作业 4

Log Creative

2023 年 11 月 15 日

1. **解**

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 5 & & \\ & 4 & \\ & & 10 \end{pmatrix} - \begin{pmatrix} & & \\ 1 & & \\ -2 & 3 & \end{pmatrix} - \begin{pmatrix} & -2 & -1 \\ & & -2 \\ & & \end{pmatrix} = D - L - U$$

$$b = \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix}$$

(1) 由于 A 是严格占优矩阵，所以 Jacobi 迭代法和 Guass–Seidel 迭代法解这个方程组都是收敛的。

(2) **Jacobi 迭代法** 对于 Jacobi 迭代法，其迭代矩阵

$$B = D^{-1}(L + U) = \begin{pmatrix} & -\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{4} & & -\frac{1}{2} \\ -\frac{1}{5} & \frac{3}{10} & \end{pmatrix} \qquad f = D^{-1}b = \begin{pmatrix} -\frac{12}{5} \\ 5 \\ \frac{3}{10} \end{pmatrix}$$

取迭代初值 $x^{(0)} = (0, 0, 0)^{\top}$，记 $\epsilon^{(k)} = x^{(k)} - x^{(k-1)}$，有

$$x^{(1)} = Bx^{(0)} + f = (-2.4, 5, 0.3)^{\top} \qquad \|\epsilon^{(1)}\|_{\infty} = 5.$$

$$x^{(2)} = Bx^{(1)} + f = (-4.46, 4.25, 2.28)^{\top} \qquad \|\epsilon^{(2)}\|_{\infty} = 2.06$$

$$x^{(3)} = Bx^{(2)} + f = (-4.556, 2.745, 2.467)^{\top} \qquad \|\epsilon^{(3)}\|_{\infty} = 1.505$$

$$x^{(4)} = Bx^{(3)} + f = (-3.9914, 2.6275, 2.0347)^{\top} \qquad \|\epsilon^{(4)}\|_{\infty} = 0.5646$$

$$x^{(5)} = Bx^{(4)} + f = (-3.85794, 2.9848, 1.88653)^{\top} \qquad \|\epsilon^{(5)}\|_{\infty} = 0.3573$$

$$x^{(6)} = Bx^{(5)} + f = (-3.971226, 3.09225, 1.967028)^{\top} \qquad \|\epsilon^{(6)}\|_{\infty} = 0.113286$$

$$x^{(7)} = Bx^{(6)} + f = (-4.0303056, 3.0236795, 2.0219202)^{\top} \qquad \|\epsilon^{(7)}\|_{\infty} = 0.0685705$$

$$x^{(8)} = Bx^{(7)} + f = (-4.01385584, 2.9814635, 2.01316497)^{\top} \qquad \|\epsilon^{(8)}\|_{\infty} = 0.042216$$

$$x^{(9)} = Bx^{(8)} + f = (-3.99521839, 2.98995356, 1.99721022)^{\top} \qquad \|\epsilon^{(9)}\|_{\infty} = 0.01863745$$

$$x^{(10)} = Bx^{(9)} + f = (-3.99542347, 3.00259029, 1.99602975)^{\top} \qquad \|\epsilon^{(10)}\|_{\infty} = 0.01263674$$

$$x^{(11)} = Bx^{(10)} + f = (-4.00024207, 3.00312926, 1.99986178)^{\top} \qquad \|\epsilon^{(11)}\|_{\infty} = 0.0048186$$

$$\boldsymbol{x}^{(12)} = \boldsymbol{B}\boldsymbol{x}^{(11)} + \boldsymbol{f} = (-4.00122406, 3.00000859, 2.00098719)^{\top} \qquad \|\boldsymbol{\epsilon}^{(12)}\|_{\infty} = 0.00312067$$

$$\boldsymbol{x}^{(13)} = \boldsymbol{B}\boldsymbol{x}^{(12)} + \boldsymbol{f} = (-4.00020088, 2.99920039, 2.00024739)^{\top} \qquad \|\boldsymbol{\epsilon}^{(13)}\|_{\infty} = 0.00102319$$

$$\boldsymbol{x}^{(14)} = \boldsymbol{B}\boldsymbol{x}^{(13)} + \boldsymbol{f} = (-3.99972963, 2.99982609, 1.99980029)^{\top} \qquad \|\boldsymbol{\epsilon}^{(14)}\|_{\infty} = 0.0006257$$

