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New network entropy: The domination entropy of graphs

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ABSTRACT

In this study, we introduce a new graph entropy measure, based on the dominating sets of graphs. To obtain the number of dominating sets, we use the domination polynomials of graphs. Moreover, we introduce a domination polynomial for the subdivided star graphs. After calculating the domination entropy of some graphs, we compared the domination entropy with other entropy measures.

1. Introduction

Shannon introduces the entropy concept in 1948 [1]. Shannon entropy can be applied to different networks by constructing a finite probability scheme for each network. Rashevsky defined the graph entropy concept in 1955 [2]. This entropy measure is based on the partitioning of vertices with respect to equivalent classes of vertex degrees. Trucco [3,4] extended this definition by using automorphism groups of graphs. Mowshowitz applied information theory to different chemical structures and mathematical structures in 1968 [5–7].

The chemical applications of information theory have been made in the study of different molecular properties [8–10], quantum chemistry [11,12], determination of the electronic structures of atoms [13], and in the explanation of the Pauli and Hund rules [14].

Many molecular properties of materials are obtained by molecular topologies [15]. These measures are called topological indices or molecular descriptors in the chemical graph theory. The chemical, physical, and biological properties of the molecules correlate well with these topological indices. Therefore, researchers from a wide range of disciplines study this topic. The first topological index was introduced by Wiener in 1947 [16]. The Wiener index equals one-half of the total distance between every pair of vertices in a graph. Moreover, Hosoya introduced the Hosoya index in 1971 [17]. The Merrifield-Simmons index was introduced by Richard Merrifield and Howard Simmons [18]. The Hosoya index equals the total number of matchings, and the Merrifield-Simmons index equals the total number of independent sets [19].

Many topological indices have been introduced in the last 50 years. It is understood that they usually correlate with the relative molecular properties of molecules, but the same index cannot have a high discrimination ability for different molecules [15]. Bonchev and Trinajstic introduced an entropy measure, based on distances to interpret the molecular branching of molecular graphs [20]. Later, they applied information theory, characterising chemical structures [21,22]. These

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molecular descriptors are called information indices, and it was shown that the information indices have greater discriminating power for molecules than the respective topological indices [23].

Entropy measures have been applied in biological networks. For example, Viol et al. applied the Shannon entropy to analyse the functional connectivity of the human brain [24].

Outside of classical entropy measures, Dehmer [25] introduced some new information functionals of the vertices of graphs. Various graph entropy measures were defined, which are based on certain graph invariants, such as the number of vertices, edges, degree sequences of vertices, and degree powers of vertices [25–27]. The maximum and minimum values of the first degree-based entropy were obtained for trees and unicyclic and bicyclic graphs [28]. The graph entropy measures, which are based on matchings and independent sets, were defined in [29] and further studied in [30]. Moreover, the Hosoya entropy is based on distance-related partitions of vertices [31]. The computation of Hosoya entropy requires partitions of vertices with respect to the number of vertices at the same distance to a vertex [32]. More details about the graph entropies can be found in [33].

Domination is one of the most important types of graph invariants. Haynes et al. demonstrated that this graph invariant is very sensitive to even slight changes in trees [34]. Therefore, they showed that this graph invariant is suitable for studies analysing RNA structures.

Because domination is a sensitive graph theoretical invariant and, as mentioned above, the information content versions of some parameters have greater discrimination power than classical versions of these parameters; as such, we decided to define a new graph entropy measure which is based on the dominating sets of graphs.

In this study, we investigated the domination entropy of the graphs. To obtain the entropy, we used the domination polynomials of graphs. In addition to some well-known domination polynomials, we define the domination polynomial of subdivided star graphs. Moreover, we compared the domination entropy with the graph entropy measures based on matchings, independent sets, vertex degrees, and automorphism groups of 21 graphs given in [30]. These 21 graphs of order five were ordered with respect to their topological complexity.

2. Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$, the notation $N_G(u) = \{v|uv \in E(G)\}$ denotes the vertices which are adjacent to u, and $N_G[u] = \{u\} \cup N_G(u)$. The degree of a vertex u is the cardinality of $N_G(u)$, which is denoted by $deg_G(u)$ or simply $deg(u)$.

The number of vertices of a graph G is called an order and is denoted by $|V(G)| = n$. The paths, cycles, and stars of order n are denoted by P_n, C_n, and S_n, respectively. Moreover, complete graphs of order n are denoted by K_n. The double star graphs $S_{p,q}$ of order n consist of the stars $S_{1,p}$ and $S_{1,q}$, such that $n = p + q + 2$. The comb graph E_k is obtained from a path P_k by adding a vertex to every vertex of the path. Thus, the order of the comb graph E_k is equal to $2k$. The friendship graph F_k consists of k copies of the cycle graph C_3, such that all of them share a common vertex. Thus, the order of the friendship graph F_k is equal to $2k + 1$. The subdivided star graph S_k^* is obtained from a star S_{k+1} by adding a vertex to the k vertices of the star with degree one. This means that the order of the subdivided star S_k^* is equal to $2k + 1$.

