



作业 7

Log Creative

2023 年 12 月 21 日

1. 解 (1)

$$\int_{-h}^h f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

根据 Simpson 公式, 有

$$\begin{aligned} A_{-1} &= \frac{h - (-h)}{6} = \frac{h}{3} \\ A_0 &= \frac{2(h - (-h))}{3} = \frac{4h}{3} \\ A_1 &= \frac{h - (-h)}{6} = \frac{h}{3} \end{aligned}$$

Simpson 公式具有 3 阶代数精度。

(2)

$$\int_{-2h}^{2h} f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h)$$

设 $f(x) = 1, x, x^2$, 有

$$\begin{cases} 4h = A_{-1} + A_0 + A_1 \\ 0 = A_{-1}(-h) + A_1h \\ \frac{16}{3}h^3 = A_{-1}h^2 + A_1h^2 \end{cases}$$

解得

$$A_{-1} = \frac{8}{3}h \quad A_0 = -\frac{4}{3}h \quad A_1 = \frac{8}{3}h$$

当 $f(x) = x^3$ 时,

$$\int_{-2h}^{2h} x^3 dx = \frac{1}{4}[(2h)^4 - (-2h)^4] = 0 = A_{-1}(-h)^3 + A_1h^3$$

当 $f(x) = x^4$ 时,

$$\int_{-2h}^{2h} x^4 dx = \frac{1}{5}[(2h)^5 - (-2h)^5] = \frac{64}{5}h^5 \neq A_{-1}(-h)^4 + A_1h^4$$

所以它具有 3 阶代数精度。



(3)

$$\int_{-1}^1 f(x) dx \approx \frac{f(-1) + 2f(x_1) + 3f(x_2)}{3}$$

对 $f(x) = 1, x, x^2$ 均能准确成立, 有

$$\begin{cases} 2 = \frac{1}{3}(1 + 2 + 3) \\ 0 = \frac{1}{3}(-1 + 2x_1 + 3x_2) \\ \frac{2}{3} = \frac{1}{3}(1 + 2x_1^2 + 3x_2^2) \end{cases}$$

解得

$$\begin{cases} x_1 = \frac{1-\sqrt{6}}{5} \\ x_2 = \frac{3+2\sqrt{6}}{15} \end{cases} \quad \text{或} \quad \begin{cases} x_1 = \frac{1+\sqrt{6}}{5} \\ x_2 = \frac{3-2\sqrt{6}}{15} \end{cases}$$

当 $f(x) = x^3$ 时,

$$\frac{1}{3}(-1 + 2x_1^3 + 3x_2^3) \neq \int_{-1}^1 x^3 dx = 0$$

所以它具有 2 阶代数精度。

(4)

$$\int_0^h f(x) dx \approx \frac{h}{2}[f(0) + f(h)] + ah^2[f'(0) - f'(h)]$$

对于 $f(x) = 1, x, x^2$ 而言均能准确成立, 有

$$\begin{cases} h = h \\ \frac{1}{2}h^2 = \frac{h^2}{2} \\ \frac{1}{3}h^3 = \frac{h^3}{2} - 2ah^3 \end{cases}$$

解得

$$a = \frac{1}{12}$$

当 $f(x) = x^3, x^4$ 时,

$$\begin{aligned} \frac{1}{4}h^4 &= \frac{1}{2}h^4 - 3ah^4 \\ \frac{1}{5}h^5 &\neq \frac{1}{2}h^5 - 4ah^5 \end{aligned}$$

故它具有 3 阶代数精度。

2. 解 (1) 使用复化梯形公式,

$$\int_a^b f(x) dx \approx \frac{h}{2}[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b)]$$



$$n = 8, \quad x_k = \frac{k}{8} (k = 0, 1, \dots, 8), \quad h = \frac{1}{8}$$

$$f(x_k) = \frac{k/8}{4 + (k/8)^2} = \frac{8k}{256 + k^2}$$

$$\int_0^1 \frac{x}{4 + x^2} dx = \frac{1}{16} \left(\frac{1}{4} + 2 \sum_{k=1}^7 \frac{8k}{256 + k^2} + \frac{1}{5} \right) = 0.12703$$

