Chapter 3

Sobolev inequalities on manifolds

3.1 Introduction

3.1.1 Notation concerning Riemannian manifolds

We want now to replace the Euclidean space $\mathbb{R}^n$ by a Riemannian manifold M and consider the possibility of having some kind of Sobolev inequalities. This brings in a whole new point of view. On Euclidean space, we could only discuss whether inequalities were true or not. In the more general setting of Riemannian manifolds, we can investigate the relations between various functional inequalities and the relations between these functional inequalities and the geometry of the manifold. We can search for necessary and/or sufficient conditions for a given Sobolev-type inequality to hold true. This leads to a better understanding of what information about M is encoded in various Sobolev-type inequalities.

Sobolev inequalities are useful when developing analysis on Riemannian manifolds, even more so than on Euclidean space, because other tools such as Fourier analysis are not available any more. This is particularly true when one studies large scale behavior of solutions of partial differential equations such as the Laplace and heat equations.

In the sequel, we will focus on complete, non-compact Riemannian manifolds. For compact manifolds, local Euclidean-type Sobolev inequalities are always satisfied and the interesting questions have to do with controlling the constants arising in these inequalities in geometric terms. We refer the interested reader to [5, 39, 40] where this is discussed at length.

Let us briefly introduce notation concerning Riemannian manifolds. Let M be a Riemannian manifold of dimension n with tangent space TM and co-tangent space T^*M. The tangent space TM is the union of the tangent spaces T_x where $x \in M$. T_x^* is the dual of the n-dimensional linear space T_x and T^*M is the union of these spaces. Smooth sections of TM are called vector fields and smooth sections of T^*M are called forms (1-forms). There is a natural pairing $TM \times T^*M \ni (\xi, \eta) \mapsto \eta(\xi) \in \mathbb{R}$ induced by the natural pairing of T_x and T_x^*, $x \in M$. It is a basic fact that vector fields can equivalently be defined as derivations, i.e., maps $\xi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that $\xi(fg) = f\xi g + g\xi f$ for all $f, g \in \mathcal{C}^\infty(M)$. If f is a smooth function on M, the relation between its derivative df which is a form and ξf where ξ is a vector field is given by

$$df(\xi) = \xi f.$$

Because M is a Riemannian manifold, we are given on each T_x a scalar product $\langle \cdot, \cdot \rangle_x$. If $f \in \mathcal{C}^\infty(M)$, its gradient is defined as the unique vector field ∇f such that

$$\forall\, x \in M,\ \ \forall\, \xi \in TM,\quad \langle \nabla f(x), \xi(x) \rangle_x = df(\xi)(x).$$

There is a canonical distance function associated to the Riemannian structure of M. We will denote it by $(x, y) \mapsto d(x, y)$. It can be defined as the shortest length of all piecewise $\mathcal{C}^1$ curves from x to y. The topology of (M, d) as a metric space is the same as that of M as a manifold. See, e.g., [13, §1.6]. There is also a canonical Riemannian measure on M which we denote by either dx or μ depending on which is more convenient. See, e.g., [13, §3.3]. We denote by $V(x, t)$ the volume of the ball of radius $t > 0$ around $x \in M$, i.e.,

$$V(x, t) = \mu(B(x, t)).$$

Thus $V(x, t)$ describes the volume growth of M.

The divergence $\mathrm{div}\xi$ of a vector field ξ is defined as the unique smooth function on M such that

$$\forall\, f \in \mathcal{C}_0^\infty(M),\ \ \int_M f \mathrm{div}\xi d\mu = -\int_M df(\xi) d\mu.$$

The Laplace–Beltrami operator Δ on M is the second order differential operator defined by

$$\forall\, f \in \mathcal{C}_0^\infty(M),\quad \Delta f = -\mathrm{div}(\nabla f).$$

Note that, with this definition,

$$\forall f, g \in \mathcal{C}_0^\infty(M),\quad \int_M f \Delta g d\mu = \int_M \langle \nabla f, \nabla g \rangle d\mu,$$

so that, in particular,

$$\int_M f \Delta f d\mu = \int_M \langle \nabla f, \nabla f \rangle d\mu \geq 0.$$

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All the objects introduced above can of course be computed in local coordinates. See, e.g., [12, 13].

We will always work under the assumption that M is complete. Although this could a priori be interpreted to have various metric or geometric meanings (e.g., geodesically complete), the different interpretations turn out to be equivalent. Thus, M is complete means that (M, d) is a complete metric space. In particular, all bounded closed sets are compact. See [13, §1.7].

3.1.2 Isoperimetry

On a Riemannian manifold, any smooth $(n-1)$-submanifold (i.e., hypersurface of co-dimension 1) inherits a Riemannian measure which we will denote by μ_{n-1}. The isoperimetric problem on M asks for the maximal volume that can be enclosed in a hypersurface of prescribed $(n-1)$-volume and for a description of the extremal sets, if they exist.

The first part of this problem can be interpreted as the search for some function Φ (depending on M) such that

$$\Phi(\mu_n(\Omega)) \leq \mu_{n-1}(\partial\Omega)$$

for all bounded sets $\Omega \subset M$ with smooth boundary $\partial\Omega$.

Solving the second part of the isoperimetric problem of course yields such an inequality. For instance, if M is n-dimensional Euclidean space, balls are extremal sets for the isoperimetric problem and this leads to the optimal function Φ given by

$$\Phi_{\mathbb{R}^n}(t) = \frac{\omega_{n-1}}{\Omega_n^{1-1/n}} t^{1-1/n}.$$

In view of this fundamental example, it is natural to consider the possibility that a Riemannian manifold M satisfies

$$\mu_n(\Omega)^{1-1/\nu} \leq C(M,\nu)\mu_{n-1}(\partial\Omega) \tag{3.1.1}$$

for some constant $\nu > 0$ and $C(M,\nu) > 0$. Note that this could possibly be satisfied for a number of different values of ν and that the set of possible values of ν is either empty or an interval.

Theorem 3.1.1 *Assume that M satisfies* (3.1.1) *for some positive finite ν and $C(M,\nu)$. Then*

$$V(x,r) \geq c(M,\nu)r^{\nu}$$

where $c(M,\nu) = [\nu C(M,\nu)]^{-\nu}$. In particular, if M is n-dimensional and satisfies (3.1.1) *then $\nu \geq n$.*

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The proof is straightforward if one observes that $\partial_r V(x,r)$ is the $(n-1)$-dimensional Riemannian volume of the boundary of $B(x,r)$. Indeed, we then have

$$V(x,r)^{1-1/\nu} \leq C(M,\nu)\partial_r V(x,r).$$

That is

$$\partial_r \left[V(x,r)^{1/\nu}\right] \geq [\nu C(M,\nu)]^{-1}.$$

This obviously yields

$$V(x,r) \geq [\nu C(M,\nu)]^{-\nu} r^\nu.$$

However, this proof is not quite complete because the boundary of a ball need not be a smooth $(n-1)$-submanifold (for large radius r). Here, we will ignore this difficulty and refer the reader to [13, §3.3,3.5] for details justifying the computation above. A different proof of this lemma, avoiding this difficulty, will be given later on (see Theorem 3.1.5 below).

In Euclidean space, we noticed the formal equivalence of the isoperimetric inequality with the Sobolev inequality $\|f\|_{n/(n-1)} \leq C_n \|\nabla f\|_1$. The argument, based on the co-area formula (see, e.g., [13, Theorems 3.13 and 6.3]), works as well on any Riemannian manifold. This proves the following important result.

Theorem 3.1.2 *A manifold M satisfies the inequality* (3.1.1) *for some positive ν and $C(M,\nu)$ if and only if it satisfies the inequality*

$$\forall f \in C_0^\infty(M), \quad \|f\|_{\nu/(\nu-1)} \leq C(M,\nu)\|\nabla f\|_1.$$

Let us fix p, ν such that $1 \leq p < \nu$. We say that a Riemannian manifold M satisfies an (L^p,ν)-Sobolev inequality if there exists a constant $C(M,p,\nu)$ such that

$$\forall f \in C_0^\infty(M), \quad \|f\|_{p\nu/(\nu-p)} \leq C(M,p,\nu)\|\nabla f\|_p.$$

Thus, Theorem 3.1.2 can be interpreted as saying that a manifold M satisfies an (L^1,ν)-Sobolev inequality if and only if it satisfies (3.1.1). The next result shows that the strength of an (L^p,ν)-Sobolev inequality decreases as p increases.

