作业 4

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1. **解**

$$
\boldsymbol{A}=\begin{pmatrix}5 & 2 & 1\\ -1 & 4 & 2\\ 2 & -3 & 10\end{pmatrix}=\begin{pmatrix}5 & & \\ & 4 & \\ & & 10\end{pmatrix}-\begin{pmatrix} & & \\ 1 & & \\ -2 & 3 & \end{pmatrix}-\begin{pmatrix} & -2 & -1\\ & & -2\\ & & \end{pmatrix}=\boldsymbol{D}-\boldsymbol{L}-\boldsymbol{U}
$$

$$
\boldsymbol{b}=\begin{pmatrix}-12\\ 20\\ 3\end{pmatrix}
$$

(1) 由于 $\boldsymbol{A}$ 是严格占优矩阵，所以 Jacobi 迭代法和 Guass–Seidel 迭代法解这个方程组都是收敛的。

(2) **Jacobi 迭代法** 对于 Jacobi 迭代法，其迭代矩阵

$$
\boldsymbol{B}=\boldsymbol{D}^{-1}(\boldsymbol{L}+\boldsymbol{U})=\begin{pmatrix} & -\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{4} & & -\frac{1}{2}\\ -\frac{1}{5} & \frac{3}{10} & \end{pmatrix}\qquad \boldsymbol{f}=\boldsymbol{D}^{-1}\boldsymbol{b}=\begin{pmatrix}-\frac{12}{5}\\ 5\\ \frac{3}{10}\end{pmatrix}
$$

取迭代初值 $\boldsymbol{x}^{(0)}=(0,0,0)^{\top}$，记 $\boldsymbol{\epsilon}^{(k)}=\boldsymbol{x}^{(k)}-\boldsymbol{x}^{(k-1)}$，有

$$
\begin{aligned}
\boldsymbol{x}^{(1)}&=\boldsymbol{B}\boldsymbol{x}^{(0)}+\boldsymbol{f}=(-2.4,5,0.3)^{\top} & \|\boldsymbol{\epsilon}^{(1)}\|_{\infty}&=5.\\
\boldsymbol{x}^{(2)}&=\boldsymbol{B}\boldsymbol{x}^{(1)}+\boldsymbol{f}=(-4.46,4.25,2.28)^{\top} & \|\boldsymbol{\epsilon}^{(2)}\|_{\infty}&=2.06\\
\boldsymbol{x}^{(3)}&=\boldsymbol{B}\boldsymbol{x}^{(2)}+\boldsymbol{f}=(-4.556,2.745,2.467)^{\top} & \|\boldsymbol{\epsilon}^{(3)}\|_{\infty}&=1.505\\
\boldsymbol{x}^{(4)}&=\boldsymbol{B}\boldsymbol{x}^{(3)}+\boldsymbol{f}=(-3.9914,2.6275,2.0347)^{\top} & \|\boldsymbol{\epsilon}^{(4)}\|_{\infty}&=0.5646\\
\boldsymbol{x}^{(5)}&=\boldsymbol{B}\boldsymbol{x}^{(4)}+\boldsymbol{f}=(-3.85794,2.9848,1.88653)^{\top} & \|\boldsymbol{\epsilon}^{(5)}\|_{\infty}&=0.3573\\
\boldsymbol{x}^{(6)}&=\boldsymbol{B}\boldsymbol{x}^{(5)}+\boldsymbol{f}=(-3.971226,3.09225,1.967028)^{\top} & \|\boldsymbol{\epsilon}^{(6)}\|_{\infty}&=0.113286\\
\boldsymbol{x}^{(7)}&=\boldsymbol{B}\boldsymbol{x}^{(6)}+\boldsymbol{f}=(-4.0303056,3.0236795,2.0219202)^{\top} & \|\boldsymbol{\epsilon}^{(7)}\|_{\infty}&=0.0685705\\
\boldsymbol{x}^{(8)}&=\boldsymbol{B}\boldsymbol{x}^{(7)}+\boldsymbol{f}=(-4.01385584,2.9814635,2.01316497)^{\top} & \|\boldsymbol{\epsilon}^{(8)}\|_{\infty}&=0.042216\\
\boldsymbol{x}^{(9)}&=\boldsymbol{B}\boldsymbol{x}^{(8)}+\boldsymbol{f}=(-3.99521839,2.98995356,1.99721022)^{\top} & \|\boldsymbol{\epsilon}^{(9)}\|_{\infty}&=0.01863745\\
\boldsymbol{x}^{(10)}&=\boldsymbol{B}\boldsymbol{x}^{(9)}+\boldsymbol{f}=(-3.99542347,3.00259029,1.99602975)^{\top} & \|\boldsymbol{\epsilon}^{(10)}\|_{\infty}&=0.01263674\\
\boldsymbol{x}^{(11)}&=\boldsymbol{B}\boldsymbol{x}^{(10)}+\boldsymbol{f}=(-4.00024207,3.00312926,1.99986178)^{\top} & \|\boldsymbol{\epsilon}^{(11)}\|_{\infty}&=0.0048186
\end{aligned}
$$