使用复化 Simpson 公式,

$$\int_a^b f(x) dx \approx \frac{h}{6} [f(a) + 4 \sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b)]$$

$$x_{k+\frac{1}{2}} = \frac{2k+1}{16},$$

$$f(x_{k+\frac{1}{2}}) = \frac{16 \times (2k+1)}{256 \times 4 + (2k+1)^2} = \frac{32k+16}{4k^2+4k+1025}$$

有

$$\int_0^1 \frac{x}{4+x^2} dx = \frac{1}{48} \left(\frac{1}{4} + 4 \sum_{k=0}^7 \frac{32k+16}{4k^2+4k+1025} + 2 \sum_{k=1}^7 \frac{8k}{256+k^2} + \frac{1}{5} \right) = 0.11678$$

4. 解 使用 Simpson 公式,

$$S = \frac{1}{6} \left(1 + 4 \frac{1}{\sqrt{e}} + \frac{1}{e} \right) \approx 0.63233$$

误差为

$$R_S = -\frac{1}{180} \left(\frac{1}{2} \right)^4 e^{-\eta} = -\frac{1}{2880} e^{-\eta}$$

其中 $\eta \in (0, 1)$, 故

$$|R_S| \leq \frac{1}{2880} \approx 3.47 \times 10^{-4}$$

5. 解 根据插值型求积公式的余项,

$$I_n = \sum_{k=0}^n f(x_k) \int_a^b l_k(x) dx$$

$$R[f] = I - I_n = \int_a^b \frac{f^{(n+1)}(\eta)}{(n+1)!} \prod_{i=0}^n (x - x_i) dx$$

有

$$\int_a^b f(x) dx = f(a) \int_a^b 1 dx + \int_a^b f'(\eta)(x-a) dx = (b-a)f(a) + \frac{f'(\eta)}{2} (b-a)^2$$

$$\int_a^b f(x) dx = f(b) \int_a^b 1 dx + \int_a^b f'(\eta)(x-b) dx = (b-a)f(b) - \frac{f'(\eta)}{2} (b-a)^2$$



根据泰勒展开式和积分中值定理, $(x - \frac{a+b}{2})^2$ 在 $x \in (a, b)$ 上不变号, 有

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b \left[f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{f''(\eta)}{2}\left(x - \frac{a+b}{2}\right)^2 \right] dx \\ &= \left[f\left(\frac{a+b}{2}\right)x + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)^2 + \frac{f''(\eta)}{6}\left(x - \frac{a+b}{2}\right)^3 \right] \Big|_a^b \\ &= (b-a)f\left(\frac{a+b}{2}\right) + \frac{f''(\eta)}{24}(b-a)^3\end{aligned}$$

7. 解 复化梯形公式的积分余项为

$$R = -\frac{b-a}{12}h^2 f''(\eta)$$

令 $M = \max_{x \in (a,b)} f''(x)$, 则

$$|R| \leq \frac{b-a}{12}h^2 M \leq \epsilon$$

则

$$h = \frac{b-a}{n} \leq \sqrt{\frac{12\epsilon}{(b-a)M}}$$

即

$$n \geq \frac{b-a}{2} \sqrt{\frac{(b-a)M}{3\epsilon}}$$

8. 解 梯形值递推公式

$$T_0^{(k+1)} = \frac{1}{2}T_0^{(k)} + \frac{h^{(k+1)}}{2} \sum_{i=0}^{2^k-1} f(x_{i+\frac{1}{2}})$$

以及外推方法

$$T_m^{(k)} = T_{m-1}^{(k+1)} + \frac{1}{4^m - 1}(T_{m-1}^{(k+1)} - T_{m-1}^{(k)})$$

对 $\frac{2}{\sqrt{\pi}} \int_0^1 e^{-x} dx = \int_0^1 \frac{2}{\sqrt{\pi}} e^{-x} dx$ 求 Romberg 积分:

k	h	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$	$T_3^{(k)}$
0	1	0.7717433			
1	$\frac{1}{2}$	0.7280699	0.7135122		
2	$\frac{1}{4}$	0.7169828	0.7132870	0.7132720	
3	$\frac{1}{8}$	0.7142002	0.7132726	0.7132717	0.7132717

可得 $\frac{2}{\sqrt{\pi}} \int_0^1 e^{-x} dx \approx 0.7132717$ 。

10. 证明 由于泰勒展开式,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

故

$$n \sin \frac{\pi}{n} = n \left(\frac{\pi}{n} - \frac{\pi^3}{3!n^3} + \frac{\pi^5}{5!n^5} - \cdots \right) = \pi - \frac{\pi^3}{3!n^2} + \frac{\pi^5}{5!n^4} - \cdots$$