Theorem 3.1.3 *If M satisfies an (L^p,ν)-Sobolev inequality then it satisfies an (L^q,ν)-Sobolev inequality for all $p \leq q < \nu$.*

Apply the (L^p,ν) inequality to $|f|^\gamma$ for some $\gamma > 1$ to be fixed later. Thus

$$\|f\|^\gamma_{\gamma p\nu/(\nu-p)} \leq \gamma C(M,p,\nu)\left(\int_M |f|^{p(\gamma-1)}|\nabla f|^p d\mu\right)^{1/p}.$$

Now, apply the Hölder inequality

$$\int hgd\mu \leq \|h\|_{q/(q-p)}\|g\|_{q/p}$$

with $h = |f|^{p(\gamma-1)}$ and $g = |\nabla f|^p$. This yields

$$\begin{aligned}&\left(\int_M |f|^{p(\gamma-1)}|\nabla f|^p d\mu\right)^{1/p}\\ &\leq \left(\int |f|^{pq(\gamma-1)/(q-p)}d\mu\right)^{1/p-1/q}\left(\int |\nabla f|^q d\mu\right)^{1/q}.\end{aligned}$$

Hence,

$$\|f\|^{\gamma}_{\gamma p\nu/(\nu-p)} \leq \gamma C(M,p,\nu)\left(\int |f|^{pq(\gamma-1)/(q-p)}d\mu\right)^{1/p-1/q}\left(\int |\nabla f|^q d\mu\right)^{1/q}.$$

Picking $\gamma = q(n-p)/p(n-q)$ and computing $\gamma - 1 = n(q-p)/p(n-q)$ yields

$$\|f\|_{qn/(n-q)} \leq \frac{q(n-p)}{p(n-q)}C(M,p,\nu)\|\nabla f\|_q$$

as desired.

3.1.3 Sobolev inequalities and volume growth

The next lemma introduces a family of (a priori weaker) inequalities that can be deduced from a Sobolev inequality. Our aim in this section is to show that weak forms of Sobolev inequalities are sufficient to imply a lower bound for the volume growth of M.

Lemma 3.1.4 *If M satisfies an (L^p, ν)-Sobolev inequality*

$$\forall f \in C_0^\infty(M), \quad \|f\|_{p\nu/(\nu-p)} \leq C(M,p,\nu)\|\nabla f\|_p$$

then it also satisfies

$$\forall f \in C_0^\infty(M), \quad \|f\|_r \leq (C(M,p,\nu)\|\nabla f\|_p)^\theta \|f\|_s^{1-\theta}$$

for all $0 < r, s < \infty$ and $0 \leq \theta \leq 1$ such that

$$\frac{1}{r} = \theta(\frac{1}{\nu} - \frac{1}{p}) + (1-\theta)\frac{1}{s}.$$

Set $q = \nu p/(\nu - p)$. Then $1/r = \theta/q + (1-\theta)/s$ and, by the Hölder inequality,

$$\|f\|_r \leq \|f\|_q^\theta \|f\|_s^{1-\theta}.$$

This yields the desired inequality. The possible range for s is actually $(0, \nu p/(\nu - p))$ and the range of r is $(s, \nu p/(\nu - p))$. Note that when $\theta = 0$ one must have $r = s$ and the conclusion of the lemma is trivial.

The following result generalizes Theorem 3.1.1.

Theorem 3.1.5 *Assume that the inequality*

$$\forall f \in C_0^\infty(M), \quad \|f\|_r \le (C\|\nabla f\|_p)^\theta \|f\|_s^{1-\theta}$$

is satisfied for some r, s, θ *with* $0 < s \le r \le \infty$ *and* $0 < \theta \le 1$. *Assume also that*

$$\frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r} > 0.$$

Then

$$V(x,t) \ge ct^\nu$$

with ν *defined by*

$$\frac{\theta}{\nu} = \frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r} \tag{3.1.2}$$

and the constant c *given by*

$$c = 2^{-\nu^2/\theta r - \nu/\theta} C^{-\nu}.$$

In particular, if M *satisfies an* (L^p, ν)*-Sobolev inequality for some* p, ν *with* $1 \le p < \nu$, *then* $n \ge \nu$ *and the volume growth function* $V(x,t)$ *satisfies*

$$\inf_{\substack{x \in M \\ t > 0}} \{t^\nu V(x,t)\} > 0.$$

We will use the fact that the distance function has gradient bounded by 1 almost everywhere. Indeed, for any fixed $x \in M$ and $t > 0$, consider the function

$$f(y) = \max\{t - d(x,y)), 0\}.$$

Then

$$\begin{aligned} \|f\|_r &\ge (t/2)V(x,t/2)^{1/r} \\ \|f\|_s &\le tV(x,t)^{1/s} \\ \|\nabla f\|_p &\le V(x,t)^{1/p}. \end{aligned}$$

Hence

$$V(x,t)^{\theta/p + (1-\theta)/s} \ge 2^{-1}(t/C)^\theta V(x,t/2)^{1/r}.$$

If we could ignore the fact that we have the volume of the ball of radius $t/2$ instead of t on the right-hand side of this inequality, we would get $V(x,t) \ge ct^\nu$ with ν given by $\theta/\nu = \theta/p + (1-\theta)/s - 1/r$. Thus, define ν by

$$\frac{\theta}{\nu} = \frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r}$$

and write the last inequality in the form

$$V(x,t) \ge (2C^\theta)^{-r\nu/(\nu+\theta r)} t^{\theta r \nu/(\nu+\theta r)} V(x,t/2)^{\nu/(\nu+\theta r)}.$$

It follows that

$$V(x,t) \geq (2C^\theta)^{-r\sum_1^i a^j} t^{\theta r \sum_1^i a^j} 2^{-\theta r \sum_1^i (j-1)a^j} V(x,t/2^i)^{a^i} \tag{3.1.3}$$

with $a = \nu/(\nu+\theta r)$. Observe that $a < 1$ as long as $\theta \neq 0$. Moreover, in this case,

$$\sum_1^\infty a^j = a(1-a)^{-1} = \frac{\nu}{\theta r}, \quad \sum_1^\infty (j-1)a^j = a^2(1-a)^{-2} = \frac{\nu^2}{\theta^2 r^2}.$$

Furthermore, $\lim_{t\to 0} t^{-n}V(x,t) = \Omega_n$. Hence for i large enough,

$$\liminf_{i\to\infty} V(x,t/2^i)^{a_i} \geq \lim_{i\to\infty} [\Omega_n t^n/2^{in+1}]^{a^i} = \lim_{i\to\infty} 2^{-ina^i} = 1.$$

Letting i tend to infinity in (3.1.3), we obtain

$$V(x,r) \geq 2^{-\nu^2/\theta r}(2C^\theta)^{-\nu/\theta} t^\nu.$$

Note that this proof uses the (Riemannian) fact that $\lim_{t\to 0} t^{-n}V(x,t) = \Omega_n$. Later, we will give another proof avoiding the use of this fact.

It is useful to illustrate Theorem 3.1.5 by a very simple case showing how the volume parameter ν abstractly defined by (3.1.2) is computed from θ, r, s, p. Take $M = \mathbb{R}$. Then we have the calculus inequality,

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \|f\|_\infty \leq (1/2)\|\nabla f\|_1.$$

Theorem 3.1.5 applies with $\theta = 1$, $r = \infty$, $p = 1$ (s is irrelevant when $\theta = 1$) and yields $\nu = 1$ and $V(t) \geq t$ (note that this is off only by a factor of 2). Applying the calculus inequality above to $|f|^2$ and using Cauchy–Schwarz, we find that $\|f\|_\infty \leq (1/2)\|f\|_2^{1/2}\|\nabla f\|_2^{1/2}$ and thus $\|f\|_2^2 \leq (1/2)\|\nabla f\|_2^{1/2}\|f\|_2^{1/2}\|f\|_1$. That is,

$$\forall \mathcal{C}_0^\infty(\mathbb{R}), \quad \|f\|_2 \leq (1/2)^{1/2}\|\nabla f\|_2^{1/3}\|f\|_1^{2/3}.$$

This is the Nash inequality, in one dimension. In the next section, we will take a closer look at this type of inequality. For now, observe that we can apply Theorem 3.1.5 again, this time with $r = p = 2$, $s = 1$, $\theta = 1/3$. Then ν is defined by $1/(3\nu) = 1/6 + 2/3 - 1/2 = 2/6$, i.e., $\nu = 1$ again.

From the proof of Theorem 3.1.5, it is obvious that one can work under a weaker hypothesis and obtain the following statement.

