作业 5

Log Creative

2023 年 12 月 1 日

2. **解**

$$f(x) = -3 \times \frac{(x-1)(x-2)}{(-1-1)\cdot(-1-2)} + 4 \times \frac{(x-1)(x+1)}{(2-1)\cdot(2+1)} = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}$$

3. **解** 计算差商表：

x_k	$f(x_k)$	一阶差商	二阶差商
0.4	-0.916291	2.23144	-2.04115
0.5	-0.693147	1.82321	
0.6	-0.510826	1.53061	-0.922
0.7	-0.357765	1.34621	
0.8	-0.223144		

分段线性插值：

$$I_l(x) = \begin{cases} \frac{x-0.5}{-0.1}\ln(0.4) + \frac{x-0.4}{0.1}\ln(0.5) = 2.23144x - 1.808867, & 0.4 \le x \le 0.5, \\ \frac{x-0.6}{-0.1}\ln(0.5) + \frac{x-0.5}{0.1}\ln(0.6) = 1.82321x - 1.604752, & 0.5 \le x \le 0.6, \\ \frac{x-0.7}{-0.1}\ln(0.6) + \frac{x-0.6}{0.1}\ln(0.7) = 1.53061x - 1.429192, & 0.6 \le x \le 0.7, \\ \frac{x-0.8}{-0.1}\ln(0.7) + \frac{x-0.7}{0.1}\ln(0.8) = 1.34621x - 1.300112, & 0.7 \le x \le 0.8. \end{cases}$$

分段二次插值：

$$\begin{aligned} I_d(x) &= \begin{cases} 2.23144x - 1.808867 - 2.04115(x-0.4)(x-0.5), & 0.4 \le x \le 0.6 \\ 1.53061x - 1.429192 - 0.922(x-0.6)(x-0.7), & 0.6 \le x \le 0.8 \end{cases} \\ &= \begin{cases} -2.04115x^2 + 4.068475x - 2.217097, & 0.4 \le x \le 0.6 \\ -0.922x^2 + 2.72921x - 1.816432, & 0.6 \le x \le 0.8 \end{cases} \end{aligned}$$

分别计算 $\ln 0.54$ 的近似值：

$$I_l(0.54) = -0.6202186$$
$$I_d(0.54) = -0.61531984$$

4. **解** 令 $f(x)=\cos x$，$h=x_{i+1}-x_i=1'=\frac{\pi}{10800}$，根据分段线性插值，当 $x_i\le x\le x_{i+1}$ 时，

$$\begin{aligned}L(x)&=\frac{x-x_{i+1}}{x_i-x_{i+1}}f(x_i)+\frac{x-x_i}{x_{i+1}-x_i}f(x_{i+1})\\&=\frac{1}{h}[-(x-x_{i+1})f(x_i)+(x-x_i)f(x_{i+1})]\\L^*(x)&=\frac{1}{h}[-(x-x_{i+1})f^*(x_i)+(x-x_i)f^*(x_{i+1})]\end{aligned}$$

则

$$\begin{aligned}|f(x)-L^*(x)|&=|f(x)-L(x)+L(x)-L^*(x)|\\&\le|f(x)-L(x)|+|L(x)-L^*(x)|\end{aligned}$$

而根据插值余项，有 $\xi\in(x_i,x_{i+1})$ 使得，

$$\begin{aligned}|f(x)-L(x)|&=\left|\frac{f''(\xi)}{2!}(x-x_i)(x-x_{i+1})\right|\\&=\left|-\frac{\cos\xi}{2}(x-x_i)(x-x_{i+1})\right|\\&\le\frac{1}{2}\left|(x-x_i)(x-x_{i+1})\right|\\&\le\frac{1}{2}\frac{h^2}{4}=\frac{h^2}{8}\end{aligned}$$