对于 $f(n) = n \sin \frac{\pi}{n}$, 考虑到 Richardson 外推加速方法

$$f_m(2n) = \frac{4^m}{4^m - 1} f_{m-1}(n) - \frac{1}{4^m - 1} f_{m-1}(2n)$$

有

n	$f_0(n)$	$f_1(n)$	$f_2(n)$
3	2.598076		
6	3.000000	3.133975	
12	3.105829	3.141105	3.141580

得到 $\pi \approx 3.141580$ 。 ■

11. 解 (1) 使用 Romberg 法得到 $\int_1^3 \frac{dy}{y} \approx 1.098612$ 。

k	h	$T_0^{(k)}$	$T_1^{(k)}$	$T_2^{(k)}$	$T_3^{(k)}$	$T_4^{(k)}$	$T_5^{(k)}$
0	2	1.333333					
1	1	1.166667	1.111111				
2	$\frac{1}{2}$	1.116667	1.100000	1.099259			
3	$\frac{1}{4}$	1.103211	1.098725	1.098640	1.098631		
4	$\frac{1}{8}$	1.099768	1.098620	1.098613	1.098613	1.098613	
5	$\frac{1}{16}$	1.098902	1.098613	1.098612	1.098612	1.098612	1.098612

(2) 使用变换 $y = x + 2$, 有

$$\int_1^3 \frac{dy}{y} = \int_{-1}^1 \frac{dx}{x+2}$$

令 $f(x) = \frac{1}{x+2}$, 使用三点 Gauss-Legendre 公式, 有

$$\int_{-1}^1 \frac{dx}{x+2} \approx \frac{5}{9} f\left(-\frac{\sqrt{15}}{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{15}}{5}\right) = \frac{56}{51} \approx 1.098039$$

使用五点 Gauss-Legendre 公式, 查表 4.5 有

$$\begin{aligned} \int_{-1}^1 \frac{dx}{x+2} &\approx 0.2369269 f(-0.9061793) + 0.2369269 f(0.9061793) \\ &\quad + 0.4786287 f(-0.5384693) + 0.4786287 f(0.5384693) + 0.5688889 f(0) \\ &\approx 1.098609 \end{aligned}$$



(3) 区间等分 4 等份, $[1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$, 复化两点 Gauss 公式

$$\int_1^{1.5} \frac{dy}{y} = \frac{1}{4} \int_{-1}^1 \frac{dx}{\frac{1}{4}x + \frac{5}{4}} = \frac{1}{4} \left(\frac{1}{\frac{1}{4}(-\frac{1}{\sqrt{3}}) + \frac{5}{4}} + \frac{1}{\frac{1}{4}(\frac{1}{\sqrt{3}}) + \frac{5}{4}} \right) = 0.40540541$$

$$\int_{1.5}^2 \frac{dy}{y} = \frac{1}{4} \int_{-1}^1 \frac{dx}{\frac{1}{4}x + \frac{7}{4}} = \frac{1}{4} \left(\frac{1}{\frac{1}{4}(-\frac{1}{\sqrt{3}}) + \frac{7}{4}} + \frac{1}{\frac{1}{4}(\frac{1}{\sqrt{3}}) + \frac{7}{4}} \right) = 0.28767123$$

$$\int_2^{2.5} \frac{dy}{y} = \frac{1}{4} \int_{-1}^1 \frac{dx}{\frac{1}{4}x + \frac{9}{4}} = \frac{1}{4} \left(\frac{1}{\frac{1}{4}(-\frac{1}{\sqrt{3}}) + \frac{9}{4}} + \frac{1}{\frac{1}{4}(\frac{1}{\sqrt{3}}) + \frac{9}{4}} \right) = 0.22314050$$

$$\int_{2.5}^3 \frac{dy}{y} = \frac{1}{4} \int_{-1}^1 \frac{dx}{\frac{1}{4}x + \frac{11}{4}} = \frac{1}{4} \left(\frac{1}{\frac{1}{4}(-\frac{1}{\sqrt{3}}) + \frac{11}{4}} + \frac{1}{\frac{1}{4}(\frac{1}{\sqrt{3}}) + \frac{11}{4}} \right) = 0.18232044$$