A subset $D \subseteq V(G)$ is a dominating set, if every vertex of $V \setminus D$ is adjacent to at least one vertex in D. The notation $\gamma(G)$ is used to show the domination number of a graph G, which is the cardinality of a dominating set with minimum order [35]. Moreover, a subset $D \subseteq V(G)$ is a total dominating set, if every vertex of $V(G)$ is adjacent to at least one vertex in D. The notation $\gamma_t(G)$ is used to show the total domination number of a graph G, which is the cardinality of a total dominating set with minimum order [36].

Alikhani and Peng introduced the domination polynomial concept [37]. Moreover, domination polynomials were obtained for paths [38], cycles [39], friendship graphs [40], and caterpillar graphs [41]. The use of domination polynomials is a useful method to characterise graphs. In this study, we also use domination polynomials to determine the domination entropy of graphs.

Definition 1. The notation $D(G, i)$ denotes the family of dominating sets of G with cardinality i, and the notation d_i denotes the cardinality of $D(G, i)$ such that $d_i(G) = |D(G, i)|$. Therefore, the domination polynomial $D(G, x)$ of G is introduced by the following equation [37]:

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d_i(G)x^i.$$

We use the notation γ_s to denote the total number of dominating sets. This notation can be mixed with the total domination number γ_t of the graphs. To clarify these concepts, we provide an example of this.

Consider a path graph $P_4 : v_1v_2v_3v_4$. To dominate the graph P_4, it is sufficient to take two vertices that dominate the other two vertices. Thus $\gamma(P_4) = 2$. The total dominating set of P_4 is $\{v_2, v_3\}$ and $\gamma_t(P_4) = 2$. The set of dominating sets of P_4 with cardinality two is $D(G, 2) = \big\{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\big\}$, and $d_2(G) = 4$. Moreover, the set of dominating sets of P_4 with cardinality three is $D(G, 3) = \big\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\big\}$, and $d_3(G) = 4$. Finally, the dominating set of P_4 with cardinality four is $D(G, 4) = \{v_1, v_2, v_3, v_4\}$, and $d_4(G) = 1$. Consequently, the domination polynomial of P_4 is $D(G, P_4) = x^4 + 4x^3 + 4x^2$. It is observed that the total number of dominating sets for P_4 is nine, that is, $\gamma_s(P_4) = 9$, and it is obtained directly from the coefficients of the domination polynomial. Therefore, we can express the following definition:

Definition 2. Let $\gamma_s(G)$ be the total number of dominating sets of a graph G. It is clear that γ_s is equal to the sum of the coefficients of the domination polynomials of graph G such that

$$\gamma_s(G) = \sum_{i=\gamma(G)}^{|V(G)|} d_i(G).$$

Definition 3. The entropy of a graph G can be defined using Dehmer's information functional approach [25]. Let G be a graph and $f : S \longrightarrow R_+$ be an information functional defined on $S = \{s_1, s_2, \ldots, s_k\}$ such that S is a set of elements of G. Then, the entropy is defined as follows:

$$\begin{aligned} I_f(G) &= -\sum_{i=1}^{k} \frac{f(s_i)}{\sum_{j=1}^{k} f(s_j)} \log\left(\frac{f(s_i)}{\sum_{j=1}^{k} f(s_j)}\right) \\ &= \log\left(\sum_{i=1}^{k} f(s_i)\right) - \frac{\sum_{i=1}^{k} f(s_i)\log f(s_i)}{\sum_{j=1}^{k} f(s_j)}, \end{aligned}$$

where logarithmic phrases have base 2.

Now, we can define the domination entropy, using a new information functional.

Definition 4. For a graph G with $|V| = n$ without an isolated vertex, we introduce the information functional such that

$$f := d_i(G)$$

where $d_i(G) = |D(G, i)|$. Thus, for each $i < \gamma(G)$ $d_i(G) = 0$, $d_{n-1}(G) = n$, and $d_n(G) = 1$. Then by using Definition 3, we obtain the domination entropy

$$\begin{aligned} I_{dom}(G) = I_f(G) &= -\sum_{i=1}^{n} \frac{d_i(G)}{\gamma_s(G)} \log\left(\frac{d_i(G)}{\gamma_s(G)}\right) \\ &= \log\left(\gamma_s(G)\right) - \frac{1}{\gamma_s(G)} \sum_{i=1}^{n} d_i(G)\log\left(d_i(G)\right) \\ &= \log\left(\gamma_s(G)\right) - \frac{1}{\gamma_s(G)} \sum_{i=1}^{n-2} d_i(G)\log\left(d_i(G)\right) - \frac{n\log n}{\gamma_s(G)}. \end{aligned}$$

Let G be a graph of order n with m edges. A matching graph G is a subset of $E(G)$, such that no two edges share a common vertex [19]. The number of matchings with k edges is denoted by $z_k(G)$. The empty set is assumed to be a matching, and $z_0(G) = 1$. Therefore, the Hosoya index (or Z-index) of graph G is computed as $Z(G) = \sum_{k=0}^{m} Z_k(G)$. If the information functional is defined as $f := z_k(G)$, the following graph entropy measure is obtained:

