# Filter Function Formalism

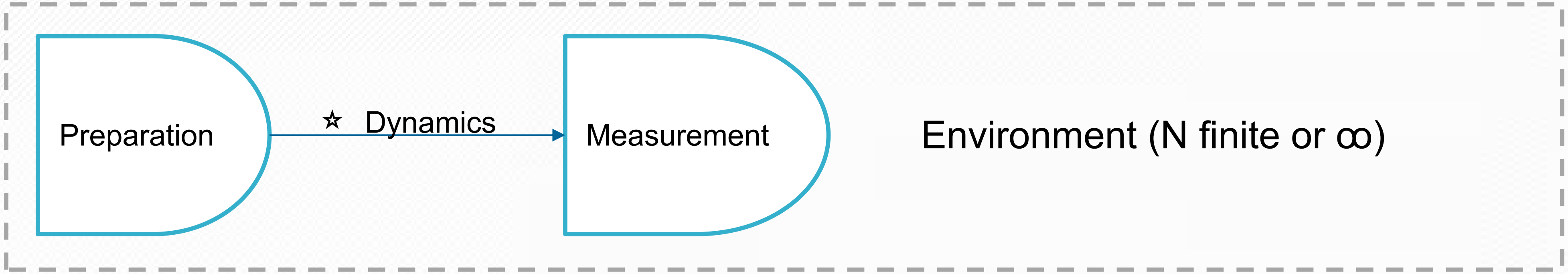
Junzhe Chen

# Intro

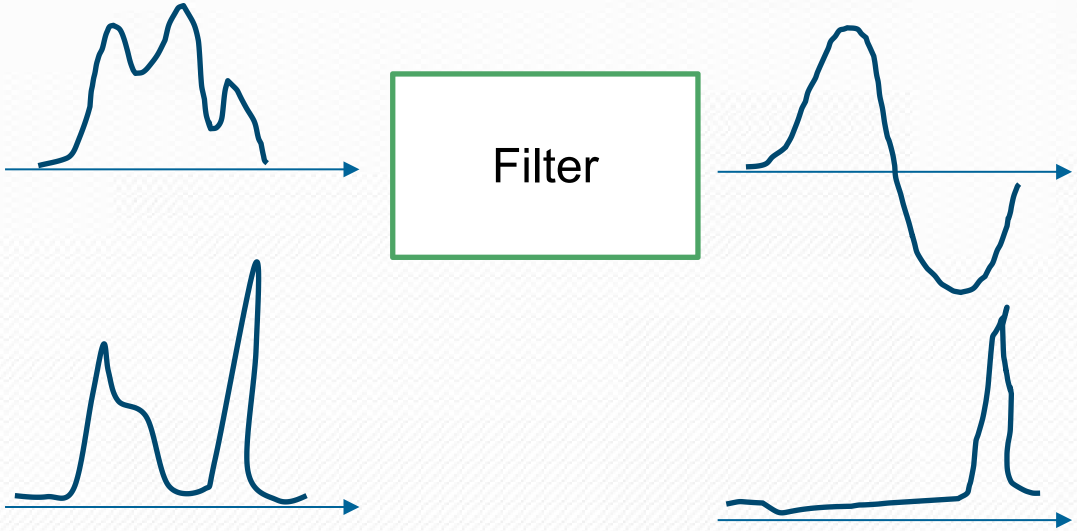
- ✓ Since last century

# Intro: Noise and Quantum Process

- General Physics Experiment




- Noise filtering




Time domain

Freq domain

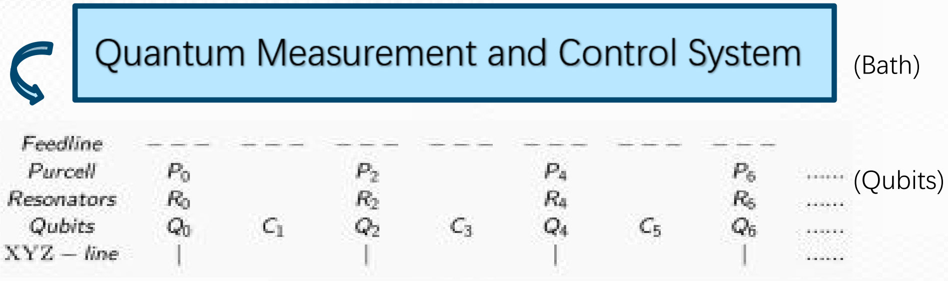
# Why FF?

‘Although some of these results are identical to that found in previous theoretical work for Josephson systems using environmental and spin-boson models, we believe this paper (Discussion on the noise of SC-qubits) is especially useful because **the noise model gives a more physical description to the origins of the decoherence and can thus be generalized readily to more complex experimental situations**. Since **the performance of electronic systems is typically evaluated using noise models and noise can be classically understood and measured**, we believe this approach will be a particularly insightful for the Josephson, and indeed many other qubit systems.’

