Formal Aspects of Designing Fractal line Pied-de-poules: The Formula, Its Properties, and The Proof of Its Properties

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1 Introduction

To formalise the fractal line pied-de-poule as described by Feijs and Toeters in their Bridges 2015 paper (archive.bridgesmathart.org/2015/bridges2015-223.pdf) we deploy rules using *two* main formal symbols. These are F and **F** where the former is interpreted as a turtle graphics forward step and where the latter can either expand to a full pied-de-poule-like zigzag line or just act as a forward step. The idea that the turtle writes either thin lines or thick lines (implemented by recursion) is reflected in the typography of these two symbols. For the aesthetic and cultural implications of the formalised constructs, we refer to the forthcoming paper by Loe Feijs, to appear in the Journal of Humanistic Mathematics in 2020.

The principle of zigzagging a pied-de-poule figure is demonstrated in Figure 1 for $N = 1, 2, 3$ and $N = 4$. The same zigzagging can be done for any $N > 0$. Please note that the total effect of the entire zigzagging action is to move the turtle over a fixed vertical distance. This means that each of the one-block diagonal steps could be replaced by one zigzagging. In this way a fractal is obtained. The procedure is not difficult, but requires some tedious formalisation to define precisely what is going on. We aim at a compact formula to describe the fractal. Once we have that, we can ask for the formal properties of the fractal, such in what sense the fractal is related to the original pied-de-poule figure. The goals of the present note are to present such a formalisation and explore some of its formal properties.

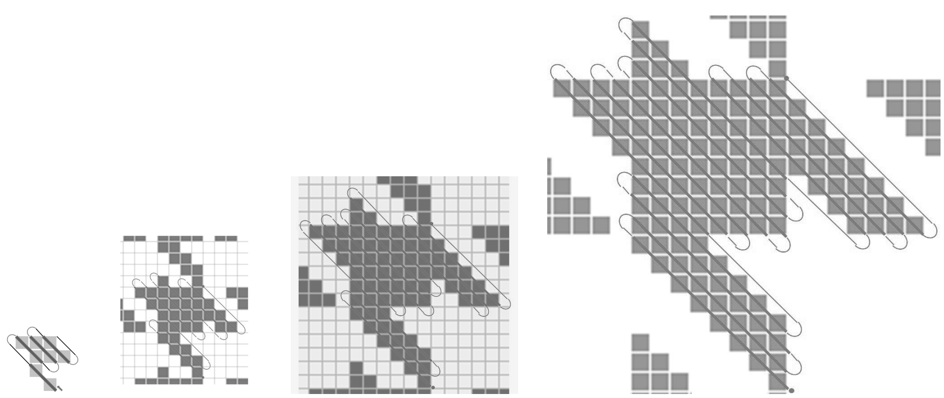

Figure 1: *Drawing the diagonals of a classic pied-de-poule with outer loops drawn with a thinner pen for* $N = 1$ *(left),* $N = 2$*,* $N - 3$ *and* $N = 4$ *(right).*

2 Principles behind the formula for fractal line pied-de-poule

In our case, we need certain symbols such as $+$, $-$ to carry parameters. In particular we interpret $+_\varphi$ as turn right over φ and $-_\varphi$ as turn left over φ. If the $+$ or $-$ have no subscript we take by default φ to be $\pi/4$, that is, 45 degrees. So $+$ abbreviates $+_{\pi/4}$ and $-$ abbreviates $-_{\pi/4}$. In the same way F is interpreted as moving forward over a certain distance, say L. We need two distinct symbols for forward, F being interpreted as the usual forward command of turtle graphics, and **F**, being a formal symbol during the Lindenmayer substitutions, yet interpreted as F when taking an approximating snapshot after a certain number of simultaneous substitutions (in practice we make recursive Processing programs calling Oogway commands (see github.com/iddi/oogway-processing/tree/master/oogway) and then this number is the recursion depth n).

