Esercizi di Analisi Matematica II

Esercizio 1

- tipologia: polinomiale con esp come denominatore

$$\begin{cases} y' = \dfrac{y^2+2y+1}{e^{2x}} \\ y(0)=0 \end{cases} \qquad f(x) = e^{-2x} \qquad g(y) = y^2+2y+1$$

Soluzioni costanti: $g(y)=0 \to y^2+2y+1=0 \quad y_{1,2} = \dfrac{-2\pm\sqrt{4-4}}{2} \to -1$

Soluzioni non costanti: $\dfrac{dy}{dx} = f(x)g(y) \to \dfrac{dy}{y^2+2y+1} = \dfrac{1}{e^{2x}}\,dx$

$$\int \frac{1}{(y+1)^2}\,dy = \int e^{-2x}\,dx \xrightarrow[dy=du]{u=y+1} \int \frac{1}{u^2}\,du = \frac{e^{-2x}}{-2}+c \to \int u^{-2}\,du = \frac{e^{-2x}}{-2}+c$$

$$\frac{u^{-2+1}}{-2+1} = \frac{e^{-2x}}{-2}+c \to -u^{-1} = \frac{e^{-2x}}{-2}+c \to \frac{1}{y+1} = \frac{e^{-2x}}{-2}+c$$

$$\frac{1}{y+1} = \frac{e^{-2x}}{2}-c \to \frac{1}{\frac{1}{y+1}} = \frac{1}{\frac{e^{-2x}}{2}-c} \to y+1 = \frac{1}{\frac{e^{-2x}}{2}-c} \to y = \frac{1}{\frac{e^{-2x}}{2}-c}-1$$

$$\to \frac{1}{\frac{1}{2e^{2x}}-c}-1 \to \frac{1}{\frac{-2ce^{2x}+1}{2e^{2x}}}-1 \to \frac{2e^{2x}}{-2ce^{2x}+1}-1 \to \frac{2e^{2x}-(-2ce^{2x}+1)}{-2ce^{2x}+1}$$

$$\to \frac{2e^{2x}+2ce^{2x}-1}{-2ce^{2x}+1} \to \frac{(2+2c)e^{2x}-1}{-2ce^{2x}+1} \xrightarrow[x=0]{\text{con}} \frac{2+2c-1}{-2c+1} \to \frac{2c+1}{-2c+1} \quad 2c+1=0 \quad \frac{2c}{2}=\frac{-1}{2}$$

$$c=-\frac{1}{2} \text{ dunque } \frac{(2+2\cdot -\frac{1}{2})e^{2x}-1}{-2\cdot -\frac{1}{2}e^{2x}+1} \to \boxed{\frac{e^{2x}-1}{e^{2x}+1}}$$

- tipologia: polinomiale moltiplicato con x

$$\begin{cases} y' = (y^2+3y-4)x \\ y(0)=0 \end{cases} \qquad f(x)=x \qquad g(y) = y^2+3y-4$$

Soluzioni costanti: $g(y)=0 \to y^2+3y-4=0 \quad y_{1,2} = \dfrac{-3\pm\sqrt{9+16}}{2}$ → -4, → 1

Soluzioni non costanti: $\dfrac{dy}{dx} = f(x)g(y) \to \dfrac{dy}{y^2+3y-4} = x\,dx$

$$\int \frac{1}{y^2+3y-4}\,dy = \int x\,dx$$

$$\frac{1}{(y+4)(y-1)} = \frac{A}{(y+4)} + \frac{B}{(y-1)} = \frac{A(y-1)+B(y+4)}{(y+4)(y-1)} = \frac{(A+B)y - A + 4B}{(y+4)(y-1)} =$$

$$\boxed{\begin{matrix} A = -1/5 \\ B = 1/5 \end{matrix}}$$

$$\begin{cases} A+B=0 \to 4B-1+B=0 \to 5B=1 \to B=1/5 \\ -A+4B=1 \to -A=1-4B \to A=4B-1 \to A=4\cdot\frac{1}{5}-1=\frac{4-5}{5}=-\frac{1}{5} \end{cases}$$

$$-\frac{1}{5}\int\frac{1}{(y+4)}dy + \frac{1}{5}\int\frac{1}{(y-1)}dy = \int x\,dx \to \frac{1}{5}\int\frac{1}{y-1}dy - \frac{1}{5}\int\frac{1}{y+4}dy = \int x\,dx$$

$$\frac{1}{5}\log|y-1| - \frac{1}{5}\log|y+4| = \frac{x^{1+1}}{1+1} \to \frac{1}{5}\log\left|\frac{y-1}{y+4}\right| = \frac{x^2}{2}+c$$

$$\log\left|\frac{y-1}{y+4}\right| = \frac{5x^2}{2}+5c \to \left|\frac{y-1}{y+4}\right| = e^{\frac{5}{2}x^2+5c} \to \frac{y-1}{y+4} = \underbrace{\pm e^{5c}}_{k}\cdot e^{\frac{5}{2}x^2}$$

$$y-1 = ke^{\frac{5}{2}x^2}(y+4) \to y-1 = ke^{\frac{5}{2}x^2}y + 4ke^{\frac{5}{2}x^2} \to y - ke^{\frac{5}{2}x^2}y = 4ke^{\frac{5}{2}x^2}+1$$

$$y(1-ke^{\frac{5}{2}x^2}) = 4ke^{\frac{5}{2}x^2}+1 \to y = \frac{4ke^{\frac{5}{2}x^2}+1}{1-ke^{\frac{5}{2}x^2}}$$

$$y(0) = \frac{4k+1}{1-k} \to 4k+1=0 \to 4k=-1 \to k=-\frac{1}{4}$$

Dunque $y = \frac{4\cdot -\frac{1}{4}e^{\frac{5}{2}x^2}+1}{1-(-\frac{1}{4})e^{\frac{5}{2}x^2}} = \frac{-e^{5/2x^2}+1}{1+\frac{1}{4}e^{\frac{5}{2}x^2}}$