—— John Martinis (2003)

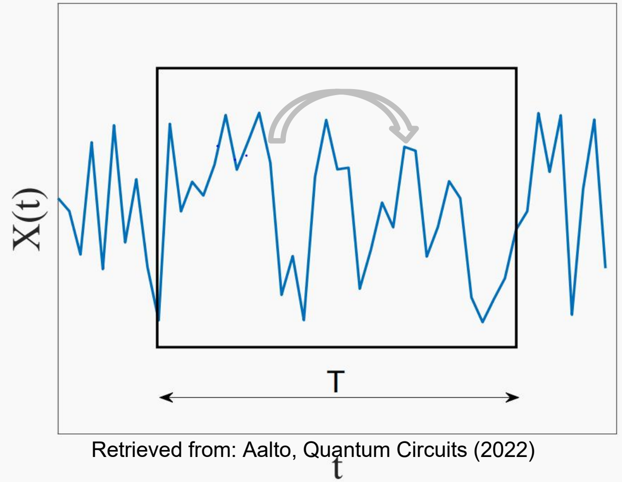
# Intro: Noise and Quantum Dyanmics

Dynamics: $H_S = \sum_j -\frac{\omega_j}{2}\sigma_j^z + \sum_{i>j} g_{ij}(\sigma_i^- - \sigma_i^+)(\sigma_j^- - \sigma_j^+) + H_d$




- How the noise act on dynamics?
  - Single-qubit: Longitudinal(Z), transverse(XY), …
  - Multi-qubits: correlation

- Statistics of random, time-dependent variable: X(t)
  - t: two-point correlation described by $\langle \beta_n(\tau)\beta_n(0)\rangle$
  - X(t): values are governed by a specific distribution

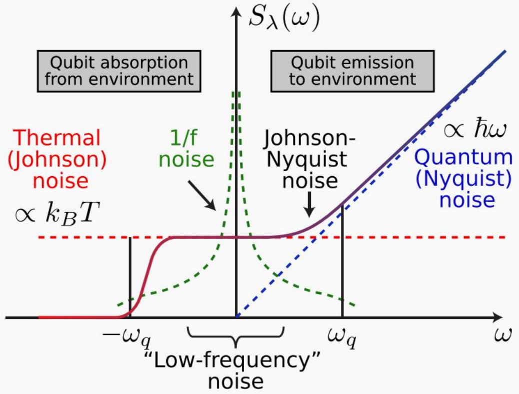



Retrieved from: Aalto, Quantum Circuits (2022)

Wiener-Khinchin theorem: $S_n(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \, \langle \beta_n(t_1)\beta_n(t_2)\rangle e^{-i\omega(t_1 - t_2)}$

Cross-correlation: $\langle \beta_n(t_n)\ldots\beta_1(t_1)\rangle$

- Noise spectrum (classically understood)
  - Thermal & Quantum Fluctuation (Nyquist, Callen, Welton, Koch…)
  - +ω & -ω
  - $1/f^{\alpha}$ & High frequency (Kogan, …)




Retrieved from: Krantz et al, Appl. Phys. Rev. 6, 021318 (2019)

# Intro: Problems

- Non-Markovian noise
- $1/f^{\alpha}$ noise
- Applications on quantum control:

→ Both are Non-Gaussian in general

Early works: Dynamical Decoupling

# Dynamical Decoupling

- ✓ 2001~2017

# DD and Filter Function: noise filtering

DD protected sequence & Free Induction Decay

$$|\psi(0)\rangle = \frac{|0\rangle + e^{i\phi_0}|1\rangle}{\sqrt{2}}$$

$|\psi(t)\rangle$ under free evolution and pulse sequences?

- Noisy Hamiltonian:

$$H = \begin{pmatrix} \omega + \delta\omega(t) & J + \delta J(t) \\ J + \delta J(t) & -\omega - \delta\omega(t) \end{pmatrix}$$

- Dynamics:

$$U(t) = \mathcal{T} e^{-i\int_0^T dt\ H(t)}$$

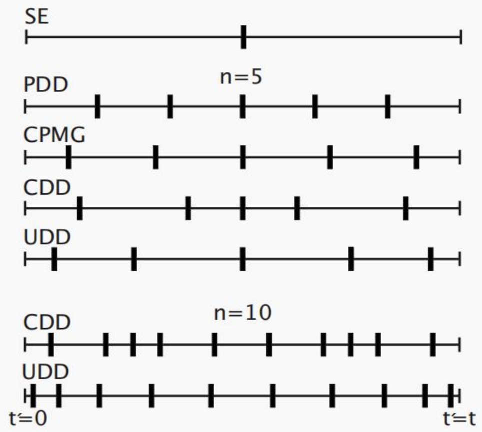
- Decoherence rate (with Gaussian noise assumption):

$$W(t) = \frac{|\rho_{01}(t)|}{|\rho_{01}(0)|} = \langle\, \exp\left(-\chi(t)\right)\, \rangle$$

- Filtered noise dynamics:

$$\chi(t) = \int_0^\infty \frac{d\omega}{2\pi}\, S_n(\omega) \frac{F(\omega t)}{\omega^2}$$

| Sequence | $F(z)$ |
|---|---|
| FID | $2\sin^2\frac{z}{2}$ |
| SE | $8\sin^4\frac{z}{4}$ |
| PDD (odd $n$) | $2\tan^2\frac{z}{2n+2}\sin^2\frac{z}{2}$ |
| CPMG (even $n$) | $8\sin^4\frac{z}{4n}\sin^2\frac{z}{2}/\cos^2\frac{z}{2n}$ |
| CDD | $2^{2l+1}\sin^2\frac{z}{2^{l+1}}\Pi_{k=1}^{l}\sin^2\frac{z}{2^{k+1}}$ |
| UDD | $\frac{1}{2}\lvert\Sigma_{k=-n-1}^{n}(-1)^k\exp[\frac{iz}{2}\cos\frac{\pi k}{n+1}]\rvert^2$ |