We give additional preparations. As before, we use exponentiation notation for repeated symbols, for example $\mathbf{F}^4$ means **FFFF**. For making half-circles inside zigzags we define R_d be a clockwise half turn with diameter d and L_d be a counter-clockwise half turn with diameter d. These half turns can be approximated using k steps for the half circle as follows:

$$
\begin{array}{rcl}
\mathrm{R}_d & \to & (+_\varphi\, F_{d'} +_\varphi)^k \\
\mathrm{L}_d & \to & (-_\varphi\, F_{d'} -_\varphi)^k
\end{array}
$$

where d' is $d \times \sin(\pi/2k)$, and φ is $\pi/2k$. For example, taking $k = 10$ each $+_\varphi$ means turning 9 degrees. By increasing k this R_d becomes a very good half circle in Lindenmayer notation. Note that these R and L are *not* the basic turtle graphics right and left turn commands (these are denoted by $+$ and $-$ respectively).

We define what it means to execute a path backwards, using negative exponent notation: $(c_1 \cdots c_k)^{-1}$ means ${c_k}^{-1} \cdots {c_1}^{-1}$, $(-_\varphi)^{-1}$ is $+_\varphi$ and $(+_\varphi)^{-1}$ is $-_\varphi$. Similarly ${\mathrm{L}_d}^{-1}$ means R_d, ${\mathrm{R}_d}^{-1}$ means L_d and finally F^{-1} is just F. For Lindenmayer rule $\mathbf{F} \to c_1 \cdots c_k$ tacitly add $\mathbf{F}^{-1} \to (c_1 \cdots c_k)^{-1}$ (treating $\mathbf{F}^{-1}$ as a symbol).

First we focus on the leftmost zigzag of Fig. 1, which is the case $N = 1$. When using recursion to make a single pied de poule figure such that the distance between the begin and end points equals L, we need an equation: $\mathbf{F}_L = -\mathbf{F}_s^4\mathrm{F}_s^3\mathrm{R}_s\mathbf{F}_s^{-4}\mathrm{F}_s\mathrm{L}_s\mathbf{F}_s^4\mathrm{R}_s\mathrm{F}_s\mathbf{F}_s^{-4}\mathrm{L}_s\mathrm{F}_s^3+$ where $s = L\sqrt{2}/8$ (the general rule is $s = L\sqrt{2}/8N$). In the Lindenmayer approach we omit the size parameters and just let the term expand by n-fold rule-application (recursion leven n). It is implicitly understood that the halfcircles have a diameter which equals the stepsize of the adjacent F steps. We give the formulas for $N = 1$, $N = 2$ and $N = 3$ now.

$(N = 1)$ $\mathbf{F} \to -\mathbf{F}^4\mathrm{F}^3\mathrm{R}\mathbf{F}^{-4}\mathrm{F}\mathrm{L}\mathbf{F}^4\mathrm{R}\mathrm{F}\mathbf{F}^{-4}\mathrm{L}\mathrm{F}^3+$

$(N = 2)$ $\mathbf{F} \to -\mathbf{F}^8\mathrm{F}\mathrm{R}\mathbf{F}^{-8}\mathrm{L}\mathrm{F}^5\mathbf{F}^8\mathrm{R}\mathrm{F}\mathbf{F}^{-8}\mathrm{F}\mathrm{L}\mathbf{F}^8\mathrm{F}\mathrm{R}\mathbf{F}^{-8}\mathrm{F}^3\mathrm{L}\mathbf{F}^8\mathrm{R}\mathrm{F}_s\mathbf{F}^{-8}\mathrm{L}\mathrm{F}^7+$

$(N = 3)$ $\mathbf{F} \to -\mathbf{F}^{12}\mathrm{F}\mathrm{R}\mathbf{F}^{-12}\mathrm{L}\mathrm{F}\mathbf{F}^{12}\mathrm{F}^7\mathrm{R}\mathbf{F}^{-12}\mathrm{F}\mathrm{L}\mathbf{F}^{12}\mathrm{R}\mathrm{F}\mathbf{F}^{-12}\mathrm{F}\mathrm{L}\mathbf{F}^{12}\mathrm{F}\mathrm{R}\mathbf{F}^{-12}\mathrm{L}\mathrm{F}\mathbf{F}^{12}\mathrm{R}\mathrm{F}^5\mathbf{F}^{-12}\mathrm{F}\mathrm{L}\mathbf{F}^{12}\mathrm{R}\mathrm{F}\mathbf{F}^{-12}\mathrm{L}\mathrm{F}^{11}+$

Formulas like these are easy to read as zigzags. Each $\mathbf{F}_s^4$ is a "zig" and each $\mathbf{F}_s^{-4}$ is a zag. For the orientation adopted throughout all the drawings such as Fig. 2, a zig goes up, a zag goes down. Everything else builds the outer loops that connects the zigs and the zags.