Definition 5. For a graph G with $|E(G)| = m$, the entropy measure—which is based on the number of matchings—is denoted by I_{nm} and computed by [29]

$$I_{nm}(G) = -\sum_{k=0}^{m} \frac{z_k(G)}{Z(G)} \log\left(\frac{z_k(G)}{Z(G)}\right).$$

Let G be a graph of order n with m edges. A subset $W \subset V(G)$ is called independent if the vertices of W are not adjacent. $\sigma_k(G)$ denotes the number of independent sets with cardinality k. The empty set can be considered as an independent set and $\sigma_0(G) = 1$. Therefore, the Merrifield-Simmons index (or σ-index) of graph G is computed as $\sigma(G) = \sum_{k=0}^{n} \sigma_k(G)$. If the information functional is introduced as $f := \sigma_k(G)$, we obtain the following graph entropy measure:

Definition 6. For a graph G of order n, the entropy measure — which is based on the number of independent sets — is denoted by I_{nis} and computed by [29]

$$I_{nis}(G) = -\sum_{k=0}^{n} \frac{\sigma_k(G)}{\sigma(G)} \log\left(\frac{\sigma_k(G)}{\sigma(G)}\right).$$

The entropy measure, which is based on degree power, is an important entropy [27]. This was introduced by

$$I_d^k(G) = -\sum_{i=1}^{n} \frac{deg(v_i)^k}{\sum_{j=1}^{n} deg(v_j)^k} \log\left(\frac{deg(v_i)^k}{\sum_{j=1}^{n} deg(v_j)^k}\right).$$

If the parameter is chosen as $k=1$, first-degree entropy is obtained. The first-degree entropy is denoted by I_{fd} in this study. Then, we obtain the following definition:

Definition 7. For a graph G of order n, the first-degree entropy is defined as

$$I_{fd}(G) = -\sum_{i=1}^{n} \frac{deg(v_i)}{\sum_{j=1}^{n} deg(v_j)} \log\left(\frac{deg(v_i)}{\sum_{j=1}^{n} deg(v_j)}\right).$$

Another important entropy measure, the topological information content, is defined as follows: [7].

Definition 8. Let G be a simple graph of order n and n_i $(1 \le i \le k)$ be the cardinality of the i-th orbit of G. Then, the topological information content I_α is defined as follows:

$$I_\alpha(G) = -\sum_{i=1}^{k} \frac{n_i}{n} \log\left(\frac{n_i}{n}\right).$$

It is clear that $I_\alpha(G)$ has the maximum value for graphs, which have no symmetry [32]. This means that the automorphism group of G is trivial.

3. Domination entropy of some graphs

We first provide an essential theorem about the entropy of a graph, which is the union of two connected graphs.

Theorem 1. *Assume that H_1 and H_2 are two connected graphs and $H = H_1 \cup H_2$ is the disjoint union of H_1 and H_2, respectively. We obtain that*

$$I_{dom}(G) = \log\left(\sum_{i=\gamma(H)}^{n} d_i(H)\right) - \frac{1}{\sum_{i=\gamma(H)}^{n} d_i(H)} \sum_{i=1}^{n} d_i(H) \log\left(d_i(H)\right)$$

such that $\gamma(H) = \gamma(H_1) + \gamma(H_2)$ and $d_i(H) = \sum_{j=\gamma(H_1)}^{i} d_j(H_1) d_{i-j}(H_2)$.

Theorem 2. *For a complete graph K_n of order n, we obtain*

$$I_{dom}(K_n) = \log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} \log\left(\binom{n}{i}\right).$$

Proof. Let G be a complete graph K_n. The domination polynomial of G is defined as $D(G,x) = (1+x)^n - 1$ [37]. Therefore, we obtain $\gamma_s(G) = 2^n - 1$. By using the binomial expansion of the domination polynomial, we obtain $d_i(G) = \binom{n}{i}$ for $1 \le i \le n$. Then

$$I_{dom}(K_n) = \log(2^n - 1) - \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n}{i} \log\left(\binom{n}{i}\right). \quad \square$$

Theorem 3. *For a star graph S_n with n vertices, we obtain the following domination entropy:*

$$I_{dom}(S_n) = \log(2^{n-1} + 1) - \frac{1}{2^{n-1} + 1} \left(\sum_{i=1}^{n-3} \binom{n-1}{i} \log\left(\binom{n-1}{i}\right)\right) - \frac{n \log n}{2^{n-1} + 1}.$$

Proof. Let G be a star graph S_n. We know that the domination polynomial of S_n is

$$D(S_n, x) = x^{n-1} + x(1+x)^{n-1}$$

in [37]. Because $\gamma(S_n) = 1$, by Definition 2, we obtain

$$\gamma_s(S_n) = \sum_{i=1}^{n} d_i(G) = 2^{n-1} + 1.$$

Using the binomial expansion of $D(S_n, x)$, we find that $d_1(G) = 1$, $d_i(S_n) = \binom{n-1}{i-1}$ for $2 \le i \le n-2$, $d_{n-1}(G) = n$, and $d_n(G) = 1$. Consequently, we obtain

$$I_{dom}(S_n) = \log(2^{n-1}+1) - \frac{1}{2^{n-1}+1}\left(\sum_{i=1}^{n-3}\binom{n-1}{i}\log\left(\binom{n-1}{i}\right)\right) - \frac{n\log n}{2^{n-1}+1}. \quad \square$$