$$x^{(12)} = Bx^{(11)} + f = (-4.00122406, 3.00000859, 2.00098719)^\top \quad \|\epsilon^{(12)}\|_\infty = 0.00312067$$
$$x^{(13)} = Bx^{(12)} + f = (-4.00020088, 2.99920039, 2.00024739)^\top \quad \|\epsilon^{(13)}\|_\infty = 0.00102319$$
$$x^{(14)} = Bx^{(13)} + f = (-3.99972963, 2.99982609, 1.99980029)^\top \quad \|\epsilon^{(14)}\|_\infty = 0.0006257$$
$$x^{(15)} = Bx^{(14)} + f = (-3.99989049, 3.00016745, 1.99989375)^\top \quad \|\epsilon^{(15)}\|_\infty = 0.00034136$$
$$x^{(16)} = Bx^{(15)} + f = (-4.00004573, 3.0000805, 2.00002833)^\top \quad \|\epsilon^{(16)}\|_\infty = 0.00015524$$
$$x^{(17)} = Bx^{(16)} + f = (-4.00003787, 2.9999744, 2.0000333,)^\top \quad \|\epsilon^{(17)}\|_\infty = 0.0001061$$
$$x^{(18)} = Bx^{(17)} + f = (-3.99999642, 2.99997389, 1.99999989)^\top \quad \|\epsilon^{(18)}\|_\infty = 4.14468074 \times 10^{-5}$$

总共需要 18 次满足要求。

Guass–Seidel 迭代法 对于 Guass–Seidel 迭代法，

$$G = (D - L)^{-1}U = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} & -2 & -1 \\ & & -2 \\ & & \end{pmatrix} = \begin{pmatrix} 0 & -0.4 & -0.2 \\ 0 & -0.1 & -0.55 \\ 0 & 0.05 & -0.125 \end{pmatrix}$$

$$f = (D - L)^{-1}b = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 4.4 \\ 2.1 \end{pmatrix}$$

取迭代初值 $x^{(0)} = (0, 0, 0)^\top$，记 $\epsilon^{(k)} = x^{(k)} - x^{(k-1)}$，有

$$x^{(1)} = Gx^{(0)} + f = (-2.4, 4.4, 2.1)^\top \quad \|\epsilon^{(1)}\|_\infty = 4.4$$
$$x^{(2)} = Gx^{(1)} + f = (-4.58, 2.805, 2.0575)^\top \quad \|\epsilon^{(2)}\|_\infty = 2.18$$
$$x^{(3)} = Gx^{(2)} + f = (-3.9335, 2.987875, 1.9830625)^\top \quad \|\epsilon^{(3)}\|_\infty = 0.6465$$
$$x^{(4)} = Gx^{(3)} + f = (-3.9917625, 3.01052813, 2.00151094)^\top \quad \|\epsilon^{(4)}\|_\infty = 0.0582625$$
$$x^{(5)} = Gx^{(4)} + f = (-4.00451344, 2.99811617, 2.00033754)^\top \quad \|\epsilon^{(5)}\|_\infty = 0.01275094$$
$$x^{(6)} = Gx^{(5)} + f = (-3.99931398, 3.00000274, 1.99986362)^\top \quad \|\epsilon^{(6)}\|_\infty = 0.00519946$$
$$x^{(7)} = Gx^{(6)} + f = (-3.99997382, 3.00007474, 2.00001718)^\top \quad \|\epsilon^{(7)}\|_\infty = 0.00065984$$
$$x^{(8)} = Gx^{(7)} + f = (-4.00003333, 2.99998307, 2.00000159)^\top \quad \|\epsilon^{(8)}\|_\infty = 9.16628308 \times 10^{-5}$$

