MSO 203B - PDE

Lecture 4: Fourier Series

\# $y'' + \lambda y = 0$
$y(\pi) = y(-\pi)$
$y'(\pi) = y'(-\pi)$ — PSLBVP

$\lambda_n = n^2 \quad (n \in \mathbb{Z})$.

$\phi_n(x) = \begin{cases} \sin nx \\ \cos nx \end{cases}$

$$f(x) = \sum_0^\infty [a_n \cos nx + b_n \sin nx]$$
$$= a_0 + \sum_{n=1}^\infty [a_n \cos nx + b_n \sin nx]$$

$Ly + \lambda r y = 0$ in I. $\quad L = (py')' + qy$

+ Periodic B.C

$(\lambda_n)_{n \in \mathbb{N}} \quad -\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \; \cdots \leq \lambda_n < \cdots$

$\lambda_1 \to u_1, \quad \lambda_2 \to u_2, \quad \lambda_3 \to u_3 \; \cdots$

$$\int_I u_1 u_2 r \, dx = 0$$

$$f(x) = \sum_0^\infty c_n u_i$$

$$\Rightarrow \int_I f u_j r = \sum_0^\infty \int_I c_n u_i u_j r \, dx = c_j \int_I u_j^2 r \, dx$$

$$\Rightarrow c_j = \frac{\int f r u_j \, dx}{\int u_j^2 r \, dx}$$

$$f(x) = a_0 + \sum_{1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \text{in } [-\pi, \pi]$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\int_{-\pi}^{\pi} 1\cdot dx + \sum_{1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right] \quad (\text{To find } a_0).$$

$$\int_{-\pi}^{\pi}\cos nx\,dx = 0$$

$$\int_{-\pi}^{\pi}\sin nx\,dx = 0$$

$$\Rightarrow 2\pi a_0 = \int_{-\pi}^{\pi} f(x)\,dx \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

To find a_m

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx.$$

$$= 0 + a_m\cdot\pi + 0 = a_m\pi$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx.$$

$$\int_{-\pi}^{\pi}\sin nx\sin mx\,dx = \begin{cases} 0, & m\neq n \\ \pi, & m=n. \end{cases}$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \begin{cases} 0, & m\neq n \\ \pi, & m=n \end{cases}$$

$$\int_{-\pi}^{\pi}\cos nx\sin mx\,dx = 0.$$

To find b_m

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx$$

$$\Rightarrow b_m\pi = \int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

$$\Rightarrow b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

Defn:- Let f be a piecewise continuous fcn on $[-\pi, \pi]$. Then the fourier series $\boxed{\text{corresponding}}$ to f is given by

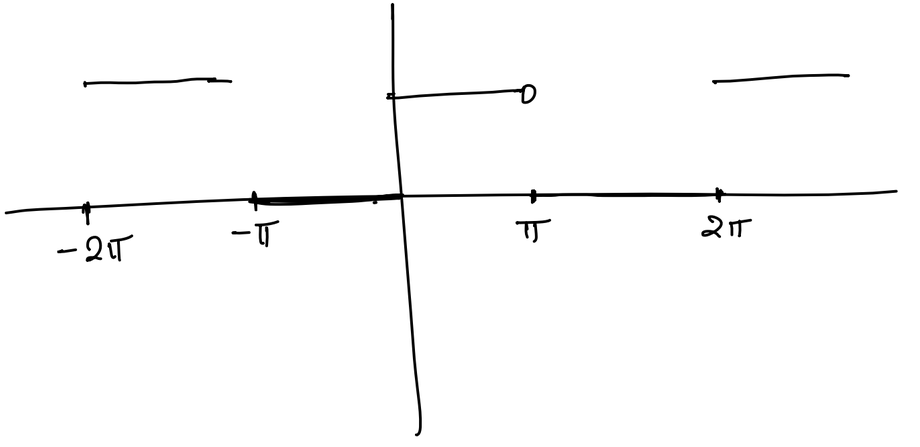
$$f(x) \approx a_0 + \sum_{1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

are called the Fourier coefficients.

Ex1:- (Square Wave Function).

$$f(x) = \begin{cases} 0 & , -\pi \le x < 0 \\ 1 & , 0 \le x < \pi \end{cases} \quad \& \quad f(x+2\pi) = f(x).$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx = \frac{1}{\pi}\left.\frac{\sin nx}{n}\right|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx = \frac{1}{n\pi}\left[\cos 0 - \cos n\pi\right] = \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd} \end{cases}$$

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

Fourier Convergence Theorem :- If 'f' is a piecewise continuous and periodic fxn with period 2π on $[-\pi, \pi]$. Then the Fourier series of 'f' is convergent and the sum is equal to $f(x)$ provided x is pt of continuity and, At the pt of discontinuity the sum of the series is given by $\frac{f(x^+) + f(x^-)}{2}$.

