Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Since $n > 2$,
   $$n! > n! - 1 > 2n - 1 > 2n - n = n > 2$$
   Then there exists a prime factor p for the positive integer $n! - 1$. We consider this prime factor $p \le n! - 1 < n!$ and prove it to be the specified prime satisfying $p > n$ as well.

   Prove by contradiction. Assume the constrcted prime $p \le n$. As $n! = \Pi_{i=1}^{n} i$ contains all the integer factors less than or equal to n,
   $$p \mid n!$$
   However, it is constrcuted that
   $$p \mid n! - 1$$
   so that
   $$p = 1$$
   because $-1 \not\equiv 0 \pmod p \Rightarrow n! - 1 \not\equiv n! \pmod p$ when $p > 1$. This contradicts the fact that p is a prime and the assumption does not hold.

   As a result, The requested prime p satisfying $n < p < n!$ is found and the proposition holds. □

2. Use the minimal counterexample principle to prove that for any integer $n \ge 7$, there exists integers $i_n \ge 0$ and $j_n \ge 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. Construct the minimal counterexample set. It is to be proved that $\varnothing = S \subset A = \{n \in \mathbb{N} \mid n \ge 7\}$ of the numbers $n \in A$ for which $\forall i \ge 0, j \ge 0, n \ne i \times 2 + j \times 2$. **Prove by contradiction.** Assume $S \ne \varnothing$, then S has a least element followed by well-ordering principle. Let $m \in S$ be the least element.

   It is observed that $7 = 2 \times 2 + 1 \times 3, 8 = 4 \times 2, 9 = 3 \times 3$, which results in $m \ge 9$. Consider the number $m - 2 \ge 7$. If $m - 2 \notin S$, $\exists i_{m-2}, j_{m-2} \in \mathbb{N}$ such that
   $$m - 2 = i_{m-2} \times 2 + j_{m-2} \times 3$$
   which leads to
   $$m = (i_{m-2} + 1) \times 2 + j_{m-2} \times 3$$
   and let $i_m \leftarrow i_{m-2} + 1, j_m \leftarrow j_{m-2}$ so that $i_m, j_m \in \mathbb{N}$ are constructed to satisfy $m = i_m \times 2 + j_m \times 3$, i.e., $m \notin S$, which could not be true as m is the element in S as provided.

   Then, $m - 2 \in S$. However, $7 \le m - 2 < m$ contradicts m is the least number in S. So the assumption fails to hold and $S = \varnothing$. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \ge 2$. Use the strong principle of mathematical induction to prove that for all $n \in \mathbb{N}$, $f(n) = 3^n - 2^n$.

Proof. Prove by strong principle of mathematical induction. To show that $n \in \mathbb{N}$, $P(n) : f(n) = 3^n - 2^n$ holds.

Basic step. As defined,

$$f(0) = 0 = 3^0 - 2^0$$
$$f(1) = 1 = 3^1 - 2^1$$

leads to $P(0)$ and $P(1)$ are correct.

Induction hypothesis. $k \geq 2$, and $P(n)$ holds for every $0 \leq n \leq k$.

Statements to be shown in induction step. $P(k+1)$ holds.

Proof of induction step. As it is defined that $f(n) = 5f(n-1) - 6f(n-2)$ for $n \geq 2$,

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5\left(3^k - 2^k\right) - 6\left(3^{k-1} - 2^{k-1}\right) \\ &= (15-6) \times 3^{k-1} - (10-6) \times 2^{k-1} \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

Therefore, $P(k+1)$ holds.

By the strong principle of mathematical induction, $\forall n \in \mathbb{N}$, $f(n) = 3^n - 2^n$. $\square$

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a *champion* team that beats every other team, or there is a 3-cycle.

Proof.

Definition 1 (beat)**.** *Denote $p \leftarrow q$ iff p beats team q; denote $q \rightarrow p$ iff q does not beat p.*

Lemma 1 (ranking)**.** *There is always a ranking $p_1, p_2, \cdots, p_n$ for n-team tournament, where $n \geq 2$.*

Proof of Lemma 1. Construct the ranking sequence by **mathematical induction** and inserting. To show that $n \in \{n \in \mathbb{N} \mid n \geq 2\}$, it is correct that $Q(n)$: there is always a ranking for the n-team.

