MSO 203B - PDE   Lecture 7 : 1^{st} order PDE (Method of Characteristics etc).

$(u \in C^1(\Omega) \Rightarrow u_x$ and u_y exists and they are continuous$)$

$$a(x,y,u)u_x + b(x,y,u)u_y + c(x,y,u) = 0, \text{ in } \Omega (\subseteq \mathbb{R}^2) \quad \text{--- (1)}$$

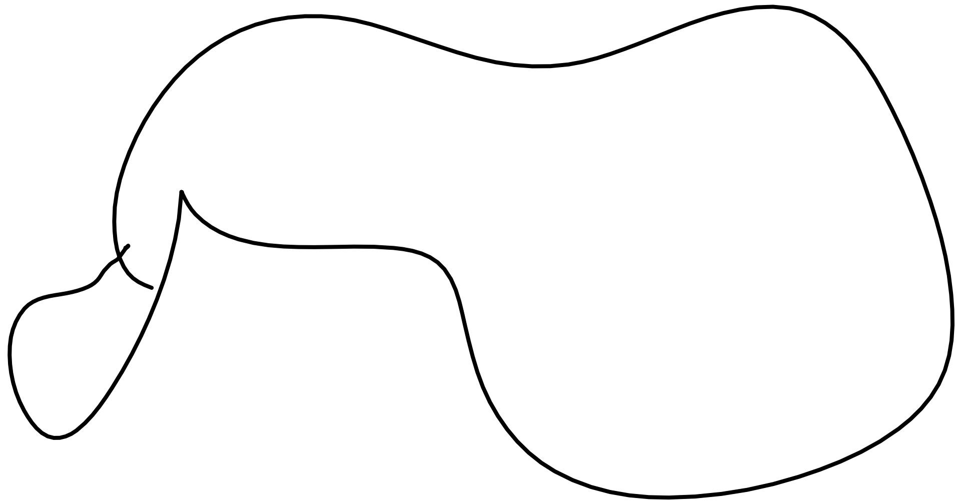
given a, b, c are smooth fxns.

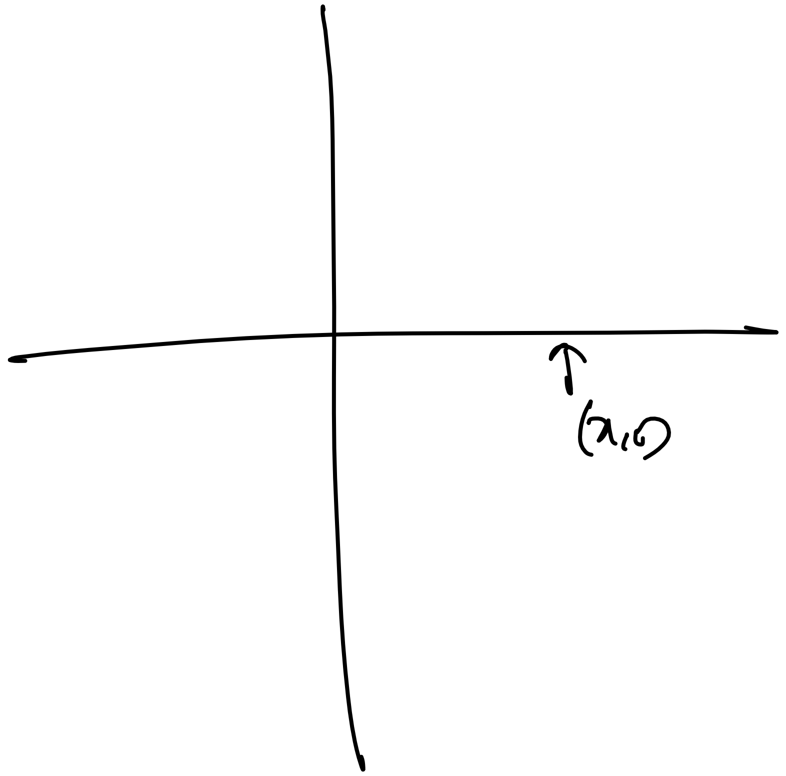
Solution to (1) :- $u \in C^1(\Omega)$ s.t u satisfies (1) for all $x \in \Omega$

