上海交通大学
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# 作业 4

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1. **解**

$$\boldsymbol{A}=\begin{pmatrix}5&2&1\\-1&4&2\\2&-3&10\end{pmatrix}=\begin{pmatrix}5&&\\&4&\\&&10\end{pmatrix}-\begin{pmatrix}&&\\1&&\\-2&3&\end{pmatrix}-\begin{pmatrix}&-2&-1\\&&-2\\&&\end{pmatrix}=\boldsymbol{D}-\boldsymbol{L}-\boldsymbol{U}$$

$$\boldsymbol{b}=\begin{pmatrix}-12\\20\\3\end{pmatrix}$$

(1) 由于 $\boldsymbol{A}$ 是严格占优矩阵，所以 Jacobi 迭代法和 Guass–Seidel 迭代法解这个方程组都是收敛的。

(2) **Jacobi 迭代法** 对于 Jacobi 迭代法，其迭代矩阵

$$\boldsymbol{B}=\boldsymbol{D}^{-1}(\boldsymbol{L}+\boldsymbol{U})=\begin{pmatrix}&-\frac{2}{5}&-\frac{1}{5}\\\frac{1}{4}&&-\frac{1}{2}\\-\frac{1}{5}&\frac{3}{10}&\end{pmatrix}\qquad \boldsymbol{f}=\boldsymbol{D}^{-1}\boldsymbol{b}=\begin{pmatrix}-\frac{12}{5}\\5\\\frac{3}{10}\end{pmatrix}$$

取迭代初值 $\boldsymbol{x}^{(0)}=(0,0,0)^\top$，有

$$\begin{aligned}
\boldsymbol{x}^{(1)}&=\boldsymbol{B}\boldsymbol{x}^{(0)}+\boldsymbol{f}=(-2.4,5,0.3)^\top & \|\boldsymbol{\epsilon}^{(1)}\|_\infty&=5.\\
\boldsymbol{x}^{(2)}&=\boldsymbol{B}\boldsymbol{x}^{(1)}+\boldsymbol{f}=(-4.46,4.25,2.28)^\top & \|\boldsymbol{\epsilon}^{(2)}\|_\infty&=2.06\\
\boldsymbol{x}^{(3)}&=\boldsymbol{B}\boldsymbol{x}^{(2)}+\boldsymbol{f}=(-4.556,2.745,2.467)^\top & \|\boldsymbol{\epsilon}^{(3)}\|_\infty&=1.505\\
\boldsymbol{x}^{(4)}&=\boldsymbol{B}\boldsymbol{x}^{(3)}+\boldsymbol{f}=(-3.9914,2.6275,2.0347)^\top & \|\boldsymbol{\epsilon}^{(4)}\|_\infty&=0.5646\\
\boldsymbol{x}^{(5)}&=\boldsymbol{B}\boldsymbol{x}^{(4)}+\boldsymbol{f}=(-3.85794,2.9848,1.88653)^\top & \|\boldsymbol{\epsilon}^{(5)}\|_\infty&=0.3573\\
\boldsymbol{x}^{(6)}&=\boldsymbol{B}\boldsymbol{x}^{(5)}+\boldsymbol{f}=(-3.971226,3.09225,1.967028)^\top & \|\boldsymbol{\epsilon}^{(6)}\|_\infty&=0.113286\\
\boldsymbol{x}^{(7)}&=\boldsymbol{B}\boldsymbol{x}^{(6)}+\boldsymbol{f}=(-4.0303056,3.0236795,2.0219202)^\top & \|\boldsymbol{\epsilon}^{(7)}\|_\infty&=0.0685705\\
\boldsymbol{x}^{(8)}&=\boldsymbol{B}\boldsymbol{x}^{(7)}+\boldsymbol{f}=(-4.01385584,2.9814635,2.01316497)^\top & \|\boldsymbol{\epsilon}^{(8)}\|_\infty&=0.042216\\
\boldsymbol{x}^{(9)}&=\boldsymbol{B}\boldsymbol{x}^{(8)}+\boldsymbol{f}=(-3.99521839,2.98995356,1.99721022)^\top & \|\boldsymbol{\epsilon}^{(9)}\|_\infty&=0.01863745\\
\boldsymbol{x}^{(10)}&=\boldsymbol{B}\boldsymbol{x}^{(9)}+\boldsymbol{f}=(-3.99542347,3.00259029,1.99602975)^\top & \|\boldsymbol{\epsilon}^{(10)}\|_\infty&=0.01263674\\
\boldsymbol{x}^{(11)}&=\boldsymbol{B}\boldsymbol{x}^{(10)}+\boldsymbol{f}=(-4.00024207,3.00312926,1.99986178)^\top & \|\boldsymbol{\epsilon}^{(11)}\|_\infty&=0.0048186
\end{aligned}$$