• *Tipologia: polinomiale*

$$\begin{cases} y' = y^2+3y-4 & f(x)=1 \\ y(1) = -2 & g(y) = y^2+3y-4 \end{cases}$$

Soluzioni costanti: $g(y)=0 \to y^2+3y-4=0 \to y_{1,2} = \frac{-3\pm\sqrt{25}}{2}$ $\begin{matrix} -4 \\ 1 \end{matrix}$

Soluzioni non costanti: $\frac{dy}{dx} = f(x)g(y) \to \frac{dy}{y^2+3y-4} = 1\,dx$

$\int\frac{1}{y^2+3y-4} = \int 1\,dx$ (guardare es prec...)

$$\frac{1}{5}\ln\left|\frac{y-1}{y+4}\right| = x+c \to \ln\left|\frac{y-1}{y+4}\right| = 5x+5c \to \left|\frac{y-1}{y+4}\right| = e^{5x+5c} \to$$

$$\frac{y-1}{y+4} = \underbrace{\pm e^{5c}}_{k}\cdot e^{5x} \to y-1 = ke^{5x}(y+4) \to y-1 = ke^{5x}y + 4ke^{5x}$$

$y - ke^{5x}y = 4ke^{5x}+1 \rightarrow y(1-ke^{5x}) = 4ke^{5x}+1 \rightarrow y = \dfrac{4ke^{5x}+1}{1-ke^{5x}}$

$\lim_{x\to 1} y = \dfrac{4ke^{5}+1}{1-ke^{5}} \rightarrow -2 \rightarrow 4ke^5+1 = -2(1-ke^5) \rightarrow 4ke^5+1 = -2+2ke^5$

$\rightarrow \dfrac{2ke^5}{2e^5} = \dfrac{-3}{2e^5} \rightarrow k = -\dfrac{3}{2}e^{-5}$

dunque $y = \dfrac{4\cdot -\frac{3}{2}e^{-5}e^{5x}+1}{1-\left(-\frac{3}{2}\right)e^{-5}e^{5x}} = \dfrac{-6e^{-5}e^{5x}+1}{1+\frac{3}{2}e^{-5}e^{5x}} = \dfrac{6e^{5x-5}+1}{1+\frac{3}{2}e^{5x-5}}$

- Tipologia: x al N e al D

$$\begin{cases} y' = \dfrac{x^2}{1+x^3} & f(x) = \dfrac{x^2}{1+x^3} \\ y(0) = 2 & g(y) = 1 \end{cases}$$

Soluzioni costanti: Non presenti

Soluzioni non costanti: $\dfrac{dy}{dx} = f(x)g(y) \rightarrow \dfrac{dy}{1} = \dfrac{x^2}{1+x^3}dx$

$\int 1\,dy = \int \dfrac{x^2}{1+x^3}dx \rightarrow \int 1\,dy = \dfrac{1}{3}\int\dfrac{3x^2}{1+x^3}dx \rightarrow y = \dfrac{1}{3}\ln|1+x^3| + c$

pongo $x=0$ e $y=2 \rightarrow \dfrac{1}{3}\ln|1| + c = 2 \rightarrow c = 2$

quindi $y = \dfrac{1}{3}\ln|1+x^3|+2$

- Tipologia: y^n e x^n moltiplicati:

$$\begin{cases} y' = y^2x & f(x) = x \\ y(0) = 1 & g(y) = y^2 \end{cases}$$

Soluzioni costanti: $g(y) = 0 \rightarrow y^2 = 0 \rightarrow y = 0$

Soluzioni non costanti: $\dfrac{dy}{dx} = f(x)g(y) \rightarrow \dfrac{dy}{y^2} = x\,dx$

$\int y^{-2}dy = \int x\,dx \rightarrow \dfrac{y^{-2+1}}{-2+1} = \dfrac{x^{1+1}}{1+1} + c \rightarrow -y^{-1} = \dfrac{x^2}{2}+c \rightarrow$

$-\dfrac{1}{y} = \dfrac{x^2}{2}+c \rightarrow -1 = y\left(\dfrac{x^2}{2}+c\right) \rightarrow y = -\dfrac{1}{\frac{x^2}{2}+c} \rightarrow y = -\dfrac{1}{\frac{x^2+2c}{2}}$