Method 1 :- (Seperation of variable). (Gauss)

$u_x + u_y = 0$ --- (1) $\longleftarrow$ $\boxed{\begin{aligned} a(x,y,u) &= 1. \\ b(x,y,u) &= 1. \\ c(x,y,u) &= 0 \end{aligned}}$

(1) + $u(x,0) = e^x$





$u(x,y) = X(x)Y(y)$

$u_x = X'Y$

$u_y = XY'$

Substituting in (1) $\Rightarrow$ $X'Y + XY' = 0 \Rightarrow \frac{X'}{X} = -\frac{Y'}{Y} = \lambda$

Then, $X' = \lambda X$ --- (I) $\Rightarrow X(x) = Ae^{\lambda x}$ (A- constant)

$Y' = -\lambda Y$ --- (II) $\Rightarrow Y(y) = Be^{-\lambda y}$ (B- constant)

$u(x,y) = Ce^{\lambda(x-y)}$.

$e^x = u(x,0) = Ce^{\lambda x} \Rightarrow C=1$ and $\lambda = 1$. $\Rightarrow u(x,y) = e^{x-y}$.

$u_x^2 + u_y^2 = 1$.

$u(x,y) = X(x) + Y(y)$

$u_x = X'$; $u_y = Y'$.

Substituting

$X'^2 + Y'^2 = 1$.

$\Rightarrow X'^2 = 1 - Y'^2 = \lambda^2$.

$\Rightarrow X(x) = \lambda x + c_1$ & $Y(y) = \sqrt{1-\lambda^2}\, y + c_2$.

$u(x,y) = \lambda x + \sqrt{1-\lambda^2}\, y + c$.

Method 2:- Method of Characteristic :-

$\tilde{a} u_x + \tilde{b} u_y = 0$ ($\tilde{a}, \tilde{b}$ are constant).

Assume

$u_x + b u_y = 0$. —— (*)

Q:- Find $u \in C^1$

Assume, $\exists\, u \in C^1$ solving (*).