$$\boldsymbol{x}^{(15)} = \boldsymbol{B}\boldsymbol{x}^{(14)} + \boldsymbol{f} = (-3.99989049, 3.00016745, 1.99989375)^{\top} \qquad \|\boldsymbol{\epsilon}^{(15)}\|_{\infty} = 0.00034136$$

$$\boldsymbol{x}^{(16)} = \boldsymbol{B}\boldsymbol{x}^{(15)} + \boldsymbol{f} = (-4.00004573, 3.0000805, 2.00002833)^{\top} \qquad \|\boldsymbol{\epsilon}^{(16)}\|_{\infty} = 0.00015524$$

$$\boldsymbol{x}^{(17)} = \boldsymbol{B}\boldsymbol{x}^{(16)} + \boldsymbol{f} = (-4.00003787, 2.9999744, 2.0000333,)^{\top} \qquad \|\boldsymbol{\epsilon}^{(17)}\|_{\infty} = 0.0001061$$

$$\boldsymbol{x}^{(18)} = \boldsymbol{B}\boldsymbol{x}^{(17)} + \boldsymbol{f} = (-3.99999642, 2.99997389, 1.99999989)^{\top} \qquad \|\boldsymbol{\epsilon}^{(18)}\|_{\infty} = 4.14468074 \times 10^{-5}$$

总共需要 18 次满足要求。

Guass–Seidel 迭代法 对于 Guass–Seidel 迭代法，

$$\boldsymbol{G} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{U} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} & -2 & -1 \\ & & -2 \\ & & \end{pmatrix} = \begin{pmatrix} 0 & -0.4 & -0.2 \\ 0 & -0.1 & -0.55 \\ 0 & 0.05 & -0.125 \end{pmatrix}$$

$$\boldsymbol{f} = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{b} = \begin{pmatrix} 5 & & \\ -1 & 4 & \\ 2 & -3 & 10 \end{pmatrix}^{-1} \begin{pmatrix} -12 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 4.4 \\ 2.1 \end{pmatrix}$$

取迭代初值 $\boldsymbol{x}^{(0)} = (0, 0, 0)^{\top}$，记 $\boldsymbol{\epsilon}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}$，有

$$\boldsymbol{x}^{(1)} = \boldsymbol{G}\boldsymbol{x}^{(0)} + \boldsymbol{f} = (-2.4, 4.4, 2.1)^{\top} \qquad \|\boldsymbol{\epsilon}^{(1)}\|_{\infty} = 4.4$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{G}\boldsymbol{x}^{(1)} + \boldsymbol{f} = (-4.58, 2.805, 2.0575)^{\top} \qquad \|\boldsymbol{\epsilon}^{(2)}\|_{\infty} = 2.18$$

$$\boldsymbol{x}^{(3)} = \boldsymbol{G}\boldsymbol{x}^{(2)} + \boldsymbol{f} = (-3.9335, 2.987875, 1.9830625)^{\top} \qquad \|\boldsymbol{\epsilon}^{(3)}\|_{\infty} = 0.6465$$

$$\boldsymbol{x}^{(4)} = \boldsymbol{G}\boldsymbol{x}^{(3)} + \boldsymbol{f} = (-3.9917625, 3.01052813, 2.00151094)^{\top} \qquad \|\boldsymbol{\epsilon}^{(4)}\|_{\infty} = 0.0582625$$

$$\boldsymbol{x}^{(5)} = \boldsymbol{G}\boldsymbol{x}^{(4)} + \boldsymbol{f} = (-4.00451344, 2.99811617, 2.00033754)^{\top} \qquad \|\boldsymbol{\epsilon}^{(5)}\|_{\infty} = 0.01275094$$

$$\boldsymbol{x}^{(6)} = \boldsymbol{G}\boldsymbol{x}^{(5)} + \boldsymbol{f} = (-3.99931398, 3.00000274, 1.99986362)^{\top} \qquad \|\boldsymbol{\epsilon}^{(6)}\|_{\infty} = 0.00519946$$

$$\boldsymbol{x}^{(7)} = \boldsymbol{G}\boldsymbol{x}^{(6)} + \boldsymbol{f} = (-3.99997382, 3.00007474, 2.00001718)^{\top} \qquad \|\boldsymbol{\epsilon}^{(7)}\|_{\infty} = 0.00065984$$

$$\boldsymbol{x}^{(8)} = \boldsymbol{G}\boldsymbol{x}^{(7)} + \boldsymbol{f} = (-4.00003333, 2.99998307, 2.00000159)^{\top} \qquad \|\boldsymbol{\epsilon}^{(8)}\|_{\infty} = 9.16628308 \times 10^{-5}$$