Theorem 3.1.6 *Assume that the inequality*

$$\forall f \in \mathcal{C}_0^\infty(M), \ \sup_{t>0}\left\{t\mu(|f|>t)^{1/r}\right\} \leq (C\|\nabla f\|_p)^\theta \left(\|f\|_\infty \mu(\text{supp}(f))^{1/s}\right)^{1-\theta}$$

is satisfied for some r, s, θ *with* $0 < s, r \leq \infty$ *and* $0 < \theta \leq 1$. *Assume also that*

$$\frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r} > 0.$$

Then

$$V(x,t) \geq ct^{\nu}$$

with $0 < \nu < \infty$ *defined by*

$$\frac{\theta}{\nu} = \frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r}$$

and the constant c *given by*

$$c = 2^{-\nu^2/\theta r - \nu/\theta} C^{-\nu}.$$

In this theorem, if $r = \infty$, the weak L^r-norm must be understood as $\|f\|_\infty$.

3.2 Weak and strong Sobolev inequalities

3.2.1 Examples of weak Sobolev inequalities

Any Sobolev inequality can be used to deduce a priori weaker inequalities through the use of the Hölder inequality and related inequalities. For instance, the Sobolev inequality

$$\|f\|_{2\nu/(\nu-2)} \leq C\|\nabla f\|_2$$

obviously implies the weak Sobolev inequality

$$\sup_{t>0} \left\{ t\mu(\{|f| > t\})^{(\nu-2)/2\nu} \right\} \leq C\|\nabla f\|_2.$$

We have also seen in Section 2.1.3 that the Sobolev inequality

$$\|f\|_{2\nu/(\nu-2)} \leq C\|\nabla f\|_2$$

implies

$$\|f\|_{2(1+2/\nu)} \leq (C\|\nabla f\|_2)^{\nu/(\nu+2)} \|f\|_2^{2/(\nu+2)}.$$

A different use of the Hölder inequality leads to

$$\|f\|_2 \leq (C\|\nabla f\|_2)^{\nu/(\nu+2)} \|f\|_1^{2/(\nu+2)}$$

which one often writes

$$\|f\|_2^{2(1+2/\nu)} \leq C^2 \|\nabla f\|_2^2 \|f\|_1^{4/\nu}.$$

This last inequality is called a Nash inequality. It first appeared in the 1958 paper of Nash [67] concerning the regularity of solutions of parabolic uniformly elliptic equations in divergence form in $\mathbb{R}^n$. Nash did not deduce this inequality from the Sobolev inequality. Instead, he gave the following direct proof.

Theorem 3.2.1 *There is a constant C_n such that any smooth compactly supported function f on $\mathbb{R}^n$ satisfies*

$$\|f\|_2^{2(1+2/n)} \le C_n^2 \|\nabla f\|_2^2 \|f\|_1^{4/n}.$$

Let $\hat{f}$ be the Fourier transform of f. Then

$$\begin{aligned}\|f\|_2^2 &= \|\hat{f}\|_2^2 = \int_{|\xi|\le R} |\hat{f}(\xi)|^2 d\xi + \int_{|\xi|>R} |\hat{f}(\xi)|^2 d\xi \\ &\le \Omega_n R^n \|\hat{f}\|_\infty^2 + R^{-2} \int |\hat{f}(\xi)|^2 |\xi|^2 d\xi \\ &\le \Omega_n R^n \|f\|_1^2 + (2\pi R)^{-2} \|\nabla f\|_2^2.\end{aligned}$$

Optimizing in R yields

$$\|f\|_2^2 \le (2+n)(\Omega_n/4\pi)^{2/(n+2)} \|\nabla f\|_2^{2n/(n+2)} \|f\|_1^{4/(n+2)},$$

which gives the desired inequality with

$$C_n = (2+n)^{1+2/n} (\Omega_n/4\pi)^{2/n}.$$

Nash's proof is given here to point out how different it is from the proofs of the corresponding Sobolev inequality that we have seen. The question naturally arises of whether or not this is yet another proof of the L^2-Sobolev inequality in $\mathbb{R}^n$. That is, can we easily deduce the Sobolev inequality

$$\|f\|_{2\nu/(\nu-2)} \le C \|\nabla f\|_2$$

from the Nash inequality

$$\|f\|_2^{2(1+2/\nu)} \le C^2 \|\nabla f\|_2^2 \|f\|_1^{4/\nu},$$

maybe with different constants C?

There is an interesting twist here because the Nash inequality of Theorem 3.2.1 is valid for all $n \ge 1$ whereas the corresponding Sobolev inequality is valid only for $n \ge 3$. In some sense, this means that, if Nash inequality implies Sobolev inequality, it cannot be in a completely obvious way.

In general, one can ask: do various weak forms of Sobolev inequality imply their stronger counterparts? The next few sections show that, somewhat surprisingly, the answer is yes. Moreover, this can be proved by some elementary and widely applicable arguments.

3.2.2 $(S^\theta_{r,s})$-inequalities: the parameters q and ν

Let us fix the parameter p, $1 \le p < \infty$. We want to discuss functional inequalities of Sobolev type for smooth compactly supported functions under the hypothesis that we can control $\|\nabla f\|_p$. The weakest type of Sobolev

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inequality we have encountered so far is that of Theorem 3.1.6 which states that, for all $f \in \mathcal{C}_0^\infty(M)$,

$$\sup_{t>0}\left\{t\mu(\{|f|>t\})^{1/r}\right\} \le (C\|\nabla f\|_p)^\theta \left(\|f\|_\infty \mu(\mathrm{supp}(f))^{1/s}\right)^{1-\theta}.$$

We call this inequality $(S_{r,s}^{*,\theta})$. Recall that it has a slightly stronger version (see Theorem 3.1.5) which reads

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_r \le (C\|\nabla f\|_p)^\theta \|f\|_s^{1-\theta}. \qquad (S_{r,s}^\theta)$$

In these inequalities, we can think of $0 < r, s \le \infty$, $0 < \theta \le 1$ as parameters and we would like to understand the meaning of these parameters.

For the time being, let us simply observe that $(S_{r,s}^1)$ (the parameter s plays no role when $\theta = 1$) is the classical Sobolev inequality

$$\|f\|_r \le C\|\nabla f\|_p.$$

Similarly, $(S_{r,s}^{*,1})$ (again, the parameter s plays no role when $\theta = 1$) is the weak Sobolev inequality

$$\sup_{t>0}\left\{t\mu(|f|>t)^{1/r}\right\} \le C\|\nabla f\|_p.$$

Finally, when $p = 2$, $(S_{2,1}^\theta)$ is the Nash inequality

$$\|f\|_2^{2(1+(1-\theta)/\theta)} \le C^2\|\nabla f\|_2^2\|f\|_1^{2(1-\theta)/\theta}.$$

Observe also that, by Theorem 3.1.6, each of these inequalities implies

$$\inf_{t>0}\{t^{-\nu}V(x,t)\} > 0$$

where the parameter ν is defined by $1/r = 1/p - 1/\nu$ for the first two inequalities (assuming $p < r$) and by $2/\nu = \theta/(1-\theta)$ for the Nash inequality.

It turns out that one of the keys to understanding these inequalities is to consider yet another parameter, call it $q = q(r,s,\theta)$, which belongs to $(-\infty,0)\cup(0,+\infty)\cup\{\infty\}$ and which is defined by

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}. \qquad (3.2.1)$$

Observe that q is related to the parameter ν of Theorems 3.1.5, 3.1.6 by

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{\nu}. \qquad (3.2.2)$$

It is a fundamental difference between q and ν that q can be computed from (r,s,θ) without explicit reference to p whereas ν cannot.

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We will prove below that, roughly speaking, any weak inequality $(S^{*,\theta_0}_{r_0,s_0})$ for some fixed r_0, s_0, θ_0 implies the full collection of strong inequalities

$$(S^{\theta}_{r,s}) \text{ for all } 0 < r, s \leq \infty,\ 0 < \theta \leq 1$$

with

$$q(r,s,\theta) = q(r_0,s_0,\theta_0).$$

More precisely, this is correct when

$$1/q(r_0,s_0,\theta_0) \leq 0.$$

When $1/q(r_0,s_0,\theta_0) > 0$, one needs to assume also that $q \geq p$.

The proof will depend on elementary functional analysis arguments and repeated application of the given inequality to functions of the type $(f - u)_+ \wedge v$ with $u, v > 0$, $u \wedge v = \min\{u, v\}$, $u_+ = \max\{u, 0\}$, where f is a given fixed function in $\mathcal{C}^\infty_0(M)$.

Here, we consider the parameter p as fixed once and for all. Later, in Section 3.2.6, we will also consider what happens when p varies. Then the parameter ν will play a crucial role. Roughly speaking, any of the inequalities $(S^{*,\theta_0}_{r_0,s_0})$ for some fixed r_0, s_0, θ_0 and p_0 implies all the inequalities $(S^{\theta}_{r,s})$ for all $p \geq p_0$ and where the parameters r, s, θ, p satisfy *both* (3.2.1) and (3.2.2).

