

## 作业 8

Log Creative

2024 年 1 月 3 日

1. 解

$$\begin{cases} y' = ax + b \\ y(0) = 0 \end{cases}$$

**Euler 方法:**

$$y_{n+1} = y_n + h(ax_n + b)$$

取步长  $h = 0.1$ , 计算结果如表所示。

$x_n$	$y_n$	$y(x_n)$	$x_n$	$y_n$	$y(x_n)$
0.1	$0.1b$	$0.005a + 0.1b$	0.6	$0.15a + 0.6b$	$0.18a + 0.6b$
0.2	$0.01a + 0.2b$	$0.02a + 0.2b$	0.7	$0.21a + 0.7b$	$0.245a + 0.7b$
0.3	$0.03a + 0.3b$	$0.045a + 0.3b$	0.8	$0.28a + 0.8b$	$0.32a + 0.8b$
0.4	$0.06a + 0.4b$	$0.08a + 0.4b$	0.9	$0.36a + 0.9b$	$0.405a + 0.9b$
0.5	$0.1a + 0.5b$	$0.125a + 0.5b$	1.0	$0.45a + 1.0b$	$0.5a + 1.0b$

表达式为

$$y_n - y_0 = \sum_{k=0}^{n-1} h(ax_k + b) = \sum_{k=0}^{n-1} h(akh + b) = \sum_{k=0}^{n-1} akh^2 + bh = \frac{n(n-1)}{2}ah^2 + nbh$$

误差为

$$y(x_n) - y_n = \frac{1}{2}a(nh)^2 + b(nh) - \frac{n(n-1)}{2}ah^2 - nbh = \frac{1}{2}nah^2$$

**改进 Euler 方法:**

$$y_{n+1} = y_n + \frac{h}{2}[ax_n + b + a(x_n + h) + b] = y_n + ahx_n + hb + \frac{1}{2}ah^2$$

取步长  $h = 0.1$ , 计算结果如表所示。

表达式为

$$y_n - y_0 = \sum_{k=0}^{n-1} \left( ahx_k + hb + \frac{1}{2}ah^2 \right) = \frac{n(n-1)}{2}ah^2 + nbh + \frac{1}{2}ah^2n = \frac{1}{2}an^2h^2 + nbh$$

$x_n$	$y_n$	$y(x_n)$	$x_n$	$y_n$	$y(x_n)$
0.1	$0.005a + 0.1b$	$0.005a + 0.1b$	0.6	$0.18a + 0.6b$	$0.18a + 0.6b$
0.2	$0.02a + 0.2b$	$0.02a + 0.2b$	0.7	$0.245a + 0.7b$	$0.245a + 0.7b$
0.3	$0.045a + 0.3b$	$0.045a + 0.3b$	0.8	$0.32a + 0.8b$	$0.32a + 0.8b$
0.4	$0.08a + 0.4b$	$0.08a + 0.4b$	0.9	$0.405a + 0.9b$	$0.405a + 0.9b$
0.5	$0.125a + 0.5b$	$0.125a + 0.5b$	1.0	$0.5a + 1.0b$	$0.5a + 1.0b$

误差为

$$y(x_n) - y_n = 0$$

可见改进 Euler 方法的结果没有误差，Euler 方法在一次项系数上有一定误差。

2. 解 改进 Euler 方法的表达式为

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2} [x_n + y_n + x_n + h + y_n + h(x_n + y_n)] \\ &= x_n \left( h + \frac{1}{2}h^2 \right) + y_n \left( 1 + h + \frac{1}{2}h^2 \right) + \frac{1}{2}h^2 \end{aligned}$$

取  $h = 0.1$ ，有

$$y_{n+1} = 0.105x_n + 1.105y_n + 0.005$$

计算结果如表所示

$x_n$	$y_n$	$y(x_n)$	$x_n$	$y_n$	$y(x_n)$
0.1	1.1100	1.1103	0.6	2.0409	2.0442
0.2	1.2420	1.2428	0.7	2.3231	2.3275
0.3	1.3985	1.3997	0.8	2.6456	2.6511
0.4	1.5818	1.5836	0.9	3.0124	3.0192
0.5	1.7949	1.7974	1.0	3.4282	3.4366

与精确解比较后可知有 2 位小数精度。

4. 证明 根据梯形计算格式，

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2} [-y_n - y_{n+1}] \\ y_{n+1} &= \frac{2-h}{2+h} y_n \end{aligned}$$

根据此迭代公式，

$$y_n = \left( \frac{2-h}{2+h} \right)^n y_0 = \left( \frac{2-h}{2+h} \right)^n$$

当  $h \rightarrow 0$  时， $x = x_0 + nh = nh$ ，

$$y_n = \lim_{h \rightarrow 0} \left( \frac{2-h}{2+h} \right)^n = \lim_{h \rightarrow 0} \left( 1 - \frac{2h}{2+h} \right)^{\frac{x}{h}} = \lim_{h \rightarrow 0} \left( 1 - \frac{2h}{2+h} \right)^{\frac{2+h}{2h} \cdot \frac{2h}{2+h} \cdot \frac{x}{h}} = \lim_{h \rightarrow 0} \left[ \left( 1 - \frac{2h}{2+h} \right)^{\frac{2+h}{2h}} \right]^{\frac{2}{2+h} x} = e^{-x}$$

■



5. 解 根据微积分基本定理, 有

$$y = \int_0^x e^{t^2} dt \Rightarrow \begin{cases} y' = e^{x^2} \\ y(0) = 0 \end{cases}$$

