

## 作业 5

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2023 年 12 月 1 日

2. 解

$$f(x) = -3 \times \frac{(x-1)(x-2)}{(-1-1) \cdot (-1-2)} + 4 \times \frac{(x-1)(x+1)}{(2-1) \cdot (2+1)} = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}$$

3. 解 计算差商表:

$x_k$	$f(x_k)$	一阶差商	二阶差商
0.4	-0.916291	2.23144	-2.04115
0.5	-0.693147	1.82321	
0.6	-0.510826	1.53061	-0.922
0.7	-0.357765	1.34621	
0.8	-0.223144		

分段线性插值:

$$I_l(x) = \begin{cases} \frac{x-0.5}{-0.1} \ln(0.4) + \frac{x-0.4}{0.1} \ln(0.5) = 2.23144x - 1.808867, & 0.4 \leq x \leq 0.5, \\ \frac{x-0.6}{-0.1} \ln(0.5) + \frac{x-0.5}{0.1} \ln(0.6) = 1.82321x - 1.604752, & 0.5 \leq x \leq 0.6, \\ \frac{x-0.7}{-0.1} \ln(0.6) + \frac{x-0.6}{0.1} \ln(0.7) = 1.53061x - 1.429192, & 0.6 \leq x \leq 0.7, \\ \frac{x-0.8}{-0.1} \ln(0.7) + \frac{x-0.7}{0.1} \ln(0.8) = 1.34621x - 1.300112, & 0.7 \leq x \leq 0.8. \end{cases}$$

分段二次插值:

$$I_d(x) = \begin{cases} 2.23144x - 1.808867 - 2.04115(x-0.4)(x-0.5), & 0.4 \leq x \leq 0.6 \\ 1.53061x - 1.429192 - 0.922(x-0.6)(x-0.7), & 0.6 \leq x \leq 0.8 \end{cases}$$

$$= \begin{cases} -2.04115x^2 + 4.068475x - 2.217097, & 0.4 \leq x \leq 0.6 \\ -0.922x^2 + 2.72921x - 1.816432, & 0.6 \leq x \leq 0.8 \end{cases}$$

分别计算  $\ln 0.54$  的近似值:

$$I_l(0.54) = -0.6202186$$

$$I_d(0.54) = -0.61531984$$



4. 解 令  $f(x) = \cos x$ ,  $h = x_{i+1} - x_i = 1' = \frac{\pi}{10800}$ , 根据分段线性插值, 当  $x_i \leq x \leq x_{i+1}$  时,

$$\begin{aligned} L(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) \\ &= \frac{1}{h} [-(x - x_{i+1}) f(x_i) + (x - x_i) f(x_{i+1})] \\ L^*(x) &= \frac{1}{h} [-(x - x_{i+1}) f^*(x_i) + (x - x_i) f^*(x_{i+1})] \end{aligned}$$

则

$$\begin{aligned} |f(x) - L^*(x)| &= |f(x) - L(x) + L(x) - L^*(x)| \\ &\leq |f(x) - L(x)| + |L(x) - L^*(x)| \end{aligned}$$

而根据插值余项, 有  $\xi \in (x_i, x_{i+1})$  使得,

$$\begin{aligned} |f(x) - L(x)| &= \left| \frac{f''(\xi)}{2!} (x - x_i)(x - x_{i+1}) \right| \\ &= \left| -\frac{\cos \xi}{2} (x - x_i)(x - x_{i+1}) \right| \\ &\leq \frac{1}{2} |(x - x_i)(x - x_{i+1})| \\ &\leq \frac{1}{2} \frac{h^2}{4} = \frac{h^2}{8} \end{aligned}$$