3.2.3 The case $0 < q < \infty$

In this section the number p, $1 \leq p < \infty$, is fixed and all Sobolev-type inequalities are relative to $\|\nabla f\|_p$. The main result of this section is described in the following theorem.

Theorem 3.2.2 *Assume that $(S^{*,\theta_0}_{r_0,s_0})$ is satisfied for some $0 < r_0, s_0 \leq \infty$ and $0 < \theta_0 \leq 1$ and let $q = q(r_0,s_0,\theta_0)$ be defined as in (3.2.1). Assume that $p \leq q < \infty$. Then all the inequalities $(S^{\theta}_{r,s})$ with $0 < r, s \leq \infty$, $0 < \theta \leq 1$ and $q(r,s,\theta) = q$ are also satisfied. In particular, there exists a finite constant A such that*

$$\forall f \in \mathcal{C}^\infty_0(M), \quad \|f\|_q \leq AC\|\nabla f\|_p$$

where C is the constant appearing in $(S^{*,\theta_0}_{r_0,s_0})$. That is, M satisfies an (L^p,ν)-Sobolev inequality with ν defined by $1/q = 1/p - 1/\nu$.

Fix a non-negative function $f \in \mathcal{C}^\infty_0(M)$ and set

$$f_{\rho,k} = (f - \rho^k)_+ \wedge \rho^k(\rho - 1) \tag{3.2.3}$$

for any $\rho > 1$ and $k \in \mathbb{Z}$. This function has the following properties. Its support is contained in $\{f \geq \rho^k\}$. Moreover,

$$\{f_{\rho,k} \geq (\rho-1)\rho^k\} = \{f \geq \rho^{k+1}\}. \tag{3.2.4}$$

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Finally, $f_{\rho,k}$ has the same "profile" as f on $\{\rho^k < f < \rho^{k+1}\}$ and is flat outside this set. In particular,

$$|\nabla f_{\rho,k}| \le |\nabla f|. \tag{3.2.5}$$

By (3.2.4), (3.2.5) and our hypothesis applied to $f_{\rho,k}$, we have

$$(\rho-1)\rho^k \mu(f \ge \rho^{k+1})^{1/r_0} \le (C\|\nabla f\|_p)^{\theta_0}\left((\rho-1)\rho^k\mu(f \ge \rho^k)^{1/s_0}\right)^{1-\theta_0}.$$

Let us set

$$N(f) = \sup_k \left\{\rho^k \mu(f \ge \rho^k)^{1/q}\right\}.$$

Using the definition of q we then get

$$\begin{aligned} &\mu(f \ge \rho^{k+1})^{1/r_0} \\ &\le \rho^{-k(\theta_0+(1-\theta_0)q/s_0)}\left(\frac{C\|\nabla f\|_p}{\rho-1}\right)^{\theta_0} N(f)^{q(1-\theta_0)/s_0} \\ &\le \rho^{-kq/r_0}\left(\frac{C\|\nabla f\|_p}{\rho-1}\right)^{\theta_0} N(f)^{q(1-\theta_0)/s_0}. \end{aligned}$$

Thus

$$N(f)^{q/r_0} \le \rho^{q/r_0}\left(\frac{C\|\nabla f\|_p}{\rho-1}\right)^{\theta_0} N(f)^{q(1-\theta_0)/s_0}.$$

Simplifying and using the definition of q again yields

$$N(f) \le \frac{\rho^{q/r_0}}{\rho-1} C\|\nabla f\|_p \tag{3.2.6}$$

and thus

$$\sup_{t>0}\{t\mu(f \ge t)^{1/q}\} \le \rho N(f) \le \frac{\rho^{1+q/r_0}}{\rho-1} C\|\nabla f\|_p.$$

Setting $\rho = 1 + r_0\theta_0/q$, we get

$$\sup_{t>0}\{t\mu(f \ge t)^{1/q}\} \le e(1+q/(r_0\theta_0))C\|\nabla f\|_p. \tag{3.2.7}$$

This is the weak form (in the sense of weak L^r spaces) of the desired (L^p,ν)-Sobolev inequality

$$\|f\|_{p\nu/(\nu-p)} \le AC\|\nabla f\|_p$$

where ν is given by $1/q = 1/p - 1/\nu$. Thus, to finish the proof of Theorem 3.2.2, it suffices to prove the following lemma.

Lemma 3.2.3 *Assume that for some* $1 \le p < q < \infty$,

$$\forall f \in \mathcal{C}_0^\infty(M),\ \sup_{t>0}\{t\mu(f \ge t)^{1/q}\} \le C_1\|\nabla f\|_p.$$

Then

$$\forall f \in \mathcal{C}_0^\infty(M),\ \|f\|_q \le 2(1+q)C_1\|\nabla f\|_p.$$

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For this lemma, we need to improve (3.2.5) and observe that, actually,

$$|\nabla f_k| \leq |\nabla f| \mathbf{1}_{\{\rho^k < f \leq \rho^{k+1}\}}. \tag{3.2.8}$$

Now, applying the hypothesis to $f_{\rho,k}$ yields

$$(\rho - 1)\rho^k \mu(f \geq \rho^{k+1})^{1/q} \leq C_1 \|\nabla f_k\|_p.$$

Raise this inequality to the power q and sum over $k \in \mathbb{Z}$ to get

$$(\rho - 1)^q \sum_k \rho^{kq} \mu(f \geq \rho^{k+1}) \leq C_1^q \sum_k \left(\int_{\{\rho^k < f \leq \rho^{k+1}\}} |\nabla f|^p \right)^{q/p}. \tag{3.2.9}$$

As $q/p \geq 1$, we have $\sum a_k^{q/p} \leq (\sum a_k)^{q/p}$ for any sequence (a_k) of non-negative reals. Also

$$\begin{aligned}
\sum \rho^{kq} \mu(f \geq \rho^{k+1}) &\geq [\rho^q(\rho^q - 1)]^{-1} q \sum \int_{\rho^{k+1}}^{\rho^{k+2}} t^{q-1} \mu(f \geq t) dt \\
&= [\rho^q(\rho^q - 1)]^{-1} q \int_0^\infty t^{q-1} \mu(f \geq t) dt \\
&= [\rho^q(\rho^q - 1)]^{-1} \|f\|_q^q.
\end{aligned}$$

Hence (3.2.9) yields

$$\|f\|_q \leq \frac{\rho(\rho^q - 1)^{1/q}}{(\rho - 1)} C_1 \|\nabla f\|_p.$$

Using $(\rho^q - 1)^{1/q} \leq \rho$ and picking $\rho = 1 + 1/q \leq 2$ yields the desired inequality.

Corollary 3.2.4 *For $\nu > 2$, the Nash-type inequality*

$$\forall f \in C_0^\infty(M), \quad \|f\|_2^{2(1+2/\nu)} \leq C^2 \|\nabla f\|_2^2 \|f\|_1^{4/\nu}$$

implies the (L^2, ν)-Sobolev inequality

$$\forall f \in C_0^\infty(M), \quad \|f\|_{2\nu/(\nu-2)} \leq A_\nu C \|\nabla f\|_2$$

for some constant A_ν.

Indeed, the postulated Nash-type inequality is exactly $(S_{2,1}^{\nu/(\nu+2)})$ raised to the power $2(\nu+2)/\nu$. Furthermore, $q = q(2, 1, \nu/(\nu+2))$ is equal to $2\nu/(\nu - 2)$ which is finite and larger than 2 if ν is larger than 2. Thus we can apply Theorem 3.2.2 which yields the desired Sobolev inequality.

3.2.4 The case $q = \infty$

As in the previous section, the number p, $1 \le p < \infty$, is fixed and all Sobolev-type inequalities are relative to $\|\nabla f\|_p$.