$y = \dfrac{-2}{x^2+2c} \rightarrow$ pongo $x=0$ e $y=1 \rightarrow \dfrac{-2}{2c} = 1 \rightarrow 2c = -2 \rightarrow c = -1$

quindi $y = -\dfrac{2}{x^2-2}$

~ Risolvibili con la primitiva ~

• Tipologia: $f(x) \neq 0$ e $a(x) = ny$

$\begin{cases} y' - 7y = 2x \\ y(0) = -1 \end{cases}$ $a(x) = -7$ $f(x) = 2x$

$A(x) = -7x \ (+c)$

$(e^{-7x} y)' = e^{-7x} \cdot 2x$

$e^{-7x} y = \int e^{-7x}\, 2x \, dx + c$

· integrazione per parti

$f(x) = 2x \quad g'(x) = e^{-7x}$
$f'(x) = 2 \quad g(x) = \frac{e^{-7x}}{-7}$

$e^{-7x} y = 2x \cdot \frac{e^{-7x}}{7} - \int 2 \cdot \frac{e^{-7x}}{7} + c$

$= -\frac{2}{7} x e^{-7x} - 2\int \frac{e^{-7x}}{7} + c$

$= -\frac{2}{7} x e^{-7x} - \frac{2}{49} e^{-7x} + D + c \quad (k = D + c)$

$e^{-7x} y = -\frac{2}{7} x e^{-7x} - \frac{2}{49} e^{-7x} + k$

$y = -\frac{2}{7} x - \frac{2}{49} + e^{7x} k \quad (y = -1,\ x = 0)$

$y(0) = 1 = -\frac{2}{49} + k e^{0} \rightarrow -\frac{2}{49} + k = -1$

$k = \frac{-49+2}{49} \rightarrow -\frac{47}{49}$

$y = -\frac{2}{7} x - \frac{2}{49} - \frac{47}{49} e^{7x}$

$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases} \rightarrow \begin{cases} y' - y = x \\ y(0) = 1 \end{cases} \rightarrow$ $a(x) = -1$, $f(x) = x$

$A(x) = -x \ (+c)$

$(e^{-x} y)' = e^{-x} \cdot x$

$e^{-x} y = \int e^{-x} \cdot x \, dx + c$

· integrazione per parti

$f(x) = x \quad g'(x) = e^{-x}$
$f'(x) = 1 \quad g(x) = -e^{-x}$

$e^{-x} y = -x e^{-x} - \int -e^{-x} \, dx + c$

$= -x e^{-x} - e^{-x} + D + c \quad (k = D + c)$

$e^{-x} y = -x e^{-x} - e^{-x} + k$

$y = -x - 1 + k e^{x} \quad (y = 1 \text{ e } x = 0)$

$-1 + k = 1 \rightarrow k = 2$

$y = -x - 1 + 2e^{x}$

• Tipologia: $f(x) \neq 0$ con trigonometrica

$\begin{cases} y' + \cos(x)\, y = \cos(x) \\ y(0) = 0 \end{cases}$ $a(x) = \cos(x)$, $f(x) = \cos(x)$

$A(x) = \sin(x)$

$(e^{\sin(x)} y)' = e^{\sin(x)} \cdot \cos(x)$

$e^{\sin(x)} y = \int e^{\sin(x)} \cdot \cos(x) + c$

$e^{\sin(x)} y = e^{\sin(x)} + k$

$y = 1 + e^{-\sin(x)} k \quad x = 0 \text{ e } y = 0$

$\rightarrow 1 + e^{-\sin(0)} k = 0$

$y = 1 - e^{-\sin(x)}$

Esercizio ②

~ CASO 1 ~

$$\begin{cases} y'' + 4y' = x^2 + 5x + 1 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$ ⋯→ controcaso 2

$y^2 + 4y \to y(y+4) = 0$ $\boxed{\begin{matrix} y = 0 \\ y = -4 \end{matrix}}$

$U = \{ c_1 + c_2 e^{-4x} \; c_1 e \; c_2 \in \mathbb{R} \}$

$$\begin{cases} y_p = ax^3 + bx^2 + cx \\ y_p' = 3ax^2 + 2bx + c \\ y_p'' = 6ax + 2b \end{cases}$$

$(6ax + 2b) + 4(3ax^2 + 2bx + c) = x^2 + 5x + 1$

$6ax + 2b + 12ax^2 + 8bx + 4c = x^2 + 5x + 1$

$\underbrace{12ax^2}_{1} + \underbrace{6ax + 8bx}_{5} + \underbrace{2b + 4c}_{1} = x^2 + 5x + 1$

$$\begin{cases} 12a = 1 \longrightarrow \boxed{a = 1/12} \\ 6a + 8b = 5 \longrightarrow 1/2 + 8b = 5 \longrightarrow 8b = \frac{9}{2} \longrightarrow \boxed{b = 9/16} \\ 2b + 4c = 1 \longrightarrow 9/8 + 4c = 1 \longrightarrow 4c = -1/8 \longrightarrow \boxed{-1/32} \end{cases}$$

$y(x) = c_1 + c_2 e^{-4x} + \frac{1}{12}x^3 + \frac{9}{16}x^2 - \frac{1}{32}x$

$y'(x) = -4c_2 e^{-4x} + \frac{1}{4}x^2 + \frac{9}{8}x - \frac{1}{32}$

$$\begin{cases} c_1 + c_2 = 0 \to c_1 = -c_2 \\ 4c_2 = -\frac{1}{32} \to c_2 = -\frac{1}{128} \text{ dunque } c_1 = \frac{1}{128} \end{cases}$$

$y(x) = \frac{1}{128} - \frac{1}{128}e^{-4x} + \frac{1}{12}x^3 + \frac{9}{16}x^2 - \frac{1}{32}x$

~ CASO 2 ~

$$\begin{cases} y'' - 5y' + 6y = 3e^x \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$ ⋯→ no eccezioni

$y^2 - 5y + 6 = 0$ $y_{1,2} = \frac{5 \pm \sqrt{25 - 24}}{2} \begin{matrix} \nearrow 3 \\ \searrow 2 \end{matrix}$