另一方面, 考虑到函数表有五位有效数字,

$$\begin{aligned} |L(x) - L^*(x)| &= \frac{1}{h} |-(x - x_{i+1})(f^*(x_i) - f(x_i)) + (x - x_i)(f^*(x_{i+1}) - f(x_{i+1}))| \\ &\leq \frac{1}{h} (|(x - x_{i+1})(f^*(x_i) - f(x_i))| + |(x - x_i)(f^*(x_{i+1}) - f(x_{i+1}))|) \\ &\leq \frac{1}{h} \cdot \frac{1}{2} \times 10^{-5} \cdot (|x - x_{i+1}| + |x - x_i|) \\ &= \frac{1}{h} \cdot \frac{1}{2} \times 10^{-5} \cdot h \\ &= \frac{1}{2} \times 10^{-5} \end{aligned}$$

综上, 有总误差限

$$|f(x) - L^*(x)| \leq \frac{h^2}{8} + \frac{1}{2} \times 10^{-5} = 0.501 \times 10^{-5}$$

5. 解

$$\begin{aligned} l_2(x) &= \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\ &= -\frac{1}{2h^3} (x - x_0)(x - x_1)(x - x_3) \end{aligned}$$

令  $t = x - x_0 \in [0, 3h]$ , 有

$$\begin{aligned} L_2(t) &= -\frac{1}{2h^3} t(t - h)(t - 3h) \\ &= -\frac{1}{2h^3} (t^3 - 4ht^2 + 3h^2t) \end{aligned}$$

为了求  $\max_{x_0 \leq x \leq x_3} l_2(x)$ , 也就是求  $\min_{0 \leq t \leq 3h} \phi(t)$ , 其中

$$\phi(t) = t^3 - 4ht^2 + 3h^2t$$

考虑到

$$\phi'(t) = 3t^2 - 8ht + 3h^2 = 0$$

有

$$t_{1,2} = \frac{4h \pm \sqrt{7}h}{3}$$

则最小值应该在  $t = 0$  或  $t = \frac{4+\sqrt{7}}{3}h$  处取到, 代入有

$$\min_{0 \leq t \leq 3h} \phi(t) = \phi\left(\frac{4+\sqrt{7}}{3}h\right) = -\frac{2(7\sqrt{7}+10)h^3}{27}$$

则

$$\min_{x_0 \leq x \leq x_3} l_2(x) = l_2\left(x_0 + \frac{4+\sqrt{7}}{3}h\right) = \frac{7\sqrt{7}+10}{27}$$

## 6. 证明 (1) Lagrange 插值多项式

$$L(x) = \sum_{j=0}^n x_j^k l_j(x)$$

就是  $x^k (k = 0, 1, \dots, n)$  的插值多项式, 根据插值余项

$$x^k - \sum_{j=0}^n x_j^k l_j(x) = R_n(x) = \frac{(\xi^k)^{(n+1)}}{(n+1)!} \omega_{n+1}(x) \equiv 0$$

其中  $\xi \in (\min\{x_0, \dots, x_n\}, \max\{x_0, \dots, x_n\})$ , 由于  $k < n$ , 所以导数  $(\xi^k)^{(n+1)}$  为 0, 最后一个等号成立。

## (2) 根据二项式定理展开

$$\begin{aligned} \sum_{j=0}^n (x_j - x)^k l_j(x) &= \sum_{j=0}^n \sum_{l=0}^k \binom{k}{l} x_j^l (-x)^{k-l} l_j(x) \\ &= \sum_{l=0}^k \binom{k}{l} (-x)^{k-l} \sum_{j=0}^n x_j^l l_j(x) \\ &= \sum_{l=0}^k \binom{k}{l} (-x)^{k-l} x^l && \text{由 (1) 可得} \\ &= (-x + x)^k \\ &\equiv 0 \end{aligned}$$

## 7. 证明 对 $(a, f(a) = 0), (b, f(b) = 0)$ 建立 $f(x)$ 的插值多项式, 有

$$L(x) = f(a) \frac{x-b}{a-b} + f(b) \frac{x-a}{b-a} \equiv 0$$

考虑插值余项, 存在  $\xi \in (a, b)$  满足

$$f(x) = f(x) - L(x) = R(x) = \frac{f''(\xi)}{2}(x-a)(x-b)$$

立刻有

$$\max_{a \leq x \leq b} |f(x)| \leq \frac{1}{8}(b-a)^2 \max_{a \leq x \leq b} |f''(x)|$$