Theorem 4. *For a double star graph $S_{a,b}$, we obtain the following domination entropy:*

$$I_{dom}(S_{a,b}) = \log\left(\gamma_s(S_{a,b})\right) - \frac{1}{\gamma_s(S_{a,b})}\left(\sum_{i=2}^{a+b-3} d_i(G)\log\left(d_i(G)\right)\right) - \frac{\left(\binom{a+b-2}{2} + a + b - 1\right)\log\left(\binom{a+b-2}{2} + a + b - 1\right) + (a+b)\log(a+b)}{\gamma_s(S_{a,b})}.$$

Proof. Let G be a double-star graph $S_{a,b}$. We assume that $a \le b$. We know that the domination polynomial of $S_{a,b}$ is

$$D(S_{a,b}, x) = x^2(1+x)^{a+b-2} + x\left(x^{a-1}(1+x)^{b-1} + x^{b-1}(1+x)^{a-1}\right) + x^{a+b-2}$$

in [41]. Note that $\gamma(S_{a,b}) = 2$. By Definition 2, we obtain:

$$\gamma_s(S_{a,b}) = \sum_{i=2}^{n} d_i(G) = 2^{a+b-2} + 2^{a-1} + 2^{b-1} + 1.$$

Using the binomial expansion of $D(S_{a,b}, x)$, we can generalise the cardinality of the dominating sets $d_i(G)$ in decreasing order $(n-3) \ge i \ge 2$ by $3 \le j \le a+b-2$ as follows:

$$d_{(a+b)-j}(G) = \binom{a+b-2}{j} + \binom{a-1}{a-j} + \binom{b-1}{b-j}.$$

We accept that $\binom{k}{l} = 0$ for $k < l$. For the remaining dominating sets, we obtain $d_{a+b-2}(G) = \binom{a+b-2}{a+b-4} + \binom{a-1}{a-2} + \binom{b-1}{b-2} + 1 = \binom{a+b-2}{2} + a + b - 1$, $d_{a+b-1}(G) = a+b$, and $d_{a+b}(G) = 1$.

Finally, we obtain that

$$I_{dom}(S_{a,b}) = \log\left(\gamma_s(S_{a,b})\right) - \frac{1}{\gamma_s(S_{a,b})}\left(\sum_{i=2}^{a+b-3} d_i(G)\log\left(d_i(G)\right)\right) - \frac{\left(\binom{a+b-2}{2} + a + b - 1\right)\log\left(\binom{a+b-2}{2} + a + b - 1\right) + (a+b)\log(a+b)}{\gamma_s(S_{a,b})}. \quad \square$$

Theorem 5. *For a comb graph E_k with cardinality $n = 2k$, we obtain the domination entropy*

$$I_{dom}(E_k) = \log(3^k) - \frac{1}{3^k}\left(\sum_{i=k}^{2k}\binom{k}{i-k}2^{2k-i}\log\left(\binom{k}{i-k}2^{2k-i}\right)\right).$$

Proof. The domination polynomial of the comb graphs which is depicted in Fig. 1 is [37]

$$D(E_k, x) = x^k(x+2)^k.$$

The domination number of the comb graphs is $\frac{n}{2}$, which is the maximum value of the domination number for a graph G. By Definition 2, we obtain the following:

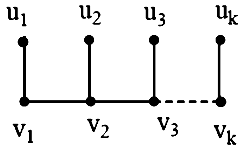

Fig. 1. Comb graph E_k.

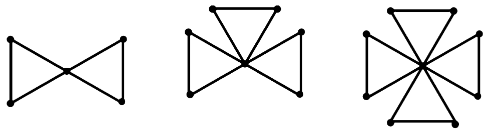

Fig. 2. Friendship graphs F_2, F_3 and F_4.

$$\gamma_s(E_k) = \sum_{i=k=\frac{n}{2}}^{n} d_i(G) = 3^k.$$

By expanding the polynomial $D(E_k, x)$, it is obtained that

$$d_i(E_k) = \binom{k}{i-k} 2^{2k-i}$$

for $k \le i \le 2k$, and $d_i(E_k) = 0$, otherwise. Therefore, the domination entropy of the comb graph E_k is expressed as:

$$I_{dom}(E_k) = \log(3^k) - \frac{1}{3^k}\left(\sum_{i=k}^{2k} \binom{k}{i-k} 2^{2k-i} \log\left(\binom{k}{i-k} 2^{2k-i}\right)\right). \quad \square$$