$U = \{ c_1 \cdot e^{3x} + c_2 e^{2x} \text{ con } c_1 e \, c_2 \in \mathbb{R} \}$

$y_p = Ce^x$ $\quad Ce^x - 5Ce^x + 6Ce^x = 3e^x \rightarrow C = 3/2$

$y'_p = Ce^x$ $\quad$ dunque $y_p = \frac{3}{2}e^x$

$y''_p = Ce^x$

$y(x) = c_1e^{3x} + c_2e^{2x} + \frac{3}{2}e^x$

$y'(x) = 3c_1e^{3x} + 2c_2e^{2x} + \frac{3}{2}e^x$

$\rightarrow \begin{cases} c_1 + c_2 + \frac{3}{2} = 0 \rightarrow c_1 + c_2 = -\frac{3}{2} \rightarrow c_1 = -\frac{3}{2} - c_2 \\ 3c_1 + 2c_1 + \frac{3}{2} = 0 \end{cases}$

$3c_1 + 2c_2 = -\frac{3}{2}$

$3\left(-\frac{3}{2} - c_2\right) + 2c_2 = -\frac{3}{2}$

$-\frac{9}{2} - 3c_2 + 2c_2 = -\frac{3}{2}$

$-c_2 = 3 \rightarrow c_2 = -3$

$c_1 = -\frac{3}{2} + 3 = \frac{-3+6}{2} = \frac{3}{2}$

$\boxed{\begin{matrix} c_1 = 3/2 \\ c_2 = -3 \end{matrix}}$

$y(x) = \frac{3}{2}e^{3x} - 3e^{2x} + \frac{3}{2}e^x$

- $\begin{cases} y'' - 3y' + 2y = \boxed{e^x} \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ $\rightarrow$ 2ª cucumme

$y^2 - 3y + 2 = 0 \quad y_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} \begin{matrix} \nearrow 2 \\ \searrow 1 \end{matrix}$

$U = \{c_1e^{2x} + c_2\boxed{e^x}$ con $c_1 e\, c_2 \in \mathbb{R}\}$

$y_p = Cxe^x$

$y'_p = Ce^x + Cxe^x$

$y''_p = Ce^x + Ce^x + Cxe^x$

$Ce^x + Ce^x + Cxe^x - 3(Ce^x + Cxe^x) + 2Cxe^x = e^x$

$Ce^x + Ce^x - 3Ce^x = e^x$

$-Ce^x = e^x \rightarrow C = -1$

dunque $y_p = -xe^x$

$y(x) = c_1e^{2x} + c_2e^x - xe^x$

$y'(x) = 2c_1e^{2x} + c_2e^x - xe^x - e^x$

$\begin{cases} c_1 + c_2 = 1 \rightarrow c_2 = 1 - c_1 \\ 2c_1 + c_2 = 0 \rightarrow 2c_1 + 1 - c_1 - 1 = 0 \end{cases}$

$\boxed{\begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}}$

$c_1 = 0$ dunque $c_2 = 1$

$y(x) = e^x - xe^x$

~ CASO 3 ~

$$\begin{cases} y'' + y' = \sin(2x) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

• Soluzioni dell'omogenea:

$\lambda^2 + \lambda = 0 \rightarrow \lambda(\lambda+1) = 0$ $\rightarrow 0$, $\rightarrow -1$

$V = \{ c_1 e^0 + c_2 e^{-x} \} = \{ c_1 + c_2 e^{-x} \quad c_1, c_2 \in \mathbb{R} \}$

• soluzioni della non omogenea, trovo la particolare:

$y_p(x) = C\cos(2x) + D\sin(2x)$

$y'_p(x) = -2C\sin(2x) + 2D\cos(2x)$

$y''_p(x) = -4C\cos(2x) - 4D\sin(2x)$

$-4C\cos(2x) - 4D\sin(2x) - 2C\sin(2x) + 2D\cos(2x) = \sin(2x)$

$\underbrace{(-4D - 2C)}_{1}\sin(2x) + \underbrace{(-4C + 2D)}_{0}\cos(2x) = \sin(2x)$

$$\begin{cases} -4D - 2C = 1 \quad -4(2C) - 2C = 1 \rightarrow -8C - 2C = 1 \rightarrow C = -\frac{1}{10} \\ -4C + 2D = 0 \rightarrow \frac{{}^2\cancel{4}C}{\cancel{2}} = \frac{2D}{\cancel{2}} \rightarrow D = 2C \quad D = 2 \cdot -\frac{1}{10} \rightarrow -\frac{1}{5} \end{cases}$$

$$\boxed{\begin{matrix} C = -\frac{1}{10} \\ D = -\frac{1}{5} \end{matrix}}$$

$y = c_1 + c_2 e^{-x} - \frac{1}{10}\cos(2x) - \frac{1}{5}\sin(2x)$

$y' = -c_2 e^{-x} + \frac{2}{\cancel{10}_5}\sin(2x) - \frac{2}{5}\cos(2x)$

pongo $y(0) = 0$ e $y'(0) = 0$

$$\begin{cases} c_1 + c_2 - \frac{1}{10} = 0 \quad c_1 - \frac{2}{5} - \frac{1}{10} = 0 \rightarrow c_1 = \frac{4+1}{10} \Rightarrow c_1 = \frac{1}{2} \\ -c_2 - \frac{2}{5} = 0 \rightarrow c_2 = -\frac{2}{5} \end{cases}$$

$$\boxed{\begin{matrix} c_1 = \frac{1}{2} \\ c_2 = -\frac{2}{5} \end{matrix}}$$

$y = \frac{1}{2} - \frac{2}{5}e^{-x} - \frac{1}{10}\cos(2x) - \frac{1}{5}\sin(2x)$