等号在  $x = \frac{a+b}{2}$  取到。 ■

8. 解 根据分段二次插值插值余项, 对于  $x_0 \leq x \leq x_2$ , 存在  $\xi \in (x_0, x_2)$  使得

$$R(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2) = \frac{e^\xi}{6}(x-x_0)(x-x_1)(x-x_2)$$

则

$$|R(x)| \leq \frac{1}{6}|x-x_0||x-x_1||x-x_2| \max_{-4 \leq x \leq x} |e^x| = \frac{e^4}{6}|x-x_0||x-x_1||x-x_2|$$

令  $t = |x-x_1| \in [0, h]$ , 有

$$r(t) = \frac{e^4}{6}(h-t)t(h+t) = \frac{e^4}{6}(-t^3 + h^2t) \leq r\left(\frac{h}{\sqrt{3}}\right) = \frac{e^4}{6} \frac{2h^3}{3\sqrt{3}} = \frac{\sqrt{3}e^4h^3}{27}$$

根据总误差限不超过  $10^{-6}$  有,

$$\frac{\sqrt{3}e^4h^3}{27} \leq 10^{-6}$$

$$h \leq \frac{3 \times 10^{-2}}{e} \sqrt[3]{\frac{1}{\sqrt{3}e}} = 0.00658$$

15. 证明 (1) 起步

$$F[x_0, x_1] = \frac{F(x_1) - F(x_0)}{x_1 - x_0} = \frac{cf(x_1) - cf(x_0)}{x_1 - x_0} = cf[x_0, x_1]$$

步进 假设

$$F[x_0, x_1, \dots, x_k] = cf[x_0, x_1, \dots, x_k]$$

则

$$\begin{aligned} F[x_0, x_1, \dots, x_k, x_{k+1}] &= \frac{F[x_1, \dots, x_k, x_{k+1}] - F[x_0, \dots, x_k]}{x_{k+1} - x_0} \\ &= \frac{cf[x_1, \dots, x_k, x_{k+1}] - cf[x_0, \dots, x_k]}{x_{k+1} - x_0} \\ &= cf[x_0, x_1, \dots, x_k, x_{k+1}] \end{aligned}$$

结论 根据数学归纳法原理, 结论成立:

$$F[x_0, x_1, \dots, x_n] = cf[x_0, x_1, \dots, x_n] \quad (1)$$



(2) 起步

$$\begin{aligned} F[x_0, x_1] &= \frac{F(x_1) - F(x_0)}{x_1 - x_0} \\ &= \frac{[f(x_1) + g(x_1)] - [f(x_0) + g(x_0)]}{x_1 - x_0} \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{g(x_1) - g(x_0)}{x_1 - x_0} \\ &= f[x_0, x_1] + g[x_0, x_1] \end{aligned}$$

步进 假设

$$F[x_0, \dots, x_k] = f[x_0, \dots, x_k] + g[x_0, \dots, x_k]$$

则

$$\begin{aligned} F[x_0, \dots, x_k, x_{k+1}] &= \frac{F[x_1, \dots, x_{k+1}] - F[x_0, \dots, x_k]}{x_{k+1} - x_0} \\ &= \frac{(f[x_1, \dots, x_{k+1}] + g[x_1, \dots, x_{k+1}]) - (f[x_0, \dots, x_k] + g[x_0, \dots, x_k])}{x_{k+1} - x_0} \\ &= \frac{f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k]}{x_{k+1} - x_0} + \frac{g[x_1, \dots, x_{k+1}] - g[x_0, \dots, x_k]}{x_{k+1} - x_0} \\ &= f[x_0, \dots, x_k, x_{k+1}] + g[x_0, \dots, x_k, x_{k+1}] \end{aligned}$$

