$, where as $t \to -\infty$:

$$u_j, u_{j,free} \sim t^{-n/2}\left(\varphi_j(\gamma(x,t))e^{im\sqrt{t^2-r^2}} + \psi_j(\gamma(x,t))e^{-im\sqrt{t^2-r^2}}\right)$$

and as $t \to +\infty$,

$$u_j \sim t^{-n/2}e^{i\theta_j}\left(\varphi_j(\gamma(x,t))e^{im\sqrt{t^2-r^2}} + \psi_j(\gamma(x,t))e^{-im\sqrt{t^2-r^2}}\right)$$

The incoming data φ_j, ψ_j undergo a phase shift when scattered by V.

Open question: Do phase shifts exist for a time-dependent central potential?

Answering this requires proof of conjecture $(\star)$.

Motivation: The existence of phase shifts is also the first step in demonstrating there is a meaningful connection between V and S — that under V, S depends continuously on its incoming data.

The conjecture - forward propagators

To prove conjecture $(\star)$, and thus demonstrate the existence of phase shifts of S, we use forward propagators.

Definition: P_V has an associated operator called the *forward propagator* G_{for}, s.t. $u = G_{for}(e)$ solves $P_V u = 0$, and when $\operatorname{supp} e \subseteq \{t \geq T_0\}$ we have $s \operatorname{supp} G_{for}(e) \subseteq \{t \geq T_0\}$, for some T_0 (see figure 2 below).

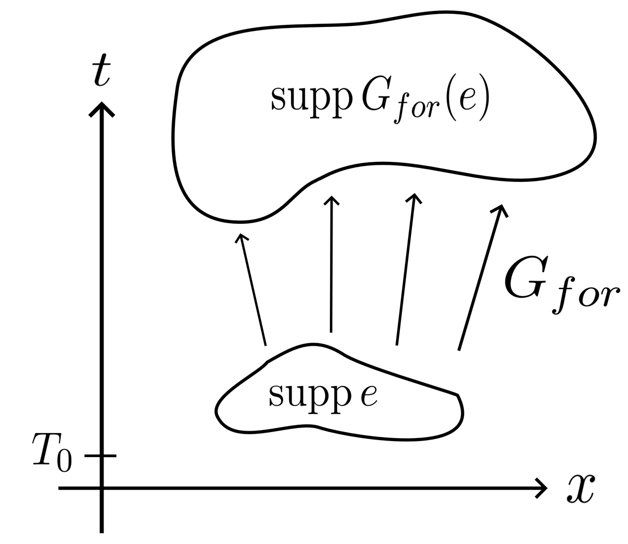

Figure 2. G_{for} propagating e forward in time.

How to use G_{for}

Let $u_{approx} = \chi_{-\infty} u_{free}(x,t)$, where $\chi_{-\infty}$ is a cut-off function to $-\infty$ ($\chi \equiv 1$ for t small enough, and $t \equiv 1$ for t large enough) and u_{free} has incoming data $\varphi^p_\pm$.

Then $e = P_V u_{approx}$ is supported in $\{t \geq T_0\}$ for some T_0, and $u = u_{approx} - G_{for}e$ solves $P_V u = 0$ and has incoming data $\varphi^p_\pm$.

Showing A is compact

We need to understand the asymptotics of u as $t \to +\infty$, which we do by proving estimates for G_{for}, such as

$$\|G_{for}e\|_X \leq C\|e\|_Y$$

in appropriate spaces X, Y. This involves calculating *weighted energy estimates*:

$$E_t = \int t^{-p}\left(|Lv|^2 + |\partial_r v|^2 + |mv|^2 + |v/r|^2\right) dtdx \qquad (1)$$

for different operators L, such as $L = \partial_t$. Here t^{-p} is the *weighting*.

These estimates for $\|G_{for}e\|$ tell us how under control $G_{for}e$ is, and can allow one to conclude that A is compact. They should be a focus for future research.

The light cone

Many of our results rely on $\gamma(r,t)$ being smooth. However, $\gamma(r,t)$ often looks like

$$\frac{tm}{\sqrt{t^2 - r^2}},$$

which is problematic when $r/t = 1$.

To rectify this, we work in **timelike infinity**, a region where $|r/t| < 1 - \epsilon$ (inside of the two cones in figure 3 below).

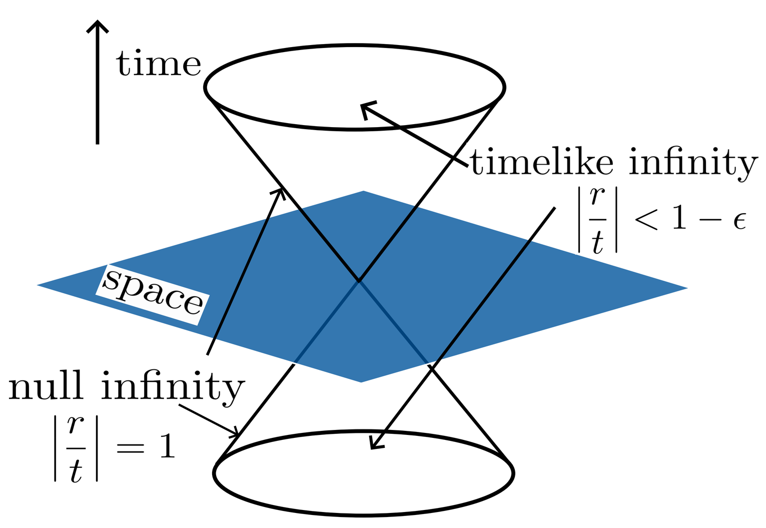

Figure 3. Diagram of a *lightcone*, with regions of timelike and null infinity.

Acknowledgements

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References

[1] S. Dyatlov and M. Zworski. *Mathematical Theory of Scattering Resonances*. Graduate Studies in Mathematics. American Mathematical Society, 2019. ISBN 9781470443665. URL `https://bookstore.ams.org/gsm-200/`.

[2] I.A. Stegun M. Abramowitz. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, chapter 9, pages 358–423. Applied mathematics series. U.S. Government Printing Office, 1964. URL `https://books.google.com.au/books?id=mlAs1jYzI_QC`.

[3] M. Zworski. *Semiclassical Analysis*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN 9780821883204. URL `https://books.google.com.au/books?id=3ZOCAQAAQBAJ`.