另一方面，考虑到函数表有五位有效数字，

$$\begin{aligned}|L(x)-L^*(x)|&=\frac{1}{h}\left|-(x-x_{i+1})(f^*(x_i)-f(x_i))+(x-x_i)(f^*(x_{i+1})-f(x_{i+1}))\right|\\&\le\frac{1}{h}\left(\left|(x-x_{i+1})(f^*(x_i)-f(x_i))\right|+\left|(x-x_i)(f^*(x_{i+1})-f(x_{i+1}))\right|\right)\\&\le\frac{1}{h}\cdot\frac{1}{2}\times10^{-5}\cdot(|x-x_{i+1}|+|x-x_i|)\\&=\frac{1}{h}\cdot\frac{1}{2}\times10^{-5}\cdot h\\&=\frac{1}{2}\times10^{-5}\end{aligned}$$

综上，有总误差限

$$|f(x)-L^*(x)|\le\frac{h^2}{8}+\frac{1}{2}\times10^{-5}=0.501\times10^{-5}$$

5. **解**

$$\begin{aligned}l_2(x)&=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\\&=-\frac{1}{2h^3}(x-x_0)(x-x_1)(x-x_3)\end{aligned}$$

令 $t=x-x_0\in[0,3h]$，有

$$\begin{aligned}L_2(t)&=-\frac{1}{2h^3}t(t-h)(t-3h)\\&=-\frac{1}{2h^3}(t^3-4ht^2+3h^2t)\end{aligned}$$

为了求 $\max_{x_0 \le x \le x_3} l_2(x)$，也就是求 $\min_{0 \le t \le 3h} \phi(t)$，其中

$$\phi(t) = t^3 - 4ht^2 + 3h^2 t$$

考虑到

$$\phi'(t) = 3t^2 - 8ht + 3h^2 = 0$$

有

$$t_{1,2} = \frac{4h \pm \sqrt{7}h}{3}$$

则最小值应该在 $t = 0$ 或 $t = \frac{4+\sqrt{7}}{3}h$ 处取到，代入有

$$\min_{0 \le t \le 3h} \phi(t) = \phi\left(\frac{4+\sqrt{7}}{3}h\right) = -\frac{2(7\sqrt{7}+10)h^3}{27}$$

则

$$\min_{x_0 \le x \le x_3} l_2(x) = l_2\left(x_0 + \frac{4+\sqrt{7}}{3}h\right) = \frac{7\sqrt{7}+10}{27}$$

6. **证明** (1) Lagrange 插值多项式

$$L(x) = \sum_{j=0}^{n} x_j^k l_j(x)$$

就是 $x^k(k = 0, 1, \cdots, n)$ 的插值多项式，根据插值余项

$$x^k - \sum_{j=0}^{n} x_j^k l_j(x) = R_n(x) = \frac{(\xi^k)^{(n+1)}}{(n+1)!}\omega_{n+1}(x) \equiv 0$$

其中 $\xi \in (\min\{x_0, \cdots, x_n\}, \max\{x_0, \cdots, x_n\})$，由于 $k < n$，所以导数 $(\xi^k)^{(n+1)}$ 为 0，最后一个等号成立。

(2) 根据二项式定理展开

$$\begin{aligned}
\sum_{j=0}^{n}(x_j - x)^k l_j(x) &= \sum_{j=0}^{n}\sum_{l=0}^{k}\binom{k}{l}x_j^l(-x)^{k-l}l_j(x) \\
&= \sum_{l=0}^{k}\binom{k}{l}(-x)^{k-l}\sum_{j=0}^{n}x_k^l l_j(x) \\
&= \sum_{l=0}^{k}\binom{k}{l}(-x)^{k-l}x^l && \text{由 (1) 可得} \\
&= (-x+x)^k \\
&\equiv 0
\end{aligned}$$

■

7. **证明** 对 $(a, f(a) = 0), (b, f(b) = 0)$ 建立 $f(x)$ 的插值多项式，有

$$L(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a} \equiv 0$$

考虑插值余项，存在 $\xi \in (a,b)$ 满足

$$f(x) = f(x) - L(x) = R(x) = \frac{f''(\xi)}{2}(x-a)(x-b)$$

立刻有

$$\max_{a\le x\le b}|f(x)| \le \frac{1}{8}(b-a)^2 \max_{a\le x\le b}|f''(x)|$$

等号在 $x=\frac{a+b}{2}$ 取到。 ■

8. **解** 根据分段二次插值插值余项，对于 $x_0 \le x \le x_2$，存在 $\xi \in (x_0,x_2)$ 使得

$$R(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2) = \frac{\mathrm{e}^{\xi}}{6}(x-x_0)(x-x_1)(x-x_2)$$