结论 根据数学归纳法原理，结论成立：

$$F[x_0, \dots, x_n] = f[x_0, \dots, x_n] + g[x_0, \dots, x_n] \quad (2)$$

■

16. 解 令  $\mathcal{F}_k(x) = x^k$ ，下面证明：

$$\mathcal{F}_k[2^0, 2^1, \dots, 2^m] = \prod_{i=0}^{m-1} \frac{2^{k-i} - 1}{2^{i+1} - 1} \quad (m \geq 1) \quad (3)$$

起步

$$\mathcal{F}_k[2^0, 2^1] = \frac{(2^1)^k - (2^0)^k}{2^1 - 2^0} = \frac{(2 \cdot 2^0)^k - (2^0)^k}{2^1 - 2^0} = \frac{2^k - 1}{2^1 - 1}$$

步进 假设

$$\mathcal{F}_k[2^0, \dots, 2^n] = \prod_{i=0}^{n-1} \frac{2^{k-i} - 1}{2^{i+1} - 1}$$

则根据差商的性质，有

$$\begin{aligned} \mathcal{F}_k[2^1, \dots, 2^{n+1}] &= \sum_{j=1}^{n+1} \frac{\mathcal{F}_k(2^j)}{\prod_{\substack{i=1 \\ i \neq j}}^{n+1} (2^j - 2^i)} \\ &= \sum_{j=1}^{n+1} \frac{2^k \mathcal{F}_k(2^{j-1})}{2^n \prod_{\substack{i=1 \\ i \neq j}}^{n+1} (2^{j-1} - 2^{i-1})} \\ &= \frac{2^k}{2^n} \sum_{j=0}^n \frac{\mathcal{F}_k(2^j)}{\prod_{\substack{i=0 \\ i \neq j}}^n (2^j - 2^i)} \\ &= 2^{k-n} \mathcal{F}_k[2^0, \dots, 2^n] \end{aligned}$$

故

$$\begin{aligned}\mathcal{F}_k[2^0, \dots, 2^{n+1}] &= \frac{\mathcal{F}_k[2^1, \dots, 2^{n+1}] - \mathcal{F}_k[2^0, \dots, 2^n]}{2^{n+1} - 2^0} \\ &= \frac{2^{k-n} - 1}{2^{n+1} - 1} \mathcal{F}_k[2^0, \dots, 2^n] \\ &= \prod_{i=0}^n \frac{2^{k-i} - 1}{2^{i+1} - 1}\end{aligned}$$

结论 根据数学归纳法原理，有结论成立：

$$\mathcal{F}_k[2^0, 2^1, \dots, 2^m] = \prod_{i=0}^{m-1} \frac{2^{k-i} - 1}{2^{i+1} - 1} \quad (m \geq 1)$$

实际上有下面结论的成立，此处不证：

$$\mathcal{F}_k[2^a, 2^{a+1}, \dots, 2^{a+m}] = (2^a)^{k-m} \prod_{i=0}^{m-1} \frac{2^{k-i} - 1}{2^{i+1} - 1} \quad (a \geq 0, m \geq 1)$$

在结论式子 (3) 注意到：当  $m \geq k+1$  时，分子存在 0 因子，

$$\mathcal{F}_k[2^0, 2^1, \dots, 2^m] = 0 \quad (m \geq k+1) \quad (4)$$

考虑到结论式子 (1) 和 (2) 的线性性质，有

$$\begin{aligned}f[2^0, \dots, 2^7] &= \mathcal{F}_7[2^0, \dots, 2^7] + \mathcal{F}_4[2^0, \dots, 2^7] + 3\mathcal{F}_3[2^0, \dots, 2^7] + \mathcal{F}_0[2^0, \dots, 2^7] \\ &= \mathcal{F}_7[2^0, \dots, 2^7] \\ &= \prod_{i=0}^6 \frac{2^{7-i} - 1}{2^{i+1} - 1} \\ &= 1\end{aligned}$$

