

作业 4

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1. 解

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 4 & \\ & & 10 \end{pmatrix} - \begin{pmatrix} & & \\ 1 & & \\ -2 & 3 & \end{pmatrix} - \begin{pmatrix} & -2 & -1 \\ & & -2 \\ & & \end{pmatrix} = D - L - U$$

$$b = \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix}$$

(1) 由于 A 是严格占优矩阵, 所以 Jacobi 迭代法和 Gauss-Seidel 迭代法解这个方程组都是收敛的。

(2) Jacobi 迭代法 对于 Jacobi 迭代法, 其迭代矩阵

$$B = D^{-1}(L + U) = \begin{pmatrix} & -\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{4} & & -\frac{1}{2} \\ -\frac{1}{5} & \frac{3}{10} & \end{pmatrix} \quad f = D^{-1}b = \begin{pmatrix} -\frac{12}{5} \\ 5 \\ \frac{3}{10} \end{pmatrix}$$

取迭代初值 $x^{(0)} = (0, 0, 0)^T$, 记 $\epsilon^{(k)} = x^{(k)} - x^{(k-1)}$, 有

$x^{(1)} = Bx^{(0)} + f = (-2.4, 5, 0.3)^T$	$\ \epsilon^{(1)}\ _{\infty} = 5.$
$x^{(2)} = Bx^{(1)} + f = (-4.46, 4.25, 2.28)^T$	$\ \epsilon^{(2)}\ _{\infty} = 2.06$
$x^{(3)} = Bx^{(2)} + f = (-4.556, 2.745, 2.467)^T$	$\ \epsilon^{(3)}\ _{\infty} = 1.505$
$x^{(4)} = Bx^{(3)} + f = (-3.9914, 2.6275, 2.0347)^T$	$\ \epsilon^{(4)}\ _{\infty} = 0.5646$
$x^{(5)} = Bx^{(4)} + f = (-3.85794, 2.9848, 1.88653)^T$	$\ \epsilon^{(5)}\ _{\infty} = 0.3573$
$x^{(6)} = Bx^{(5)} + f = (-3.971226, 3.09225, 1.967028)^T$	$\ \epsilon^{(6)}\ _{\infty} = 0.113286$
$x^{(7)} = Bx^{(6)} + f = (-4.0303056, 3.0236795, 2.0219202)^T$	$\ \epsilon^{(7)}\ _{\infty} = 0.0685705$
$x^{(8)} = Bx^{(7)} + f = (-4.01385584, 2.9814635, 2.01316497)^T$	$\ \epsilon^{(8)}\ _{\infty} = 0.042216$
$x^{(9)} = Bx^{(8)} + f = (-3.99521839, 2.98995356, 1.99721022)^T$	$\ \epsilon^{(9)}\ _{\infty} = 0.01863745$
$x^{(10)} = Bx^{(9)} + f = (-3.99542347, 3.00259029, 1.99602975)^T$	$\ \epsilon^{(10)}\ _{\infty} = 0.01263674$
$x^{(11)} = Bx^{(10)} + f = (-4.00024207, 3.00312926, 1.99986178)^T$	$\ \epsilon^{(11)}\ _{\infty} = 0.0048186$



$$\begin{aligned}
 \mathbf{x}^{(12)} &= \mathbf{B}\mathbf{x}^{(11)} + \mathbf{f} = (-4.00122406, 3.00000859, 2.00098719)^\top & \|\boldsymbol{\epsilon}^{(12)}\|_\infty &= 0.00312067 \\
 \mathbf{x}^{(13)} &= \mathbf{B}\mathbf{x}^{(12)} + \mathbf{f} = (-4.00020088, 2.99920039, 2.00024739)^\top & \|\boldsymbol{\epsilon}^{(13)}\|_\infty &= 0.00102319 \\
 \mathbf{x}^{(14)} &= \mathbf{B}\mathbf{x}^{(13)} + \mathbf{f} = (-3.99972963, 2.99982609, 1.99980029)^\top & \|\boldsymbol{\epsilon}^{(14)}\|_\infty &= 0.0006257 \\
 \mathbf{x}^{(15)} &= \mathbf{B}\mathbf{x}^{(14)} + \mathbf{f} = (-3.99989049, 3.00016745, 1.99989375)^\top & \|\boldsymbol{\epsilon}^{(15)}\|_\infty &= 0.00034136 \\
 \mathbf{x}^{(16)} &= \mathbf{B}\mathbf{x}^{(15)} + \mathbf{f} = (-4.00004573, 3.0000805, 2.00002833)^\top & \|\boldsymbol{\epsilon}^{(16)}\|_\infty &= 0.00015524 \\
 \mathbf{x}^{(17)} &= \mathbf{B}\mathbf{x}^{(16)} + \mathbf{f} = (-4.00003787, 2.9999744, 2.0000333)^\top & \|\boldsymbol{\epsilon}^{(17)}\|_\infty &= 0.0001061 \\
 \mathbf{x}^{(18)} &= \mathbf{B}\mathbf{x}^{(17)} + \mathbf{f} = (-3.99999642, 2.99997389, 1.99999989)^\top & \|\boldsymbol{\epsilon}^{(18)}\|_\infty &= 4.14468074 \times 10^{-5}
 \end{aligned}$$