故

$$\int_1^3 \frac{dy}{y} = \int_1^{1.5} \frac{dy}{y} + \int_{1.5}^2 \frac{dy}{y} + \int_2^{2.5} \frac{dy}{y} + \int_{2.5}^3 \frac{dy}{y} = 1.098538$$

真实值为

$$\int_1^3 \frac{dy}{y} = \ln(y) \Big|_1^3 = 1.098612$$

可以看到 Romberg 法结果误差最小。

补充 1. **解** 权函数 $\frac{1}{\sqrt{x}}$, 设正交多项式零点为 x_0 和 x_1 , 有正交多项式 $w(x) = (x-x_0)(x-x_1) = x^2 + bx + c$ 满足与 1 和 x 的带权正交

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} w(x) dx &= \int_0^1 x^{\frac{3}{2}} + bx^{\frac{1}{2}} + cx^{-\frac{1}{2}} dx = \frac{2}{5} + \frac{2}{3}b + 2c = 0 \\ \int_0^1 \frac{1}{\sqrt{x}} x w(x) dx &= \int_0^1 x^{\frac{5}{2}} + bx^{\frac{3}{2}} + cx^{\frac{1}{2}} dx = \frac{2}{7} + \frac{2}{5}b + \frac{2}{3}c = 0 \end{aligned}$$

解得

$$b = -\frac{6}{7} \quad c = \frac{3}{35}$$

故 $w(x) = x^2 - \frac{6}{7}x + \frac{3}{35} = 0$ 解得

$$x_0 = \frac{1}{7} \left(3 - 2\sqrt{\frac{6}{5}} \right) = 0.115587 \quad x_1 = \frac{1}{7} \left(3 + 2\sqrt{\frac{6}{5}} \right) = 0.741556$$

考虑

$$\int_0^1 \frac{1}{\sqrt{x}} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$$



对 $f(x) = 1, x$ 都准确成立, 有

$$\begin{aligned} 2 &= A_0 + A_1 \\ \frac{2}{3} &= A_0 x_0 + A_1 x_1 \end{aligned}$$

解得

$$A_0 = \frac{2 - 6x_1}{3x_0 - 3x_1} = 1.304290 \quad A_1 = \frac{6x_0 - 2}{3x_0 - 3x_1} = 0.695710$$

所以构造出的 Gauss 积分公式为

$$\int_0^1 \frac{1}{\sqrt{x}} f(x) dx \approx 1.304290 f(0.115587) + 0.695710 f(0.741556)$$

补充 2. 解 (1) 由于是等分节点, 所以三点插值型积分公式即 Simpson 公式,

$$\begin{aligned} I_n &= \frac{x_4 - x_0}{6} [f(x_0) + 4f(x_2) + f(x_4)] \\ &= \frac{1.4 - 1.0}{6} \times [0.2500 + 4 \times 0.2066 + 0.1736] \\ &= 0.0833333 \end{aligned}$$

五点插值型积分公式即 Cotes 公式,

$$\begin{aligned} I_n &= \frac{x_4 - x_0}{90} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] \\ &= \frac{1.4 - 1.0}{90} \times [7 \times 0.2500 + 32 \times 0.2268 + 12 \times 0.2066 + 32 \times 0.1890 + 7 \times 0.1736] \\ &= 0.0833333 \end{aligned}$$

(2) 复化梯形公式求解,

$$\begin{aligned} T_n &= \frac{0.1}{2} [f(1.0) + 2f(1.1) + 2f(1.2) + 2f(1.3) + f(1.4)] \\ &= \frac{0.1}{2} \times (0.2500 + 2 \times 0.2268 + 2 \times 0.2066 + 2 \times 0.1890 + 0.1736) \\ &= 0.0834200 \end{aligned}$$

(3) 使用复化 Simpson 公式求解,

$$\begin{aligned} S_n &= \frac{0.2}{6} [f(1.0) + 4f(1.1) + f(1.2) + f(1.2) + 4f(1.3) + f(1.4)] \\ &= \frac{0.2}{6} \times [0.2500 + 4 \times 0.2268 + 2 \times 0.2066 + 4 \times 0.1890 + 0.1736] \\ &= 0.0833333 \end{aligned}$$