Lab01-Algorithm Analysis

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

∗ If there is any problem, please contact TA Haolin Zhou. Also please use English in homework.

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1. *Complexity Analysis.* Please analyze the time and space complexity of Alg. 1 and Alg. 2.

Algorithm 1: QuickSort

Input: An array $A[1, \cdots, n]$

Output: $A[1, \cdots, n]$ sorted nondecreasingly

```
pivot ← A[n]; i ← 1;
for j ← 1 to n − 1 do
    if A[j] < pivot then
        swap A[i] and A[j];
        i ← i + 1;
swap A[i] and A[n];
if i > 1 then QuickSort(A[1, ··· , i − 1]);
if i < n then QuickSort(A[i + 1, ··· , n]);
```

Algorithm 2: CocktailSort

Input: An array $A[1, \cdots, n]$

Output: $A[1, \cdots, n]$ sorted nonincreasingly

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i ← 1; j ← n; sorted ← false;
while not sorted do
    sorted ← true;
    for k ← i to j − 1 do
        if A[k] < A[k + 1] then
            swap A[k] and A[k + 1];
            sorted ← false;
    j ← j − 1;
    for k ← j downto i + 1 do
        if A[k − 1] < A[k] then
            swap A[k − 1] and A[k];
            sorted ← false;
    i ← i + 1;
```

(a) Fill in the blanks and **explain** your answers. You need to answer when the best case and the worst case happen.

Algorithm	Time Complexity[1]	Space Complexity
QuickSort		
CocktailSort		

[1] The response order can be given in *best*, *average*, and *worst*.

(b) For Alg. 1, how to modify the algorithm to achieve the same expected performance as the **average** case when the **worst** case happens?

Solution. (a) Algorithm 1 – QuickSort:

Best Case

Average Case

Worst Case

□

2. *Growth Analysis.* Rank the following functions by order of growth with brief explanations: that is, find an arrangement $g_1, g_2, \ldots, g_{15}$ of the functions $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{14} =$

$\Omega(g_{15})$. Partition your list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$. Use symbols "=" and "$\prec$" to order these functions appropriately. Here $\log n$ stands for $\ln n$.

1	n	$\log n$	$\log(\log n)$	$n \log n$
$\log_4 n$	2^n	4^n	$2^{\log n}$	2^{2^n}
$\log(n!)$	$n!$	$(2n)!$	$n^{1/2}$	n^2

Solution. Arrangement:

$$2^{2^n}, 2^{n^2}, (2n)!, n!, 4^n, 2^n, n^2, n\log n, \log(n!), n, 2^{\log n}, n^{1/2}, \log n, \log\log n, 1$$

Partition:

$$\begin{aligned} &1 \prec \log\log n \prec \log n \prec n^{1/2} \prec 2^{\log n} \prec n \prec \log(n!) \\ &= n\log n \prec n^2 \prec 2^n \prec 4^n \prec n! \prec (2n)! \prec 2^{n^2} \prec 2^{2^n} \end{aligned}$$

Along the proof, relations (6) and (7) are considered as fundamentals. And Stirling's approximation is used:

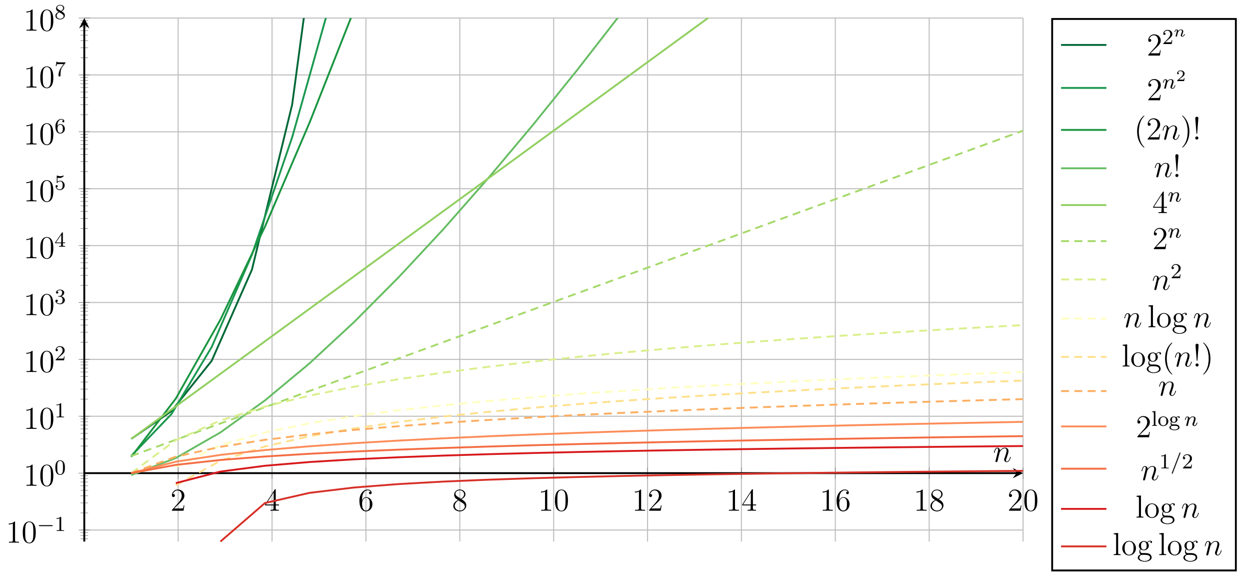
$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

L'Hôpital's rule is used:

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = +\infty, g'(n) \neq 0, \exists \lim_{n\to\infty} \frac{f'(n)}{g'(n)} \Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

The transformation rule of $\omega(\cdot)$ is used:

$$g(n) \neq 0 \Rightarrow \frac{\omega(f(x))}{g(x)} = \omega\left(\frac{f(x)}{g(x)}\right)$$