以及

$$\begin{aligned}f[2^0, \dots, 2^8] &= \mathcal{F}_7[2^0, \dots, 2^8] + \mathcal{F}_4[2^0, \dots, 2^8] + 3\mathcal{F}_3[2^0, \dots, 2^8] + \mathcal{F}_0[2^0, \dots, 2^8] \\ &= 0\end{aligned}$$

#### 19. 解 使用 Newton-Hermite 插值，差商表

$x_i$	$f(x_i)$	一阶差商	二阶差商	三阶差商	四阶差商
0	0	$f[0, 0] = 0$	$f[0, 0, 1] = 1$	$f[0, 0, 1, 1] = -1$	$f[0, 0, 1, 1, 2] = \frac{1}{4}$
0	0	$f[0, 1] = 1$	$f[0, 1, 1] = 0$	$f[0, 1, 1, 2] = -\frac{1}{2}$	
1	1	$f[1, 1] = 1$	$f[1, 1, 2] = -1$		
1	1	$f[1, 2] = 0$			
2	1				

$$\begin{aligned}
 F(x) &= f(0) + f[0, 0](x - 0) + f[0, 0, 1](x - 0)^2 + f[0, 0, 1, 1](x - 0)^2(x - 1) + f[0, 0, 1, 1, 2](x - 0)^2(x - 1)^2 \\
 &= 0 + 0 \times (x - 0) + 1 \times (x - 0)^2 + (-1) \times (x - 0)^2(x - 1) + \frac{1}{4}(x - 0)^2(x - 1)^2 \\
 &= \frac{1}{4}x^4 - \frac{3}{2}x^3 + \frac{9}{4}x^2
 \end{aligned}$$

21. 解 等距节点函数值

$x$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	0.038462	0.058824	0.1	0.2	0.5	1	0.5	0.2	0.1	0.058824	0.038462

分段线性插值函数

$$\begin{aligned}
 I_h(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) \\
 &= \frac{1}{h} [-(x - x_{i+1})f(x_i) + (x - x_i)f(x_{i+1})] \\
 &= \frac{1}{h} (f(x_{i+1}) - f(x_i))x + \frac{1}{h} [x_{i+1}f(x_i) - x_i f(x_{i+1})]
 \end{aligned}$$

令  $I_h(x) = ax + b$ , 有

$x$	[-5, -4]	[-4, -3]	[-3, -2]	[-2, -1]	[-1, 0]	[0, 1]	[1, 2]	[2, 3]	[3, 4]	[4, 5]
$a$	0.02036	0.04118	0.1	0.3	0.5	-0.5	-0.3	-0.1	-0.04118	-0.02036
$b$	0.14027	0.22353	0.4	0.8	1	1	0.8	0.4	0.22353	0.14027

得到中点值与真实值, 并计算误差

$x$	-4.5	-3.5	-2.5	-1.5	-0.5	0.5	1.5	2.5	3.5	4.5
$I_h(x)$	0.04864	0.07941	0.15	0.35	0.75	0.75	0.35	0.15	0.07941	0.04864
$f(x)$	0.04706	0.07547	0.13793	0.30769	0.8	0.8	0.30769	0.13793	0.07547	0.04706
$\Delta$	0.00158	0.00394	0.01207	0.04231	-0.05	-0.05	0.04231	0.01207	0.00394	0.00158

误差估计, 根据线性插值余项

$$|R(x)| = \left| \frac{f''(\xi)}{2!} (x - x_i)(x - x_{i+1}) \right| \leq \frac{1}{8} \max_{-5 \leq x \leq 5} |f''(x)|$$

其中

$$f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}$$



而

$$f'''(x) = \frac{-24x(x^2 - 1)}{(x^2 + 1)^4} = 0$$

$$x = -1, 0, 1$$

$$f''(0) = -2$$

$$f''(-1) = f''(1) = 0.5$$

$$f''(-5) = f''(5) = 0.008421$$

故

$$|R(x)| \leq \frac{1}{4} = 0.25$$

补充题 解 (1) 根据三转角方程, 有

$$\begin{pmatrix} 2 & \mu_1 \\ \lambda_2 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} g_1 - \lambda_1 m_0 \\ g_2 - \mu_2 m_3 \end{pmatrix} \quad (5)$$