Theorem 6. *For a friendship graph F_k with cardinality $n = 2k+1$, we obtain the domination entropy*

$$I_{dom}(F_k) = \log(3^k + 2^{2k}) - \frac{1}{3^k + 2^{2k}}\left(\sum_{i=0}^{2k}\left(\binom{2k}{i-1} + \binom{k}{i-k}2^{2k-i}\right)\log\left(\binom{2k}{i-1} + \binom{k}{i-k}2^{2k-i}\right)\right).$$

Proof. The domination polynomial of a friendship graph $G = F_k$ which is depicted in Fig. 2 is [40]

$$D(F_k, x) = (x^2 + 2x)^k + x(x+1)^{2k}.$$

The domination number of a friendship graph is one. By Definition 2, we obtain:

$$\gamma_s(F_k) = \sum_{i=1}^{n} d_i(G) = 3^k + 2^{2k}.$$

Using the binomial expansion of $D(F_k, x)$, the number of dominating sets is obtained as $d_i(G) = \binom{2k}{i-1}$ for $1 \le i \le k-1$, and $d_i(G) = \binom{2k}{i-1} + \binom{k}{i-k}2^{2k-i}$ for $k \le i \le 2k$. It is known that $d_{2k+1}(G) = 1$. Therefore, the domination entropy of the friendship graph F_k is

$$I_{dom}(F_k) = \log(3^k + 2^{2k}) - \frac{1}{3^k + 2^{2k}}\left(\sum_{i=0}^{2k}\left(\binom{2k}{i-1} + \binom{k}{i-k}2^{2k-i}\right)\log\left(\binom{2k}{i-1} + \binom{k}{i-k}2^{2k-i}\right)\right).$$

We note that $\binom{k}{i-k} = 0$ for $i < k$. $\square$

Theorem 7. *For a subdivided star graph S_k^* with cardinality $n = 2k+1$, we obtain the domination entropy*

$$I_{dom}(S_k^*) = \log(2 \times 3^k - 1) - \frac{1}{2 \times 3^k - 1}\Bigg((2^k - 1)\log(2^k - 1)$$

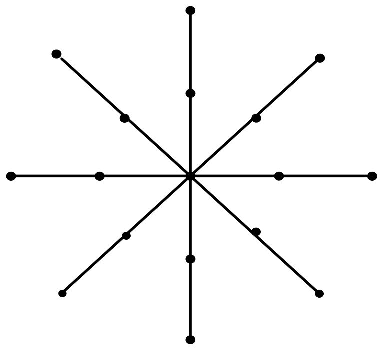

Fig. 3. The subdivided star graph S_k^*.

$$+\sum_{i=0}^{k}\left(\binom{k}{i}2^{k-i}+\binom{k}{i+1}2^{k-i-1}\right)\log\left(\binom{k}{i}2^{k-i}+\binom{k}{i+1}2^{k-i-1}\right)\Bigg).$$

Proof. The domination polynomial of a subdivided star graph S_k^*, which is depicted in Fig. 3, is obtained in the following way.

To dominate S_k^*, at least one of the vertices for each arm is contained in a dominating set. Thus $\gamma(S_k^*)=k$. The domination polynomial for subdivided star graphs can be expressed as follows:

$$D(S_k^*,x)=\sum_{i=k}^{n=2k+1} d_i(G)x^i = x^k(x+2)^k(x+1)-x^k.$$

Two cases were investigated for the domination polynomial of S_k^*.

Case 1. Every dominating set must contain at least one vertex in each of the k arms. The generating polynomial of the sets satisfying this case is $(x^2+2x)^k(x+1)$: for each arm, there is one set of cardinality 2 and two sets of cardinality 1, whereas the central vertex is either taken or not.

Case 2. Out of the sets satisfying Case 1, only one is not a dominating set: the one consisting of vertices having degree one only. Hence, we need to subtract x^k to derive the domination polynomial.

The domination polynomial of S_k^* can be expressed in an open form, as follows:

$$D(S_k^*,x)=(2^k-1)x^k+\sum_{i=0}^{k}\left(\binom{k}{i}2^{k-i}+\binom{k}{i+1}2^{k-i-1}\right)x^{k+i+1}.$$

By using the sum of the coefficients of this polynomial we obtain that

$$\gamma_s(S_k^*)=\sum_{i=k}^{2k+1} d_i(G)=2\times 3^k-1.$$

Therefore, the domination entropy of the subdivided star graph S_k^* is

$$\begin{aligned}I_{dom}(S_k^*)&=\log(2\times 3^k-1)\\&\quad-\frac{1}{2\times 3^k-1}\Bigg((2^k-1)\log(2^k-1)\\&\quad+\sum_{i=0}^{k}\left(\binom{k}{i}2^{k-i}+\binom{k}{i+1}2^{k-i-1}\right)\log\left(\binom{k}{i}2^{k-i}+\binom{k}{i+1}2^{k-i-1}\right)\Bigg). \quad \square\end{aligned}$$

4. Comparison of the domination entropy with the other entropy measures

In this section, we compare the domination entropy with the entropy measures I_α, I_{fd}, I_{nis}, and I_{nm}. We conducted numerical experiments for the 21 graphs depicted in Fig. 4. These graphs have increasing topological complexity (TC)