总共需要 8 次满足要求。

8. **解**

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} & & & \\ & & & \\ \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \end{pmatrix} - \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ & & & \\ & & & \end{pmatrix} = D - L - U$$

$$b = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

(1) 对于 Jacobi 迭代法，

$$\boldsymbol{B}_0=\boldsymbol{D}^{-1}(\boldsymbol{L}+\boldsymbol{U})=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}^{-1}\begin{pmatrix}&&\frac14&\frac14\\&&\frac14&\frac14\\\frac14&\frac14&&\\\frac14&\frac14&&\end{pmatrix}=\begin{pmatrix}&&\frac14&\frac14\\&&\frac14&\frac14\\\frac14&\frac14&&\\\frac14&\frac14&&\end{pmatrix}$$

求它的特征值

$$0=|\lambda\boldsymbol{E}-\boldsymbol{B}_0|=\begin{vmatrix}\lambda&&-\frac14&-\frac14\\&\lambda&-\frac14&-\frac14\\-\frac14&-\frac14&\lambda&\\-\frac14&-\frac14&&\lambda\end{vmatrix}=\lambda^2\left(\lambda-\frac12\right)\left(\lambda+\frac12\right)$$

得到 $\lambda_{1,2}=0,\lambda_3=\frac12,\lambda_4=-\frac12$，故谱半径 $\rho(\boldsymbol{B}_0)=\frac12$。

(2) 对于 Guass–Seidel 迭代法，

$$\boldsymbol{B}_0=(\boldsymbol{D}-\boldsymbol{L})^{-1}\boldsymbol{U}=\begin{pmatrix}1&&&\\0&1&&\\-\frac14&-\frac14&1&\\-\frac14&-\frac14&0&1\end{pmatrix}^{-1}\begin{pmatrix}&&\frac14&\frac14\\&&\frac14&\frac14\\&&&\\&&&\end{pmatrix}=\begin{pmatrix}&&\frac14&\frac14\\&&\frac14&\frac14\\&&\frac18&\frac18\\&&\frac18&\frac18\end{pmatrix}$$

求它的特征值

$$0=|\lambda\boldsymbol{E}-\boldsymbol{B}_0|=\begin{vmatrix}\lambda&&-\frac14&-\frac14\\&\lambda&-\frac14&-\frac14\\&&\lambda-\frac18&-\frac18\\&&-\frac18&\lambda-\frac18\end{vmatrix}=\lambda^3\left(\lambda-\frac14\right)$$

得到 $\lambda_{1,2,3}=0,\lambda_4=\frac14$，故谱半径 $\rho(\boldsymbol{B}_0)=\frac14$。

(3) 由于谱半径都小于 1，所以 Jacobi 迭代法和 Guass–Seidel 迭代法均收敛。

9. **解**

$$\boldsymbol{A}=\begin{pmatrix}4&-1&\\-1&4&-1\\&-1&4\end{pmatrix}=\boldsymbol{D}-\boldsymbol{L}-\boldsymbol{U}=\begin{pmatrix}4&&\\&4&\\&&4\end{pmatrix}-\begin{pmatrix}&&\\1&&\\&1&\end{pmatrix}-\begin{pmatrix}&1&\\&&1\\&&\end{pmatrix}$$

$$\boldsymbol{b}=\begin{pmatrix}1\\4\\-3\end{pmatrix}$$

对于 SOR 迭代法，松弛因子为 ω，有

$$\boldsymbol{L}_\omega=(\boldsymbol{D}-\omega\boldsymbol{L})^{-1}[(1-\omega)\boldsymbol{D}+\omega\boldsymbol{U}]=\begin{pmatrix}4&&\\-\omega&4&\\&-\omega&4\end{pmatrix}^{-1}\begin{pmatrix}4(1-\omega)&\omega&\\&4(1-\omega)&\omega\\&&4(1-\omega)\end{pmatrix}$$

$$\boldsymbol{f}=\omega(\boldsymbol{D}-\omega\boldsymbol{L})^{-1}\boldsymbol{b}=\omega\begin{pmatrix}4&&\\-\omega&4&\\&-\omega&4\end{pmatrix}^{-1}\begin{pmatrix}1\\4\\-3\end{pmatrix}$$

取迭代初值 $\boldsymbol{x}^{(0)} = (0,0,0)^{\top}$，记 $\boldsymbol{\epsilon}^{(k)} = \boldsymbol{x}^{*} - \boldsymbol{x}^{(k)}$，有