其中

$$\lambda_i = \frac{h_i}{h_{i-1} + h_i} = \frac{1}{2}$$

$$\mu_i = \frac{h_{i-1}}{h_{i-1} + h_i} = \frac{1}{2}$$

$$g_i = 3(\lambda_i f[x_i, x_{i+1}] + \mu_i f[x_{i-1}, x_i]) = 0$$

考虑到已知  $m_0 = 1, m_3 = 0$ , 故式 (5) 可化为

$$\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

解得

$$m_1 = -\frac{4}{15} \quad m_2 = \frac{1}{15}$$

则根据分段三次 Hermite 插值有

$$\begin{aligned} S(x) &= y_i \alpha_i(x) + y_{i+1} \alpha_{i1}(x) + m_i \beta_i(x) + m_{i+1} \beta_{i1}(x) \\ &= m_i \beta_i(x) + m_{i+1} \beta_{i1}(x) \end{aligned} \quad x \in \Delta_i = [x_i, x_{i+1}]$$

其中

$$\alpha_i(x) = \left(1 - 2 \frac{x - x_i}{x_i - x_{i+1}}\right) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)^2 = (1 + 2(x - x_i))(x - x_{i+1})^2$$

$$\alpha_{i1}(x) = \left(1 - 2 \frac{x - x_{i+1}}{x_{i+1} - x_i}\right) \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2 = (1 - 2(x - x_{i+1}))(x - x_i)^2$$

$$\beta_i(x) = (x - x_i) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)^2 = (x - x_i)(x - x_{i+1})^2$$

$$\beta_{i1}(x) = (x - x_{i+1}) \left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2 = (x - x_{i+1})(x - x_i)^2$$



将下述参数代入

$x_i$	0	1	2	3
$y_i$	0	0	0	0
$m_i$	1	$-\frac{4}{15}$	$\frac{1}{15}$	0

有

$$S(x) = \begin{cases} \frac{11}{15}x^3 - \frac{26}{15}x^2 + x, & x \in [0, 1], \\ -\frac{1}{5}x^3 + \frac{16}{15}x^2 - \frac{9}{5}x + \frac{14}{15}, & x \in [1, 2], \\ \frac{1}{15}x^3 - \frac{8}{15}x^2 + \frac{7}{5}x - \frac{6}{5}, & x \in [2, 3]. \end{cases}$$

(2) 根据三弯矩方程, 已知  $M_0 = 1, M_3 = 0$ , 有

$$\begin{aligned} 2m_0 + m_1 &= 3f[x_0, x_1] - \frac{h_0}{2}M_0 = -\frac{1}{2} \\ m_2 + 2m_3 &= 3f[x_2, x_3] + \frac{h_2}{2}M_3 = 0 \end{aligned}$$

结合式 (5), 即解方程组

$$\begin{pmatrix} 2 & 1 & & \\ \frac{1}{2} & 2 & \frac{1}{2} & \\ & \frac{1}{2} & 2 & \frac{1}{2} \\ & & 1 & 2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

解得参数

$x_i$	0	1	2	3
$y_i$	0	0	0	0
$m_i$	$-\frac{13}{45}$	$\frac{7}{90}$	$-\frac{1}{45}$	$\frac{1}{90}$

有

$$S(x) = \begin{cases} -\frac{19}{90}x^3 + \frac{1}{2}x^2 - \frac{13}{45}x, & x \in [0, 1], \\ \frac{1}{18}x^3 - \frac{3}{10}x^2 + \frac{23}{45}x - \frac{4}{15}, & x \in [1, 2], \\ -\frac{1}{90}x^3 + \frac{1}{10}x^2 - \frac{13}{45}x + \frac{4}{15}, & x \in [2, 3]. \end{cases}$$