Esercizio 3

~ Esercizi sul dominio ~

- $f(x,y) = \ln(x^2-y^2-1)$

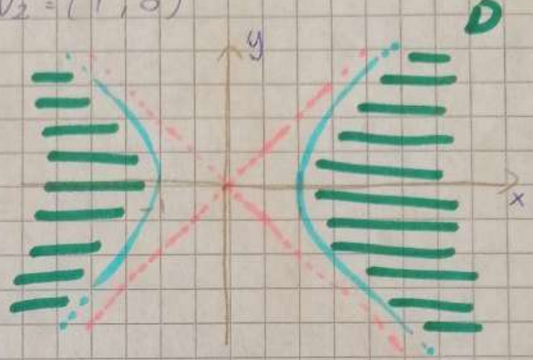
$D = \{(x,y) \in \mathbb{R} : x^2-y^2-1>0\} \to \boxed{x^2-y^2>1}$

Rappresentiamo $\boxed{x^2-y^2=1}$ (Iperbole) con $a=1$ e $b=1$

$C=(0,0)$ $V_1=(-1,0)$ $V_2=(1,0)$

asintoto$_1$ $\Rightarrow$ $\boxed{y=x}$

asintoto$_2$ $\Rightarrow$ $\boxed{y=-x}$



- $f(x,y) = \dfrac{1}{4x^2-8x+y^2-6y+9}$

$D = \{(x,y) \in \mathbb{R} : 4x^2-8x+y^2-6y+9 \neq 0\}$

$4x^2-8x+y^2-6y+9=0$

$4(x^2-2x)+y^2-6y+9=0$

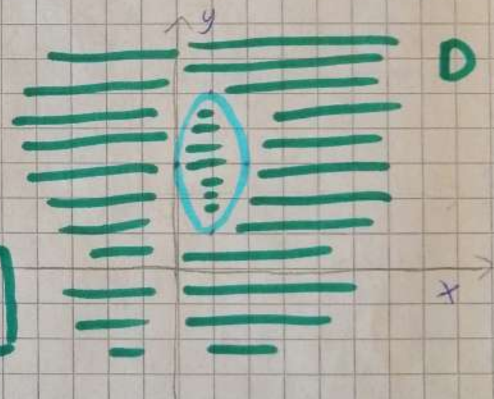
$4(\underbrace{x^2-2x+1}-1)+\underbrace{y^2-6y+9}=0$

$4[(x-1)^2-1]+(y-3)^2=0$

$4(x-1)^2+(y-3)^2=4$

$\boxed{(x-1)^2+\dfrac{(y-3)^2}{4}=1}$ (Ellisse) con $a=1$ e $b=2$

$\boxed{\text{D: tutto ciò che non è } \square}$



$C=(1,3)$ $V_1=(0,3)$ $V_2=(2,3)$

$V_3=(1,1)$ $V_4=(1,5)$

- $f(x,y) = \dfrac{1}{\ln(x^2-y^2+4)}$

$D = \begin{cases} x^2-y^2+4>0 & (1) \\ \ln(x^2-y^2+4) \neq 0 \end{cases}$ $\to$ quand'è che $\ln(f(x))=0$?

- quando $f(x)=1$, dunque $x^2-y^2+4 \neq 1$ (2)

① $\boxed{x^2 - y^2 + 4 > 0}$

$\boxed{x^2 - y^2 + 4 = 0}$

$x^2 - y^2 = -4$

$\frac{x^2}{4} - \frac{y^2}{4} = -1$

$V_1(0, -2)$

$V_2(0, +2)$

② $\boxed{x^2 - y^2 + 4 \neq 1}$

$\boxed{x^2 - y^2 + 4 = 1}$

$x^2 - y^2 = -3$

$\frac{x^2}{3} - \frac{y^2}{3} = -1$

$V_3(0, -\sqrt{3})$

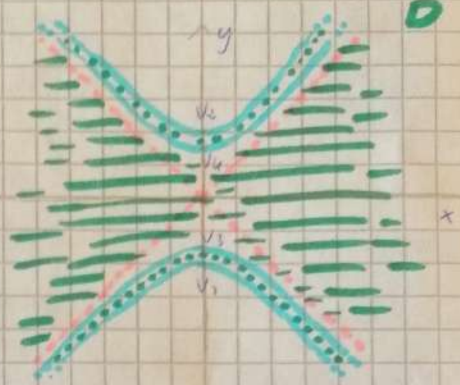
$V_4(0, +\sqrt{3})$

Iperboli che intersecano l'asse y e con $C(0,0)$ e con

asintoto$_1$ ⇒ $\boxed{y = x}$

asintoto$_2$ ⇒ $\boxed{y = -x}$

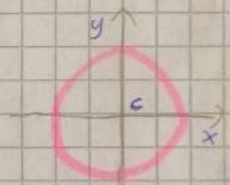
D



~ Esercizi sulle curve di livello ~

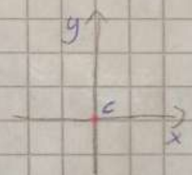
- $L_1(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Circonferenza con $c = (0, 0)$ e $r = 1$