(1) $\omega = 1.03$ 需要 5 次迭代:

$$
\begin{aligned}
\boldsymbol{x}^{(1)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.2575, 1.09630625, -0.49020114)^{\top} & \|\boldsymbol{\epsilon}^{(1)}\|_{\infty} &= 0.2425\\
\boldsymbol{x}^{(2)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.53207386, 1.00789304, -0.49826151)^{\top} & \|\boldsymbol{\epsilon}^{(2)}\|_{\infty} &= 0.03207386\\
\boldsymbol{x}^{(3)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.50107024, 1.00048646, -0.49992689)^{\top} & \|\boldsymbol{\epsilon}^{(3)}\|_{\infty} &= 0.00107024\\
\boldsymbol{x}^{(4)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50009316, 1.00002822, -0.49999493)^{\top} & \|\boldsymbol{\epsilon}^{(4)}\|_{\infty} &= 9.31555814 \times 10^{-5}\\
\boldsymbol{x}^{(5)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50000447, 1.00000161, -0.49999974)^{\top} & \|\boldsymbol{\epsilon}^{(5)}\|_{\infty} &= 4.47176785 \times 10^{-6}
\end{aligned}
$$

(2) $\omega = 1$ 需要 6 次迭代:

$$
\begin{aligned}
\boldsymbol{x}^{(1)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.25, 1.0625, -0.484375)^{\top} & \|\boldsymbol{\epsilon}^{(1)}\|_{\infty} &= 0.25\\
\boldsymbol{x}^{(2)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.515625, 1.0078125, -0.49804688)^{\top} & \|\boldsymbol{\epsilon}^{(2)}\|_{\infty} &= 0.015625\\
\boldsymbol{x}^{(3)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.50195312, 1.00097656, -0.49975586)^{\top} & \|\boldsymbol{\epsilon}^{(3)}\|_{\infty} &= 0.00195312\\
\boldsymbol{x}^{(4)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50024414, 1.00012207, -0.49996948)^{\top} & \|\boldsymbol{\epsilon}^{(4)}\|_{\infty} &= 0.00024414\\
\boldsymbol{x}^{(5)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50003052, 1.00001526, -0.49999619)^{\top} & \|\boldsymbol{\epsilon}^{(5)}\|_{\infty} &= 3.05175781 \times 10^{-5}\\
\boldsymbol{x}^{(6)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(5)} + \boldsymbol{f} = (0.50000381, 1.00000191, -0.49999952)^{\top} & \|\boldsymbol{\epsilon}^{(6)}\|_{\infty} &= 3.81469727 \times 10^{-6}
\end{aligned}
$$

(3) $\omega = 1.1$ 需要 6 次迭代:

$$
\begin{aligned}
\boldsymbol{x}^{(1)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.275, 1.175625, -0.50170313)^{\top} & \|\boldsymbol{\epsilon}^{(1)}\|_{\infty} &= 0.225\\
\boldsymbol{x}^{(2)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.57079688, 1.00143828, -0.49943416)^{\top} & \|\boldsymbol{\epsilon}^{(2)}\|_{\infty} &= 0.07079688\\
\boldsymbol{x}^{(3)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.49331584, 0.99817363, -0.50055883)^{\top} & \|\boldsymbol{\epsilon}^{(3)}\|_{\infty} &= 0.00668416\\
\boldsymbol{x}^{(4)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50016617, 1.00007465, -0.49992359)^{\top} & \|\boldsymbol{\epsilon}^{(4)}\|_{\infty} &= 0.00016617\\
\boldsymbol{x}^{(5)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50000391, 1.00001462, -0.50000362)^{\top} & \|\boldsymbol{\epsilon}^{(5)}\|_{\infty} &= 1.46243508 \times 10^{-5}\\
\boldsymbol{x}^{(6)} &= \boldsymbol{L}_{\omega}\boldsymbol{x}^{(5)} + \boldsymbol{f} = (0.50000363, 0.99999854, -0.50000004)^{\top} & \|\boldsymbol{\epsilon}^{(6)}\|_{\infty} &= 3.63040468 \times 10^{-6}
\end{aligned}
$$

11. **证明** 迭代公式

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^{(k)}) = (\boldsymbol{E} - \omega\boldsymbol{A})\boldsymbol{x}^{(k)} + \omega\boldsymbol{b}$$

