MSO203B - PDE

Lecture 2 : Sturm - Liouville Problem

$Ly = (py')' + qy \; ; \; x \in I$

$\boxed{y(x) = y \text{ is unknown}}$

$Ly = -\lambda r y$

$Ly = \mu y$

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x) \in \mathbb{R}$

Assum p and q are cont

$L : C^2(I) \to C(I)$

(weight fxn)

Problem:- 1. $Ly + \lambda r y = 0$ in $I = [a,b]$

$\left.\begin{array}{l} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \; ; \; \alpha_1^2 + \alpha_2^2 \neq 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \; ; \; \beta_1^2 + \beta_2^2 \neq 0 \end{array}\right\} \longrightarrow (I)$

a b

$L : C^1(I) \to C(I)$

$L(f) = f'$

$L : C^2(I) \to C(I)$

$L(f) = f'' + f$

and moreover if $p > 0$ and $r > 0$ then we call (I) as a Regular SLBVP.

Q:- Find $\lambda \in \mathbb{C}$ s.t (I) admits a nontrivial soln

These λ are called eigenvalues and the corresponding soln u is called an eigen fxn.

$\boxed{\text{Remark:- If } p \geq 0 \text{ or } r \geq 0 \text{ then the problem (I) is a singular SLP}}$

2. $\left.\begin{array}{l} Ly + \lambda r y = 0 \text{ in } [a,b] \; ; \; p > 0, r > 0. \\ y(a) = y(b) \\ y'(a) = y'(b) \end{array}\right\} \longrightarrow (II)$ is called a periodic SLBVP.

Ex 1. $y'' + \lambda y = 0$; $y(0) = 0$ and $y'(\pi) = 0$ $\quad \lambda \in \mathbb{R}$ $\quad$ Regular.

Case 1:- $\lambda = 0$

$$y'' = 0 \Rightarrow y(x) = Ax + B$$

$$0 = y(0) = A \cdot 0 + B \Rightarrow B = 0.$$

$$\therefore y(x) = Ax \Rightarrow y'(x) = A$$

$$0 = y'(\pi) = A \Rightarrow A = 0.$$

For $\lambda = 0$ we have $y = 0$.

$\therefore$ $\lambda = 0$ is not an eigenvalue.

Case 2:- $\lambda < 0$; $\lambda = -\mu^2$ $(\mu \neq 0)$

$$y'' - \mu^2 y = 0.$$

$$y(x) = Ae^{\mu x} + Be^{-\mu x}.$$

$$0 = y(0) = A + B \quad \text{— (i)}.$$

$$0 = A\mu e^{\mu\pi} - B\mu e^{-\mu\pi} = y'(\pi) \quad \cancel{\text{— (ii)}}$$

$$Ae^{\mu\pi} - Be^{-\mu\pi} = 0 \quad (\mu \neq 0). \text{— (ii)}$$

$$\Rightarrow A = B = 0.$$

Hence $\nexists$ any negative e.v

Alternative

$\lambda < 0$ — y_λ

$$y_\lambda'' + \lambda y_\lambda = 0$$

$$\Rightarrow \int_a^b y_\lambda'' y_\lambda + \lambda \int_a^b y_\lambda^2 = 0.$$

$$\Rightarrow -\int_a^b |y_\lambda'|^2 + \cancel{y_\lambda y_\lambda' \Big|_a^b} + \lambda \int_a^b y_\lambda^2 = 0$$

$$\Rightarrow \lambda \int_a^b y_\lambda^2 = \int_a^b |y_\lambda'|^2$$

— a contradiction

Case 3

$\lambda > 0$; $\lambda = \mu^2$ $(\mu \neq 0)$

$y'' + \mu^2 y = 0$.

$y(x) = A\cos\mu x + B\sin\mu x$

$0 = y(0) = A \Rightarrow A = 0$.

$y(x) = B\sin\mu x \Rightarrow y'(x) = B\mu\cos\mu x$.

$0 = y'(\pi) = B\mu\cos\mu\pi$.

$\Rightarrow \cos\mu\pi = 0$. $(\because B \neq 0)$.

$\Rightarrow \mu = \left(\frac{2n-1}{2}\right)$; $n \in \mathbb{Z}$.

$\lambda_n = \left(\frac{2n-1}{2}\right)^2$; $n \in \mathbb{Z}$

$y_n = \cancel{B}\sin\left(\frac{2n-1}{2}x\right)$

Simple e.v

Ex 2:- $y'' + \lambda y = 0$

$y(0) - y(\pi) = 0$. ← Periodic.

$y'(0) - y'(\pi) = 0$.

$\boxed{(\lambda - y)} \nrightarrow$ (e.v, e.f)

$\int_0^\pi y''y + \lambda\int_0^\pi y^2 = 0$.

$\Rightarrow -\int |y'|^2 + \cancel{yy'\Big|_0^\pi} + \lambda\int_0^\pi y^2 = 0$.

$y(\pi)y'(\pi) - y(0)y'(0) = y(0)y'(0) - y(0)y'(0) = 0$

$\Rightarrow \lambda\int_0^\pi y^2 = \int |y'|^2 \Rightarrow \lambda \geq 0$

$\therefore$ Noneg e.v.

Case 1:- $\lambda = 0$; $y'' = 0 \Rightarrow y(x) = Ax + B$.

$y(0) = y(\pi) \Rightarrow A \cdot 0 + B = A\pi + B \Rightarrow B = A\pi + B \Rightarrow A = 0$.

$y'(x) = A. \Rightarrow y'(0) = y'(\pi) =$

$\lambda = 0 \leftarrow$ E.value

$\downarrow$

$y_\lambda = 1$

Case 2:- $\lambda > 0 \Rightarrow \lambda = \mu^2$ $(\mu \neq 0)$

$y'' + \mu^2 y = 0$.

$y(x) = A\cos\mu x + B\sin\mu x. \Rightarrow y'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$

$[A \cdot 1 + B \cdot 0] - [A\cos\mu\pi + B\sin\mu\pi] = 0$.

$[-A\mu \cdot 0 + B\mu \cdot 1] - [-A\mu\sin\mu\pi + B\mu\cos\mu\pi] = 0$

$\Rightarrow$

$A(1 - \cos\mu\pi) - B\sin\mu\pi = 0$.

$A\sin\mu\pi + B(1 - \cos\mu\pi) = 0$ $[\mu \neq 0]$

$$\Rightarrow \begin{vmatrix} 1 - \cos\mu\pi & -\sin\mu\pi \\ \sin\mu\pi & 1 - \cos\mu\pi \end{vmatrix} = 0.$$

$\Rightarrow 2 - 2\cos\mu\pi = 0 \Rightarrow \cos\mu\pi = 1$.

$\Rightarrow \mu = 2n.$; $n \in \mathbb{Z}$.

E.V $\Rightarrow \lambda_n = 4n^2$; $n \in \mathbb{Z}$

$$y_n \begin{cases} \sin 2nx \\ \cos 2nx \end{cases}$$

$(py')' + qy + \lambda ry = 0$ ← Self-adjoint form

$Ly = py'' + qy' + ry$ ← normal form.

Integrating Factor