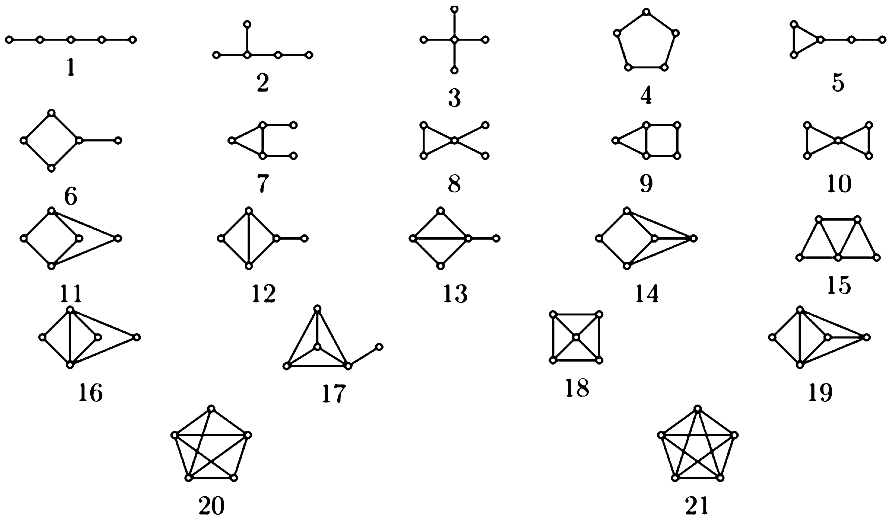

Fig. 4. All connected graphs with 5 vertices which have increased topological complexity measures.

Table 1
Measures of I_α, I_{fd}, I_{nis}, I_{nm}, I_{dom} and TC of 21 graphs.

Graphs	I_α	I_{fd}	I_{nis}	I_{nm}	I_{dom}	TC
1	1.522	2.25	1.643	1.406	1.712	60
2	1.922	2.156	1.727	1.379	1.688	76
3	0.722	2	2.022	0.722	2.022	100
4	0	2.322	1.349	1.349	1.704	160
5	1.922	2.242	1.349	1.361	1.741	172
6	1.922	2.246	1.650	1.361	1.713	290
7	1.522	2.171	1.650	1.352	1.712	212
8	1.522	2.122	1.760	1.299	1.952	230
9	1.522	2.292	1.361	1.314	1.719	482
10	0.722	2.252	1.361	1.325	1.890	292
11	0.971	2.292	1.677	1.314	1.719	504
12	1.922	2.292	1.361	1.314	1.741	511
13	1.922	2.189	1.677	1.322	1.927	566
14	1.522	2.306	1.352	1.272	1.709	1278
15	1.522	2.271	1.352	1.287	1.890	1316
16	0.971	2.236	1.685	1.296	1.719	1394
17	1.371	2.217	1.352	1.296	1.923	1396
18	0.722	2.311	1.298	1.236	1.863	3216
19	1.522	2.281	1.299	1.252	1.709	3290
20	0.971	2.308	1.149	1.207	2.001	7806
21	0	2.322	0.650	1.169	2.060	18180

measures [30]. The values of I_α, I_{fd}, I_{nis}, and I_{nm} are taken from [30]. Moreover, the domination entropy of the twenty-one (21) graphs is listed in Table 1 with the other entropies.

We use the domination polynomials of the graphs [42] depicted in Fig. 4.

$$D(1,x) = x^5 + 5x^4 + 8x^3 + 3x^2$$
$$D(2,x) = x^5 + 5x^4 + 7x^3 + 2x^2$$
$$D(3,x) = x^5 + 5x^4 + 6x^3 + 4x^2 + x$$
$$D(4,x) = x^5 + 5x^4 + 10x^3 + 5x^2$$
$$D(5,x) = x^5 + 5x^4 + 9x^3 + 6x^2$$
$$D(6,x) = x^5 + 5x^4 + 9x^3 + 4x^2$$
$$D(7,x) = x^5 + 5x^4 + 8x^3 + 3x^2$$
$$D(8,x) = x^5 + 5x^4 + 8x^3 + 4x^2 + x$$
$$D(9,x) = x^5 + 5x^4 + 10x^3 + 7x^2$$

$$D(10,x) = x^5 + 5x^4 + 10x^3 + 8x^2 + x$$
$$D(11,x) = x^5 + 5x^4 + 10x^3 + 7x^2$$
$$D(12,x) = x^5 + 5x^4 + 9x^3 + 6x^2$$
$$D(13,x) = x^5 + 5x^4 + 9x^3 + 5x^2 + x$$
$$D(14,x) = x^5 + 5x^4 + 10x^3 + 9x^2$$
$$D(15,x) = x^5 + 5x^4 + 10x^3 + 8x^2 + x$$
$$D(16,x) = x^5 + 5x^4 + 10x^3 + 7x^2$$
$$D(17,x) = x^5 + 5x^4 + 9x^3 + 7x^2 + x$$
$$D(18,x) = x^5 + 5x^4 + 10x^3 + 10x^2 + x$$
$$D(19,x) = x^5 + 5x^4 + 10x^3 + 9x^2$$
$$D(20,x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 3x$$
$$D(21,x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x.$$