Basic step. When $n = 2$,

$$\begin{aligned} p_1 &\leftarrow p_2 & \quad & p_1, p_2 \text{ is a ranking} \\ \text{or } p_2 &\leftarrow p_1 & & p_2, p_1 \text{ is a ranking} \end{aligned}$$

which shows $Q(2)$ holds.

Induction hypothesis. $k \geq 3$, $Q(k)$ holds.

Statements to be shown in induction step. $Q(k+1)$ holds.

Proof of induction step. Consider inserting p_{k+1} into the remaining k-team ranking $p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_k$. Begin inserting from p_1, if $p_{k+1} \leftarrow p_1$, then $p_{k+1} \leftarrow p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_k$ is the new ranking. Otherwise $p_1 \leftarrow p_{k+1}$, compare p_2 with p_{k+1}, if $p_{k+1} \leftarrow p_2$, then $p_1 \leftarrow p_{k+1} \leftarrow p_2 \leftarrow \cdots \leftarrow p_k$ is the new ranking and so on. The inserting algorithm could be interpreted as follows:

Algorithm 1: Construct New Ranking by Inserting

Input: A k-team ranking $p_1, p_2, \cdots, p_k$, a new team p_{k+1}
Output: New $(k+1)$-team ranking

if $p_{k+1} \leftarrow p_1$ **then**
    return $p_{k+1} \leftarrow p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_k$;
for $i = 2$ *to* k **do**
    if $p_{k+1} \leftarrow p_i$ **then**
        return $p_1 \leftarrow \cdots \leftarrow p_{i-1} \leftarrow p_{k+1} \leftarrow p_i \leftarrow \cdots \leftarrow p_k$;
return $p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_k \leftarrow p_{k+1}$;

By such, we construct the new ranking and $Q(k+1)$ holds.

By the principle of mathematical induction, the Lemma 1 holds. $\square$

Lemma 2 (champion)**.** *If there is a* champion *team among the ranking* $p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_n$*, the* champion *team must be the head one, namely* p_1*.*

Proof of Lemma 2. Prove by contradiction. Assume the *champion* team is p_i other than p_1, i.e., $2 \leq i \leq k$. Then there always exists a no-beating senario:

$$p_{i-1} \leftarrow p_i$$

which contradicts the definition of beating. The assumption fails to hold and the Lemma 2 is correct. $\square$

Back to the original problem. According to Lemma 1, there always exists a ranking for n-team tournament:

$$p_1 \leftarrow p_2 \leftarrow \cdots \leftarrow p_n$$

Case 1: A *champion* team. If there is a *champion* team, it must be p_1 according to Lemma 2.

Case 2: A 3-cycle. Otherwise, no one could be the champion, which indicates that there exists p_m such that

$$p_1 \rightarrow p_m$$

where $3 \leq m \leq n$ because if $m = 2$, the following statements are contradictory:

$$\begin{aligned} &p_1 \rightarrow p_2 && \text{(assumption)} \\ &p_1 \leftarrow p_2 && \text{(ranking)} \end{aligned}$$

By recusively investigating p_{m-1}, a 3-cycle could be constructed. If $p_{m-1} \rightarrow p_1$, then there is a 3-cycle p_1, p_{m-1}, p_m shown as follows:

$$\begin{tikzcd} p_1 \arrow[rrr, bend right] & \cdots \arrow[l] & p_{m-1} \arrow[l] \arrow[ll, bend right] & p_m \arrow[l] & \cdots p_n \arrow[l] \end{tikzcd}$$

Otherwise, investigate p_{m-2} and attempt to constrct the 3-cycle p_{m-2}, p_{m-1}, p_1. Until p_3 is under investigating and now p_1, p_2, p_3 must be a 3-cycle.

$$\begin{tikzcd} p_1 \arrow[rr, bend right] & p_2 \arrow[l] & p_3 \arrow[l] & \cdots \arrow[l] & p_n \arrow[l] \end{tikzcd}$$

The procedure above could be summarized by the following algorithm.

Algorithm 2: Construct 3-cycle by Shrinking the Range

Input: A k-team ranking $p_1, p_2, \cdots, p_k$, a winner p_m that $p_1 \to p_m$
Output: A 3-cycle

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for j = m to 3 do
    if p_{j-1} → p_1 then
        return p_1, p_{j-1}, p_j;
return p_1, p_2, p_3;
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As a result, there must be a 3-cycle in this case.

□

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