总共需要 8 次满足要求。

8. **解**

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} & & & \\ & & & \\ \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \end{pmatrix} - \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ & & & \\ & & & \end{pmatrix} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U}$$

$$\boldsymbol{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

(1) 对于 Jacobi 迭代法，

$$\boldsymbol{B}_0 = \boldsymbol{D}^{-1}(\boldsymbol{L}+\boldsymbol{U}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \end{pmatrix} = \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{4} & \frac{1}{4} & & \end{pmatrix}$$

求它的特征值

$$0 = |\lambda \boldsymbol{E} - \boldsymbol{B}_0| = \begin{vmatrix} \lambda & & -\frac{1}{4} & -\frac{1}{4} \\ & \lambda & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \lambda & \\ -\frac{1}{4} & -\frac{1}{4} & & \lambda \end{vmatrix} = \lambda^2 \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)$$

得到 $\lambda_{1,2} = 0, \lambda_3 = \frac{1}{2}, \lambda_4 = -\frac{1}{2}$，故谱半径 $\rho(\boldsymbol{B}_0) = \frac{1}{2}$。

(2) 对于 Guass–Seidel 迭代法，

$$\boldsymbol{B}_0 = (\boldsymbol{D} - \boldsymbol{L})^{-1}\boldsymbol{U} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -\frac{1}{4} & -\frac{1}{4} & 1 & \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{8} & \frac{1}{8} \\ & & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

求它的特征值

$$0 = |\lambda \boldsymbol{E} - \boldsymbol{B}_0| = \begin{vmatrix} \lambda & & -\frac{1}{4} & -\frac{1}{4} \\ & \lambda & -\frac{1}{4} & -\frac{1}{4} \\ & & \lambda - \frac{1}{8} & -\frac{1}{8} \\ & & -\frac{1}{8} & \lambda - \frac{1}{8} \end{vmatrix} = \lambda^3 \left(\lambda - \frac{1}{4}\right)$$

得到 $\lambda_{1,2,3} = 0, \lambda_4 = \frac{1}{4}$，故谱半径 $\rho(\boldsymbol{B}_0) = \frac{1}{4}$。

(3) 由于谱半径都小于 1，所以 Jacobi 迭代法和 Guass–Seidel 迭代法均收敛。

9. **解**

$$\boldsymbol{A} = \begin{pmatrix} 4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4 \end{pmatrix} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U} = \begin{pmatrix} 4 & & \\ & 4 & \\ & & 4 \end{pmatrix} - \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix} - \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$$

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

对于 SOR 迭代法，松弛因子为 ω，有

$$\boldsymbol{L}_\omega = (\boldsymbol{D} - \omega\boldsymbol{L})^{-1}[(1-\omega)\boldsymbol{D} + \omega\boldsymbol{U}] = \begin{pmatrix} 4 & & \\ -\omega & 4 & \\ & -\omega & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4(1-\omega) & \omega & \\ & 4(1-\omega) & \omega \\ & & 4(1-\omega) \end{pmatrix}$$

$$\boldsymbol{f} = \omega(\boldsymbol{D} - \omega\boldsymbol{L})^{-1}\boldsymbol{b} = \omega\begin{pmatrix} 4 & & \\ -\omega & 4 & \\ & -\omega & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

取迭代初值 $\boldsymbol{x}^{(0)} = (0,0,0)^\top$，记 $\boldsymbol{\epsilon}^{(k)} = \boldsymbol{x}^* - \boldsymbol{x}^{(k)}$，有