- $L_0(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$

Circonferenza con $c = (0, 0)$ e $r = 0$ → $\boxed{\text{punto}}$



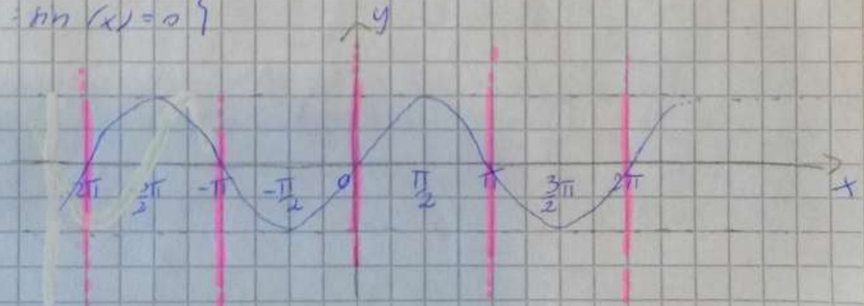
- $L_{-1}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = -1\}$ $\emptyset$ non esiste

- $L_0(f) = \{(x, y) \in \mathbb{R}^2 : \lfloor x \rfloor = 0\}$ $\boxed{0 \leq x < 1}$

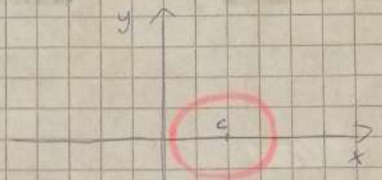
Tutti i numeri minori di 1 e maggiori/uguali a 0

- $L_0(f) = \{(x, y) \in \mathbb{R}^2 : \sin(x) = 0\}$

Quando $\sin(x) = 0$



• $Lg(f) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2-6x+13=9\}$



$x^2+y^2-6x+13=9$

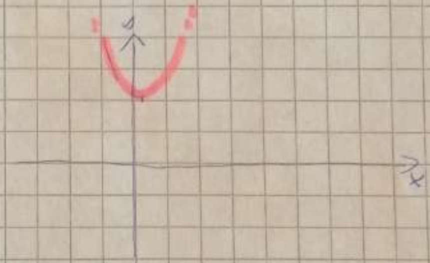
$(x^2-6x)+y^2+13=9$

$(\underline{x^2-6x+9}-9)+y^2+13=9$

$(x-3)^2+y^2=5$

Circonferenza con $C(3,0)$ e $r=\sqrt{5}$

• $Lg(f) = \{(x,y) \in \mathbb{R}^2 : y-x^2+x=9\}$



$y-x^2+x=9$

$y=x^2-x+9$

Parabola con $V=\left(-\frac{-1}{2}; -\frac{1-36}{4}\right)$
$=\left(\frac{1}{2}, \frac{35}{4}\right)$

Esercizio 4

1) Verifica di esistenza o non esistenza di un limite

• $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$?

$f(0,y) = \frac{0\cdot y}{0\cdot y^2} = 0$

$f(x,0) = \frac{x^2\cdot 0}{x^4\cdot 0} \xrightarrow[x\to 0]{} 0$

pongo $y=x^2$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^2\cdot x^2}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$ Non esiste

• $\lim\limits_{(x,y)\to(0,0)} x e^{-\frac{y}{x}} \to \frac{e^{-\frac{y}{x}}}{\frac{1}{x}}$

$f(0,y)=0$ quando $x\to 0$, il D $\to +\infty$, dunque devo far tendere N più velocemente di D a $+\infty$

$f(x,0)=0$

pongo $y=-x^\alpha$, se $1-\alpha>0$, $f(x,y)\to+\infty$ dunque pongo

$\frac{e^{-\frac{y}{x}}}{\frac{1}{x}} \to \frac{e^{-\frac{-x^\alpha}{x}}}{\frac{1}{x}} \to \frac{e^{\frac{x^\alpha}{x}}}{\frac{1}{x}} \to \frac{e^{\frac{1}{x^{1-\alpha}}}}{\frac{1}{x}}$ $\alpha=\frac{1}{2}$

Dunque $y = -\sqrt{x}$

$f(x, -\sqrt{x}) = x e^{\frac{1}{\sqrt{x}}}$

$\lim_{x \to 0} \frac{e^{\frac{1}{\sqrt{x}}}}{\frac{1}{x}} \to \lim_{t \to +\infty} \frac{e^t}{t^2} = +\infty$ Non esiste

$\left(t = \frac{1}{\sqrt{x}} \right)$

2) Lunghezza dell'arco di curva

- $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$t \mapsto (2\cos(t), 2\sin(t))$

$\gamma'(t) = (-2\sin(t), 2\cos(t))$

$L(\gamma) = \int_0^{2\pi} \sqrt{4\sin^2(t) + 4\cos^2(t)}\, dt$

$= 2\int_0^{2\pi} 1\, dt$

$= [2t]_0^{2\pi} \to \boxed{4\pi}$

- $\gamma : [0, 1] \to \mathbb{R}^3$

$t \mapsto (2t, 3t, t^{\frac{3}{2}})$

$\gamma'(t) = (2, 3, \frac{3}{2} t^{\frac{1}{2}})$

$L(\gamma) = \int_0^1 \sqrt{4 + 9 + \frac{9}{4} t}\, dt$

$= \int_0^1 \sqrt{13 + \frac{9}{4} t}\, dt$ $\qquad u = 13 + \frac{9}{4} t$

$= \frac{4}{9} \int_0^1 u^{\frac{1}{2}}\, du$ $\qquad du = \frac{9}{4} dt$

$= \frac{4}{9} \cdot \frac{2}{3} \left[u^{\frac{3}{2}} \right]_0^1$

$= \left[\frac{8}{27} \left(13 + \frac{9}{4} t \right)^{\frac{3}{2}} \right]_0^1$