Now we want a formula for arbitrary N. We make a number of observations. First, the main skeleton of the formula has to be a "$-$" followed by N of $\mathbf{F}^{4N}$ or $\mathbf{F}^{-4N}$, alternating, followed by one final "$+$". Between each zig $\mathbf{F}^{4N}$ and its subsequent zag $\mathbf{F}^{-4N}$ there is one R turn. Between each zag $\mathbf{F}^{-4N}$ and its subsequent zig $\mathbf{F}^{4N}$ there is one L turn. Moreover there is always an F, either immediately before, or after each turn: mostly it is a single F, but we also see a few multistep forward moves, such as the F^3s for $N = 1$, the F^3, F^5, F^7 for $N = 2$, and the F^5, F^7, F^{11} for $N = 3$.

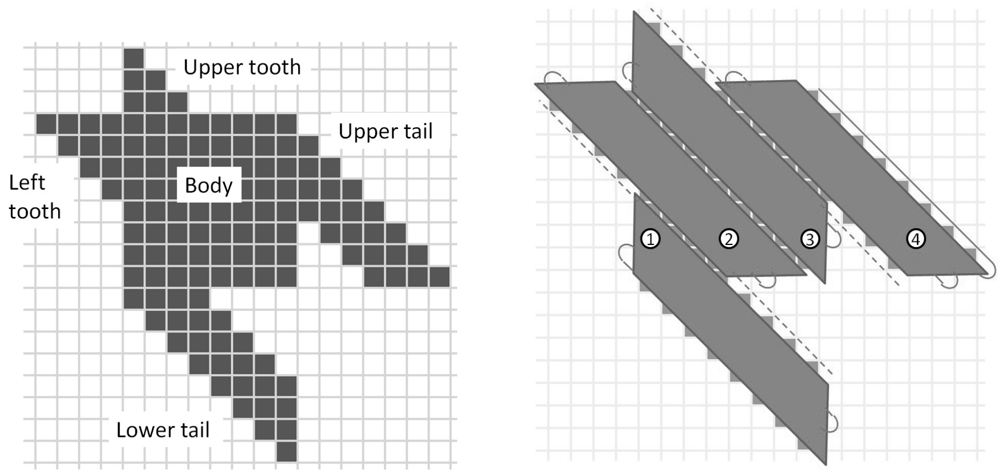

Figure 2: *Main geometric parts of the classic pied-de-poule figure.*

In order to discuss the classic pied de poule figure, which has to be zigzagged, we need some terminology, which is given in Fig. 2 (left). We label the four main areas as ①, ②, ③ and ④ as in Fig. 2. The zigzags have to cover these four areas, in the given order, and connect them. These are the four areas:

① the lower tail of the classic pied de poule (including the lower left body-corner),
② most of the leftmost (diagonally cut) half of the body together with the left tooth,
③ most of the rightmost (diagonally cut) half of the body together with the upper tooth,
④ the upper tail of the classic pied de poule (including the upper right body-corner).

Our first formal formula describes the skeleton of our Lindenmayer system:

$$\mathbf{F} \to -\overbrace{\underbrace{\mathbf{F}^{4N}\mathbf{F}^{-4N}}_{\textcircled{1}} \;\; \underbrace{\qquad}_{\textcircled{2}} \cdots \underbrace{\qquad}_{\textcircled{3}} \;\; \underbrace{\mathbf{F}^{4N}\mathbf{F}^{-4N}}_{\textcircled{4}}}^{4N \text{ of } \mathbf{F}^{4N} \text{ or } \mathbf{F}^{-4N}\text{, alternating}} \; \mathrm{LF}^{4N-1}+ \qquad \leftarrow \text{four parts of equal size}$$

The underbraced parts of the formula correspond to the four main geometric parts of the classic pied de poule figure. We assume the parts label becomes a formal part of the symbols such as $\mathbf{F}_{\textcircled{1}}$, $\mathbf{F}_{\textcircled{2}}$, etcetera. We avoid having different formulas for odd and even N and therefore we adopt the following logic for reading the above skeleton: there is a strict alternation between $\mathbf{F}^{4N}$ and $\mathbf{F}^{-4N}$ and in case of odd N the subdivision into the four groups works such that groups ② and ④ actually begin with an $\mathbf{F}^{-4N}$. For example for $N = 1$ the skeleton amounts to $-\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{4}}^{-1}\mathbf{F}_{\textcircled{4}}^{-1}\mathbf{F}_{\textcircled{4}}^{-1}\mathbf{F}_{\textcircled{4}}^{-1}\mathrm{F}^{3}+$. This gives the skeleton; now we must explore the missing details.

