Lab10-Turing Machine

CS214-Algorithm and Complexity, Xiaofeng Gao & Lei Wang, Spring 2021.

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1. Design a one-tape TM M that computes the function $f(x, y) = \lfloor x/y \rfloor$, where x and y are positive integers ($x > y$). The alphabet is $\{1, 0, \square, \rhd, \lhd\}$, and the inputs are x "1"s, $\square$ and y "1"s. Below is the initial configuration for input $x = 7$ and $y = 3$. The result $z = f(x, y)$ should also be represented in the form of z "1"s on the tape with pattern of $\rhd 111 \cdots 111 \lhd$, which is $\rhd 11 \lhd$ for the example.

Initial Configuration

	$\rhd$	1	1	1	1	1	1	1	$\square$	1	1	1	$\lhd$
	$\uparrow$ q_S												

(a) Please describe your design and then write the specifications of M in the form like $\langle q_S, \rhd\rangle \to \langle q_1, \rhd, R\rangle$. Explain the transition functions in detail.

Solution. This solution will eliminate y by x and output one bit once one round is completed. In the final state, the tape will be cleaned for output.

$\langle q_S, \rhd\rangle \to \langle q_1, \rhd, R\rangle$	Start state
$\langle q_1, 1\rangle \to \langle q_1, 1, R\rangle$	Skip x
$\langle q_1, \square\rangle \to \langle q_2, \square, R\rangle$	At the split of x and y
$\langle q_2, 1\rangle \to \langle q_3, 0, L\rangle$	Begin eliminating on y
$\langle q_3, \square\rangle \to \langle q_4, \square, L\rangle$	Spliter on x is detected
$\langle q_4, 0\rangle \to \langle q_4, 0, L\rangle$	Skip the eliminated bit on x
$\langle q_4, 1\rangle \to \langle q_5, 0, R\rangle$	Eliminate on x
$\langle q_4, \rhd\rangle \to \langle q_t, \square, R\rangle$	Eliminating on x is completed
$\langle q_5, 0\rangle \to \langle q_5, 0, R\rangle$	Skip the eliminated bit on x
$\langle q_5, \square\rangle \to \langle q_5, \square, R\rangle$	Spliter on y is detected
$\langle q_5, 1\rangle \to \langle q_3, 0, L\rangle$	Continue eliminating on y
$\langle q_5, \lhd\rangle \to \langle q_6, \lhd, R\rangle$	Finish eliminating on y
$\langle q_6, 1\rangle \to \langle q_6, 1, R\rangle$	Skip the outputed bit
$\langle q_6, \square\rangle \to \langle q_7, 1, L\rangle$	Output the result bit
$\langle q_7, 1\rangle \to \langle q_7, 1, L\rangle$	Returning to y
$\langle q_7, \lhd\rangle \to \langle q_8, \lhd, L\rangle$	Spliter on y is detected
$\langle q_8, 0\rangle \to \langle q_8, 1, L\rangle$	Flush the digit of y to original state
$\langle q_8, \square\rangle \to \langle q_2, \square, R\rangle$	Begin eliminating on y
$\langle q_t, 0\rangle \to \langle q_t, \square, R\rangle$	Replacing x, y to empty
$\langle q_t, \square\rangle \to \langle q_t, \square, R\rangle$	Spliter on y is detected
$\langle q_t, 1\rangle \to \langle q_t, \square, R\rangle$	Replacing y to empty
$\langle q_t, \lhd\rangle \to \langle q_f, \rhd, R\rangle$	Spliter on result is detected
$\langle q_f, 1\rangle \to \langle q_f, 1, R\rangle$	Skip the result bit
$\langle q_f, \square\rangle \to \langle q_H, \lhd, S\rangle$	Place the terminating symbol

$\square$

(b) Please draw the state transition diagram.