$= \frac{8}{27} \left[\left(13 + \frac{9}{4} \right)^{\frac{3}{2}} - 13^{\frac{3}{2}} \right]$

$= \boxed{\frac{8}{27} \left[\left(\frac{61}{4} \right)^{\frac{3}{2}} - 13^{\frac{3}{2}} \right]}$

- $\gamma : [1,5] \to \mathbb{R}^2$

$$t \mapsto \left(t, 2(t-1)^{\frac{3}{2}}\right)$$

$$\gamma'(t) = \left(1, 3(t-1)^{\frac{1}{2}}\right)$$

$$L(\gamma) = \int_1^5 \sqrt{1 + 9t - 9}\, dt$$

$$= \int_1^5 \sqrt{9t - 8}\, dt$$

$u = 9t - 8$

$du = 9\,dt$

$$= \frac{1}{9} \int_1^5 u^{\frac{1}{2}}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} \left[u^{\frac{3}{2}} \right]_1^5$$

$$= \left[\frac{2}{27} u^{\frac{3}{2}} \right]_1^5$$

$$= \left[\frac{2}{27} (9t-8)^{\frac{3}{2}} \right]_1^5$$

$$= \frac{2}{27} \left[\left(45^{\frac{3}{2}} - 8^{\frac{3}{2}}\right) - \left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \right]$$

$$= \boxed{\frac{2}{27} \left[37^{\frac{3}{2}} - 1 \right]}$$

Altri esercizi

~ Trovare la parametrizzazione ~

- $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$t \mapsto (\sin(4t)\cos(t), \sin(4t)\sin(t))$ in $P = \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$

1) $\begin{cases} \sin(4t)\cos(t) = \frac{3}{4} \\ \sin(4t)\sin(t) = \frac{\sqrt{3}}{4} \end{cases}$ $\cancel{\sin(4t)}\cos(t) = \sqrt{3}\,\cancel{\sin(4t)}\sin(t)$

$\frac{\sqrt{3}\,\cancel{\sin(t)}}{\cancel{\sin(t)}} = \frac{\cos(t)}{\sin(t)} \to \frac{\sin(t)}{\cos(t)} = \tan(t) = \frac{1}{\sqrt{3}}$

dunque $t = 30$

2) $\gamma'(t) = (4\cos(4t)\cos(t) + \sin(4t)\cdot -\sin(t), 4\cos(4t)\sin(t) + \sin(4t)\cos(t))$

3) $\gamma'(30) = \left(-\sqrt{3} - \frac{\sqrt{3}}{4}, -1 + \frac{3}{4}\right) = \left(-\frac{5\sqrt{3}}{4}, -\frac{1}{4}\right)$

4) $r(t) = \left(\frac{3}{4} - \frac{5\sqrt{3}}{4}s, \frac{\sqrt{3}}{4} - \frac{1}{4}s\right)$

- $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$t \mapsto (\cos(t), \sin(t))$ in $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

1) $\begin{cases} \cos(t) = \frac{1}{2} \\ \sin(t) = \frac{\sqrt{3}}{2} \end{cases}$ dunque $t = 30$

2) $\gamma'(t) = (-\sin(t), \cos(t))$

3) $\gamma'(30) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

4) $r(t) = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}s, \frac{\sqrt{3}}{2} + \frac{1}{2}s\right)$

- $\mathbb{R} \to \mathbb{R}^2$

$t \to (t^2, t^3)$ $P(1, 1)$

2) $\begin{cases} t^2 = 1 \to t = \pm 1 \\ t^3 = 1 \to t = +1 \end{cases}$ dunque $t = +1$

2) $\gamma'(t) = (2t, 3t^2)$

3) $\gamma'(1) = (2, 3)$

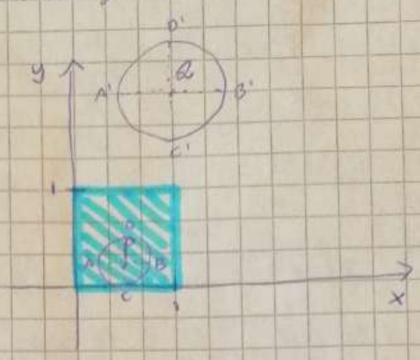
4) $r(t) = (1 + 2s, 1 + 3s)$

~ Punti interni, esterni e di frontiera ~

• $S = [0,1] \times [0,1] \subseteq \mathbb{R}^2$

$P = \left(\frac{1}{2}, \frac{1}{4}\right)$

$Q = (1,2)$



• Dimostro che P è interno.

Prendo U_r con $r = \frac{1}{4} \to A\left(\frac{1}{2} - \frac{1}{4}, \frac{1}{4}\right)$ $B\left(\frac{1}{2} + \frac{1}{4}, \frac{1}{4}\right) \to A\left(\frac{1}{4}, \frac{1}{4}\right)$ $B\left(\frac{3}{4}, \frac{1}{4}\right)$

$C\left(\frac{1}{2}, \frac{1}{4} - \frac{1}{4}\right)$ $D\left(\frac{1}{2}, \frac{1}{4} + \frac{1}{4}\right) \to C\left(\frac{1}{2}, 0\right)$ $D\left(\frac{1}{2}, \frac{1}{2}\right)$

tutti punti interni

• Dimostro che Q è esterno.