Retrieved from: 10.1103/PhysRevB.77.174509

Implementations on:

- Semiconductor Quantum Dot
  - Pure dephasing: 10.1103/PhysRevB.77.174509
  - Decoherence: 10.1103/PhysRevB.93.121407
  - Notch filtering: 10.1038/NNANO.2016.170
  - Gates： 10.1103/PhysRevB.101.205307
- Superconducting Qubit
  - Decoherence in SCqubits: 10.1103/PhysRevB.77.174509
  - Noise Sensor: Nat Commun 4, 2337 (2013)

Effectively varies the $\omega$ distribution ….of noise

# DD-assisted Two-qubit Cross-correlation Spectroscopy

Das Sarma's Group @ Maryland: 10.1103/PhysRevA.94.012109, 2016

*Characterization of noise spectra matrix*:

$$S_{\alpha\beta}(\omega) = S^R_{\alpha\beta}(\omega) + i\, S^I_{\alpha\beta}(\omega) = S^*_{\beta\alpha}(\omega)$$

For two-qubits, decoherence rate and filtered noise dynamics:

$$\rho_{\sigma_1\sigma_2,\bar{\sigma}_1\bar{\sigma}_2}(t) \propto \exp(-\,\chi_{11}(t) - \chi_{22}(t) - 2\sigma_1\sigma_2\,\chi_{12}(t))$$

$$\chi_{12}(t) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, S_{12}(\omega)\widetilde{f_1}(-\omega)\,\widetilde{f_2}^T(\omega)$$

With two distinct pulse

Theoretical Protocol:

State Initialization -> DD pulse evolution -> Tomographic Measurement -> Reconstruct correlated spectra: $C_{\alpha\beta}(t)$

# Intro: Problems

- Non-Markovian noise
- $1/f^{\alpha}$ noise

→ Both are Non-Gaussian in general

- Applications on quantum control:

    Dynamical Decoupling (Early works)

    Quantum Gate

    *Characterization of multiple system*

    *Experimental Schemes / Devices*

E.g.

- Geometric Description + Magnus expansion
- Correlated Gate Process
- XY4 code and ZZ crosstalk

# Correlated Process

✓ Recent works

# Correlated Quantum Process: Device

**System**: $H = H_S + H_{SB} + \underbrace{H_C + H_{CE} + H_E}_{H_B}$

- $H_S(t) = \sum_{j=1,2}\left[\frac{\omega_j}{2}\sigma_j^z + \Omega_j \cos\left(\omega_{dj}\, t\right)\sigma_j^x\right]$
- $H_{SB} = \sum_j B_j \sigma_j^z$

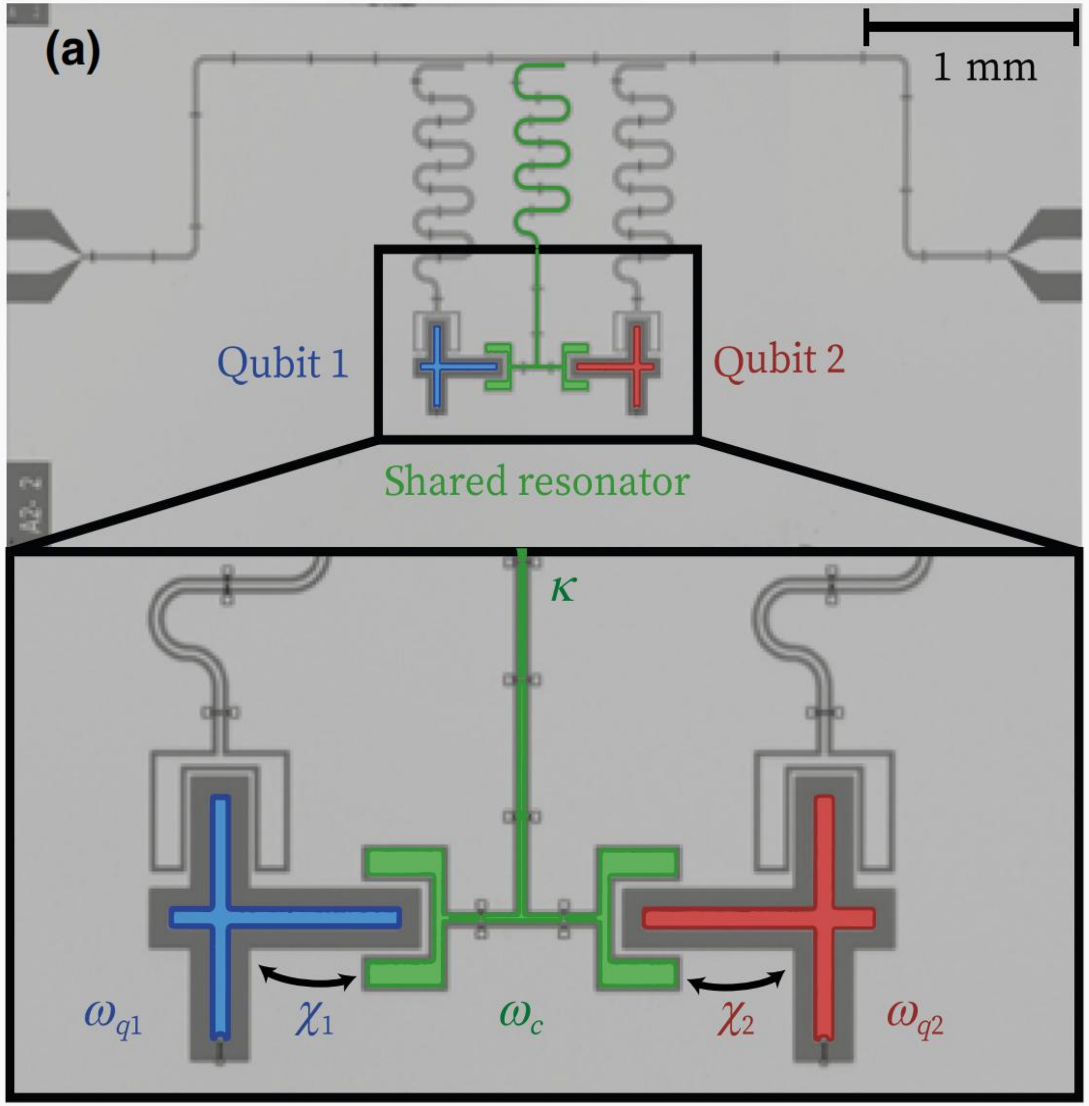
**Aim**: Construct the cross-correlation spectra between Q1, Q2