构造 Euler 格式,

$$y_{n+1} = y_n + h e^{x_n^2}$$

取  $h = 0.5$ , 有

$$y(0.5) = 0 + 0.5 \times e^{0^2} = 0.5$$

$$y(1) = 0.5 + 0.5 \times e^{0.5^2} = 1.1420$$

$$y(1.5) = 1.1420 + 0.5 \times e^{1^2} = 2.5011$$

$$y(2) = 2.5011 + 0.5 \times e^{1.5^2} = 7.2450$$

6. 解 经典的四阶 Runge-Kutta 方法是

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_1) \\ K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_2) \\ K_4 = f(x_n + h, y_n + h K_3) \end{cases}$$

(1)

$$\begin{cases} y' = x + y, & 0 < x < 1, \\ y(0) = 1 \end{cases}$$

解得

$x_n$	$y_n$
0.2	1.2428
0.4	1.5836
0.6	2.0442
0.8	2.6510
1.0	3.4365

(2)

$$\begin{cases} y' = 3y/(1+x), & 0 < x < 1 \\ y(0) = 1 \end{cases}$$

解得



$x_n$	$y_n$
0.2	1.7275
0.4	2.7430
0.6	4.0942
0.8	5.8292
1.0	7.9960

## 7. 证明

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(K_2 + K_3) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + th, y_n + thK_1) \\ K_3 = f(x_n + (1-t)h, y_n + (1-t)hK_1) \end{cases}$$

根据 Taylor 格式,

$$\begin{aligned} K_1 &= f_n \\ K_2 &= f_n + th(f_x + f \cdot f_y)_n + \cdots \\ K_3 &= f_n + (1-t)h(f_x + f \cdot f_y)_n + \cdots \end{aligned}$$

代入第一个式子,

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}(f_x + f \cdot f_y)_n + \cdots$$

这与二阶 Taylor 格式的前两阶相同, 所以误差是  $O(h^3)$ , 也就是它是二阶 Runge-Kutta 格式。■

## 8. 证明 对于形如

$$\begin{cases} y_{n+1} = y_n + h(\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + ph, y_n + phK_1) \\ K_3 = f(x_n + qh, y_n + qh(rK_1 + sK_2)) \end{cases}$$

的格式, 只要满足

$$\begin{cases} \lambda_2 p + \lambda_3 q = \frac{1}{2} \\ \lambda_2 p^2 + \lambda_3 q^2 = \frac{1}{3} \\ \lambda_3 pqs = \frac{1}{6} \end{cases} \quad (1)$$

就是三阶 Runge-Kutta 格式。

(1)

$$\begin{cases} y_{n+1} = y_n + \frac{h}{4}(K_1 + 3K_3), \\ K_1 = f(x_n, y_n), \\ K_2 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}K_1\right), \\ K_3 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hK_2\right) \end{cases}$$



这里,

$$\lambda_1 = \frac{1}{4}, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{3}{4}, \quad p = \frac{1}{3}, \quad q = \frac{2}{3}, \quad r = 0, \quad s = 1$$

容易验证满足式 (1)。

(2)

$$\begin{cases} y_{n+1} = y_n + \frac{h}{9}(2K_1 + 3K_2 + 4K_3), \\ K_1 = f(x_n, y_n), \\ K_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1\right), \\ K_3 = f\left(x_n + \frac{3}{4}h, y_n + \frac{3}{4}hK_2\right) \end{cases}$$

这里,

$$\lambda_1 = \frac{2}{9}, \quad \lambda_2 = \frac{1}{3}, \quad \lambda_3 = \frac{4}{9}, \quad p = \frac{1}{2}, \quad q = \frac{3}{4}, \quad r = 0, \quad s = 1$$

容易验证满足式 (1)。

12. 解 令  $y_1 = y, y_2 = y'$ ,

(1)

$$\begin{cases} y_1' = y_2 \\ y_2' = 3y_2 - 2y \end{cases} \xrightarrow{z=y_2} \begin{cases} y' = z, y(0) = 1 \\ z' = 3z - 2y, z(0) = 1 \end{cases}$$

(2)

$$\begin{cases} y_1' = y_2 \\ y_2' = 0.1(1 - y^2)y_2 - y \end{cases} \xrightarrow{z=y_2} \begin{cases} y' = z, y(0) = 1 \\ z' = 0.1(1 - y^2)z - y, z(0) = 0 \end{cases}$$

(3) 令  $x_1 = x, x_2 = x', r = \sqrt{x^2 + y^2}$ ,

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{x}{r^3} \\ y_1' = y_2 \\ y_2' = -\frac{y}{r^3} \end{cases} \xrightarrow{u=x_2, z=y_2} \begin{cases} x' = u, x(0) = 0.4 \\ y' = z, y(0) = 0 \\ u' = -\frac{x}{r^3}, u(0) = 0 \\ z' = -\frac{y}{r^3}, z(0) = 2 \end{cases}$$

补充题. 解 对于隐式中点公式,

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})\right)$$

对于模型方程  $y' = \lambda y$ , 有

$$y_{n+1} = y_n + \frac{1}{2}\lambda h(y_n + y_{n+1})$$

也就是

$$y_{n+1} = \frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h} y_n$$

则为了使其绝对稳定, 需要满足

$$\left| \frac{1 + \frac{1}{2}\lambda h}{1 - \frac{1}{2}\lambda h} \right| \leq 1$$