We are ready to compute the domination entropy of the twenty-one (21) graphs depicted in Fig. 4. We use the equation shown in Definition 4:

$$I_{dom}(G) = -\sum_{i=1}^{n} \frac{d_i(G)}{\gamma_s(G)} \log\left(\frac{d_i(G)}{\gamma_s(G)}\right).$$

Example 8. Let $G = P_5$ which is the first graph in Fig. 4. Thus, $\gamma_s(P_5) = 17$, and

$$I_{dom}(P_5) = -\frac{1}{17}\log(\frac{1}{17}) - \frac{5}{17}\log(\frac{5}{17}) - \frac{8}{17}\log(\frac{8}{17}) - \frac{3}{17}\log(\frac{3}{17}) = 1.712$$

It is possible to obtain many results from the table. The first important result is that I_{nis} of graph 3 is equal to I_{dom} of graph 3. Graph 3 is a star graph S_5. From this result, we can investigate the I_{nis} and I_{dom} of star graphs. I_{dom} of the stars is defined in Theorem 3, such that

$$\begin{aligned} I_{dom}(S_n) &= \log(2^{n-1}+1) - \frac{1}{2^{n-1}+1}\left(\sum_{i=1}^{n-3}\binom{n-1}{i}\log\left(\binom{n-1}{i}\right)\right) - \frac{n\log n}{2^{n-1}+1} \\ &= \log(2^{n-1}+1) - \frac{n\log n}{2^{n-1}+1} \\ &\quad -\frac{1}{2^{n-1}+1}\left[\binom{n-1}{1}\log\binom{n-1}{1} + \cdots + \binom{n-1}{n-3}\log\binom{n-1}{n-3}\right]. \end{aligned}$$

I_{nis} of the stars is defined in [29], such that

$$\begin{aligned} I_{nis}(S_n) &= \log(2^{n-1}+1) - \frac{n\log n}{2^{n-1}+1} - \frac{1}{2^{n-1}+1}\left(\sum_{i=2}^{n-1}\binom{n-1}{i}\log\left(\binom{n-1}{i}\right)\right) \\ &= \log(2^{n-1}+1) - \frac{n\log n}{2^{n-1}+1} \\ &\quad -\frac{1}{2^{n-1}+1}\left[\binom{n-1}{2}\log\binom{n-1}{2} + \cdots + \binom{n-1}{n-2}\log\binom{n-1}{n-2}\right]. \end{aligned}$$

Because the equation $\binom{n-1}{2} = \binom{n-1}{n-2}$ is attained, the third terms of I_{dom} and I_{nis} are equal. Therefore, we obtain $I_{dom}(S_n) = I_{nis}(S_n)$.

Now, we can investigate the behaviours of different star-like graphs for I_{dom} and I_{nis}. If the two vertices of a star with degree one are combined by an edge, a graph denoted by S_n^+ is obtained. The domination polynomial of S_n^+ is obtained as follows:

$$D(S_n^+, x) = x(1+x)^{n-1} + x^{n-3}(x^2 + 2x).$$

The first term corresponds to all subsets containing the central vertex, whereas the second term corresponds to the remaining dominating sets. They must contain all $(n-3)$ vertices with degree one, plus at least one out of the two vertices incident to the extra edge. If the domination polynomial is written in an open form, we obtain

$$D(S_n^+, x) = \sum_{i=0}^{n-4} \binom{n-1}{i} x^{i+1} + \left[\binom{n-1}{2} + 2\right] x^{n-2} + nx^{n-1} + x^n.$$

Therefore, we obtain the total number of the dominating sets of S_n^+ such that $\gamma_s(S_n^+) = 2^{n-1} + 3$ and the domination entropy of S_n^+ is obtained as

$$I_{dom}(S_n^+) = \log(2^{n-1} + 3) - \frac{1}{2^{n-1} + 3} \left(\sum_{i=0}^{n-4} \binom{n-1}{i} \log\left(\binom{n-1}{i}\right)\right) - \frac{\left(\binom{n-1}{2} + 2\right) \log\left(\binom{n-1}{2} + 2\right)}{2^{n-1} + 3} - \frac{n \log n}{2^{n-1} + 3}.$$

I_{nis} of S_n^+ is defined in [30], such that

$$I_{nis}(S_n^+) = \log(3 \times 2^{n-3} + 1) - \frac{n \log n + \sum_{i=2}^{n-2} \left(\binom{n-3}{i} + 2\binom{n-3}{i-1}\right) \log\left(\binom{n-3}{i} + 2\binom{n-3}{i-1}\right)}{3 \times 2^{n-3} + 1}.$$

The equality $I_{dom}(S_n) = I_{nis}(S_n)$ is not attained after a simple change in a star graph.