**Engineered Noise**: Photon shot noise

- $\overline{n} = \Delta \mathrm{n}^2 = |\alpha|^2$
- Lead qubits to pure dephasing
- $S_{jk}(\omega) = \chi_j \chi_k \overline{n} \frac{\kappa}{(\omega + \Delta_c)^2 + \left(\frac{\kappa}{2}\right)^2}$

*Oliver's Group @ MIT and Viola's Group @ Dartmouth*

10.1103/PRXQuantum.1.010305

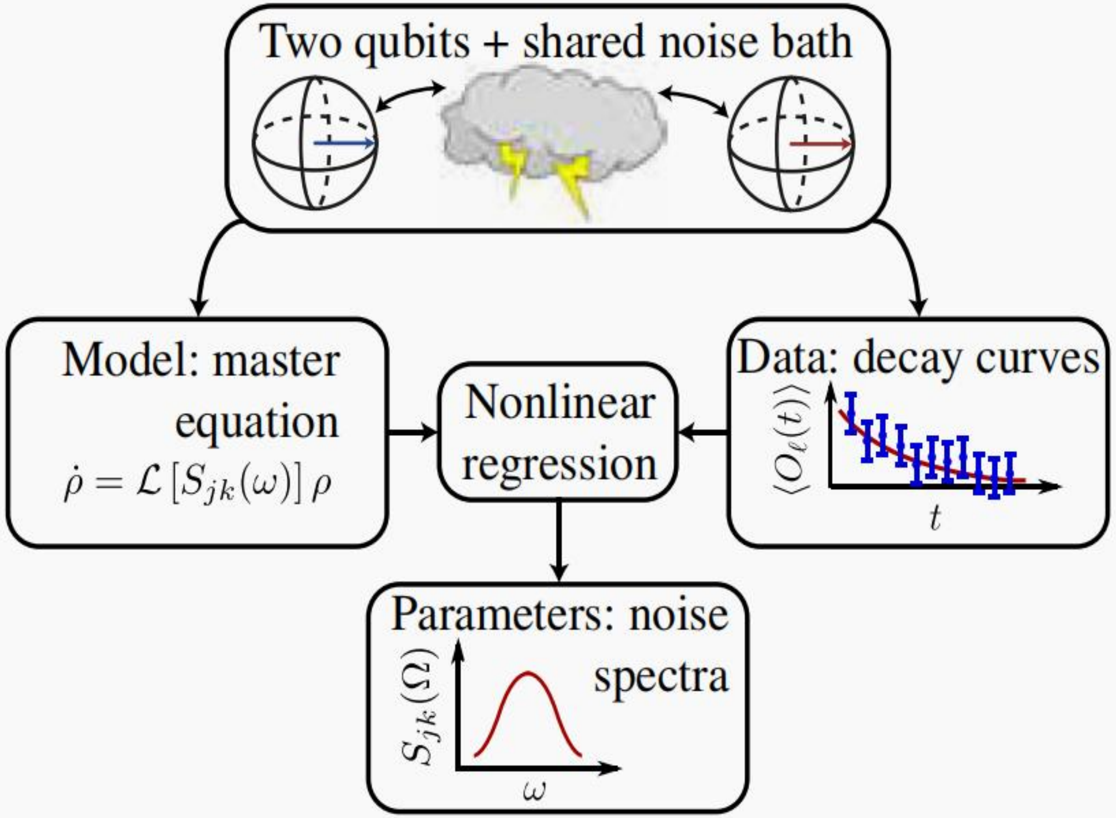



| | Qubit 1 | Qubit 2 | Common resonator |
|---|---|---|---|
| $\omega/2\pi$ (GHz) | 3.483 | 4.600 | 7.471 |
| $T_1$ ($\mu$s) | 87 | 54 | |
| $T_2^{\mathrm{echo}}$ ($\mu$s) | 54 | 68 | |
| $\kappa/2\pi$ (kHz) | | | 198 |
| $\chi_j/2\pi$ (kHz) | −29.1 | −59.5 | |
| $\Delta_{qj}/2\pi$ (kHz) | −1265 | 299 | |
| $\gamma_{\phi_j}$ ($\times 10^3$ rad/s) | 87.7 | 31.0 | |