总共需要 18 次满足要求。

Guass-Seidel 迭代法 对于 Guass-Seidel 迭代法,

$$\begin{aligned}
 \mathbf{G} &= (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} -2 & -1 & \\ & -2 & \\ & & \end{pmatrix} = \begin{pmatrix} 0 & -0.4 & -0.2 \\ 0 & -0.1 & -0.55 \\ 0 & 0.05 & -0.125 \end{pmatrix} \\
 \mathbf{f} &= (\mathbf{D} - \mathbf{L})^{-1}\mathbf{b} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 4.4 \\ 2.1 \end{pmatrix}
 \end{aligned}$$

取迭代初值 $\mathbf{x}^{(0)} = (0, 0, 0)^\top$, 记 $\boldsymbol{\epsilon}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}$, 有

$$\begin{aligned}
 \mathbf{x}^{(1)} &= \mathbf{G}\mathbf{x}^{(0)} + \mathbf{f} = (-2.4, 4.4, 2.1)^\top & \|\boldsymbol{\epsilon}^{(1)}\|_\infty &= 4.4 \\
 \mathbf{x}^{(2)} &= \mathbf{G}\mathbf{x}^{(1)} + \mathbf{f} = (-4.58, 2.805, 2.0575)^\top & \|\boldsymbol{\epsilon}^{(2)}\|_\infty &= 2.18 \\
 \mathbf{x}^{(3)} &= \mathbf{G}\mathbf{x}^{(2)} + \mathbf{f} = (-3.9335, 2.987875, 1.9830625)^\top & \|\boldsymbol{\epsilon}^{(3)}\|_\infty &= 0.6465 \\
 \mathbf{x}^{(4)} &= \mathbf{G}\mathbf{x}^{(3)} + \mathbf{f} = (-3.9917625, 3.01052813, 2.00151094)^\top & \|\boldsymbol{\epsilon}^{(4)}\|_\infty &= 0.0582625 \\
 \mathbf{x}^{(5)} &= \mathbf{G}\mathbf{x}^{(4)} + \mathbf{f} = (-4.00451344, 2.99811617, 2.00033754)^\top & \|\boldsymbol{\epsilon}^{(5)}\|_\infty &= 0.01275094 \\
 \mathbf{x}^{(6)} &= \mathbf{G}\mathbf{x}^{(5)} + \mathbf{f} = (-3.99931398, 3.00000274, 1.99986362)^\top & \|\boldsymbol{\epsilon}^{(6)}\|_\infty &= 0.00519946 \\
 \mathbf{x}^{(7)} &= \mathbf{G}\mathbf{x}^{(6)} + \mathbf{f} = (-3.99997382, 3.00007474, 2.00001718)^\top & \|\boldsymbol{\epsilon}^{(7)}\|_\infty &= 0.00065984 \\
 \mathbf{x}^{(8)} &= \mathbf{G}\mathbf{x}^{(7)} + \mathbf{f} = (-4.00003333, 2.99998307, 2.00000159)^\top & \|\boldsymbol{\epsilon}^{(8)}\|_\infty &= 9.16628308 \times 10^{-5}
 \end{aligned}$$

