HW5

1. The initial-value problem

   $y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact

   solution $y(t) = t\tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and

   compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the

   solution, and compare it with the actual values of y.

$$y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2, \quad 1 \le t \le 2, \; y(1) = 0.$$

$$y(t) = t \cdot \tan(\ln t)$$

Euler's method: $y_{i+1} = y_i + h f(t_i, y_i)$

$\Rightarrow t_0 = 1.0, \; y_0 = 0$

$t_1 = 1.1 \quad y_1 = y_0 + 0.1 f(t_0, y_0)$

$= 0 + 0.1 [1 + 0 + 0] = 0.1$

$t_2 = 1.2, \quad y_2 = y_1 + 0.1 f(t_0, y_0),$

$= 0.1 + 0.1 \times [1 + \frac{0.1}{1.1} + \frac{0.1^2}{1.1^2}] \approx 0.2099$

v.s. ~~精確解~~ $y = t \cdot \tan(\ln t)$

$t_0 = 1.0$, $y = 1.0 \times \tan(\ln t) = 0$

$t_1 = 1.1$, $y = 1.1 \times \tan(\ln t) = 0.1052$

$t_2 = 1.2$, $y = 1.2 \times \tan(\ln t) = 0.2212$

HW5_1.1				
h=	0.1	t0=	0	
ti	yi	精確解	誤差	誤差百分比
1.0	0.0000	0	0.00000	0.000%
1.1	0.1000	0.10516	0.00516	4.907%
1.2	0.2099	0.22124	0.01133	5.119%
1.3	0.3305	0.34912	0.01865	5.342%
1.4	0.4624	0.48968	0.02733	5.581%
1.5	0.6063	0.64388	0.03759	5.838%
1.6	0.7630	0.81275	0.04971	6.116%
1.7	0.9335	0.99749	0.06402	6.418%
1.8	1.1185	1.19944	0.08090	6.745%
1.9	1.3193	1.42012	0.10082	7.100%
2.0	1.5369	1.66128	0.12434	7.484%

隨著 Euler 法的 t 增大，誤差百分比也隨即上升.

二階 Taylor method:

$$y_{i+1} = y_i + h\,T^{(2)}(t_i, y_i)$$

$$T^{(2)} = f(y_i, t_i) + \frac{h}{2}\cdot\frac{df}{dt}(t_i, y_i)$$

其中, $\frac{df}{dt} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}$

$$\left\{\begin{aligned}
&\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right] = -\frac{y}{t^2} - \frac{y^2}{t^3}\\
&\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right] = \frac{1}{t} + \frac{2y}{t^2}\\
&f\frac{\partial f}{\partial y} = \left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right]\left(\frac{1}{t} + \frac{2y}{t^2}\right)\\
&\frac{df}{dt} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = \left(-\frac{y}{t^2} - \frac{2y^2}{t^3}\right) + \left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right]\left(\frac{1}{t} + \frac{2y}{t^2}\right)
\end{aligned}\right.$$

$\Rightarrow t_0 = 1.0,\quad y_0 = 0$

$t_1 = 1.1 \quad y_1 = 0.105$

$t_2 = 1.2, \quad y_2 = 0.2209.$

v.s. 精確解 $y = t \cdot \tan(\ln t)$

$t_0 = 1.0$, $y = 1.0 \times \tan(\ln t) = 0$

$t_1 = 1.1$, $y = 1.1 \times \tan(\ln t) = 0.1052$

$t_2 = 1.2$, $y = 1.2 \times \tan(\ln t) = 0.2212$

HW5_1.2				
h=	0.1	t0=	0	
ti	yi	精確解	誤差	誤差百分比
1.0	0.0000	0	0.00000	0.000%
1.1	0.1050	0.10516	0.00016	0.152%
1.2	0.2209	0.22124	0.00032	0.146%
1.3	0.3486	0.34912	0.00051	0.146%
1.4	0.4890	0.48968	0.00073	0.149%
1.5	0.6429	0.64388	0.00099	0.154%
1.6	0.8114	0.81275	0.00131	0.162%
1.7	0.9958	0.99749	0.00171	0.171%
1.8	1.1973	1.19944	0.00219	0.182%
1.9	1.4173	1.42012	0.00277	0.195%
2.0	1.6578	1.66128	0.00349	0.210%

由此可知，泰勒法雖有一定誤差，但相較於歐拉法誤差百分比小了約一個數量級。

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3},$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3},$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.

$$w_{i,j+1} = w_{i,j} + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i})$$

$$k_{1,i} = h \cdot f_i(t_i, w_{1,j}, \cdots, w_{n,j})$$

$$= 0.1 \times [9 \times \frac{4}{3} + 24 \times \frac{2}{3} + 5\cos(0) - \frac{1}{3} \cdot \sin(0)] = 3.3.$$

$$y' = f(t, y), \quad y(0) = y_0 = u_0.$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t_0, u_0)$$

$$k_2 = f(t_0 + \frac{h}{2}, u_0 + \frac{h}{2} \times k_1)$$

$$k_3 = f(t_0 + \frac{h}{2}, u_0 + \frac{h}{2} \times k_2)$$

$$k_4 = f(t_0 + h, u_0 + hk_3)$$

HW5_2					
h=	0.1	u1(0)=	1.3333333	u2(0)=	0.666667
t0+h/2=	0.05	u1+0.5k1,1=	1.4983333	u2+0.5*k1,2=	0.288333
		u1+0.5k2,1=	1.4602438	u2+0.5*k2,2=	0.36704
		u1+k3,1=	1.6026157	u2+k3,1=	0.035627
	k1,1	k1,2	k2,1	k2,2	
0	3.3	-7.566666667	2.5382092	-5.992542595	
	k3,1	k3,2	k4,1	k4,2	
	2.6928235	-6.310399953	2.0220324	-4.923712289	
	u1(0.1)=	3.964016276	精確解	1.793062585	
	u2(0.1)=	-5.516044009	精確解	-1.032002453	

HW5_2					
h=	0.05	u1(0)=	1.3333333	u2(0)=	0.666667
t0+h/2=	0.025	u1+0.5k1,1=	2.1583333	u2+0.5*k1,2=	-1.225
		u1+0.5k2,1=	1.208711	u2+0.5*k2,2=	0.739445
		u1+k3,1=	3.0140929	u2+k3,2=	-3.15579
t0	k1,1	k1,2	k2,1	k2,2	
0	33	-75.66666667	-4.984895	2.911144819	
	k3,1	k3,2	k4,1	k4,2	
	33.615191	-76.44907318	-43.63496	82.7906022	
	u1(0.1)=	1.721880259	精確解	1.912058635	
	u2(0.1)=	-0.499599343	精確解	-0.909076587	

可见h越小，误差越小，贴合精确，