Theorem 3.2.5 *Assume that* $(S^{*,1-s_0/r_0}_{r_0,s_0})$ *is satisfied for some* $0 < s_0 < r_0 < \infty$ *(this has* $q(r_0, s_0, 1 - s_0/r_0) = \infty$ *for* q *given at* (3.2.1)*). Then all the inequalities* $(S^{1-s/r}_{r,s})$ *with* $0 < s < r < \infty$ *are satisfied.*

Fix a non-negative $f \in C_0^\infty(M)$ and apply the hypothesis to $f_{\rho,k}$ defined at (3.2.3). Thanks to (3.2.4), (3.2.5), this gives

$$\rho^{kr_0}\mu(f \ge \rho^{k+1}) \le \left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{r_0 - s_0} \rho^{ks_0}\mu(f \ge \rho^k). \tag{3.2.10}$$

Fix $0 < s < r < \infty$ and observe that it is enough to prove the desired inequality for some set S of pairs (r, s), $0 < s < r < \infty$ such that $\sup\{s : (r, s) \in S\} = \infty$. Indeed, the Hölder inequality shows that, if $r \ge s$, then $(S^\theta_{r,s})$ implies $(S^{\theta'}_{r',s'})$ for all $r' \ge s'$ such that $r' \le r$ and $s' \le s$. For instance, it is enough to prove $(S^{1-s/r}_{r,s})$ for $(r, s) \in S = \{(r_0 + t, s_0 + t) : t > 0\}$. Now, fix any $t > 0$ and set $r = r_0 + t$, $s = s_0 + t$. Multiply (3.2.10) by q^{kt} to obtain

$$\rho^{kr}\mu(f \ge \rho^{k+1}) \le \left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{r_0 - s_0} \rho^{ks}\mu(f \ge \rho^k).$$

Summing over all k, we get

$$\sum_k \rho^{kr}\mu(f \ge \rho^{k+1}) \le \left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{r-s} \sum_k \rho^{ks}\mu(f \ge \rho^k).$$

But

$$\frac{1}{\rho^r(\rho^r - 1)}\|f\|_r^r \le \frac{1}{\rho^r(\rho^r - 1)} \sum_k r \int_{\rho^{k+1}}^{\rho^{k+2}} t^{s-1}\mu(f \ge t)dt \le \rho^{kr}\mu(f \ge \rho^{k+1})$$

and, similarly,

$$\sum_k \rho^{ks}\mu(f \ge \rho^k) \le \frac{\rho^s}{\rho^s - 1}\|f\|_s^s.$$

Hence

$$\|f\|_r^r \le \frac{\rho^{r+s}(\rho^r - 1)}{(\rho^s - 1)(\rho - 1)^{r-s}}(C\|\nabla f\|_p)^{r-s}\|f\|_s^s$$

for all $t > 1$ and $r = r_0 + t$, $s = s_0 + t$. For any fixed $\rho > 1$, this gives the desired inequality for all $0 < s < r < \infty$. This proves Theorem 3.2.5.

We now want to obtain a result that complements Theorem 3.2.5 under the same hypothesis. Observe that we cannot take $r = \infty$ in Theorem 3.2.5.

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Actually, the case of $\mathbb{R}^n$ shows that we cannot hope to have an inequality of the form

$$\|f\|_\infty \le (C\|\nabla f\|_p)^\theta \|f\|_s^{1-\theta}$$

when $q = \infty$. What we can hope for is some form of local exponential integrability. This can be obtained by looking more closely at the proof given above. By Theorem 3.2.5, we can now assume without loss of generality that $(S_{2,1}^{1/2})$ is satisfied. Repeating the argument above and using $\gamma(x-1) \le x^\gamma - 1 \le \gamma(x-1)x^{\gamma-1}$ which is valid for $\gamma \ge 1$, we obtain that there exists a constant C such that

$$\|f\|_r \le 2^{1/r}\rho^3(\rho-1)^{-1+s/r}(C\|\nabla f\|_p)^{1-s/r}\|f\|_s^{s/r}$$

for all $t > 0$, $r = 2+t$, $s = 1+t$. Let Ω be the support of f. As $1 \le s < r$, we have $\|f\|_s \le \mu(\Omega)^{1/s-1/r}\|f\|_r$. Hence, for any $r \ge 2$ and $s = r-1$

$$\|f\|_r \le 2^{1/r}\rho^3(\rho-1)^{-1+s/r}(C\|\nabla f\|_p)^{1-s/r}\mu(\Omega)^{1-s/r}\|f\|_r^{s/r}$$

which gives (recall that $r - s = 1$)

$$\|f\|_r \le 2\rho^{3r}(\rho-1)^{-1}C\|\nabla f\|_p\mu(\Omega)^{1/r}$$

for all $r \ge 2$. Picking $\rho = 1 + 1/r$ yields

$$\|f\|_r \le 54\, r\, C\|\nabla f\|_p\mu(\Omega)^{1/r}.$$

Hence we have obtained the following result.

Theorem 3.2.6 *Under the hypothesis of Theorem 3.2.5 there exists a constant A_1 such that for all bounded sets $\Omega \subset M$ and all $r \ge 1$,*

$$\forall\, f \in C_0^\infty(\Omega), \quad \|f\|_r \le A_1(1+r)\mu(\Omega)^{1/r}\|\nabla f\|_p.$$

Moreover, there exist two constants $\alpha > 0$ and A_2 such that for all bounded sets $\Omega \subset M$,

$$\forall\, f \in C_0^\infty(\Omega), \quad \int_\Omega e^{\alpha|f|/\|\nabla f\|_p}d\mu \le A_2\mu(\Omega).$$

The second inequality stated above is an easy consequence of the first. Note that this is not the sharpest result that can be obtained. Indeed, as in the case of $\mathbb{R}^n$, one can show that $(S_{r_0,s_0}^{*,1-s_0/r_0})$ for some fixed $0 < s_0 < r_0 < \infty$ implies the stronger integrability

$$\forall\, f \in C_0^\infty(\Omega), \quad \int_\Omega e^{(\alpha'|f|/\|\nabla f\|_p)^{p/(p-1)}}d\mu \le A'\mu(\Omega)$$

for some constants $A', \alpha' > 0$. See [6] for a proof of this sharper result using the same ideas. Interestingly enough, in order to obtain the sharper result one apparently needs to use Lorentz spaces instead of mere L^r-spaces.

3.2.5 The case $-\infty < q < 0$

As in the previous two sections, the number p, $1 \le p < \infty$, is fixed and all Sobolev-type inequalities are relative to $\|\nabla f\|_p$.

Theorem 3.2.7 *Assume that $(S^{*,\theta_0}_{r_0,s_0})$ is satisfied for some $0 < r_0, s_0 \le \infty$ and $0 < \theta_0 \le 1$ and let $q = q(r_0, s_0, \theta_0)$ be defined as in (3.2.1). Assume that $-\infty < q < 0$ (this forces $s_0 < r_0$). Then all the inequalities $(S^{\theta}_{r,s})$ with $0 < s < r \le \infty$, $0 < \theta \le 1$ and $q(r, s, \theta) = q$ are also satisfied. In particular, there exists a finite constant A such that*

$$\forall f \in C_0^\infty(M), \quad \|f\|_\infty \le A(C\|\nabla f\|_p)^{1/(1-s/q)}\|f\|_s^{1/(1-q/s)},$$

for all $0 < s < \infty$ (recall that $q < 0$). Here C is the constant appearing in $(S^{*,\theta_0}_{r_0,s_0})$.

Fix $f \in C_0^\infty(M)$, $f \ge 0$ and $\|f\|_\infty \ne 0$. Fix also $\epsilon > 0$ small enough and $\rho > 1$. Define the functions $f_{\rho,k}$ by setting

$$f_{\rho,k} = (f - (\|f\|_\infty - \epsilon - \rho^k))_+ \wedge \rho^{k-1}(\rho - 1)$$

for all $k \le k(f)$ where $k(f)$ is the largest integer k such that $\rho^k < \|f\|_\infty$. Note that $f_{\rho,k}$ is compactly supported if $0 < \epsilon \le \|f\|_\infty - \rho^{k(f)}$ and that $|\nabla f_{\rho,k}| \le |\nabla f|$ for all $k \le k(f)$. Set

$$\lambda_k = \|f\|_\infty - \epsilon - \rho^k.$$

Observe that $f_{\rho,k}$ has support in $\{f \ge \lambda_k\}$ and that

$$\{f_{\rho,k} \ge \rho^{k-1}(\rho - 1)\} = \{f \ge \lambda_{k-1}\}.$$

Applying $(S^{*,\theta}_{r_0,s_0})$ to $f_{\rho,k}$, we obtain

$$\rho^{k-1}\mu(f \ge \lambda_{k-1})^{1/r_0} \le \left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{\theta_0} \rho^{(k-1)(1-\theta_0)}\mu(f \ge \lambda_k)^{(1-\theta_0)/s_0}.$$

Multiply this inequality by $\rho^{\delta(k-1)}$ and rearrange to obtain

$$\begin{aligned}&\left(\rho^{r_0(k-1)(1+\delta)}\mu(f \ge \lambda_{k-1})\right)^{1/r_0}\\ \le\ &\left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{\theta_0} \left[\rho^{s_0(k-1)(1+\delta/(1-\theta_0))}\mu(f \ge \lambda_k)\right]^{(1-\theta_0)/s_0}.\end{aligned}$$