总共需要 8 次满足要求。

8. 解

$$\begin{aligned}
 \mathbf{A} &= \begin{pmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \\ & \\ & \end{pmatrix} = \mathbf{D} - \mathbf{L} - \mathbf{U} \\
 \mathbf{b} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
 \end{aligned}$$



(1) 对于 Jacobi 迭代法,

$$\mathbf{B}_0 = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \end{pmatrix} = \begin{pmatrix} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \end{pmatrix}$$

求它的特征值

$$0 = |\lambda \mathbf{E} - \mathbf{B}_0| = \begin{vmatrix} \lambda & -\frac{1}{4} & -\frac{1}{4} \\ & \lambda & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \lambda \end{vmatrix} = \lambda^2 \left(\lambda - \frac{1}{2} \right) \left(\lambda + \frac{1}{2} \right)$$

得到 $\lambda_{1,2} = 0, \lambda_3 = \frac{1}{2}, \lambda_4 = -\frac{1}{2}$, 故谱半径 $\rho(\mathbf{B}_0) = \frac{1}{2}$ 。

(2) 对于 Gauss-Seidel 迭代法,

$$\mathbf{B}_0 = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}^{-1} \begin{pmatrix} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & & \frac{1}{4} \\ & & \end{pmatrix} = \begin{pmatrix} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \end{pmatrix}$$

求它的特征值

$$0 = |\lambda \mathbf{E} - \mathbf{B}_0| = \begin{vmatrix} \lambda & -\frac{1}{4} & -\frac{1}{4} \\ & \lambda & -\frac{1}{4} \\ & \lambda - \frac{1}{8} & -\frac{1}{8} \end{vmatrix} = \lambda^3 \left(\lambda - \frac{1}{4} \right)$$

得到 $\lambda_{1,2,3} = 0, \lambda_4 = \frac{1}{4}$, 故谱半径 $\rho(\mathbf{B}_0) = \frac{1}{4}$ 。

(3) 由于谱半径都小于 1, 所以 Jacobi 迭代法和 Gauss-Seidel 迭代法均收敛。

9. 解

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4 \end{pmatrix} = \mathbf{D} - \mathbf{L} - \mathbf{U} = \begin{pmatrix} 4 & & \\ & 4 & \\ & & 4 \end{pmatrix} - \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix} - \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

对于 SOR 迭代法, 松弛因子为 ω , 有

$$\mathbf{L}_\omega = (\mathbf{D} - \omega \mathbf{L})^{-1}[(1 - \omega)\mathbf{D} + \omega \mathbf{U}] = \begin{pmatrix} 4 & & \\ -\omega & 4 & \\ & -\omega & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4(1 - \omega) & \omega & \\ & 4(1 - \omega) & \omega \\ & & 4(1 - \omega) \end{pmatrix}$$

$$\mathbf{f} = \omega(\mathbf{D} - \omega \mathbf{L})^{-1}\mathbf{b} = \omega \begin{pmatrix} 4 & & \\ -\omega & 4 & \\ & -\omega & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$



取迭代初值 $\mathbf{x}^{(0)} = (0, 0, 0)^\top$, 记 $\boldsymbol{\epsilon}^{(k)} = \mathbf{x}^* - \mathbf{x}^{(k)}$, 有

(1) $\omega = 1.03$ 需要 5 次迭代:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{L}_\omega \mathbf{x}^{(0)} + \mathbf{f} = (0.2575, 1.09630625, -0.49020114)^\top & \|\boldsymbol{\epsilon}^{(1)}\|_\infty &= 0.2425 \\ \mathbf{x}^{(2)} &= \mathbf{L}_\omega \mathbf{x}^{(1)} + \mathbf{f} = (0.53207386, 1.00789304, -0.49826151)^\top & \|\boldsymbol{\epsilon}^{(2)}\|_\infty &= 0.03207386 \\ \mathbf{x}^{(3)} &= \mathbf{L}_\omega \mathbf{x}^{(2)} + \mathbf{f} = (0.50107024, 1.00048646, -0.49992689)^\top & \|\boldsymbol{\epsilon}^{(3)}\|_\infty &= 0.00107024 \\ \mathbf{x}^{(4)} &= \mathbf{L}_\omega \mathbf{x}^{(3)} + \mathbf{f} = (0.50009316, 1.00002822, -0.49999493)^\top & \|\boldsymbol{\epsilon}^{(4)}\|_\infty &= 9.31555814 \times 10^{-5} \\ \mathbf{x}^{(5)} &= \mathbf{L}_\omega \mathbf{x}^{(4)} + \mathbf{f} = (0.50000447, 1.00000161, -0.49999974)^\top & \|\boldsymbol{\epsilon}^{(5)}\|_\infty &= 4.47176785 \times 10^{-6} \end{aligned}$$