(1) $\omega = 1.03$ 需要 5 次迭代：

$$\boldsymbol{x}^{(1)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.2575, 1.09630625, -0.49020114)^\top \qquad \|\boldsymbol{\epsilon}^{(1)}\|_\infty = 0.2425$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.53207386, 1.00789304, -0.49826151)^\top \qquad \|\boldsymbol{\epsilon}^{(2)}\|_\infty = 0.03207386$$

$$\boldsymbol{x}^{(3)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.50107024, 1.00048646, -0.49992689)^\top \qquad \|\boldsymbol{\epsilon}^{(3)}\|_\infty = 0.00107024$$

$$\boldsymbol{x}^{(4)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50009316, 1.00002822, -0.49999493)^\top \qquad \|\boldsymbol{\epsilon}^{(4)}\|_\infty = 9.31555814 \times 10^{-5}$$

$$\boldsymbol{x}^{(5)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50000447, 1.00000161, -0.49999974)^\top \qquad \|\boldsymbol{\epsilon}^{(5)}\|_\infty = 4.47176785 \times 10^{-6}$$

(2) $\omega = 1$ 需要 6 次迭代：

$$\boldsymbol{x}^{(1)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.25, 1.0625, -0.484375)^\top \qquad \|\boldsymbol{\epsilon}^{(1)}\|_\infty = 0.25$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.515625, 1.0078125, -0.49804688)^\top \qquad \|\boldsymbol{\epsilon}^{(2)}\|_\infty = 0.015625$$

$$\boldsymbol{x}^{(3)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.50195312, 1.00097656, -0.49975586)^\top \qquad \|\boldsymbol{\epsilon}^{(3)}\|_\infty = 0.00195312$$

$$\boldsymbol{x}^{(4)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50024414, 1.00012207, -0.49996948)^\top \qquad \|\boldsymbol{\epsilon}^{(4)}\|_\infty = 0.00024414$$

$$\boldsymbol{x}^{(5)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50003052, 1.00001526, -0.49999619)^\top \qquad \|\boldsymbol{\epsilon}^{(5)}\|_\infty = 3.05175781 \times 10^{-5}$$

$$\boldsymbol{x}^{(6)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(5)} + \boldsymbol{f} = (0.50000381, 1.00000191, -0.49999952)^\top \qquad \|\boldsymbol{\epsilon}^{(6)}\|_\infty = 3.81469727 \times 10^{-6}$$

(3) $\omega = 1.1$ 需要 6 次迭代：

$$\boldsymbol{x}^{(1)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(0)} + \boldsymbol{f} = (0.275, 1.175625, -0.50170313)^\top \qquad \|\boldsymbol{\epsilon}^{(1)}\|_\infty = 0.225$$

$$\boldsymbol{x}^{(2)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(1)} + \boldsymbol{f} = (0.57079688, 1.00143828, -0.49943416)^\top \qquad \|\boldsymbol{\epsilon}^{(2)}\|_\infty = 0.07079688$$

$$\boldsymbol{x}^{(3)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(2)} + \boldsymbol{f} = (0.49331584, 0.99817363, -0.50055883)^\top \qquad \|\boldsymbol{\epsilon}^{(3)}\|_\infty = 0.00668416$$

$$\boldsymbol{x}^{(4)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(3)} + \boldsymbol{f} = (0.50016617, 1.00007465, -0.49992359)^\top \qquad \|\boldsymbol{\epsilon}^{(4)}\|_\infty = 0.00016617$$

$$\boldsymbol{x}^{(5)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(4)} + \boldsymbol{f} = (0.50000391, 1.00001462, -0.50000362)^\top \qquad \|\boldsymbol{\epsilon}^{(5)}\|_\infty = 1.46243508 \times 10^{-5}$$

$$\boldsymbol{x}^{(6)} = \boldsymbol{L}_\omega\boldsymbol{x}^{(5)} + \boldsymbol{f} = (0.50000363, 0.99999854, -0.50000004)^\top \qquad \|\boldsymbol{\epsilon}^{(6)}\|_\infty = 3.63040468 \times 10^{-6}$$

11. **证明** 迭代公式

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^{(k)}) = (\boldsymbol{E} - \omega\boldsymbol{A})\boldsymbol{x}^{(k)} + \omega\boldsymbol{b}$$

讨论 $\boldsymbol{L}_\omega = \boldsymbol{E} - \omega\boldsymbol{A}$ 的特征值 λ，对于任意的 $\boldsymbol{y} \in \mathbf{R}^n$，有

$$\boldsymbol{L}_\omega \boldsymbol{y} = \lambda\boldsymbol{y}$$
$$(\boldsymbol{E} - \omega\boldsymbol{A})\boldsymbol{y} = \lambda\boldsymbol{y}$$
$$\boldsymbol{y} - \omega\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{y} \tag{1}$$