- inside parts ① and ③ there is a FR between every $\mathbf{F}$ and $\mathbf{F}^{-1}$ (a loop at a leftmost vertical wall);
- inside parts ① and ③ there is a LF between every $\mathbf{F}^{-1}$ and $\mathbf{F}$ (a loop at a rightmost vertical wall);
- inside parts ② and ④ there is a RF between every $\mathbf{F}$ and $\mathbf{F}^{-1}$ (a loop at a upper horizontal wall);
- inside parts ② and ④ there is a FL between every $\mathbf{F}^{-1}$ and $\mathbf{F}$ (a loop at a lower horizontal wall).

The transitions *between* the parts are trickier. The situation depends on whether the transition happens as a zig-zag type or as a zag-zig type. By inspection of Figs. 2 we find the following principles:

- between every $\mathbf{F}$ and $\mathbf{F}^{-1}$ precisely one turn R occurs, between $\mathbf{F}^{-1}$ and $\mathbf{F}$ one L.

- between ① and ② there is an upward-going outer loop having $2N+1$ steps.
- the transition from ② to ③ is of zag-zig type having always an FL transition;
- between ③ and ④ there is a downward-going outer loop which has $2N-1$ steps.

3 Coding and testing the formula

We code the above principles into the following substitution rules:

$$
\begin{array}{lll}
\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{1}}^{-1} := \mathbf{F}\mathrm{FR}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{1}}^{-1}\mathbf{F}_{\textcircled{1}} := \ \mathbf{F}^{-1}\mathrm{LF}\mathbf{F} & \text{(inside area ①)} \\
\mathbf{F}_{\textcircled{2}}\mathbf{F}_{\textcircled{2}}^{-1} := \mathbf{F}\mathrm{RF}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{2}} := \ \mathbf{F}^{-1}\mathrm{FL}\mathbf{F} & \text{(inside area ②)} \\
\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{3}}^{-1} := \mathbf{F}\mathrm{FR}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{3}}^{-1}\mathbf{F}_{\textcircled{3}} := \ \mathbf{F}^{-1}\mathrm{LF}\mathbf{F} & \text{(inside area ③)} \\
\mathbf{F}_{\textcircled{4}}\mathbf{F}_{\textcircled{4}}^{-1} := \mathbf{F}\mathrm{RF}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{4}}^{-1}\mathbf{F}_{\textcircled{4}} := \ \mathbf{F}^{-1}\mathrm{FL}\mathbf{F} & \text{(inside area ④)} \\
\mathbf{F}_{\textcircled{1}}\mathbf{F}_{\textcircled{2}}^{-1} := \mathbf{F}\mathrm{F}^{2N+1}\mathrm{R}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{1}}^{-1}\mathbf{F}_{\textcircled{2}} := \ \mathbf{F}^{-1}\mathrm{LF}^{2N+1}\mathbf{F} & \text{(from ① to ②)} \\
 & \mathbf{F}_{\textcircled{2}}^{-1}\mathbf{F}_{\textcircled{3}} := \ \mathbf{F}^{-1}\mathrm{FL}\mathbf{F} & \text{(from ② to ③)} \\
\mathbf{F}_{\textcircled{3}}\mathbf{F}_{\textcircled{4}}^{-1} := \mathbf{F}\mathrm{RF}^{2N-1}\mathbf{F}^{-1} & \mathbf{F}_{\textcircled{3}}^{-1}\mathbf{F}_{\textcircled{4}} := \ \mathbf{F}^{-1}\mathrm{F}^{2N-1}\mathrm{L}\mathbf{F} & \text{(from ③ to ④)} \\
\mathbf{F}_{\textcircled{x}}\mathbf{F}_{\textcircled{x}} := \mathbf{F}\mathbf{F} & \mathbf{F}_{\textcircled{x}}^{-1}\mathbf{F}_{\textcircled{x}}^{-1} := \ \mathbf{F}^{-1}\mathbf{F}^{-1} & \text{(inside all areas)}
\end{array}
$$

These fifteen formal rules together with the skeleton rule define the fractal line pied de poule as a single formula for all positive integer N.