(2) $\omega = 1$ 需要 6 次迭代:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{L}_\omega \mathbf{x}^{(0)} + \mathbf{f} = (0.25, 1.0625, -0.484375)^\top & \|\boldsymbol{\epsilon}^{(1)}\|_\infty &= 0.25 \\ \mathbf{x}^{(2)} &= \mathbf{L}_\omega \mathbf{x}^{(1)} + \mathbf{f} = (0.515625, 1.0078125, -0.49804688)^\top & \|\boldsymbol{\epsilon}^{(2)}\|_\infty &= 0.015625 \\ \mathbf{x}^{(3)} &= \mathbf{L}_\omega \mathbf{x}^{(2)} + \mathbf{f} = (0.50195312, 1.00097656, -0.49975586)^\top & \|\boldsymbol{\epsilon}^{(3)}\|_\infty &= 0.00195312 \\ \mathbf{x}^{(4)} &= \mathbf{L}_\omega \mathbf{x}^{(3)} + \mathbf{f} = (0.50024414, 1.00012207, -0.49996948)^\top & \|\boldsymbol{\epsilon}^{(4)}\|_\infty &= 0.00024414 \\ \mathbf{x}^{(5)} &= \mathbf{L}_\omega \mathbf{x}^{(4)} + \mathbf{f} = (0.50003052, 1.00001526, -0.49999619)^\top & \|\boldsymbol{\epsilon}^{(5)}\|_\infty &= 3.05175781 \times 10^{-5} \\ \mathbf{x}^{(6)} &= \mathbf{L}_\omega \mathbf{x}^{(5)} + \mathbf{f} = (0.50000381, 1.00000191, -0.49999952)^\top & \|\boldsymbol{\epsilon}^{(6)}\|_\infty &= 3.81469727 \times 10^{-6} \end{aligned}$$

(3) $\omega = 1.1$ 需要 6 次迭代:

$$\begin{aligned} \mathbf{x}^{(1)} &= \mathbf{L}_\omega \mathbf{x}^{(0)} + \mathbf{f} = (0.275, 1.175625, -0.50170313)^\top & \|\boldsymbol{\epsilon}^{(1)}\|_\infty &= 0.225 \\ \mathbf{x}^{(2)} &= \mathbf{L}_\omega \mathbf{x}^{(1)} + \mathbf{f} = (0.57079688, 1.00143828, -0.49943416)^\top & \|\boldsymbol{\epsilon}^{(2)}\|_\infty &= 0.07079688 \\ \mathbf{x}^{(3)} &= \mathbf{L}_\omega \mathbf{x}^{(2)} + \mathbf{f} = (0.49331584, 0.99817363, -0.50055883)^\top & \|\boldsymbol{\epsilon}^{(3)}\|_\infty &= 0.00668416 \\ \mathbf{x}^{(4)} &= \mathbf{L}_\omega \mathbf{x}^{(3)} + \mathbf{f} = (0.50016617, 1.00007465, -0.49992359)^\top & \|\boldsymbol{\epsilon}^{(4)}\|_\infty &= 0.00016617 \\ \mathbf{x}^{(5)} &= \mathbf{L}_\omega \mathbf{x}^{(4)} + \mathbf{f} = (0.50000391, 1.00001462, -0.50000362)^\top & \|\boldsymbol{\epsilon}^{(5)}\|_\infty &= 1.46243508 \times 10^{-5} \\ \mathbf{x}^{(6)} &= \mathbf{L}_\omega \mathbf{x}^{(5)} + \mathbf{f} = (0.50000363, 0.99999854, -0.50000004)^\top & \|\boldsymbol{\epsilon}^{(6)}\|_\infty &= 3.63040468 \times 10^{-6} \end{aligned}$$

11. **证明** 由于 \mathbf{A} 是对称正定矩阵, ■