# Characterize Correlation Spectra: Protocol

Interface between numerical and experimental results

1. Rough experimental calibration -> get qubit parameters
2. Simulate the dynamics by Master Equation
3. Characterization of correlation function: Spin-locking relaxometry
4. Estimation the difference between numerical and experimental results by regression



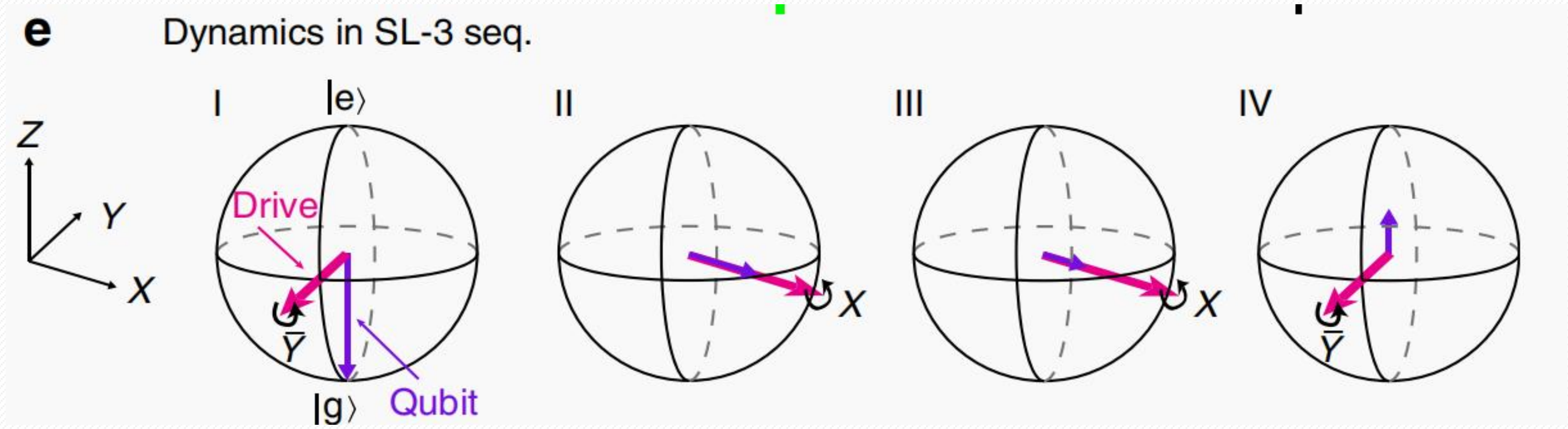
# Noise-sensitive technique: Spin-locking Relaxomtry

Transform to spin-locking frame:

$$H_{RWA} = \frac{1}{2}(\Delta_d \sigma_z + \Omega \sigma_x) \xrightarrow{spin-locking\ frame} H'_S = \frac{1}{2}\sqrt{\Delta_d^2 + \Omega^2}\,\tau_z$$

when $\Delta_d \approx 0$

$$\tau_z = |+x\rangle\langle +x| - |-x\rangle\langle -x|$$




*Yan, F et al., Nat Commun 4, 2337 (2013)*

# Dynamics of Qubits: Time-convolutionless Master Equation

Transform to spin-locking frame:

$$H_S(t) = \sum_{j=1,2}\left[\frac{\omega_j}{2}\sigma_j^z + \Omega_j \cos\left(\omega_{dj}\, t\right)\sigma_j^x\right] \qquad H'_S = \sum_j \frac{1}{2}\sqrt{\Delta_{qj}^2 + \Omega_j^2}\,\tau_j^z$$

$$H_{SB} = \sum_j B_j \sigma_j^z \xrightarrow{\text{RWA} + spin-locking\ frame} H'_{SB} = \sum_j B_j(\cos\Theta_j\,\tau_j^z - \sin\Theta_j \tau_j^x),\ \Theta_j = \arctan\left(\frac{\Delta_{qj}}{\Omega_k}\right)$$

TCL-ME (Time-local, weak coupling)

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t) = \underbrace{\mathcal{K}(t)}_{TCL\ generator}\ \mathcal{P}\rho(t) + \underbrace{I(t)}_{inhomogenity}\ \mathcal{Q}\rho(t_0)$$

Note that:

- $\mathcal{P}\rho = Tr(\rho)\otimes\rho_B = \rho_S\otimes\rho_B,\quad \mathcal{Q} = 1-\mathcal{P},\quad \mathcal{K}(t) = \sum_n \alpha^n\,\mathcal{K}_n(t),$ when $H_{tot} = H_S + H_B + \alpha V$

*Assumptions*:

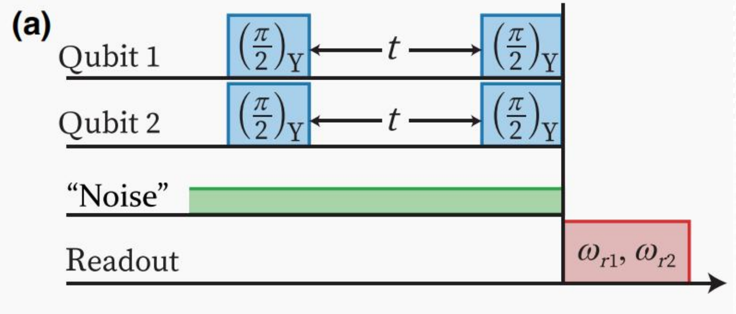
1) Weak coupling: $\mathcal{K}_1 = \mathcal{P}\mathcal{L}(t)\mathcal{P} = 0,\ \ \mathcal{K}_2(t) = \int_0^t dt_1 \mathcal{P}\mathcal{L}(t)\mathcal{L}(t_1)\mathcal{P}$
2) Large bath: $\rho_B(t) = \rho_B$
3) Separable initial state: $\rho_{SE}(0) = \rho_S(0)\otimes\rho_B$
4) Finite difference spectra: $\Omega_1 \approx \Omega_2$