Solution. The state transition diagram is shown in Figure 1. $\square$

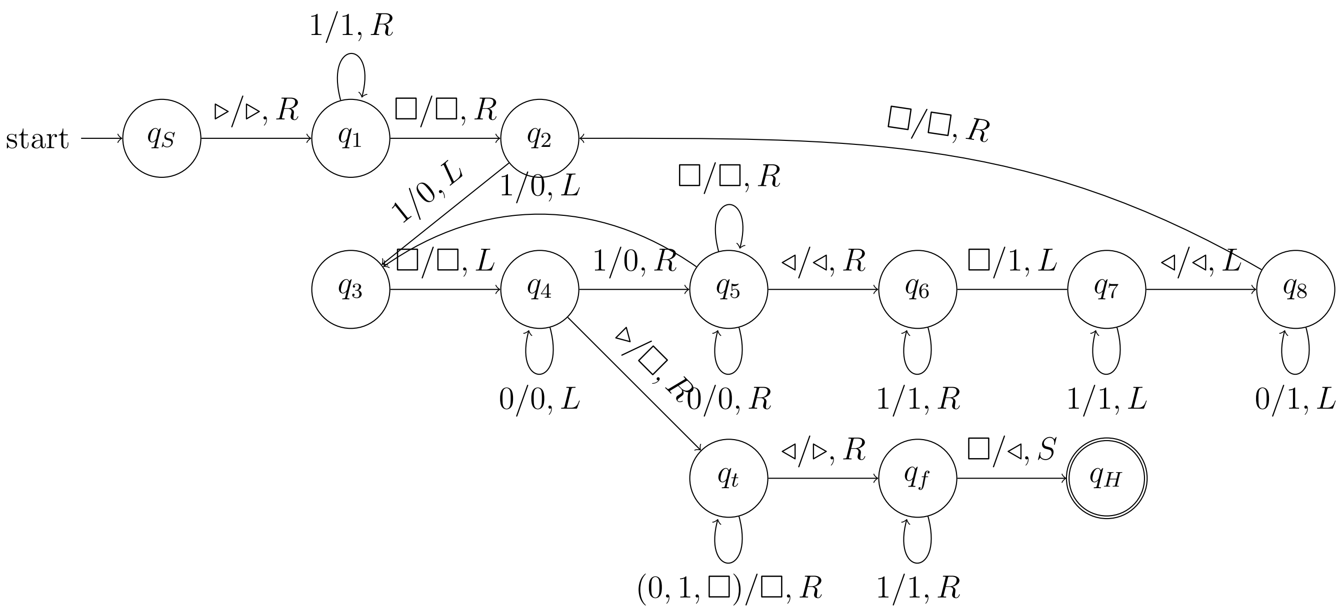

Figure 1: The state transition diagram

(c) Show briefly and clearly the whole process from initial to final configurations for input $x = 7$ and $y = 3$. You may start like this:

$(q_s, \underline{\triangleright}1111111\square111\triangleleft) \vdash (q_1, \triangleright\underline{1}111111\square111\triangleleft) \vdash^* (q_1, \triangleright1111111\underline{\square}111\triangleleft) \vdash (q_2, \triangleright1111111\square\underline{1}11\triangleleft)$

(Note that for simplicity, we write $(q_1, \triangleright\underline{1}111111\square111\triangleleft) \vdash^* (q_1, \triangleright1111111\underline{\square}111\triangleleft)$ if the corresponding transaction repeats on multiple inputs with the same state.)

Solution.

$$\begin{aligned}
(q_s, \underline{\triangleright}1111111\square111\triangleleft) &\vdash (q_1, \triangleright\underline{1}111111\square111\triangleleft) \vdash (q_1, \triangleright1111111\underline{\square}111\triangleleft) \\
&\rightarrow (q_2, \triangleright1111111\square\underline{1}11\triangleleft) \rightarrow (q_3, \triangleright1111111\underline{\square}011\triangleleft) \\
&\rightarrow (q_4, \triangleright111111\underline{1}\square011\triangleleft) \rightarrow (q_5, \triangleright1111110\underline{\square}011\triangleleft) \\
&\vdash (q_5, \triangleright1111110\square0\underline{1}1\triangleleft) \rightarrow (q_3, \triangleright1111110\square\underline{0}01\triangleleft) \\
&\vdash (q_3, \triangleright1111100\square0\underline{0}0\triangleleft) \vdash (q_5, \triangleright1111000\square000\underline{\triangleleft}) \\
&\rightarrow (q_6, \triangleright1111000\square000 \triangleleft \underline{\square}) \rightarrow (q_7, \triangleright1111000\square000\underline{\triangleleft}1) \\
&\rightarrow (q_8, \triangleright1111000\square00\underline{0} \triangleleft 1) \rightarrow (q_8, \triangleright1111000\square0\underline{0}1 \triangleleft 1) \\
&\vdash (q_8, \triangleright1111000\underline{\square}111 \triangleleft 1) \rightarrow (q_2, \triangleright1111000\square\underline{1}11 \triangleleft 1) \\
&\vdash (q_2, \triangleright1000000\square\underline{1}11 \triangleleft 11) \rightarrow (q_3, \triangleright1000000\underline{\square}011 \triangleleft 11) \\
&\vdash (q_4, \triangleright\underline{1}000000\square011 \triangleleft 11) \vdash (q_4, \underline{\triangleright}0000000\square001 \triangleleft 11) \\
&\rightarrow (q_t, \square\underline{0}000000\square001 \triangleleft 11) \vdash (q_t, \square\square\square\square\square\square\square\square\square\square\square\square\square\underline{\triangleleft}11) \\
&\rightarrow (q_f, \square\square\square\square\square\square\square\square\square\square\square\square\square \triangleright \underline{1}1) \vdash (q_f, \square\square\square\square\square\square\square\square\square\square\square\square\square \triangleright 11\underline{\square}) \\
&\rightarrow (q_H, \square\square\square\square\square\square\square\square\square\square\square\square\square \triangleright 11\underline{\triangleleft})
\end{aligned}$$

$\square$

2. Given the alphabet $\{1, 0, \Box, \triangleright, \triangleleft\}$, design a time efficient 3-tape TM M to compute $f : \{0,1\}^* \rightarrow \{0,1\}$ which verifies whether the number of 0 and the number of 1 are the same in an input consisting of only 0's and 1's. M should output 1 if the numbers are the same, and 0 otherwise. For eample, for the input tape $\triangleright 001101 \triangleleft$, M should output 1.