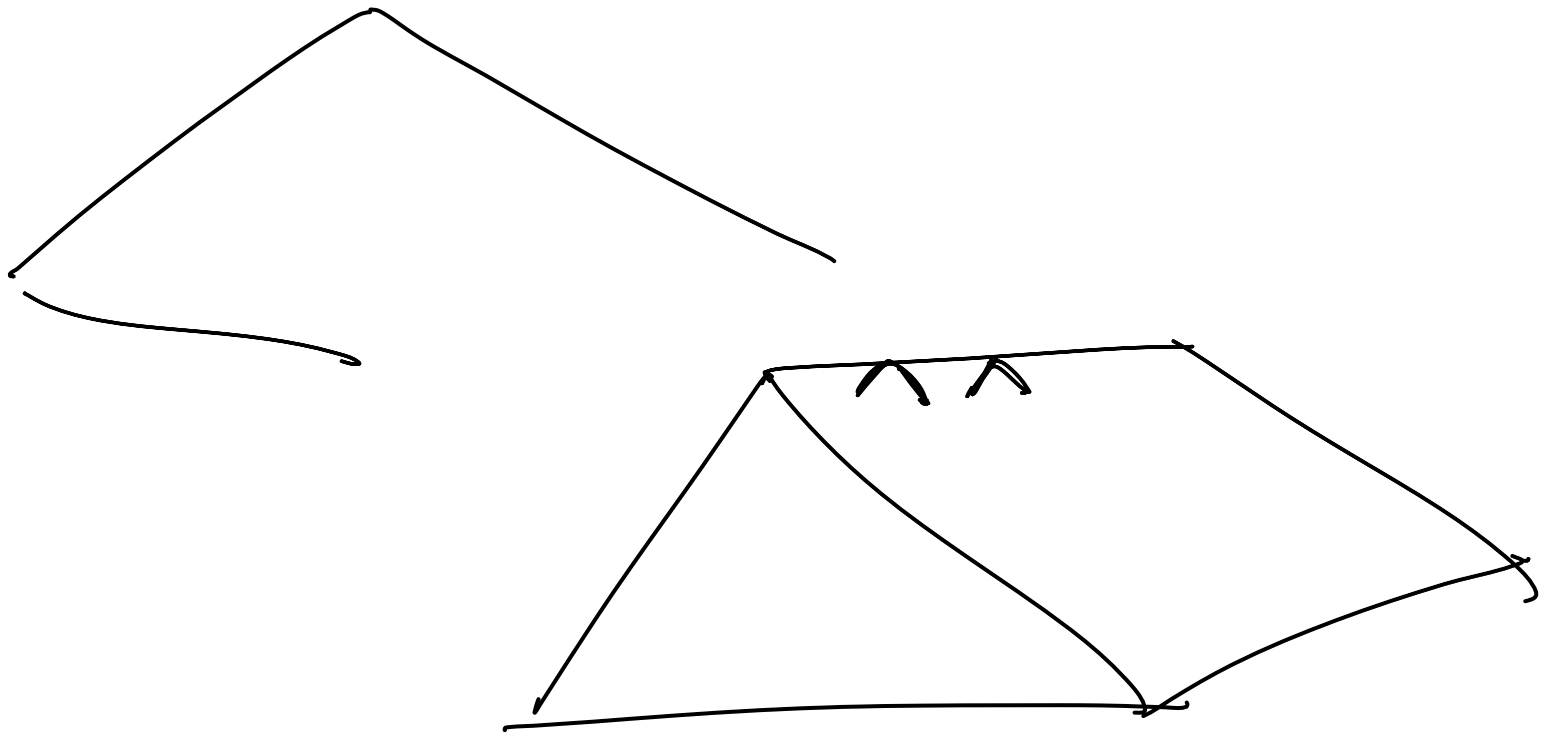
Graph of $u = S = \{(x, y, u(x,y)) : (x,y) \in \mathbb{R}^2\}$ is smooth

$$u_x + \frac{\tilde{b}}{\tilde{a}} u_y = 0$$

$$\Downarrow$$

b

$u_x + b u_y = 0$



$\hat{n} = (u_x, u_y, -1)$ to any pt in S.

$u_x + b u_y = 0$

$\Rightarrow (u_x, u_y, -1) \cdot (1, b, 0) = 0 \quad \forall (x,y) \in \mathbb{R}^2$