则

$$|R(x)| \le \frac{1}{6}|x-x_0||x-x_1||x-x_2| \max_{-4\le x\le x}|\mathrm{e}^x| = \frac{\mathrm{e}^4}{6}|x-x_0||x-x_1||x-x_2|$$

令 $t=|x-x_1|\in[0,h]$，有

$$r(t) = \frac{\mathrm{e}^4}{6}(h-t)t(h+t) = \frac{\mathrm{e}^4}{6}(-t^3+h^2t) \le r\left(\frac{h}{\sqrt{3}}\right) = \frac{\mathrm{e}^4}{6}\frac{2h^3}{3\sqrt{3}} = \frac{\sqrt{3}\mathrm{e}^4h^3}{27}$$

根据总误差限不超过 10^{-6} 有，

$$\frac{\sqrt{3}\mathrm{e}^4h^3}{27} \le 10^{-6}$$
$$h \le \frac{3\times10^{-2}}{\mathrm{e}}\sqrt[3]{\frac{1}{\sqrt{3}\mathrm{e}}} = 0.00658$$

15. **证明** (1) **起步**

$$F[x_0,x_1] = \frac{F(x_1)-F(x_0)}{x_1-x_0} = \frac{cf(x_1)-cf(x_0)}{x_1-x_0} = cf[x_0,x_1]$$

步进 假设

$$F[x_0,x_1,\cdots,x_k] = cf[x_0,x_1,\cdots,x_k]$$

则

$$\begin{aligned}F[x_0,x_1,\cdots,x_k,x_{k+1}] &= \frac{F[x_1,\cdots,x_k,x_{k+1}]-F[x_0,\cdots,x_k]}{x_{k+1}-x_0}\\ &= \frac{cf[x_1,\cdots,x_k,x_{k+1}]-cf[x_0,\cdots,x_k]}{x_{k+1}-x_0}\\ &= cf[x_0,x_1,\cdots,x_k,x_{k+1}]\end{aligned}$$

结论 根据数学归纳法原理，结论成立：

$$F[x_0,x_1,\cdots,x_n] = cf[x_0,x_1,\cdots,x_n] \tag{1}$$

(2) **起步**

$$
\begin{aligned}
F[x_0,x_1] &= \frac{F(x_1)-F(x_0)}{x_1-x_0} \\
&= \frac{[f(x_1)+g(x_1)]-[f(x_0)+g(x_0)]}{x_1-x_0} \\
&= \frac{f(x_1)-f(x_0)}{x_1-x_0}+\frac{g(x_1)-g(x_0)}{x_1-x_0} \\
&= f[x_0,x_1]+g[x_0,x_1]
\end{aligned}
$$

步进 假设

$$F[x_0,\cdots,x_k]=f[x_0,\cdots,x_k]+g[x_0,\cdots,x_k]$$

则

$$
\begin{aligned}
F[x_0,\cdots,x_k,x_{k+1}] &= \frac{F[x_1,\cdots,x_{k+1}]-F[x_0,\cdots,x_k]}{x_{k+1}-x_0} \\
&= \frac{(f[x_1,\cdots,x_{k+1}]+g[x_1,\cdots,x_{k+1}])-(f[x_0,\cdots,x_k]+g[x_0,\cdots,x_k])}{x_{k+1}-x_0} \\
&= \frac{f[x_1,\cdots,x_{k+1}]-f[x_0,\cdots,x_k]}{x_{k+1}-x_0}+\frac{g[x_1,\cdots,x_{k+1}]-g[x_0,\cdots,x_k]}{x_{k+1}-x_0} \\
&= f[x_0,\cdots,x_k,x_{k+1}]+g[x_0,\cdots,x_k,x_{k+1}]
\end{aligned}
$$