Now, choose δ so that $r_0(1 + \delta) = s_0(1 + \delta/(1 - \theta_0))$. It turns out that this is equivalent to $r_0(1 + \delta) = q$. Setting $a_k = \rho^{qk}\mu(f \ge \lambda_k)$ yields

$$a_{k-1}^{1/r_0} \le \rho^{-q(1-\theta_0)/s_0} \left(\frac{C\|\nabla f\|_p}{\rho - 1}\right)^{\theta_0} a_k^{(1-\theta_0)/s_0} \qquad (3.2.11)$$

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for all $k \le k(f)$. Observe that $a_k > 0$ for all $k \le k(f)$ and that

$$\lim_{k \to -\infty} a_k = \infty$$

because $\mu(f \ge \|f\|_\infty - \epsilon) > 0$ and $q < 0$. This is actually the reason why the parameter ϵ was introduced in the computation above. Because of these observations, it is clear that

$$a = \inf_{k \le k(f)} a_k$$

is positive. It follows that (3.2.11) implies

$$a^{1/r_0} \le \rho^{-q(1-\theta_0)/s_0} \left(\frac{C\|\nabla f\|_p}{\rho - 1} \right)^{\theta_0} a^{(1-\theta_0)/s_0},$$

that is,

$$a = \inf_{k \le k(f)} \left\{ \rho^{qk} \mu(f \ge \lambda_k) \right\} \ge \rho^{-q^2(1-\theta_0)/s_0\theta_0} \left(\frac{C\|\nabla f\|_p}{\rho - 1} \right)^{\theta_0}.$$

As $\lambda^s \mu(f \ge \lambda) \le \|f\|_s^s$, we get

$$\lambda_k^{-s} \rho^{qk} \|f\|_s^s \ge \rho^{-q^2(1-\theta_0)/s_0\theta_0} \left(\frac{C\|\nabla f\|_p}{\rho - 1} \right)^{\theta_0}.$$

Observe here that $\lambda_k = \lambda_k(\epsilon)$ depends on the small parameter ϵ. However, we can let ϵ tend to 0 in the above inequality. Choosing $k = k(f) - 1$ and observing that for this choice of k

$$\lambda_k(0) \ge \rho^k(\rho - 1) \ge \rho^{-2}(\rho - 1)\|f\|_\infty,$$

we obtain

$$\|f\|_\infty^{-s+q} \|f\|_s^s \ge (\rho - 1)^s \rho^{-2s+2q} \rho^{-q^2(1-\theta_0)/s_0\theta_0} \left(\frac{C\|\nabla f\|_p}{\rho - 1} \right)^{\theta_0}.$$

Rearranging, we obtain

$$\|f\|_\infty \le \rho^2(\rho - 1)^{-1} \left(\rho^{-q(1-\theta_0)/s_0\theta_0} C\|\nabla f\|_p \right)^{1/(1-s/q)} \|f\|_s^{1/(1-q/s)}.$$

Picking $\rho = 1 + 1/(1 + |q|)$ (recall that q is negative), we get

$$\|f\|_\infty \le 4(1 + |q|) \left(e^{(1-\theta_0)/s_0\theta_0} C\|\nabla f\|_p \right)^{1/(1-s/q)} \|f\|_s^{1/(1-q/s)}.$$

This ends the proof of Theorem 3.2.7.

Corollary 3.2.8 *Assume that* $(S^{*,\theta_0}_{r_0,s_0})$ *is satisfied for some* $0 < r_0, s_0 \leq \infty$ *and* $0 < \theta_0 \leq 1$ *and let* $q = q(r_0, s_0, \theta_0)$ *be defined as in* (3.2.1). *Assume that* $-\infty < q < 0$ *(this forces* $s_0 < r_0$*). Then there exists a constant* A *such that*

$$\forall\, f \in C_0^\infty(\Omega), \quad \|f\|_\infty \leq AC\mu(\Omega)^{-1/q}\|\nabla f\|_p$$

for all bounded domains $\Omega \subset M$. *Here* C *is the constant appearing in* $(S^{*,\theta_0}_{r_0,s_0})$.

Indeed, for any finite $s > 0$, $\|f\|_s \leq \|f\|_\infty \mu(\Omega)^{1/s}$ if f is supported in Ω. Hence, after simplifications, we have

$$\|f\|_\infty \leq AC\mu(\Omega)^{-1/q}\|\nabla f\|_p.$$

3.2.6 Increasing p

In the last three sections, we studied the inequalities $(S^{*,\theta}_{r,s})$ and $(S^{\theta}_{r,s})$ for a fixed value of p. In order to discuss what happens when the parameter p is allowed to vary, let us introduce the notation

$$(S^{*,\theta}_{r,s}(p)) \quad \text{and} \quad (S^{\theta}_{r,s}(p))$$

to refer to these inequalities with $1 \leq p < \infty$.

Theorem 3.2.9 *Assume that for some* $1 \leq p_0 < \infty$, $0 < r_0, s_0 \leq \infty$, $0 < \theta_0 \leq 1$, *the inequality* $(S^{*,\theta_0}_{r_0,s_0}(p_0))$ *is satisfied. Assume also that* $q_0 = q(r_0, s_0, \theta_0)$ *defined at* (3.2.1) *satisfies* $1/q_0 < 1/p_0$ *(which is obviously satisfied if* $q_0 < 0$ *or* $q_0 = \infty$*). Let* ν *be defined by* (3.2.2), *that is,* $1/q_0 = 1/p_0 - 1/\nu$. *Then all the inequalities*

$$(S^{\theta}_{r,s}(p))$$

where $0 < r, s \leq \infty$, $0 < \theta \leq 1$, $p_0 \leq p < \infty$, *and*

$$\frac{1}{q(r,s,\theta)} = \frac{1}{p} - \frac{1}{\nu}$$

are also satisfied.

By the results of the last three sections, we can assume without loss of generality that the inequality $(S^{\sigma_0}_{p_0,u_0}(p_0))$ is satisfied for some (or all) u_0, σ_0 such that $q(p_0, u_0, \sigma_0) = q_0$. For any $f \in C_0^\infty(M)$, we can apply this inequality to $|f|^\gamma$, $\gamma \geq 1$,

$$\|f\|^\gamma_{p_0\gamma} \leq C\left(\gamma \int_M |f|^{(\gamma-1)p_0}|\nabla f|^{p_0} d\mu\right)^{\sigma_0/p_0} \|f\|^{\gamma(1-\sigma_0)}_{u_0\gamma}.$$

By the Hölder inequality

$$\left(\int f^{(\gamma-1)r} g^r d\mu\right)^{1/r} \leq \|f\|_{r\gamma}^{\gamma-1} \|g\|_{r\gamma}$$

this yields

$$\|f\|_{p_0\gamma}^{\gamma} \leq C\gamma^{\sigma_0/p_0} \|f\|_{p_0}^{(\gamma-1)\sigma_0} \|\nabla f\|_{p_0\gamma}^{\sigma_0} \|f\|_{u_0\gamma}^{\gamma(1-\sigma_0)}.$$

Simplifying, we get that the inequality $(S_{p,u}^{\sigma}(p))$ is satisfied where $p = \gamma p_0$, with $\sigma = \sigma_0/(\sigma_0 + \gamma(1-\sigma_0))$, $u = pu_0/p_0$. Now, observe that

$$\frac{1-\sigma}{\sigma} = \gamma \frac{1-\sigma_0}{\sigma_0}.$$

It follows that the parameter ν defined by $1/\nu = 1/p_0 - 1/q_0$ satisfies

$$\begin{aligned}
\frac{1}{\nu} &= \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_0} - \frac{1}{\sigma_0}\left(\frac{1}{p_0} - \frac{1-\sigma_0}{u_0}\right) \\
&= \frac{1-\sigma_0}{\sigma_0}\left(\frac{1}{u_0} - \frac{1}{p_0}\right) = \frac{1-\sigma}{\gamma\sigma}\left(\frac{1}{u_0} - \frac{1}{p_0}\right) \\
&= \frac{1-\sigma}{\sigma}\left(\frac{1}{\gamma u_0} - \frac{1}{\gamma p_0}\right) = \frac{1-\sigma}{\sigma}\left(\frac{1}{u} - \frac{1}{p}\right) \\
&= \frac{1}{p} - \frac{1}{\sigma}\left(\frac{1}{p} - \frac{1-\sigma}{u}\right) = \frac{1}{p} - \frac{1}{q(p,u,\sigma)}.
\end{aligned}$$

In words, ν defined by $1/\nu = 1/p - 1/q$ has not changed when passing from $(S_{p_0,u_0}^{\sigma_0}(p_0))$ to $(S_{p,u}^{\sigma}(p))$.