We want to test it now for $N=4$. The skeleton rule gives us $4N$ of $\mathbf{F}^{4N}$ or $\mathbf{F}^{-4N}$, alternatingly, which is

$$\mathbf{F} \rightarrow -\underbrace{\mathbf{F}^{16}\mathbf{F}^{-16}\mathbf{F}^{16}\mathbf{F}^{-16}}_{\textcircled{1}}\,\underbrace{\mathbf{F}^{16}\mathbf{F}^{-16}\mathbf{F}^{16}\mathbf{F}^{-16}}_{\textcircled{2}}\,\underbrace{\mathbf{F}^{16}\mathbf{F}^{-16}\mathbf{F}^{16}\mathbf{F}^{-16}}_{\textcircled{3}}\,\underbrace{\mathbf{F}^{16}\mathbf{F}^{-16}\mathbf{F}^{16}\mathbf{F}^{-16}}_{\textcircled{4}}\mathrm{LF}^{15}+\ =$$

$$-\mathbf{F}_{\textcircled{1}}^{16}\mathbf{F}_{\textcircled{1}}^{-16}\mathbf{F}_{\textcircled{1}}^{16}\mathbf{F}_{\textcircled{1}}^{-16}\mathbf{F}_{\textcircled{2}}^{16}\mathbf{F}_{\textcircled{2}}^{-16}\mathbf{F}_{\textcircled{2}}^{16}\mathbf{F}_{\textcircled{2}}^{-16}\mathbf{F}_{\textcircled{3}}^{16}\mathbf{F}_{\textcircled{3}}^{-16}\mathbf{F}_{\textcircled{3}}^{16}\mathbf{F}_{\textcircled{3}}^{-16}\mathbf{F}_{\textcircled{4}}^{16}\mathbf{F}_{\textcircled{4}}^{-16}\mathbf{F}_{\textcircled{4}}^{16}\mathbf{F}_{\textcircled{4}}^{-16}\mathrm{LF}^{15}+\ =$$

$$-\mathbf{F}^{16}\mathrm{FR}\mathbf{F}^{-16}\mathrm{LF}\mathbf{F}^{16}\mathrm{FR}\mathbf{F}^{-16}\mathrm{LF}^{9}\mathbf{F}^{16}\mathrm{RF}\mathbf{F}^{-16}\mathrm{FL}\mathbf{F}^{16}\mathrm{RF}\mathbf{F}^{-16}\mathrm{FL}\mathbf{F}^{16}\mathrm{FR}\mathbf{F}^{-16}\mathrm{LF}\mathbf{F}^{16}\mathrm{FR}\mathbf{F}^{-16}\mathrm{F}^{7}\mathrm{L}\mathbf{F}^{16}\mathrm{RF}\mathbf{F}^{-16}\mathrm{FL}\mathbf{F}^{16}\mathrm{RF}\mathbf{F}^{-16}\mathrm{LF}^{15}+$$

It is a tedious but easy exercise to verify that this describes precisely the upward path shown in Fig. 1 (right).

4 Alternative representation of the formula

The following alternative representation of the formula distinguishes even and odd cases:

$$
\mathbf{F} \rightarrow \begin{cases}
-(\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF})^{N/2-1}\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF}^{2N+1}(\mathbf{F}^{4N}\mathrm{RF}\mathbf{F}^{-4N}\mathrm{FL})^{N/2} & \\
\quad(\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF})^{N/2-1}\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF}^{2N-1}(\mathbf{F}^{4N}\mathrm{RF}\mathbf{F}^{-4N}\mathrm{FL})^{N/2-1}\mathbf{F}^{4N}\mathrm{RF}\mathbf{F}^{-4N}\mathrm{LF}^{4N-1}+ & \text{(even } N\text{)} \\
-(\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF})^{(N-1)/2}\mathbf{F}^{4N}\mathrm{F}^{2N+1}\mathrm{R}(\mathbf{F}^{-4N}\mathrm{FL}\mathbf{F}^{4N}\mathrm{RF})^{(N-1)/2}\mathbf{F}^{-4N}\mathrm{FL} & \\
\quad(\mathbf{F}^{4N}\mathrm{FR}\mathbf{F}^{-4N}\mathrm{LF})^{(N-1)/2}\mathbf{F}^{4N}\mathrm{RF}^{2N-1}(\mathbf{F}^{-4N}\mathrm{FL}\mathbf{F}^{4N}\mathrm{RF})^{(N-1)/2}\mathbf{F}^{-4N}\mathrm{LF}^{4N-1}+ & \text{(odd } N\text{)}
\end{cases}
$$