$$\boldsymbol{x}^{(12)} = \boldsymbol{B}\boldsymbol{x}^{(11)} + \boldsymbol{f} = (-4.00122406, 3.00000859, 2.00098719)^\top \quad \|\boldsymbol{\epsilon}^{(12)}\|_\infty = 0.00312067$$
$$\boldsymbol{x}^{(13)} = \boldsymbol{B}\boldsymbol{x}^{(12)} + \boldsymbol{f} = (-4.00020088, 2.99920039, 2.00024739)^\top \quad \|\boldsymbol{\epsilon}^{(13)}\|_\infty = 0.00102319$$
$$\boldsymbol{x}^{(14)} = \boldsymbol{B}\boldsymbol{x}^{(13)} + \boldsymbol{f} = (-3.99972963, 2.99982609, 1.99980029)^\top \quad \|\boldsymbol{\epsilon}^{(14)}\|_\infty = 0.0006257$$
$$\boldsymbol{x}^{(15)} = \boldsymbol{B}\boldsymbol{x}^{(14)} + \boldsymbol{f} = (-3.99989049, 3.00016745, 1.99989375)^\top \quad \|\boldsymbol{\epsilon}^{(15)}\|_\infty = 0.00034136$$
$$\boldsymbol{x}^{(16)} = \boldsymbol{B}\boldsymbol{x}^{(15)} + \boldsymbol{f} = (-4.00004573, 3.0000805, 2.00002833)^\top \quad \|\boldsymbol{\epsilon}^{(16)}\|_\infty = 0.00015524$$
$$\boldsymbol{x}^{(17)} = \boldsymbol{B}\boldsymbol{x}^{(16)} + \boldsymbol{f} = (-4.00003787, 2.9999744, 2.0000333, )^\top \quad \|\boldsymbol{\epsilon}^{(17)}\|_\infty = 0.0001061$$
$$\boldsymbol{x}^{(18)} = \boldsymbol{B}\boldsymbol{x}^{(17)} + \boldsymbol{f} = (-3.99999642, 2.99997389, 1.99999989)^\top \quad \|\boldsymbol{\epsilon}^{(18)}\|_\infty = 4.14468074 \times 10^{-5}$$

总共需要 18 次满足要求。

**Guass–Seidel 迭代法** 对于 Guass–Seidel 迭代法，

$$\boldsymbol{G} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{U} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} & -2 & -1 \\ & & -2 \\ & & \end{pmatrix} = \begin{pmatrix} 0 & -0.4 & -0.2 \\ 0 & -0.1 & -0.55 \\ 0 & 0.05 & -0.125 \end{pmatrix}$$
$$\boldsymbol{f} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{b} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 4.4 \\ 2.1 \end{pmatrix}$$

取迭代初值 $\boldsymbol{x}^{(0)} = (0, 0, 0)^\top$，有

$$\boldsymbol{x}^{(1)} = \boldsymbol{G}\boldsymbol{x}^{(0)} + \boldsymbol{f} = (-2.4, 4.4, 2.1)^\top \quad \|\boldsymbol{\epsilon}^{(1)}\|_\infty = 4.4$$
$$\boldsymbol{x}^{(2)} = \boldsymbol{G}\boldsymbol{x}^{(1)} + \boldsymbol{f} = (-4.58, 2.805, 2.0575)^\top \quad \|\boldsymbol{\epsilon}^{(2)}\|_\infty = 2.18$$
$$\boldsymbol{x}^{(3)} = \boldsymbol{G}\boldsymbol{x}^{(2)} + \boldsymbol{f} = (-3.9335, 2.987875, 1.9830625)^\top \quad \|\boldsymbol{\epsilon}^{(3)}\|_\infty = 0.6465$$
$$\boldsymbol{x}^{(4)} = \boldsymbol{G}\boldsymbol{x}^{(3)} + \boldsymbol{f} = (-3.9917625, 3.01052813, 2.00151094)^\top \quad \|\boldsymbol{\epsilon}^{(4)}\|_\infty = 0.0582625$$
$$\boldsymbol{x}^{(5)} = \boldsymbol{G}\boldsymbol{x}^{(4)} + \boldsymbol{f} = (-4.00451344, 2.99811617, 2.00033754)^\top \quad \|\boldsymbol{\epsilon}^{(5)}\|_\infty = 0.01275094$$
$$\boldsymbol{x}^{(6)} = \boldsymbol{G}\boldsymbol{x}^{(5)} + \boldsymbol{f} = (-3.99931398, 3.00000274, 1.99986362)^\top \quad \|\boldsymbol{\epsilon}^{(6)}\|_\infty = 0.00519946$$
$$\boldsymbol{x}^{(7)} = \boldsymbol{G}\boldsymbol{x}^{(6)} + \boldsymbol{f} = (-3.99997382, 3.00007474, 2.00001718)^\top \quad \|\boldsymbol{\epsilon}^{(7)}\|_\infty = 0.00065984$$
$$\boldsymbol{x}^{(8)} = \boldsymbol{G}\boldsymbol{x}^{(7)} + \boldsymbol{f} = (-4.00003333, 2.99998307, 2.00000159)^\top \quad \|\boldsymbol{\epsilon}^{(8)}\|_\infty = 9.16628308 \times 10^{-5}$$

总共需要 8 次满足要求。

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