对于 $\boldsymbol{A}$ 的特征值 $0 < \alpha \leq \lambda' \leq \beta$，有

$$\boldsymbol{A}\boldsymbol{y} = \lambda'\boldsymbol{y} \tag{2}$$

结合式 (1) 和 (2) 有

$$(1 - \omega\lambda' - \lambda)\boldsymbol{y} = \boldsymbol{0}$$

考虑到对于 $\forall \boldsymbol{y} \in \mathbf{R}^n$ 都成立，有

$$\lambda = 1 - \omega\lambda'$$

当 $0 < \omega < \frac{2}{\beta}$ 时，有

$$-1 = 1 - \frac{2}{\beta}\beta < \lambda < 1 - 0 = 1$$

即 $|\lambda| < 1$，则迭代法收敛。 ∎

13. **解**　(1) 设 $\boldsymbol{z}^{(m)} = \begin{pmatrix} \boldsymbol{z}_1^{(m)} \\ \boldsymbol{z}_2^{(m)} \end{pmatrix}$，$\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$，则对于迭代方法

$$\boldsymbol{A}\boldsymbol{z}_1^{(m+1)} = \boldsymbol{b}_1 - \boldsymbol{B}\boldsymbol{z}_2^{(m)}, \quad \boldsymbol{A}\boldsymbol{z}_2^{(m+1)} = \boldsymbol{b}_2 - \boldsymbol{B}\boldsymbol{z}_1^{(m)}$$

等价为

$$\begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix}\boldsymbol{z}^{(m+1)} = \boldsymbol{b} - \begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix}\boldsymbol{z}^{(m)}$$

$$\boldsymbol{z}^{(m+1)} = -\begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix}^{-1}\begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix}\boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A} & \\ & \boldsymbol{A} \end{pmatrix}^{-1}\boldsymbol{b}$$

$$= -\begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix}\begin{pmatrix} & \boldsymbol{B} \\ \boldsymbol{B} & \end{pmatrix}\boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix}\boldsymbol{b}$$

$$= \begin{pmatrix} & -\boldsymbol{A}^{-1}\boldsymbol{B} \\ -\boldsymbol{A}^{-1}\boldsymbol{B} & \end{pmatrix}\boldsymbol{z}^{(m)} + \begin{pmatrix} \boldsymbol{A}^{-1} & \\ & \boldsymbol{A}^{-1} \end{pmatrix}\boldsymbol{b}$$

对于迭代矩阵，它的特征值为 λ，则对于 $\forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{R}^{2n}$，有

$$\begin{pmatrix} & -A^{-1}B \\ -A^{-1}B & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} -A^{-1}Bz_2 = \lambda z_1 \\ -A^{-1}Bz_1 = \lambda z_2 \end{cases}$$

$$\Rightarrow \begin{cases} (A^{-1}B)^2 z_1 = \lambda^2 z_1 \\ (A^{-1}B)^2 z_2 = \lambda^2 z_2 \end{cases}$$

即 $\rho \begin{pmatrix} & -A^{-1}B \\ -A^{-1}B & \end{pmatrix} = \rho(A^{-1}B)$，迭代法收敛的充要条件即 $\rho(A^{-1}B) < 1$。

(2) 设 $z^{(m)} = \begin{pmatrix} z_1^{(m)} \\ z_2^{(m)} \end{pmatrix}$，$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$，则对于迭代方法

$$Az_1^{(m+1)} = b_1 - Bz_2^{(m)}, \quad Az_2^{(m+1)} = b_2 - Bz_1^{(m+1)}$$

等价为

$$\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} z^{(m+1)} = b - \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} z^{(m)}$$

$$z^{(m+1)} = -\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}^{-1} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} z^{(m)} + \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}^{-1} b$$

$$= -\begin{pmatrix} A^{-1} & 0 \\ -A^{-1}BA^{-1} & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} z^{(m)} + \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}^{-1} b$$

$$= \begin{pmatrix} 0 & -A^{-1}B \\ 0 & A^{-1}BA^{-1}B \end{pmatrix} z^{(m)} + \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}^{-1} b$$