5 Formal properties

Intuitively we can say that the outer loops are a minor thing, but can we prove it in a formal sense? We shall present two theorems doing precisely that. First we need some preliminaries. We write PDP as an abbreviation of "classic pied-de-poule". We say that a set $\mathscr{P} \subseteq \mathbb{R}^2$ is a PDP of type N if it has been constructed according to the methods mentioned in Section 1 and detailed in Feijs' Bridges 2012 paper entitled: Geometry and Computation of Pied-de-poule (houndstooth), see archive.bridgesmathart.org/2012/bridges2012-299.html. Such a $\mathscr{P}$ is the union of $8N^2$ non-overlapping square regions of width d for some $d \in \mathbb{R}$. We call d the *grid size*. We say that the *size* of a PDP $\mathscr{P}$, denoted by $\mathsf{size}(\mathscr{P})$ is the width of the smallest square box which is aligned with the grid and which encloses $\mathscr{P}$. It equals $5N-1$. We write flPDP for "fractal line pied-de-poule approximation" and we say that a set $\mathscr{F} \subseteq \mathbb{R}^2$ is an flPDP of type N and recursion level n if it has been constructed according to the methods defined in Section 1. Let $X \subseteq \mathbb{R}^2$ be an arbitrary set then we define the *ε-fattening* of X, denoted as $\lceil X \rceil^{\varepsilon}$ to be a set like X, but with a band of size ε added all around it. Formally $\lceil X \rceil^{\varepsilon} = \{p \in \mathbb{R}^2 \mid \exists q \in X \cdot \|q - p\| \leq \varepsilon\}$ where $\| \ \|$ is the Euclidean length. For each PDP $\mathscr{P}$ of type N, let $\mathscr{F}_n(\mathscr{P})$ be the flPDP which runs though the diagonals of $\mathscr{P}$ and has recursion level n according to the methods defined in Section 1.

Theorem Let $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \ldots$ be a sequence of PDPs of type $1, 2, 3, \ldots$ and fixed $\mathsf{size}(\mathscr{P}_N) = s$ for all integer $N > 0$. The $\mathscr{P}_N$ thus have shrinking grid sizes $s/4$, $s/9$, $s/14, \ldots$. Then for each $n \in \mathbb{N}$ there is a sequence of corresponding flPDPs $\mathscr{F}_n(\mathscr{P}_1)$, $\mathscr{F}_n(\mathscr{P}_2)$, $\mathscr{F}_n(\mathscr{P}_3), \ldots$ and a sequence of real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ such that for all $N > 0$ we have $\mathscr{F}_n(\mathscr{P}_N) \subseteq \lceil \mathscr{P}_N \rceil^{\varepsilon_N}$ and

$$\lim_{N \to \infty} \frac{\varepsilon_N}{\mathsf{size}(\mathscr{P}_N)} = 0$$

Proof. The outer loops (see Figure 1) protrude from the classic figure by a distance of at most the diameter of the arcs, which is the distance between adjacent diagonals, viz. $\frac{1}{2}\sqrt{2}d_N$ where d_N is the grid size. The outer loops of the inner recursive figures also protrude (check in Figure **??**, for example), but they protrude even less than one arc diameter. Therefore we can take $\varepsilon_N = \frac{1}{2}\sqrt{2}d_N = \frac{1}{2}\sqrt{2}(\frac{s}{5N-1})$, which vanishes for large N. In other words: adding a vanishing ε-fattening band around the classic pied-de-poule is enough to let it cover the fractal line pied-de-poule approximation, including the protruding outer loops. In the same way we find that the other out-of-the box violations, e.g. along the "staircase" boundaries, are vanishing. **Q.E.D.**

The idea of the theorem is presented in Fig. 3.