**Main result**: $\tilde{\rho}(t) = i[H'_S, \tilde{\rho}] + \sum_{jk}\mathcal{L}_{jk}\rho(t)$, where we extract the Lindblad-like super-operator:

$$\mathcal{L}_{jk}[\rho] \equiv S_{jk}(-\Omega)\left[\tau_k^-\rho\tau_j^+ - \frac{1}{2}\{\tau_j^+\tau_k^-, \rho\}\right] + S_{jk}(\Omega)\left[\tau_k^+\rho\tau_j^- - \frac{1}{2}\{\tau_j^-\tau_k^+, \rho\}\right]$$

# 1) Pre-experiment: Ramsey Interferometry

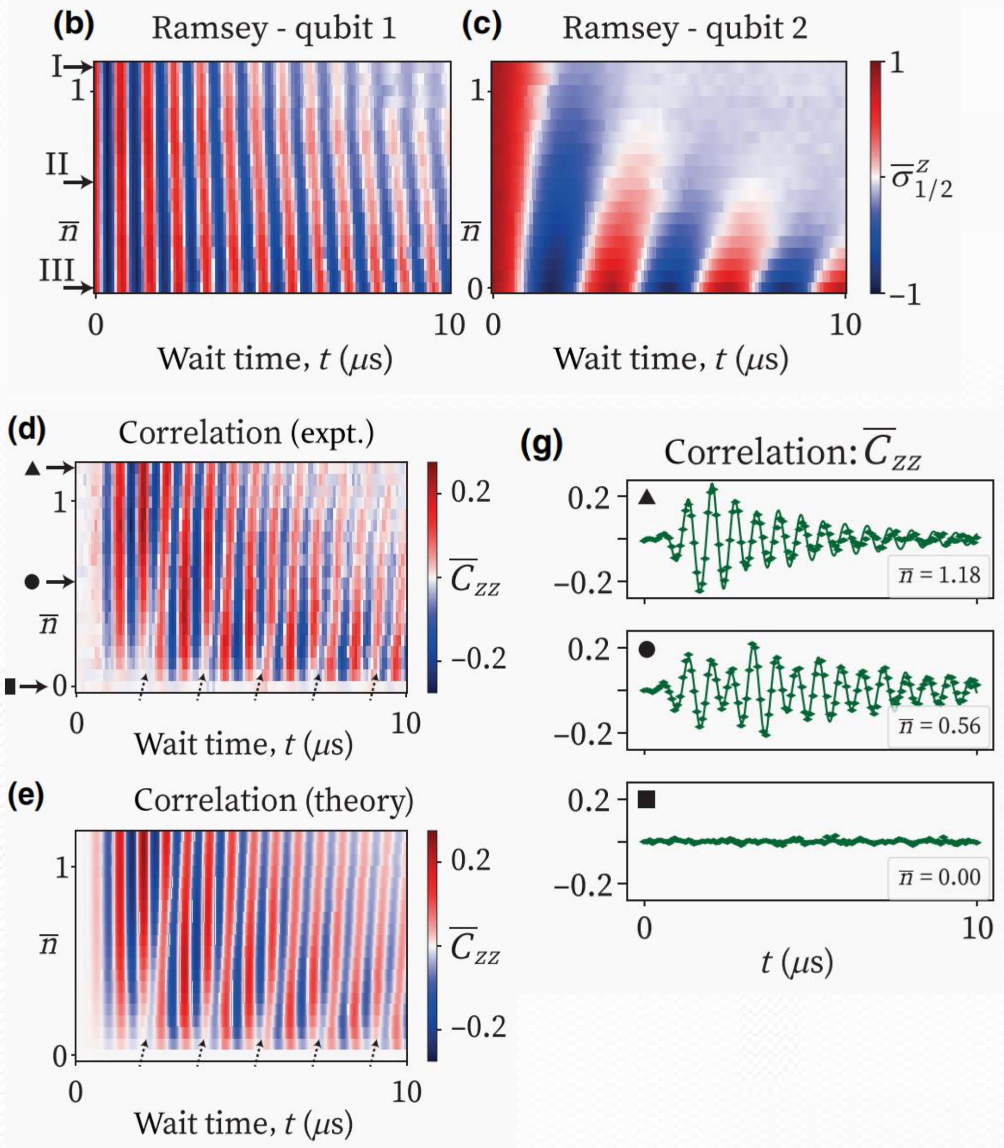
Pulse sequence




Compare the photon shot noise v.s native noise by measuring: $\overline{C}_{zz} = \langle\sigma_1^z\sigma_2^z\rangle - \langle\sigma_1^z\rangle\langle\sigma_2^z\rangle$