对于迭代矩阵，它的特征值为 λ，则对于 $\forall z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{R}^{2n}$，有

$$\begin{pmatrix} 0 & -A^{-1}B \\ 0 & A^{-1}BA^{-1}B \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} -A^{-1}Bz_2 = \lambda z_1 \\ (A^{-1}B)^2 z_2 = \lambda z_2 \end{cases}$$

$$\Rightarrow \begin{cases} (A^{-1}B)^2 z_1 = \lambda z_1 \\ (A^{-1}B)^2 z_2 = \lambda z_2 \end{cases}$$

即 $\rho \begin{pmatrix} \mathbf{0} & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B} \end{pmatrix} = \rho((\mathbf{A}^{-1}\mathbf{B})^2)$，迭代法的充要条件即 $\rho((\mathbf{A}^{-1}\mathbf{B})^2) = \rho((\mathbf{A}^{-1}\mathbf{B}))^2 < 1$。

方法 (1) 的收敛速度为 $R_1 = -\ln\rho((\mathbf{A}^{-1}\mathbf{B}))$，方法 (2) 的收敛速度为 $R_2 = -2\ln\rho((\mathbf{A}^{-1}\mathbf{B}))$，即 $R_2 = 2R_1$，方法 (2) 的收敛速度是方法 (1) 的 2 倍。

14. **证明**　若矩阵 $\mathbf{A} = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$ 是正定的，那么

$$\begin{cases} \det\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = 1 - a^2 > 0 & \Rightarrow -1 < a < 1 \\ \det\mathbf{A} = (a-1)^2(2a+1) > 0 & \Rightarrow a > -\frac{1}{2} \end{cases}$$

即当 $-\frac{1}{2} < a < 1$ 时，矩阵 $\mathbf{A}$ 是正定的。

考察

$$\mathbf{A} = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix} = \mathbf{D} - \mathbf{L} - \mathbf{U} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} & & \\ -a & & \\ -a & -a & \end{pmatrix} - \begin{pmatrix} & -a & -a \\ & & -a \\ & & \end{pmatrix}$$

在 Jacobi 迭代法中，迭代矩阵

$$\mathbf{B} = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} & -a & -a \\ -a & & -a \\ -a & -a & \end{pmatrix} = \begin{pmatrix} & -a & -a \\ -a & & -a \\ -a & -a & \end{pmatrix}$$

考察它的特征值

$$|\lambda\mathbf{E} - \mathbf{B}| = 0 \Rightarrow \begin{vmatrix} \lambda & a & a \\ a & \lambda & a \\ a & a & \lambda \end{vmatrix} = (\lambda - a)^2(\lambda + 2a) = 0 \Rightarrow \lambda_{1,2} = a, \lambda_3 = -\frac{1}{2a}$$

为了使迭代法收敛，有 $|\lambda| < 1$，则需要 $-\frac{1}{2} < a < \frac{1}{2}$。∎

19. **证明**　(1) 由于 $(\mathbf{A}^{\top}\mathbf{A})^{\top} = \mathbf{A}^{\top}(\mathbf{A}^{\top})^{\top} = \mathbf{A}^{\top}\mathbf{A}$，则 $\mathbf{A}^{\top}\mathbf{A}$ 是对称矩阵。

由于 $\mathbf{A}$ 是非奇异矩阵，对 $\forall \mathbf{x} \neq 0$，有 $\mathbf{x}^{\top}\mathbf{A}^{\top}\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^{\top}(\mathbf{A}\mathbf{x}) > 0$，所以 $\mathbf{A}^{\top}\mathbf{A}$ 是正定矩阵。

综上，$\mathbf{A}^{\top}\mathbf{A}$ 是对称正定矩阵。

(2) 由于 $\mathbf{A}^{\top}\mathbf{A}$ 是对称矩阵，所以

$$\text{cond}(\mathbf{A}^{\top}\mathbf{A})_2 = \frac{|\lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})|}{|\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})|} \tag{3}$$

另一方面，

$$\text{cond}(\mathbf{A})_2 = \sqrt{\frac{\lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})}{\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})}} \tag{4}$$

综合式 (3) 和 (4) 有

$$\text{cond}(\mathbf{A}^{\top}\mathbf{A})_2 = (\text{cond}(\mathbf{A})_2)^2$$ ∎