结论 根据数学归纳法原理，结论成立：

$$F[x_0,\cdots,x_n]=f[x_0,\cdots,x_n]+g[x_0,\cdots,x_n] \tag{2}$$

■

16. **解** 令 $\mathcal{F}_k(x)=x^k$，下面证明：

$$\mathcal{F}_k[2^0,2^1,\cdots,2^m]=\prod_{i=0}^{m-1}\frac{2^{k-i}-1}{2^{i+1}-1} \quad (m\geq 1) \tag{3}$$

起步

$$\mathcal{F}_k[2^0,2^1]=\frac{(2^1)^k-(2^0)^k}{2^1-2^0}=\frac{(2\cdot 2^0)^k-(2^0)^k}{2^1-2^0}=\frac{2^k-1}{2^1-1}$$

步进 假设

$$\mathcal{F}_k[2^0,\cdots,2^n]=\prod_{i=0}^{n-1}\frac{2^{k-i}-1}{2^{i+1}-1}$$

则根据差商的性质，有

$$
\begin{aligned}
\mathcal{F}_k[2^1,\cdots,2^{n+1}] &= \sum_{j=1}^{n+1}\frac{\mathcal{F}_k(2^j)}{\prod_{\substack{i=1\\ i\neq j}}^{n+1}(2^j-2^i)} \\
&= \sum_{j=1}^{n+1}\frac{2^k\mathcal{F}_k(2^{j-1})}{2^n\prod_{\substack{i=1\\ i\neq j}}^{n+1}(2^{j-1}-2^{i-1})} \\
&= \frac{2^k}{2^n}\sum_{j=0}^{n}\frac{\mathcal{F}_k(2^j)}{\prod_{\substack{i=0\\ i\neq j}}^{n}(2^j-2^i)} \\
&= 2^{k-n}\mathcal{F}_k[2^0,\cdots,2^n]
\end{aligned}
$$

故

$$
\begin{aligned}
\mathcal{F}_k[2^0,\cdots,2^{n+1}] &= \frac{\mathcal{F}_k[2^1,\cdots,2^{n+1}]-\mathcal{F}_k[2^0,\cdots,2^n]}{2^{n+1}-2^0}\\
&= \frac{2^{k-n}-1}{2^{n+1}-1}\mathcal{F}_k[2^0,\cdots,2^n]\\
&= \prod_{i=0}^{n}\frac{2^{k-i}-1}{2^{i+1}-1}
\end{aligned}
$$

结论 根据数学归纳法原理，有结论成立：

$$
\mathcal{F}_k[2^0,2^1,\cdots,2^m]=\prod_{i=0}^{m-1}\frac{2^{k-i}-1}{2^{i+1}-1}\quad(m\ge 1)
$$

实际上有下面结论的成立，此处不证：

$$
\mathcal{F}_k[2^a,2^{a+1},\cdots,2^{a+m}]=(2^a)^{k-m}\prod_{i=0}^{m-1}\frac{2^{k-i}-1}{2^{i+1}-1}\quad(a\ge 0, m\ge 1)
$$

在结论式子 (3) 注意到：当 $m\ge k+1$ 时，分子存在 0 因子，

$$
\mathcal{F}_k[2^0,2^1,\cdots,2^m]=0\quad(m\ge k+1) \tag{4}
$$

考虑到结论式子 (1) 和 (2) 的线性性质，有

$$
\begin{aligned}
f[2^0,\cdots,2^7] &= \mathcal{F}_7[2^0,\cdots,2^7]+\mathcal{F}_4[2^0,\cdots,2^7]+3\mathcal{F}_3[2^0,\cdots,2^7]+\mathcal{F}_0[2^0,\cdots,2^7]\\
&= \mathcal{F}_7[2^0,\cdots,2^7]\\
&= \prod_{i=0}^{6}\frac{2^{7-i}-1}{2^{i+1}-1}\\
&= 1
\end{aligned}
$$