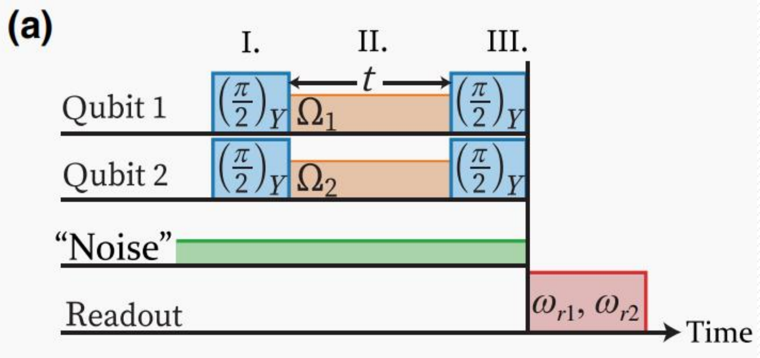
1) Create stationary noise
2) Verifies the existence of correlation
3) Display the features of the correlation

Results



# 2) Two-qubit correlation: Noise Frequency Selection

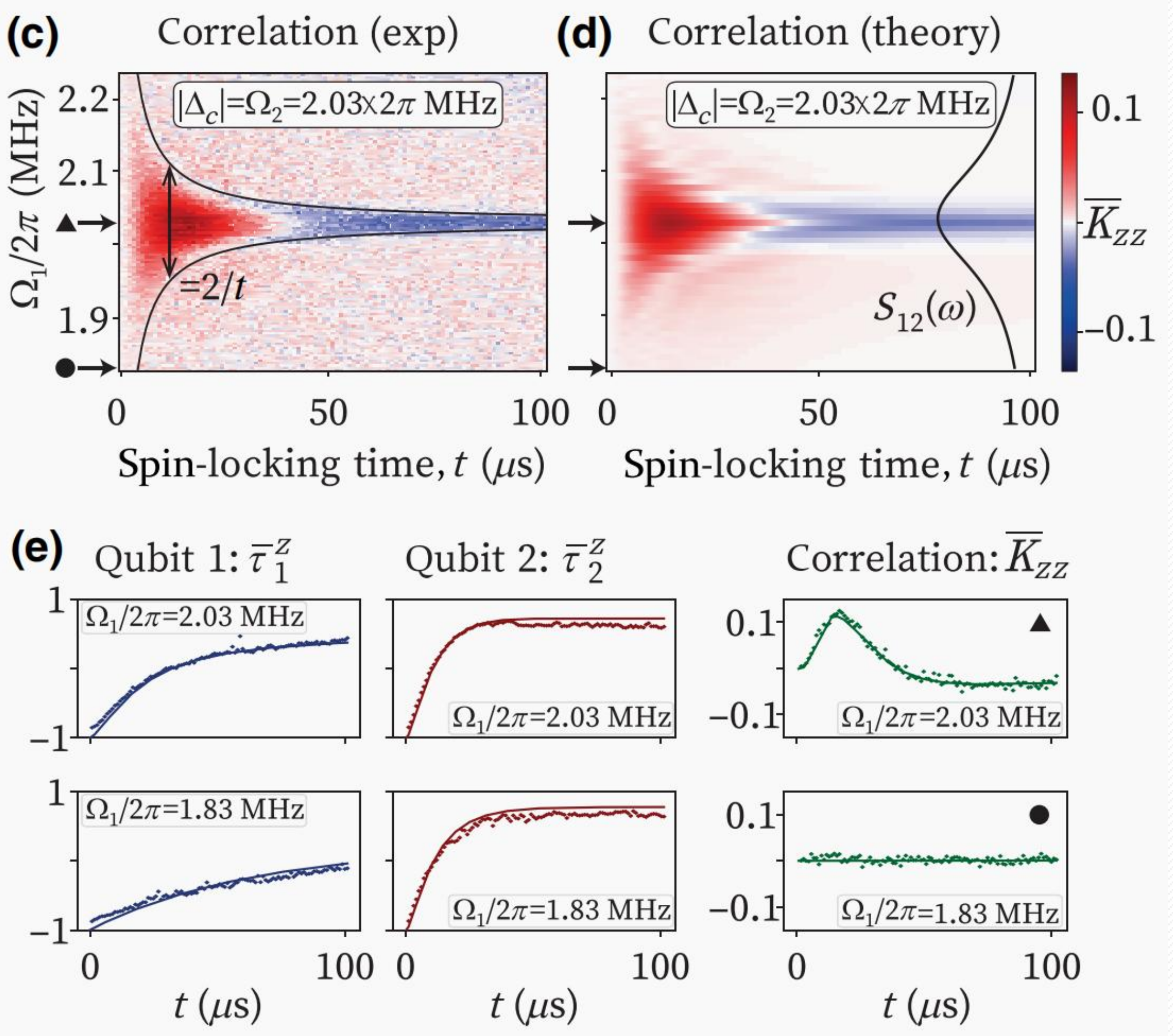
Pulse sequence




Evaluate cross-correlation by measuring: $\overline{K}_{zz} = \langle \tau_1^z \tau_2^z \rangle - \langle \tau_1^z \rangle \langle \tau_2^z \rangle$

1) Choose Rabi frequency of Q2 to maximize noise effect
2) Sweep $\Omega_1$
3) Confirm the expected non-zero correlation when $\Omega_1 \approx \Omega_2$
4) Decay channel for Q1