In any case, we have that $(S_{p,u}^{\sigma}(p))$ is satisfied for some σ, u such that

$$\frac{1}{q(p,u,\sigma)} = \frac{1}{p} - \frac{1}{\nu}.$$

Now, depending on whether this q is positive, infinity, or negative, one of Theorems 3.2.2, 3.2.5, 3.2.7 shows that *all* the inequalities $(S_{r,s}^{\theta}(p))$, $0 < r, s \leq \infty$, $0 < \theta \leq 1$, with

$$\frac{1}{q(r,s,\theta)} = \frac{1}{p} - \frac{1}{\nu}$$

are satisfied. This is the desired conclusion.

Corollary 3.2.10 *Assume that for some $1 \leq p_0 < \infty$, $0 < r_0, s_0 \leq \infty$, $0 < \theta_0 \leq 1$, the inequality $(S_{r_0,s_0}^{*,\theta_0}(p_0))$ is satisfied. Assume also that $q_0 = q(r_0, s_0, \theta_0)$ defined at (3.2.1) satisfies $1/q_0 < 1/p_0$ (which is obviously satisfied if $q_0 < 0$ or $q_0 = \infty$). Let ν be defined by (3.2.2), that is, $1/q_0 = 1/p_0 - 1/\nu$. Then there exists a constant $c > 0$ such that $V(x,t) \geq ct^{\nu}$.*

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By Theorem 3.2.9, we can assume that $(S^\theta_{r,s}(p))$ is satisfied with p so large that $q(r,s,\theta) < 0$ (indeed, $1/q(r,s,\theta) = 1/p - 1/\nu$ with $0 < \nu < \infty$). By Theorem 3.2.7, we then have the inequality

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_\infty \leq A\|\nabla f\|_p^\theta \|f\|_1^{1-\theta}$$

with $\theta = (1 - 1/p + 1/\nu)^{-1}$. Applying this to the function $f(y) = (t - d(x,y))_+$ yields

$$t \leq AV(x,t)^{\theta/p} t^{1-\theta} V(x,t)^{1-\theta},$$

that is,

$$V(x,t)^{1/\nu} \geq A^{-1} t.$$

Corollary 3.2.10 is the same as Theorem 3.1.6. Note that the proof above does not use the fact that $\lim_{t\to 0} t^{-n} V(x,t) = \Omega_n > 0$ whereas the proof of Theorem 3.1.6 used this fact. Corollary 3.2.10 shows that the volume growth lower bound $V(x,t) \geq ct^\nu$ can be interpreted as a very weak form (a vestige) of Sobolev inequality.

3.2.7 Local versions

We would like to record here two useful comments about the results obtained in the previous four sections. The first comment is that we can replace M by an open subset, say $U \subset M$, without changing anything in Theorems 3.2.2, 3.2.5, 3.2.6, 3.2.7, 3.2.9, or in Lemma 3.2.3 or in Corollaries 3.2.4, 3.2.8. In other words, we do not need to assume that M is complete in these results. Note that Corollary 3.2.10 does not extend unchanged to the case of open subsets of M.

We would like also to mention that the results listed above can be extended to the case where the inequalities $(S^{*,\theta}_{r,s}(p))$, $(S^\theta_{r,s}(p))$ are replaced by their uniform local counterparts

$$\sup_{t>0} \left\{ t\mu(\{|f| > t\})^{1/r} \right\} \leq (C\|\nabla f\|_p + T\|f\|_p)^\theta \left(\|f\|_\infty \mu(\mathrm{supp}(f))^{1/s} \right)^{1-\theta},$$

$$\|f\|_r \leq (C\|\nabla f\|_p + T\|f\|_p)^\theta \|f\|_s^{1-\theta}$$

for all $f \in \mathcal{C}_0^\infty(M)$, which we denote respectively by $(\widetilde{S}^{*,\theta}_{r,s}(p))$, $(\widetilde{S}^\theta_{r,s}(p))$.

This is more or less obvious once it has been observed that the quantity

$$W_p(f) = C\|\nabla f\|_p + T\|f\|_p$$

behaves just like $\|\nabla f\|_p$ with respect to the transformation $f \mapsto (f-t)_+ \wedge s = f_t^s$, $t,s > 0$. More precisely, we have $W_p(f_t^s) \leq W(f)$ and

$$\sum_k W_p(f_{\rho,k}) \leq W_p(f)$$

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for $\rho > 1$ and $f_{\rho,k} = (f - \rho^k)_+ \wedge \rho^k(\rho - 1)$. Note that the ratio C/T is kept constant in this process.

Here are a few statements that are easily obtained by implementing this remark.

Theorem 3.2.11 *Let M be a complete Riemannian manifold. Assume that the inequality*

$$\sup_{t>0} \left\{ t\mu(\{|f| > t\})^{1/r_0} \right\} \le (C\|\nabla f\|_p + T\|f\|_p)^{\theta_0} \left(\|f\|_\infty \mu(\mathrm{supp}(f))^{1/s_0} \right)^{1-\theta_0},$$

is satisfied for all $f \in \mathcal{C}_0^\infty(M)$ for some $0 < r_0, s_0 \le \infty$, $0 < \theta_0 \le 1$ and $1 \le p \le 1$. Define q by $1/r_0 = \theta_0/q + (1-\theta_0)/s_0$ and assume that $p \le q < \infty$. Then there exists a constant A such that

$$\forall\, f \in \mathcal{C}_0^\infty(M), \quad \|f\|_q \le A\,(C\|\nabla f\|_p + T\|f\|_p).$$

Corollary 3.2.12 *For $\nu > 2$, the Nash-type inequality*

$$\forall\, f \in \mathcal{C}_0^\infty(M), \quad \|f\|_2^{(1+2/\nu)} \le (C\|\nabla f\|_2 + T\|f\|_2)\, \|f\|_1^{2/\nu}$$

implies the (L^2, ν)-Sobolev inequality

$$\forall\, f \in \mathcal{C}_0^\infty(M), \quad \|f\|_{2\nu/(\nu-2)} \le A_\nu\,(C\|\nabla f\|_2 + T\|f\|_2)$$

for some constant A_ν.

Theorem 3.2.13 *Assume that for some $1 \le p_0 < \infty$, $0 < r_0, s_0 \le \infty$, $0 < \theta_0 \le 1$, the inequality $(\widetilde{S}^{*,\theta_0}_{r_0,s_0}(p_0))$ is satisfied for some $C, T > 0$. Assume also that $q_0 = q(r_0, s_0, \theta_0)$ defined at (3.2.1) satisfies $1/q_0 < 1/p_0$ (which is obviously satisfied if $q_0 < 0$ or $q_0 = \infty$). Let ν be defined by (3.2.2), that is, $1/q_0 = 1/p_0 - 1/\nu$. Then there exists a constant $c > 0$ such that*

$$\forall\, x \in M,\ \forall t \in (0, T^{-1}), \quad V(x,t) \ge ct^\nu.$$

3.3 Examples

3.3.1 Pseudo-Poincaré inequalities

It turns out that a useful and widely applicable tool to prove Sobolev inequalities is a set of inequalities indexed by the semi-axis $\{t > 0\}$ that we call pseudo-Poincaré inequalities. For any $f \in \mathcal{C}_0^\infty(M)$, set

$$f_t(x) = \frac{1}{V(x,t)} \int_{B(x,t)} f(y)\,d\mu(y).$$

We say that M satisfies a pseudo-Poincaré inequality in L^p if there exists a constant C such that

$$\forall f \in C_0^\infty(M), \;\; \forall t > 0, \quad \|f - f_t\|_p \le C\,t\,\|\nabla f\|_p. \tag{3.3.1}$$

Note that the L^p norms on both side are taken over the whole space M. This should be interpreted as an inequality concerning the approximation of f by the more regular functions f_t, which are averages over balls of radius t. The important fact about f_t is that

$$\|f\|_\infty \le V(x,t)^{-1}\|f\|_1. \tag{3.3.2}$$

This obvious fact should be thought of as a quantitative version of the "smoothing" effect of $f \to f_t$. Actually, in the arguments that will be developed below the only things that matter are that f_t satisfies both (3.3.1) and (3.3.2).

Theorem 3.3.1 *Assume that M satisfies* (3.3.1) *for some* $1 \le p_0 < \infty$ *and that*

$$\inf_{\substack{t>0\\ x\in M}} \left\{t^{-\nu}V(x,t)\right\} > 0 \tag{3.3.3}$$

for some $\nu > 0$. Then the inequalities of Sobolev type $(S^\theta_{r,s}(p))$ hold true for all $p \ge p_0$ and all $0 < r, s \le \infty$, $0 < \theta \le 1$ such that $q(r,s,\theta)$ defined at (3.2.1) *satisfies* $1/q = 1/p - 1/\nu$.