讨论 $\boldsymbol{L}_{\omega} = \boldsymbol{E} - \omega\boldsymbol{A}$ 的特征值 λ，对于任意的 $\boldsymbol{y} \in \mathbf{R}^{n}$，有

$$
\begin{aligned}
\boldsymbol{L}_{\omega}\boldsymbol{y} &= \lambda\boldsymbol{y}\\
(\boldsymbol{E} - \omega\boldsymbol{A})\boldsymbol{y} &= \lambda\boldsymbol{y}\\
\boldsymbol{y} - \omega\boldsymbol{A}\boldsymbol{y} &= \lambda\boldsymbol{y} \qquad (1)
\end{aligned}
$$

对于 $\boldsymbol{A}$ 的特征值 $0 < \alpha \le \lambda' \le \beta$，有

$$\boldsymbol{A}\boldsymbol{y} = \lambda'\boldsymbol{y} \qquad (2)$$

结合式 (1) 和 (2) 有

$$(1 - \omega\lambda' - \lambda)\boldsymbol{y} = \boldsymbol{0}$$

考虑到对于 $\forall \boldsymbol{y} \in \mathbf{R}^n$ 都成立，有

$$\lambda = 1 - \omega\lambda'$$

当 $0 < \omega < \frac{2}{\beta}$ 时，有

$$-1 = 1 - \frac{2}{\beta}\beta < \lambda < 1 - 0 = 1$$

即 $|\lambda| < 1$，则迭代法收敛。 ∎

13. **解** (1) 设 $\boldsymbol{z}^{(m)} = \begin{pmatrix} z_1^{(m)} \\ z_2^{(m)} \end{pmatrix}$，$\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$，则对于迭代方法

$$\boldsymbol{A}z_1^{(m+1)} = \boldsymbol{b}_1 - \boldsymbol{B}z_2^{(m)}, \quad \boldsymbol{A}z_2^{(m+1)} = \boldsymbol{b}_2 - \boldsymbol{B}z_1^{(m)}$$

等价为

$$\begin{aligned}
\begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix} \boldsymbol{z}^{(m+1)} = \boldsymbol{b} - \begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix} \boldsymbol{z}^{(m)} \\
\boldsymbol{z}^{(m+1)} &= -\begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix}^{-1} \begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix} \boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix}^{-1} \boldsymbol{b} \\
&= -\begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix} \begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix} \boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix} \boldsymbol{b} \\
&= \begin{pmatrix} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ -\boldsymbol{A}^{-1}\boldsymbol{B} & \end{pmatrix} \boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix} \boldsymbol{b}
\end{aligned}$$

对于迭代矩阵，它的特征值为 λ，则对于 $\forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{R}^{2n}$，有

$$\begin{pmatrix} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ -\boldsymbol{A}^{-1}\boldsymbol{B} & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} -\boldsymbol{A}^{-1}\boldsymbol{B}z_2 = \lambda z_1 \\ -\boldsymbol{A}^{-1}\boldsymbol{B}z_1 = \lambda z_2 \end{cases}$$

$$\Rightarrow \begin{cases} (\boldsymbol{A}^{-1}\boldsymbol{B})^2 z_1 = \lambda^2 z_1 \\ (\boldsymbol{A}^{-1}\boldsymbol{B})^2 z_2 = \lambda^2 z_2 \end{cases}$$

即 $\rho\begin{pmatrix} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ -\boldsymbol{A}^{-1}\boldsymbol{B} & \end{pmatrix} = \rho(\boldsymbol{A}^{-1}\boldsymbol{B})$，迭代法收敛的充要条件即 $\rho(\boldsymbol{A}^{-1}\boldsymbol{B}) < 1$。

(2) 设 $\boldsymbol{z}^{(m)} = \begin{pmatrix} z_1^{(m)} \\ z_2^{(m)} \end{pmatrix}$，$\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$，则对于迭代方法

$$\boldsymbol{A}z_1^{(m+1)} = \boldsymbol{b}_1 - \boldsymbol{B}z_2^{(m)}, \quad \boldsymbol{A}z_2^{(m+1)} = \boldsymbol{b}_2 - \boldsymbol{B}z_1^{(m+1)}$$