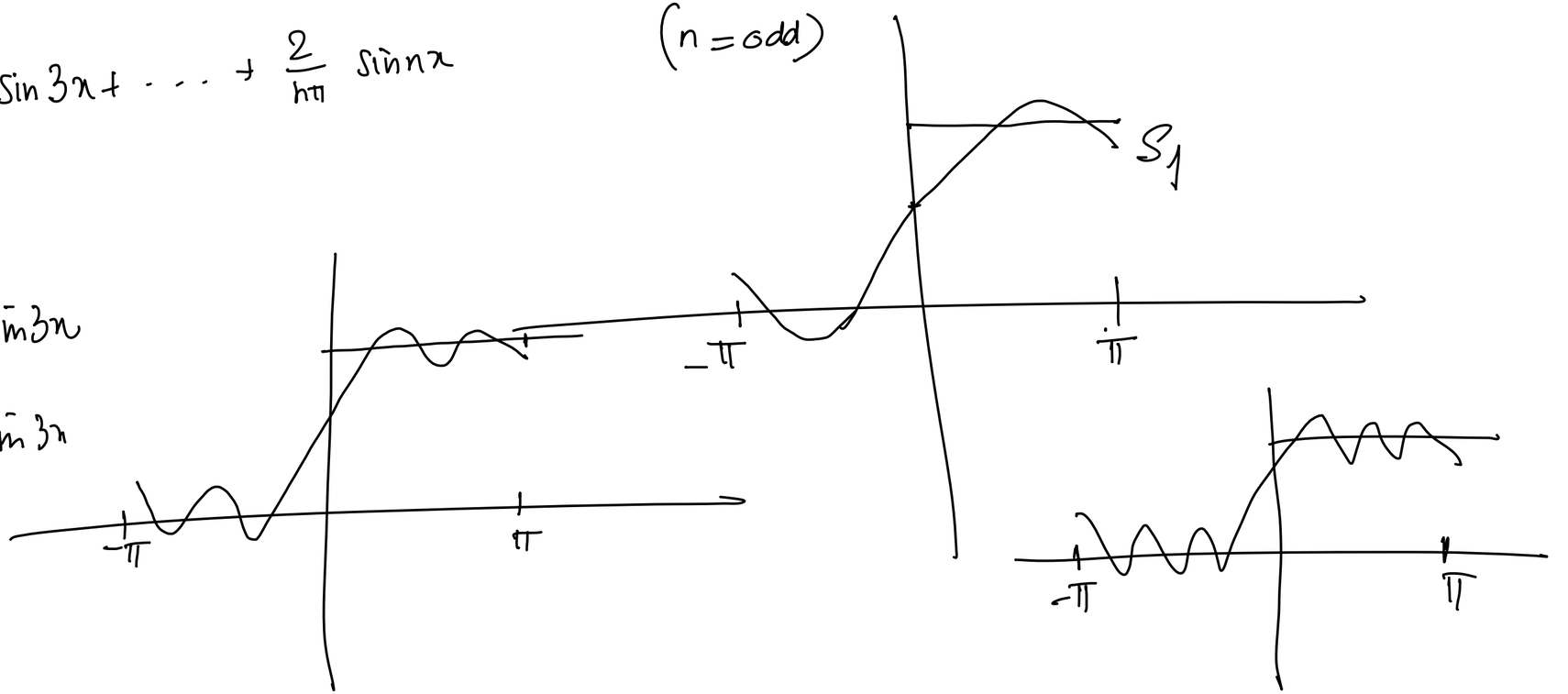
Interpretation :

$$S_n = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots + \frac{2}{n\pi}\sin nx \quad (n = \text{odd})$$

$$S_1 = \frac{1}{2} + \frac{2}{\pi}\sin x$$

$$S_2 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x$$

$$S_3 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x$$



#. Let f is a periodic fn defined on $[-\pi,\pi]$ s.t $f(x+2L)=f(x)$. (Arbitrary period Representation).

$g(y) = f(x)$. and $y = \frac{\pi x}{L}$.

$$f(x+2L) = g\left(\frac{\pi}{L}(x+2L)\right) = g\left(\frac{\pi x}{L} + 2\pi\right).$$

$$= g\left(\frac{\pi x}{L} + 2\pi\right) = g(y+2\pi)$$

Defn:-

Fourier series of f which is p.c on $[-\pi,\pi]$ and periodic of period $2L$ is given by .....

$$f(x) \underset{L}{\approx} a_0 + \sum_{1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where $a_0 = \frac{1}{2\pi}\int_{-L}^{L} f(x)\,dx$; $a_n = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$; $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$