Results



Reconstruction of two-qubit spectroscopy

- Data collection

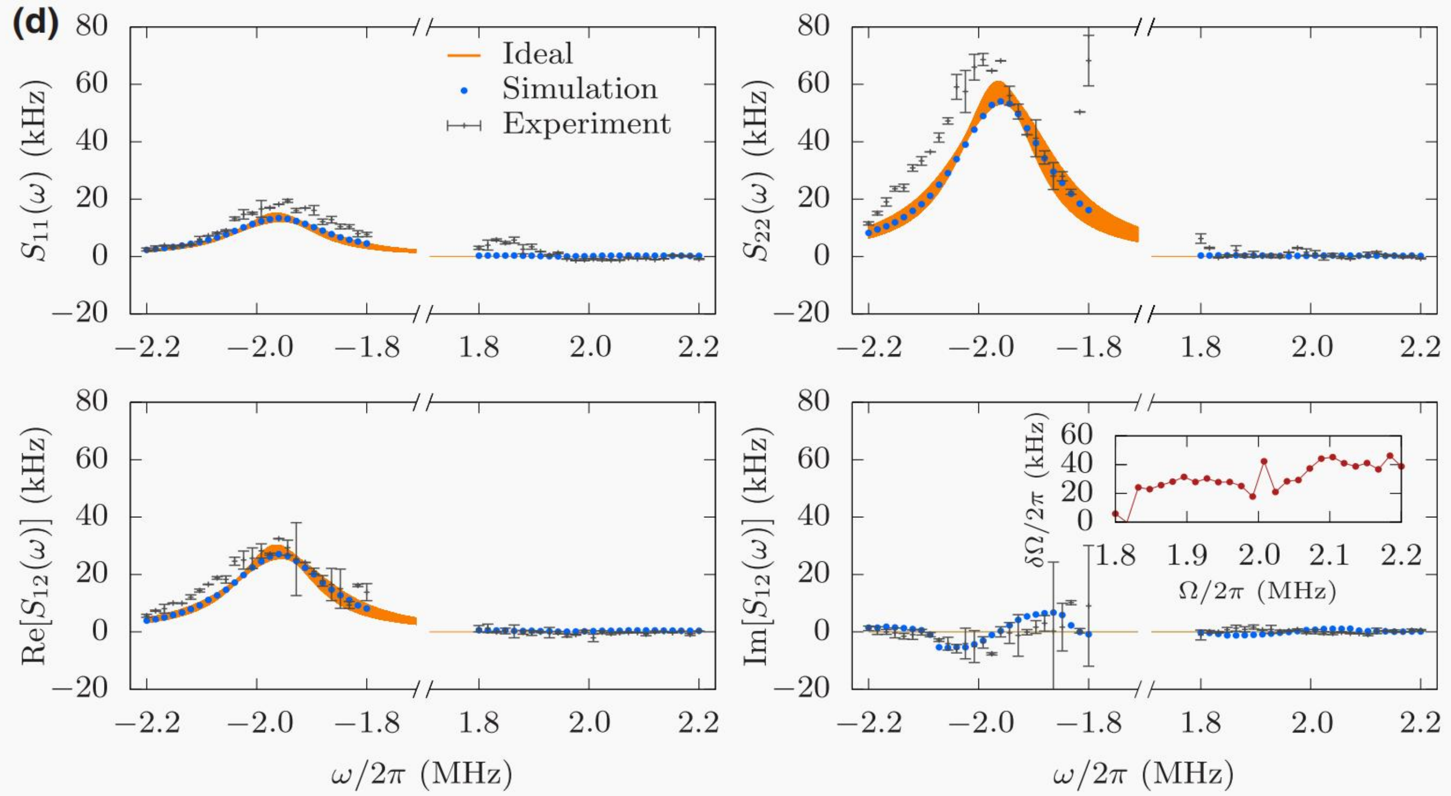
| | |
|---|---|
| Initial states $\lvert\psi_s\rangle$ | $\lvert +x,+x\rangle, \lvert +x,-x\rangle, \lvert -x,+x\rangle, \lvert -x,-x\rangle$ |
| Observables $O_r$ | $\tau_1^z, \tau_2^z, \{K_{\ell_1\ell_2}\}, \ell_1,\ell_2 \in \{x,y,z\}$ |
| Evolution times $t_q$ ($\mu$s) | $1, 3, 5, \ldots, 11, 16, 21, 26, \ldots, 71, 81, \ldots, 151$ |

$$\overline{K}_{l_1 l_2} = \langle \tau_1^{l_1}\tau_2^{l_2}\rangle - \langle \tau_1^{l_1}\rangle\langle \tau_2^{l_2}\rangle,\ l_1, l_2 = x, y, z$$

- Reconstruction

1. Initial guess: $\vec{S} = \{S_{jk}(\Omega), S_{jk}(-\Omega)\}$
2. Reconstruct $\vec{S}$ from measured $\langle O_\Omega\rangle$
3. Compare numerical prediction and experimental data
4. Calculate new $\vec{S} = \{S_{jk}(\Omega), S_{jk}(-\Omega)\}$
5. Iterate for each $\Omega$

Results




Time consumption / data dimension: d= $N_{states} \times N_{times} \times N_{observables}$

# Discussion

Advantages

1. Simple scheme for noise characterization of experimental platforms – v.s Long dynamical decoupling sequence, entanglement state preperation
2. Extensible
3. Relatively short sequence
4. Possible extension for qubit calibration protocol

Others:

1. Assumption: weak noise acting along Sigma-Z axis
2. Only include Gaussian noise so far
3. T1 effect not included