以及

$$
\begin{aligned}
f[2^0,\cdots,2^8] &= \mathcal{F}_7[2^0,\cdots,2^8]+\mathcal{F}_4[2^0,\cdots,2^8]+3\mathcal{F}_3[2^0,\cdots,2^8]+\mathcal{F}_0[2^0,\cdots,2^8]\\
&= 0
\end{aligned}
$$

19. **解** 使用 Newton–Hermite 插值，差商表

x_i	$f(x_i)$	一阶差商	二阶差商	三阶差商	四阶差商
0	0	$f[0,0]=0$	$f[0,0,1]=1$	$f[0,0,1,1]=-1$	$f[0,0,1,1,2]=\frac{1}{4}$
0	0	$f[0,1]=1$	$f[0,1,1]=0$	$f[0,1,1,2]=-\frac{1}{2}$	
1	1	$f[1,1]=1$	$f[1,1,2]=-1$		
1	1	$f[1,2]=0$			
2	1				

$$
\begin{aligned}
F(x) &= f(0) + f[0,0](x-0) + f[0,0,1](x-0)^2 + f[0,0,1,1](x-0)^2(x-1) + f[0,0,1,1,2](x-0)^2(x-1)^2 \\
&= 0 + 0\times(x-0) + 1\times(x-0)^2 + (-1)\times(x-0)^2(x-1) + \frac{1}{4}(x-0)^2(x-1)^2 \\
&= \frac{1}{4}x^4 - \frac{3}{2}x^3 + \frac{9}{4}x^2
\end{aligned}
$$

21. **解**　等距节点函数值

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	0.038462	0.058824	0.1	0.2	0.5	1	0.5	0.2	0.1	0.058824	0.038462

分段线性插值函数

$$
\begin{aligned}
I_h(x) &= \frac{x-x_{i+1}}{x_i-x_{i+1}}f(x_i) + \frac{x-x_i}{x_{i+1}-x_i}f(x_{i+1}) \\
&= \frac{1}{h}[-(x-x_{i+1})f(x_i) + (x-x_i)f(x_{i+1})] \\
&= \frac{1}{h}(f(x_{i+1}) - f(x_i))x + \frac{1}{h}[x_{i+1}f(x_i) - x_i f(x_{i+1})]
\end{aligned}
$$

令 $I_h(x) = ax + b$，有

x	$[-5,-4]$	$[-4,-3]$	$[-3,-2]$	$[-2,-1]$	$[-1,0]$	$[0,1]$	$[1,2]$	$[2,3]$	$[3,4]$	$[4,5]$
a	0.02036	0.04118	0.1	0.3	0.5	-0.5	-0.3	-0.1	-0.04118	-0.02036
b	0.14027	0.22353	0.4	0.8	1	1	0.8	0.4	0.22353	0.14027

得到中点值与真实值，并计算误差

x	-4.5	-3.5	-2.5	-1.5	-0.5	0.5	1.5	2.5	3.5	4.5
$I_h(x)$	0.04864	0.07941	0.15	0.35	0.75	0.75	0.35	0.15	0.07941	0.04864
$f(x)$	0.04706	0.07547	0.13793	0.30769	0.8	0.8	0.30769	0.13793	0.07547	0.04706
Δ	0.00158	0.00394	0.01207	0.04231	-0.05	-0.05	0.04231	0.01207	0.00394	0.00158

误差估计，根据线性插值余项

$$
|R(x)| = \left|\frac{f''(\xi)}{2!}(x-x_i)(x-x_{i+1})\right| \le \frac{1}{8}\max_{-5\le x\le 5}|f''(x)|
$$

其中

$$
f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}
$$

而

$$
\begin{aligned}
f'''(x) &= \frac{-24x(x^2-1)}{(x^2+1)^4} = 0 \\
x &= -1, 0, 1 \\
f''(0) &= -2 \\
f''(-1) &= f''(1) = 0.5 \\
f''(-5) &= f''(5) = 0.008421
\end{aligned}
$$

