MSO203B - PDE Lecture 6 : Preliminaries

ODE :- $y : [a,b] \to \mathbb{R}$ is smooth.

$F(x, y, y', y'') = 0$ is an ode. (2^{nd} Order).

$F : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

(i) $F(x,y,y',y'') = y''$

(ii) $F(x,y,y',y'') = xy^2 + y' + y''$

PDE :- $u : \Omega (\subseteq \mathbb{R}^2) \to \mathbb{R}$ is smooth.

$Du(x,y) = \nabla u(x,y)$ = Total Derivative of u at (x,y).

$= (u_x(x,y), u_y(x,y))$

Assumption :- u is smooth $\Rightarrow u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}$ are defined and are continuous.

$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ (F is smooth). (2^{nd} Order PDE in two variables provided F is not independent of the last 3 variable (either).

Ex: (i) $F(\ldots) = u_{xx} + u_{yy}$ $\Delta u = 0$

(ii) $F(\ldots) = u_x + u_y$

Given a PDE $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ —— (1)

Defn :- $u \in C^2(\Omega)$ is said to be a soln of (1) if u satisfies (1) for all $(x, y) \in \Omega$

Ex :- $u_{xx} = 0$ in $\mathbb{R}^2$ $u(x, y) \in C^2(\mathbb{R}^2)$

$u(x, y) = 0$

$u_{xx}(x, y) = 0$

Well Posedness of PDE :- We call a PDE (1) is well posed if

(i) $\exists$ at least one soln. (Existence of soln)

(ii) The soln has to be unique (Uniqueness of soln)

(iii) For small change in data the solution change is also small (Perturbation)

Classification of PDE :- (2nd Order ; 2 variable)

Linear PDE :- $a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0$ in Ω.

Here, $a, b, c, \ldots, g$ are smooth fns in Ω

Ex 1 :- $u_{xx} + u_{yy} = 0$ Ex 2 :- $x u_{xx} + u_{yy} = 0$

$L[u] = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu.$

$L[\tilde{c}u_1 + u_2] = \tilde{c}L[u_1] + L[u_2]$; $\tilde{c}$ is a constant.

Should hold for linearity of L.

Let us assume, $\exists\, u_1, u_2, \ldots, u_n \in C^2(\Omega)$ solving $L[u_n] = 0 \;\forall n \in \mathbb{N}$ — (III)

$$u = \sum_{n=1}^{\infty} \tilde{C}_n u_n$$

Hence, $u \in C^2(\Omega)$ and

$$L[u] = \sum_{n=1}^{\infty} \tilde{C}_n L[u_n] \qquad [L[u_n] = 0 \;\forall n]$$

$$= 0$$

Hence, u solves (III) (Principle of Superposition).

Let us come back to (I) $\quad Lu = g$ — (II)

Let us assume $\exists\, v$ solving (II)

$L[u] = 0$

$\Rightarrow L[u+v] = L[u] + L[v] = 0 + g = g$

$L[u] + g = 0$ — (II)

$g = 0$

Semilinear :-

$$a u_{xx} + b u_{xy} + c u_{yy} + \underbrace{F(u_x, u_y, u, x, y)} = 0.$$

where a, b, c and F are smooth. $\quad a, b, c$ .. fns of (x, y)

Ex :- (i) $u_{xx} + u_{yy} + u_x^2 + u_y^2 = 0$

(ii) $u_{xx} + \sin u = 0.$

(iii) $u_{xx} + u_y = 0$

Quasilinear :- $a(x, y, u, u_x, u_y)\, u_{xx} + b(x, y, u, u_x, u_y)\, u_{xy} + c(x, y, u, u_x, u_y)\, u_{yy} + F(u_x, u_y, u, x, y) = 0.$

Ex 1 :- ① $u^2 u_{xx} + u_{yy} = 0$

② $u_x u_{xx} + e^u = 0$

③ $u_{xx}^2 + 2 = 0$ × not a quasilinear

$u u_x + u_y = 0$ — Burger's Eqn

Nonlinear :- $F(u_{xx}, u_{xy}, u_x, u_y, u, x, y) = 0.$

① $\det(D^2 u) = 0$

$u : \mathbb{R}^2 \to \mathbb{R}$

$$D^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}$$