$\Downarrow$

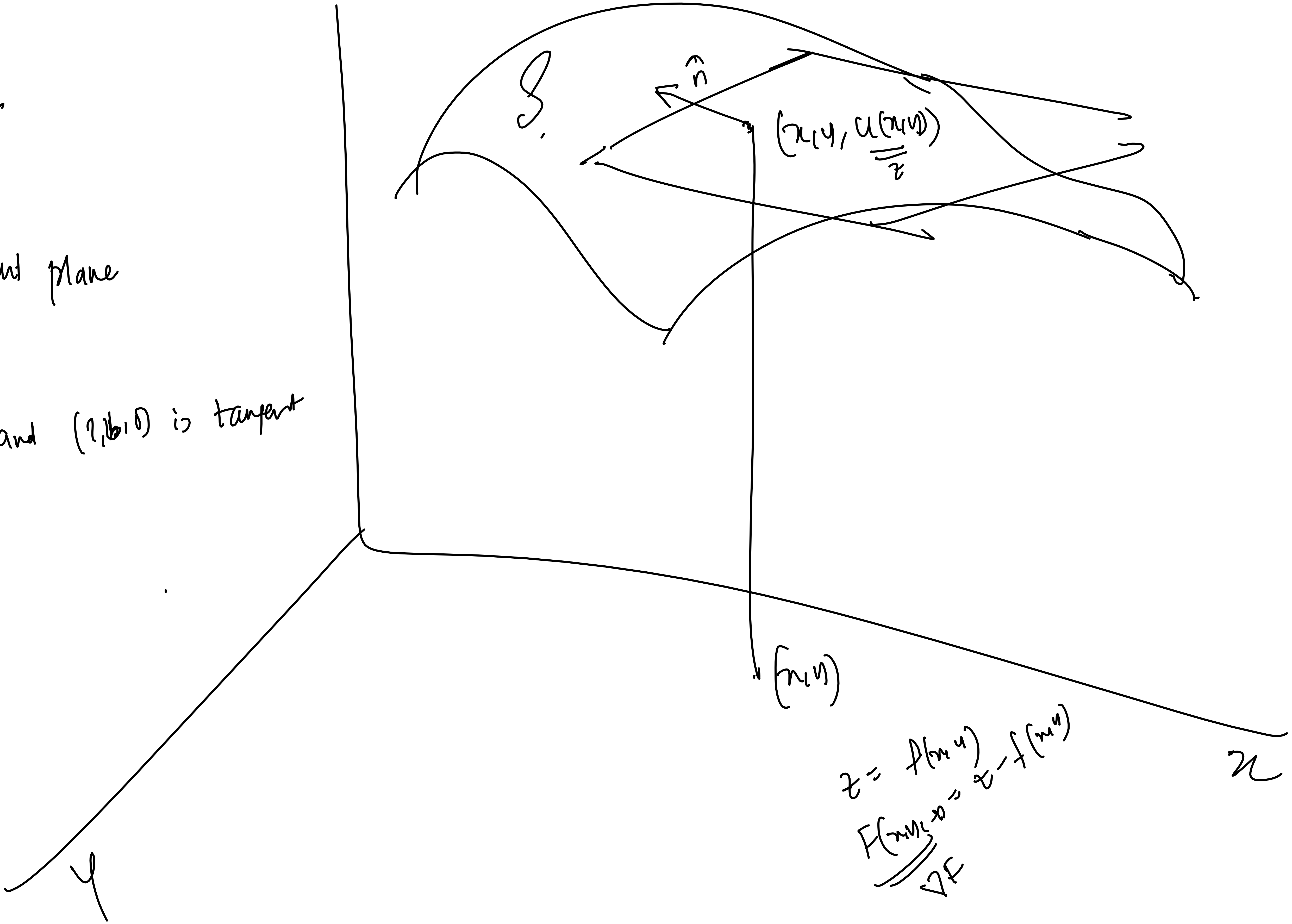
The vector $(1, b, 0)$ lies on the tangent plane

<u>Question boils down to:-</u>

Find a curve C s.t C lies on S and $(1,b,0)$ is tangent to C at any pt.

$C(s) = (x(s), y(s), z(s))$ lies on S s.t

$$\underbrace{\dot{x}(s) = 1 \; ; \; \dot{y}(s) = b \; ; \; \dot{z}(s) = 0,}_{\text{Characteristic eqn}}$$



$\dot{x}(s) = 1 \Rightarrow x(s) = s + c_1$. — (I)

$\dot{y}(s) = b \Rightarrow y(s) = bs + c_2$. — (II)

$\dot{z}(s) = 0 \Rightarrow z(s) = c_3$. — (III)

The curve $\mathcal{C}$ is called the Characteristic curve or Integral Curve

$\bigcup \mathcal{C} = S$, is called the Integral surface

From (I) and (II) $\Rightarrow y(s) = b(x(s) - c_1) + c_2 \Rightarrow y = bx + c_4$.

Define, $u(x,y) := z(x(s), y(s)) = \underline{\underline{f(y - bx)}}$ for any arbitrary $f \in \underline{\underline{\mathcal{C}^1}}(\mathbb{R} \to \mathbb{R})$

$u_x = -bf'$ | $u_y = f'$

$$\boxed{u_x + bu_y = 0}$$