故

$$
|R(x)| \le \frac{1}{4} = 0.25
$$

补充题 **解** (1) 根据三转角方程，有

$$
\begin{pmatrix} 2 & \mu_1 \\ \lambda_2 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} g_1 - \lambda_1 m_0 \\ g_2 - \mu_2 m_3 \end{pmatrix} \tag{5}
$$

其中

$$
\begin{aligned}
\lambda_i &= \frac{h_i}{h_{i-1}+h_i} = \frac{1}{2} \\
\mu_i &= \frac{h_{i-1}}{h_{i-1}+h_i} = \frac{1}{2} \\
g_i &= 3(\lambda_i f[x_i, x_{i+1}] + \mu_i f[x_{i-1}, x_i]) = 0
\end{aligned}
$$

考虑到已知 $m_0 = 1, m_3 = 0$，故式 (5) 可化为

$$
\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}
$$

解得

$$
m_1 = -\frac{4}{15} \qquad m_2 = \frac{1}{15}
$$

则根据分段三次 Hermite 插值有

$$
\begin{aligned}
S(x) &= y_i\alpha_i(x) + y_{i+1}\alpha_{i1}(x) + m_i\beta_i(x) + m_{i+1}\beta_{i1}(x) \\
&= m_i\beta_i(x) + m_{i+1}\beta_{i1}(x) \qquad\qquad x \in \Delta_i = [x_i, x_{i+1}]
\end{aligned}
$$

其中

$$
\begin{aligned}
\alpha_i(x) &= \left(1 - 2\frac{x-x_i}{x_i-x_{i+1}}\right)\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2 = (1+2(x-x_i))(x-x_{i+1})^2 \\
\alpha_{i1}(x) &= \left(1 - 2\frac{x-x_{i+1}}{x_{i+1}-x_i}\right)\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 = (1-2(x-x_{i+1}))(x-x_i)^2 \\
\beta_i(x) &= (x-x_i)\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2 = (x-x_i)(x-x_{i+1})^2 \\
\beta_{i1}(x) &= (x-x_{i+1})\left(\frac{x-x_i}{x_{i+1}-x_i}\right)^2 = (x-x_{i+1})(x-x_i)^2
\end{aligned}
$$

将下述参数代入

x_i	0	1	2	3
y_i	0	0	0	0
m_i	1	$-\frac{4}{15}$	$\frac{1}{15}$	0

有

$$S(x) = \begin{cases} \frac{11}{15}x^3 - \frac{26}{15}x^2 + x, & x \in [0,1], \\ -\frac{1}{5}x^3 + \frac{16}{15}x^2 - \frac{9}{5}x + \frac{14}{15}, & x \in [1,2], \\ \frac{1}{15}x^3 - \frac{8}{15}x^2 + \frac{7}{5}x - \frac{6}{5}, & x \in [2,3]. \end{cases}$$

(2) 根据三弯矩方程，已知 $M_0 = 1, M_3 = 0$，有

$$\begin{aligned} 2m_0 + m_1 &= 3f[x_0, x_1] - \frac{h_0}{2}M_0 = -\frac{1}{2} \\ m_2 + 2m_3 &= 3f[x_2, x_3] + \frac{h_2}{2}M_3 = 0 \end{aligned}$$

结合式 (5)，即解方程组

$$\begin{pmatrix} 2 & 1 & & \\ \frac{1}{2} & 2 & \frac{1}{2} & \\ & \frac{1}{2} & 2 & \frac{1}{2} \\ & & 1 & 2 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

解得参数

x_i	0	1	2	3
y_i	0	0	0	0
m_i	$-\frac{13}{45}$	$\frac{7}{90}$	$-\frac{1}{45}$	$\frac{1}{90}$

有

$$S(x) = \begin{cases} -\frac{19}{90}x^3 + \frac{1}{2}x^2 - \frac{13}{45}x, & x \in [0,1], \\ \frac{1}{18}x^3 - \frac{3}{10}x^2 + \frac{23}{45}x - \frac{4}{15}, & x \in [1,2], \\ -\frac{1}{90}x^3 + \frac{1}{10}x^2 - \frac{13}{45}x + \frac{4}{15}, & x \in [2,3]. \end{cases}$$