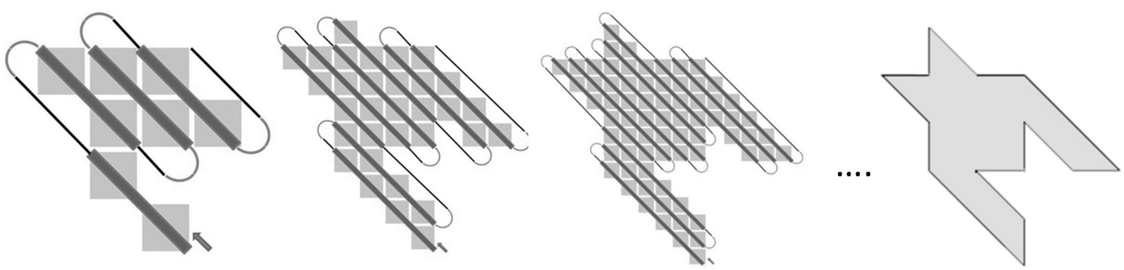

Figure 3: *Illustration of the first theorem.*

So we can neglect the outer loops for large N, but what about fixed N? The next theorem says that the outer loops are neglectable anyhow as they have infinitesimal thickness. If we make a practical drawing of an flPDP $\mathscr{F}$, the mathematical line has to become visible. Whenever we use a pen of a certain stroke width, we are in fact making an ε-fattening $\lceil \mathscr{F} \rceil^{\varepsilon}$ where ε is half the pen stroke width. How wide such pen stroke can be is naturally limited by the condition that adjacent strokes still have sufficient white space in between – even inside the smallest recursive figures embedded in $\mathscr{F}$. Without loss of generality we interpret 'sufficient white space' to mean that the white space is equally wide as the pen strokes themselves.

Theorem Let $\mathscr{F}^0, \mathscr{F}^1, \mathscr{F}^2, \ldots$ be a sequence of flPDPs of increasing recursion level $n = 0, 1, 2, \ldots$. and such that the $\mathscr{F}^n$ all run through the diagonals of a single PDP $\mathscr{P}$ of given type N and given size. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ be the sequence of values in $\mathbb{R}$ (half stroke widths) such that for each $n \in \mathbb{N}$ the band of points between the adjacent diagonal strokes of the smallest recursive figures embedded in $\mathscr{F}^n$ is equally wide as the diagonal strokes $\lceil \mathscr{F}^n \rceil^\varepsilon$ themselves, viz. $2\varepsilon_n$. Then

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\mathsf{size}(\mathscr{P})} = 0$$

Proof. The distance between adjacent diagonal lines in $\mathscr{F}^n$ is $\frac{1}{2}\sqrt{2}\,d_n$, so we should take $\varepsilon_n = \frac{1}{4}\sqrt{2}\,d_n$ where d_n is the grid size of the smallest recursive figure embedded in $\mathscr{F}^n$. The d_n are determined by the grid size at the top level $d_0 = \mathsf{size}(\mathscr{P})/(5N-1)$ and by an overall scaling factor which is $(\sqrt{2}/8N)^n$. Therefore $\varepsilon_n/\mathsf{size}(\mathscr{P}) = \frac{1}{4}\sqrt{2}(5N-1)^{-1}(\sqrt{2}/8N)^n$, which vanishes for large n. **Q.E.D.**

The idea of the second theorem is presented in Fig. 4.

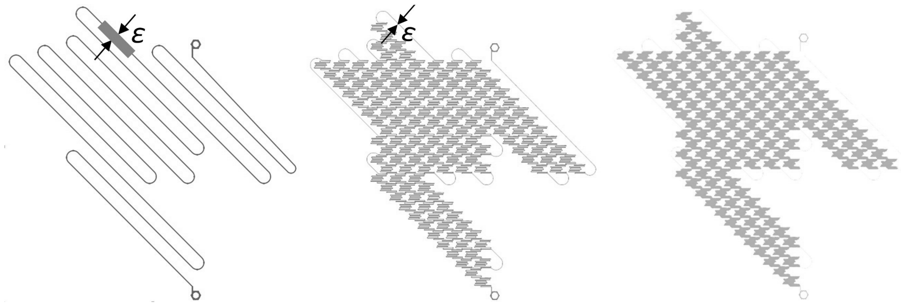

Figure 4: *Illustration of the second theorem.*