The proof is as follows:

$$
\begin{aligned}
2^{2^n} = \omega(2^{n^2}) \Leftarrow & \quad \lim_{n\to\infty} \frac{2^{2^n}}{2^{n^2}} = \lim_{n\to\infty} 2^{2^n - n^2} = \lim_{n\to\infty} 2^{\omega(n^2) - n^2} = 2^\infty & (1) \\
2^{n^2} = \omega((2n)!) \Leftarrow & \quad \lim_{n\to\infty} \frac{2^{n^2}}{(2n)!} = \lim_{n\to\infty} \frac{(2^n)^n}{\sqrt{2\pi(2n)}\left(\frac{n}{e}\right)^n} \\
& = \lim_{n\to\infty} \frac{1}{2\sqrt{\pi}} \left(e\frac{2^n}{n}\right)^n \cdot n^{-\frac{1}{2}} \\
& = \lim_{n\to\infty} \frac{1}{2\sqrt{\pi}} \left(e\frac{\omega(n^2)}{n}\right)^n \cdot n^{-\frac{1}{2}} \\
& = \lim_{n\to\infty} \frac{1}{2\sqrt{\pi}} e^n \omega\left(n^{n-\frac{1}{2}}\right) = \infty & (2) \\
(2n)! = \omega(n!) \Leftarrow & \quad \lim_{n\to\infty} \frac{(2n)!}{n!} = \lim_{n\to\infty} \prod_{i=n+1}^{2n} i = \infty & (3) \\
n! = \omega(4^n) \Leftarrow & \quad \lim_{n\to\infty} \frac{n!}{4^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{4^n} = \infty & (4) \\
4^n = \omega(2^n) \Leftarrow & \quad \lim_{n\to\infty} \frac{4^n}{2^n} = \lim_{n\to\infty} 2^n = \infty & (5) \\
2^n = \omega(n^2) \Leftarrow & \quad \lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{e^{n\log 2}}{n^2} \\
& = \lim_{n\to\infty} \frac{1 + n\log 2 + \frac{(n\log 2)^2}{2} + \frac{(n\log 2)^3}{6} + \omega((n\log 2)^3)}{n^2} \\
& = \lim_{n\to\infty} \left[\alpha + \frac{n\log^3 2}{6} + \omega(n\log^3 2)\right] \ (\alpha > 0) = \infty & (6) \\
n^2 = \omega(n\log n) \Leftarrow & \quad \lim_{n\to\infty} \frac{n^2}{n\log n} = \lim_{n\to\infty} \frac{n}{\log n} = \lim_{n\to\infty} \frac{n'}{(\log n)'} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty & (7) \\
n\log n = \Theta(\log(n!)) \Leftarrow & \quad \lim_{n\to\infty} \frac{n\log n}{\log(n!)} = \lim_{n\to\infty} \frac{n\log n}{\frac{1}{2}\log 2\pi n + n\log n - n} = 1 & (8) \\
\log(n!) = \omega(n) \Leftarrow & \quad \lim_{n\to\infty} \frac{\log(n!)}{n} = \lim_{n\to\infty} \left(\frac{\pi}{n} + \frac{\log n}{2n} + \log n - 1\right) = \infty & (9) \\
n = \omega(2^{\log n}) \Leftarrow & \quad \lim_{n\to\infty} \frac{n}{2^{\log n}} = \lim_{n\to\infty} \frac{n}{2^{\frac{\log_2 n}{\log_2 e}}} = \lim_{n\to\infty} n^{1-\frac{1}{\log_2 e}} = \lim_{n\to\infty} n^{0.31} = \infty & (10) \\
2^{\log n} = \omega(n^{1/2}) \Leftarrow & \quad \lim_{n\to\infty} \frac{2^{\log n}}{n^{1/2}} = \lim_{n\to\infty} n^{\frac{1}{\log_2 e} - \frac{1}{2}} = \lim_{n\to\infty} n^{0.19} = \infty & (11) \\
n^{1/2} = \omega(\log n) \Leftarrow & \quad \lim_{n\to\infty} \frac{n^{1/2}}{\log n} = \lim_{n\to\infty} \frac{\left(n^{1/2}\right)'}{(\log n)'} = \lim_{n\to\infty} \frac{n^{1/2}}{2} = \infty & (12) \\
\log n = \omega(\log\log n) \Leftarrow & \quad \lim_{n\to\infty} \frac{\log n}{\log\log n} = \lim_{n\to\infty} \frac{(\log n)'}{(\log\log n)'} = \lim_{n\to\infty} \frac{1/n}{1/(n\log n)} = \infty & (13) \\
\log\log n = \omega(1) \Leftarrow & \quad \lim_{n\to\infty} \log\log n = \infty & (14)
\end{aligned}
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□

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.