等价为

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix} z^{(m+1)} = \boldsymbol{b} - \begin{pmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} z^{(m)}$$

$$\begin{aligned} z^{(m+1)} &= -\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} z^{(m)} + \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^{-1} \boldsymbol{b} \\ &= -\begin{pmatrix} \boldsymbol{A}^{-1} & \boldsymbol{0} \\ -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1} & \boldsymbol{A}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} z^{(m)} + \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^{-1} \boldsymbol{b} \\ &= \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix} z^{(m)} + \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{A} \end{pmatrix}^{-1} \boldsymbol{b} \end{aligned}$$

对于迭代矩阵，它的特征值为 λ，则对于 $\forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{R}^{2n}$，有

$$\begin{pmatrix} \boldsymbol{0} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} -\boldsymbol{A}^{-1}\boldsymbol{B}z_2 = \lambda z_1 \\ (\boldsymbol{A}^{-1}\boldsymbol{B})^2 z_2 = \lambda z_2 \end{cases}$$

$$\Rightarrow \begin{cases} (\boldsymbol{A}^{-1}\boldsymbol{B})^2 z_1 = \lambda z_1 \\ (\boldsymbol{A}^{-1}\boldsymbol{B})^2 z_2 = \lambda z_2 \end{cases}$$

即 $\rho\begin{pmatrix} \boldsymbol{0} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix} = \rho((\boldsymbol{A}^{-1}\boldsymbol{B})^2)$，迭代法的充要条件即 $\rho((\boldsymbol{A}^{-1}\boldsymbol{B})^2) = \rho((\boldsymbol{A}^{-1}\boldsymbol{B}))^2 < 1$。方法 (1) 的收敛速度为 $R_1 = -\ln\rho((\boldsymbol{A}^{-1}\boldsymbol{B}))$，方法 (2) 的收敛速度为 $R_2 = -2\ln\rho((\boldsymbol{A}^{-1}\boldsymbol{B}))$，即 $R_2 = 2R_1$，方法 (2) 的收敛速度是方法 (1) 的 2 倍。

14. **证明** 若矩阵 $\boldsymbol{A} = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$ 是正定的，那么

$$\begin{cases} \det\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = 1 - a^2 > 0 & \Rightarrow -1 < a < 1 \\ \det \boldsymbol{A} = (a-1)^2(2a+1) > 0 & \Rightarrow a > -\frac{1}{2} \end{cases}$$

即当 $-\frac{1}{2} < a < 1$ 时，矩阵 $\boldsymbol{A}$ 是正定的。
考察

$$\boldsymbol{A} = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} & & \\ -a & & \\ -a & -a & \end{pmatrix} - \begin{pmatrix} & -a & -a \\ & & -a \\ & & \end{pmatrix}$$

在 Jacobi 迭代法中，迭代矩阵

$$B = D^{-1}(L+U) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} & -a & -a \\ -a & & -a \\ -a & -a & \end{pmatrix} = \begin{pmatrix} & -a & -a \\ -a & & -a \\ -a & -a & \end{pmatrix}$$

考察它的特征值

$$|\lambda E - B| = 0 \Rightarrow \begin{vmatrix} \lambda & a & a \\ a & \lambda & a \\ a & a & \lambda \end{vmatrix} = (\lambda - a)^2(\lambda + 2a) = 0 \Rightarrow \lambda_{1,2} = a, \lambda_3 = -\frac{1}{2a}$$

为了使迭代法收敛，有 $|\lambda| < 1$，则需要 $-\frac{1}{2} < a < \frac{1}{2}$。 ■

19. **证明** (1) 由于 $(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A$，则 $A^\top A$ 是对称矩阵。

由于 A 是非奇异矩阵，对 $\forall x \neq 0$，有 $x^\top A^\top A x = (Ax)^\top (Ax) > 0$，所以 $A^\top A$ 是正定矩阵。

综上，$A^\top A$ 是对称正定矩阵。

(2) 由于 $A^\top A$ 是对称矩阵，所以

$$\mathrm{cond}(A^\top A)_2 = \frac{|\lambda_{\max}(A^\top A)|}{|\lambda_{\min}(A^\top A)|} \tag{3}$$

另一方面，

$$\mathrm{cond}(A)_2 = \sqrt{\frac{\lambda_{\max}(A^\top A)}{\lambda_{\min}(A^\top A)}} \tag{4}$$

综合式 (3) 和 (4) 有

$$\mathrm{cond}(A^\top A)_2 = (\mathrm{cond}(A)_2)^2$$ ■