As shown in Table 1, the measures of I_α are ordered as 1.522 seven times, 1.922 five times, 0.971 three times, 0.722 three times, 0 twice, and 1.371 once. The graphs 4 and 21 are regular graphs, and their automorphism groups consist of one orbit based on their symmetry structure. Graph 21 has the maximum topological complexity measure in 21 graphs. However, its I_α value equals zero.

It can be seen from Table 1 that because the graphs 4 and 21 regular graphs, their first-degree entropies are equal to $logn$ for n vertices. However, it is also seen that there is a significant difference between the topological complexity of these graphs. There is no significant correlation between the first-degree entropies and the topological complexity values of these graphs. Moreover, in the table, P_5 has the minimum complexity, and the complete graph K_5 has the maximum topological complexity. To make some observations, we obtain the first-degree entropy of paths. For a path graph of order n, two vertices have degree one, and the remaining $n-2$ vertices have degree two. Then

$$I_{fd}(P_n) = -(n-2)\frac{2}{2n-2} \log \frac{2}{2n-2} - 2\frac{1}{2n-2} \log \frac{1}{2n-2}.$$

From the last equation, we can compute I_{fd} of P_n, and we add the values of I_{fd} for K_n $(3 \le n \le 6)$.

$I_{fd}(P_3) = 1.5$, $I_{fd}(K_3) = 1.585$ and $I_{fd}(K_3) - I_{fd}(P_3) = 0.085$

$I_{fd}(P_4) = 1.919$, $I_{fd}(K_4) = 2$ and $I_{fd}(K_4) - I_{fd}(P_4) = 0.081$

$I_{fd}(P_5) = 2.25$, $I_{fd}(K_5) = 2.322$ and $I_{fd}(K_5) - I_{fd}(P_5) = 0.072$

$I_{fd}(P_6) = 2.521$, $I_{fd}(K_6) = 2.585$ and $I_{fd}(K_6) - I_{fd}(P_6) = 0.064$.

It is understood that if the order of paths increases, the weight of the vertices with degree two also increases. Therefore, the difference $I_{fd}(K_n) - I_{fd}(P_n)$ decreases, while n increases.

It is time to mention the domination polynomials of the paths. For every $n \ge 4$, the domination polynomial of a path graph is defined by [38]:

$$D(P_n, x) = x\big[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)\big]$$

with initial conditions $D(P_1, x) = x$, $D(P_2, x) = x^2 + 2x$, and $D(P_3, x) = x^3 + 3x^2 + x$. Therefore, the total number of dominating sets of paths for $n \ge 4$ is attained as

$$\gamma_s(P_n) = \gamma_s(P_{n-1}) + \gamma_s(P_{n-2}) + \gamma_s(P_{n-3})$$

with initial conditions $\gamma_s(P_1) = 1$, $\gamma_s(P_2) = 3$ and $\gamma_s(P_3) = 5$.

Now, we calculate I_{dom} of P_n and K_n for $3 \le n \le 6$.

$I_{dom}(P_3) = 1.371$, $I_{dom}(K_3) = 1.45$ and $I_{dom}(K_3) - I_{dom}(P_3) = 0.079$

$I_{dom}(P_4) = 1.390$, $I_{dom}(K_4) = 1.809$ and $I_{dom}(K_4) - I_{dom}(P_4) = 0.419$

$I_{dom}(P_5) = 1.712$, $I_{dom}(K_5) = 2.060$ and $I_{dom}(K_5) - I_{dom}(P_5) = 0.348$

$I_{dom}(P_6) = 1.829$, $I_{dom}(K_6) = 3.110$ and $I_{dom}(K_6) - I_{dom}(P_6) = 1.281$.

The difference in I_{dom} between K_n and P_n is relatively clear, compared to the difference in I_{fd} between the same graphs of order from 3 to 6.

The table also shows that the graphs 21, 3, 20, 8, 13, and 17 have the maximum domination entropy measures, respectively. All of its domination numbers are one. This means that it is possible to find a dominating set with cardinality from 1 to 5. It is clear that if the numbers of dominating sets with different cardinalities are almost equal, the domination entropy measure has a high value. Consequently, the domination entropy might be suitable for enumeration of the uncertainty of dominating sets of graphs.

5. Conclusion

In this study, we define the domination entropy which is based on the dominating sets of graphs. We obtained the domination entropy of complete graphs, stars, double stars, comb graphs, friendship graphs, and subdivided star graphs. To find the number of dominating sets of a graph, we use domination polynomials. We define the domination polynomial of the subdivided star graphs. Finally, we make some observations about the five entropy measures of the twenty-one (21) graphs. We find that the I_{dom} and I_{nis} of the stars are equal.

There are some open problems in future investigations. For example, the total number of matchings and the total number of independent sets are well-known topological indices in graph theory, such as the Hosoya index and Merrifield-Simmons index. To the best of our knowledge, the total number of dominating sets has never been studied (as a molecular descriptor). It is an interesting topic to investigate the number of dominating sets of graphs which can be called the “Domination Index.”

The roots of domination polynomials have been investigated in recent years. Then, the roots of the domination polynomials of the subdivided star graphs should be investigated. Moreover, the maximal and minimal graphs should be found with respect to domination entropy.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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