Prendo U_r con $r = \frac{1}{2} \to A'\left(1 - \frac{1}{2}, 2\right)$ $B'\left(1 + \frac{1}{2}, 2\right) \to A'\left(\frac{1}{2}, 2\right)$ $B'\left(\frac{3}{2}, 2\right)$

$C'\left(1, 2 - \frac{1}{2}\right)$ $D'\left(1, 2 + \frac{1}{2}\right) \to C'\left(1, \frac{3}{2}\right)$ $D'\left(1, \frac{5}{2}\right)$

tutti punti esterni

• $x^2 + y^2 \geq 1$

$P = (1,0)$

$Q = \left(\frac{1}{2}, \frac{1}{2}\right)$

$R = (0,2)$

• Dimostro che P è un punto di frontiera: $1^2 + 0^2 = 1$

prendo un intorno $U_r(P) \to \boxed{U_r(P) \cap \mathbb{R} \setminus D \neq \emptyset}$

considero il punto $R = \left(1 + r/2, 0\right)$

$= \left(1 + r/2\right)^2$ è maggiore di 1

poiché $P \in U_r(P)$ non è compreso nel dominio ma in $U_r(P)$

perciò $\boxed{U_r(P) \cap D \neq \emptyset}$

• Dimostro che Q è esterno: $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

prendo $U_{\frac{1}{8}} \to A = \left(\frac{1}{2} - \frac{1}{8}, \frac{1}{2}\right)$ $B = \left(\frac{1}{2} + \frac{1}{8}, \frac{1}{2}\right) \to A\left(\frac{3}{8}, \frac{1}{2}\right)$ $B = \left(\frac{5}{8}, \frac{1}{2}\right)$

$C = \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{8}\right)$ $D = \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{8}\right) \to C\left(\frac{1}{2}, \frac{3}{8}\right)$ $D = \left(\frac{1}{2}, \frac{5}{8}\right)$

tutti punti esterni

• Dimostro che R è interno : $0^2+2^2=4$

prendo $U_{\frac{1}{2}}$ -> $A'\left(-\frac{1}{2},2\right)$ $B'\left(\frac{1}{2},2\right)$ tutti punti interni

$C'\left(0,\frac{3}{2}\right)$ $D'=\left(0,\frac{5}{2}\right)$

~ Dimostrare che $\lim_{(x,y)\to(0,0)} f(x,y)=0$ ~

• $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}=0$

$0\leq\left|\frac{xy^2}{x^2+y^2}\right|$ $\quad \left|\frac{xy^2}{x^2+y^2}\right|=\left|\frac{xy\cdot y}{x^2+y^2}\right|=|xy|\cdot\left|\frac{y}{x^2+y^2}\right|$

visto che $D\geq N$, allora $\left|\frac{y}{x^2+y^2}\right||xy|\leq|xy|$

$$\underset{\downarrow}{0}\leq\underset{\downarrow}{\left|\frac{xy^2}{x^2+y^2}\right|}\leq\underset{\downarrow}{|xy|}$$
$$0 \qquad 0 \qquad 0$$

• $\lim_{(x,y)\to(0,0)} \frac{x\sin(y^2)}{y^2}=0$ $\quad 0\leq\left|\frac{x\sin(y^2)}{y^2}\right|=\underset{\substack{\downarrow\\0}}{|x|}\left|\underset{\substack{\downarrow\\1}}{\frac{\sin(y^2)}{y^2}}\right|$

dunque $\lim_{(x,y)\to(0,0)} f(x,y)=0$

• $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sqrt{|xy|}}$

$0\leq\frac{|\sin(xy)|}{\sqrt{|xy|}}=\frac{|\sin(xy)|}{|xy|}\cdot\sqrt{|xy|}=\left|\underset{\substack{\downarrow\\1}}{\frac{\sin(xy)}{xy}}\right|\cdot\underset{\substack{\downarrow\\0}}{\sqrt{|xy|}}$

dunque $\lim_{(x,y)\to(0,0)} f(x,y)=0$

- $$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin(\sqrt{x^2+y^2})}{x^2+y^2}$$

$$0 \le |f(\rho,\vartheta) - 0| = \left|\frac{\rho^2\cos^2(\vartheta)\cdot \sin(\sqrt{\rho^2\cos^2(\vartheta)+\rho^2\sin^2(\vartheta)})}{\rho^2\cos^2(\vartheta)+\rho^2\sin^2(\vartheta)}\right|$$

$$= \left|\frac{\cancel{\rho^2}\cos^2(\vartheta)\cdot \sin(\rho)}{\cancel{\rho^2}}\right| = \cos^2(\vartheta)\cdot|\sin(\rho)| \le |\sin(\rho)|$$

$$0 \le \frac{x^2\sin(\sqrt{x^2+y^2})}{x^2+y^2} \le |\sin(\rho)|$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

- $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

$$0 \le |f(\rho,\vartheta) - 0| \le \left|\frac{\rho\cos(\vartheta)\cdot\rho\sin(\vartheta)}{\rho}\right| = \left|\frac{\rho^{\cancel{2}}\cos(\vartheta)\cdot\sin(\vartheta)}{\cancel{\rho}}\right|$$

$$|\rho\cos(\vartheta)\cdot\sin(\vartheta)| = |\rho|\cdot\underbrace{|\cos(\vartheta)\cdot\sin(\vartheta)|}_{\text{sempre minore di 1}} \le |\rho|$$

dunque se $\rho \to 0$, allora

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$