   (a) Please describe your design and then write the specifications of M in the form like $\langle q_S, \triangleright, \triangleright, \triangleright \rangle \rightarrow \langle q_1, \triangleright, \triangleright, R, R, S \rangle$. Explain the transition functions in detail.

   Solution. The design of the TM is scan from left to right to count 0 and scan from right to left to count 1. If the numbers are the same, the left-scanning process will terminate just at the position where both the input tape and the working tape are at $\triangleright$.

$\langle q_S, \triangleright, \triangleright, \triangleright \rangle \rightarrow \langle q_1, \triangleright, \triangleright, R, R, S \rangle$	Start State
$\langle q_1, 0, \Box, \triangleright \rangle \rightarrow \langle q_1, 0, \triangleright, R, R, S \rangle$	Count 0
$\langle q_1, 1, \Box, \triangleright \rangle \rightarrow \langle q_1, \Box, \triangleright, R, S, S \rangle$	Ignore 1
$\langle q_1, \triangleleft, \Box, \triangleright \rangle \rightarrow \langle q_2, \triangleleft, \triangleright, L, L, S \rangle$	Scan 0 Complete
$\langle q_2, 1, 0, \triangleright \rangle \rightarrow \langle q_2, 1, \triangleright, L, L, S \rangle$	Count 1
$\langle q_2, 0, 0, \triangleright \rangle \rightarrow \langle q_2, 0, \triangleright, L, S, S \rangle$	Ignore 0
$\langle q_2, 1, \triangleright, \triangleright \rangle \rightarrow \langle q_H, \triangleright, 0, S, S, S \rangle$	1 is more than 0
$\langle q_2, \triangleright, 0, \triangleright \rangle \rightarrow \langle q_H, 0, 0, S, S, S \rangle$	0 is more than 1
$\langle q_2, \triangleright, \triangleright, \triangleright \rangle \rightarrow \langle q_H, \triangleright, 1, S, S, S \rangle$	The numbers are the same

$\Box$

   (b) Show the time complexity for one-tape TM M' to compute the same function f with n symbols in the input and give a brief description of such M' .

   Solution. The complexity for two tape TM M is $2n$. And according to the **fact**:

   > If $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is computable in time $T(n)$ by a TM M using k tapes, then it is computable in time $5kT(n)^2$ by a single-tape TM M'.

   Thus, M' will have a complexity less than $5 \times 2 \times (2n)^2 = 40n^2$ because the third tape is only used for outputing the result.

   The brief description of such M' is as follows:

Algorithm 1: One tape TM M'

1 The machine M' places $\triangleright$ after the input string;

2 Copying the input bits to the imaginary input tape with one single space added. The space could be filled with 0 if it is 0 in the input, or filled with $\Box$ if it is 1. And the original input bit will be replaced by $\triangleright$;

3 When the copying on the input tape is completed, the pointer of the scan will be backward on the imaginary input string;

4 When it hits 1 on the odd position, it will replace the last 0 on the even position with 1. No matter what is scanned on the odd position, the bit will be placed by $\triangleleft$;

5 After scanning 0 on the even position, the pointer will search rightend odd position bit;

6 If there is no enough 0 or no enough 1, the result will be 0; otherwise it will be 1.

The time complexity of this TM is $3n + \sum_{i=1}^{n} 2i = n^2 + 4n = O(n^2)$. $\Box$

3. Define the corresponding decision or search problem of the following problems and give the "certificate" and "certifier" for each decision problem provided in the subquestions or defined by yourself.

   (a) *3-Dimensional Matching.* Given disjoint sets X, Y, Z all with the size of n, and a set $M \subseteq X \times Y \times Z$. Is there a subset M' of M of size n where no two elements of M' agree in any coordinate?

   Solution. This is a decision problem. The corresponding search problem is:

   > Find a subset M' of M with the maximum size where no two elements of M' agree in any coordinate.

   And for the original decision problem, the **certificate** is:

   > A subset M' of M with size n.

   and the **certifier** is:

   > Check that no two elements of M' agree in any coordinate.

   □

   (b) *Travelling Salesman Problem.* Given a list of cities and the distances between each pair of cities, find the shortest possible route that visits each city exactly once and returns to the origin city.

   Solution. This is a searching problem. The corresponding decision problem is:

   > Does there exist a shortest route of total distance $\leq k$ that visits each city exactly once and returns to the origin city.

   the **certificate** is:

   > A shortest route that visits each city exactly once and returns to the origin city.

   and the **certifier** is:

   > Check that the shortest route is of distance smaller than k.

   □

   (c) *Job Sequencing.* Given a set of unit-time jobs, each of which has an integer deadline and a nonnegative penalty for missing the deadline. Does there exist a job sequence that has a total penalty $w \leqslant k$?

   Solution. This is a decision problem. The corresponding searching problem is:

   > Find a job sequence with a minimum total penalty.

   The **certificate** of the original problem is:

   > A job sequence for the unit-time job set.

   and the **certifier** is:

   > Check that the total penalty $w \leq k$.

   □

Remark: Please include your .pdf, .tex files for uploading with standard file names.