The proof is surprisingly simple. It has its origins in [72]. By Theorem 3.2.9, it suffices to treat the case $p = p_0$. By hypothesis there exists a constant $c > 0$ such that $V(x,t) \ge (ct)^\nu$ for all $x \in M$ and $t > 0$. Fix $f \in C_0^\infty(M)$, $f \ge 0$ and a real $\lambda > 0$. For any $t > 0$, write

$$\mu(f \ge \lambda) \le \mu(|f - f_t| \ge \lambda/2) + \mu(f_t \ge \lambda/2)$$

and pick t so that $(ct)^{-\nu}\|f\|_1 = \lambda/4$. Then

$$\mu(f_t \ge \lambda/2) = 0$$

and

$$\begin{aligned}
\mu(f \ge \lambda) &\le \mu(|f - f_t| \ge \lambda/2)\\
&\le (2/\lambda)^p \|f - f_t\|_p^p\\
&\le (2Ct\|\nabla f\|_p/\lambda)^p\\
&= \left[(2^{1+2/\nu}C/c)\|f\|_1^{1/\nu}\|\nabla f\|_p \lambda^{-1-1/\nu}\right]^p.
\end{aligned}$$

That is

$$\lambda^{p(1+1/\nu)}\mu(f \ge \lambda) \le \left[(2^{1+2/\nu}C/c)\|f\|_1^{1/\nu}\|\nabla f\|_p\right]^p.$$

Raising this to the power τ/p with $\tau = 1/(1+1/\nu) = \nu/(1+\nu)$ yields

$$\lambda\mu(f \geq \lambda)^{\tau/p} \leq 4(C/c)^{\tau}\|\nabla f\|_p^{\tau}\|f\|_1^{1-\tau}$$

which is $(S^{*\tau}_{p/\tau,1}(p))$. This, together with Theorems 3.2.2, 3.2.5, 3.2.7, proves Theorem 3.3.1.

We now state some obvious but important corollaries of Theorem 3.3.1.

Corollary 3.3.2 *Assume that M satisfies* (3.3.1) *with $p = 1$ and* (3.3.3) *for some $\nu > 1$. Then the isoperimetric inequality*

$$\mu(\Omega)^{1-1/\nu} \leq C_1\mu(\partial\Omega)$$

is satisfied by all bounded sets Ω with smooth boundary. Also the (L^1, ν)-Sobolev inequality

$$\forall\, f \in C_0^\infty(M), \quad \|f\|_{\nu/(\nu-1)} \leq C_1\|\nabla f\|_1$$

is satisfied.

Corollary 3.3.3 *Fix $1 \leq p < \infty$ and $\nu > p$. Assume that M satisfies* (3.3.1) *and* (3.3.3) *for these p and ν. Then the (L^p, ν)-Sobolev inequality*

$$\forall\, f \in C_0^\infty(M), \quad \|f\|_{p\nu/(\nu-p)} \leq C_2\|\nabla f\|_p$$

is satisfied.

Corollary 3.3.4 *Assume that M satisfies* (3.3.1) *with $p = 2$ and* (3.3.3) *for some $\nu > 0$. Then the Nash inequality*

$$\forall\, f \in C_0^\infty(M), \quad \|f\|_2^{2(1+2/\nu)} \leq C_3\|\nabla f\|_2^2\|f\|_1^{4/\nu}$$

is satisfied.

3.3.2 Pseudo-Poincaré technique: local version

The argument developed in the section above can easily be adapted to the case when one only has local hypotheses. As an example, we prove the following result.

Theorem 3.3.5 *Fix $R > 0$. Assume that M satisfies*

$$\forall\, f \in C_0^\infty(M),\ \forall\, t \in (0, R), \quad \|f - f_t\|_{p_0} \leq C\, t\, \|\nabla f\|_{p_0} \tag{3.3.4}$$

for some $1 \leq p_0 < \infty$ and that

$$\inf_{\substack{t\in(0,R)\\ x\in M}} \left\{t^{-\nu}V(x,t)\right\} = c > 0 \tag{3.3.5}$$

for some $\nu > p_0$. Then the inequalities

$$\|f\|_q \leq [C(\nu,p)/c]\left(C\|\nabla f\|_p + R^{-1}\|f\|_p\right)$$

are satisfied for all $p_0 \leq p < \nu$ and $1/q = 1/p - 1/\nu$. The constant $C(\nu, p)$ is independent of C, c and R.

The proof is similar to that of Theorem 3.3.1. We can assume that $p = p_0$. By hypothesis, $V(x,t) \geq (ct)^\nu$ for all $x \in M$ and $t \in (0,R)$. Fix $f \in C_0^\infty(M)$, $f \geq 0$ and a real $\lambda > 0$. For any $0 < t < R$, write

$$\mu(f \geq \lambda) \leq \mu(|f - f_t| \geq \lambda/2) + \mu(f_t \geq \lambda/2).$$

If possible, pick t so that $(ct)^{-\nu}\|f\|_1 = \lambda/4$. This can be done if $\lambda > 4(cR)^{-\nu}\|f\|_1$. Then, by (3.3.2)

$$\mu(f_t \geq \lambda/2) = 0$$

and

$$\begin{aligned} \mu(f \geq \lambda) &\leq \mu(|f - f_t| \geq \lambda/2) \\ &\leq (2/\lambda)^p \|f - f_t\|_p^p \\ &\leq (2Ct\|\nabla f\|_p/\lambda)^p \\ &= \left[(2^{1+2/\nu}C/c)\|f\|_1^{1/\nu}\|\nabla f\|_p \lambda^{-1-1/\nu}\right]^p. \end{aligned}$$

That is

$$\lambda^{p(1+1/\nu)}\mu(f \geq \lambda) \leq \left[(2^{1+2/\nu}C/c)\|f\|_1^{1/\nu}\|\nabla f\|_p\right]^p.$$

Now, if $\lambda \leq 4(cR)^{-\nu}\|f\|_1$, simply write

$$\mu(f > \lambda) \leq \lambda^{-p}\|f\|_p^p$$

to see that, in this case,

$$\lambda^{p(1+1/\nu)}\mu(f \geq \lambda) \leq \left[4^{1/\nu}(cR)^{-1}\|f\|_1^{1/\nu}\|f\|_p\right]^p.$$

Thus, in all cases,

$$\lambda^{p(1+1/\nu)}\mu(f \geq \lambda) \leq \left[(2^{1+2/\nu}/c)\|f\|_1^{1/\nu}(C\|\nabla f\|_p + R^{-1}\|f\|_p\right]^p.$$

Raising this to the power τ/p with $\tau = 1/(1+1/\nu) = \nu/(1+\nu)$ yields

$$\lambda\mu(f \geq \lambda)^{\tau/p} \leq 4(1/c)^\tau(C\|\nabla f\|_p + R^{-1}\|f\|_p)^\tau\|f\|_1^{1-\tau}$$

which is the local version of $(S^{*\tau}_{p/\tau,1}(p))$. This, together with the local version of Theorem 3.2.2, proves Theorem 3.3.5. See Section 3.2.7.

Since we have not explained in detail the local version of Theorem 3.2.2, let us note that in the case $p_0 = p = 1$, the argument above yields

$$\lambda\mu(f \geq \lambda)^\tau \leq C_1(C\|\nabla f\|_1 + R^{-1}\|f\|_1)^\tau\|f\|_1^{1-\tau}.$$

Applying this inequality to regularizations of $f = \mathbf{1}_\Omega$ (where Ω is a bounded set having smooth boundary) with $\lambda = 1/2$ yields the isoperimetric inequality

$$\mu(\Omega)^{1-1/\nu} \leq C_1'(C\mu(\partial\Omega) + R^{-1}\mu(\Omega)).$$

This inequality can then be integrated using the co-area formula to recover the Sobolev inequality

$$\|f\|_{\nu/(\nu-1)} \leq C_1'(C\|\nabla f\|_1 + R^{-1}\|f\|_1).$$

3.3.3 Lie groups

A connected Lie group of topological dimension n is a manifold G equipped with a product $G \times G \ni (g, h) \mapsto gh$ such that $(G, \cdot)$ is a group and the application $G \times G \ni (g, h) \mapsto gh^{-1} \in G$ is analytic. The fifth Hilbert problem was to decide if every connected locally Euclidean topological group is a Lie group. It was solved with an affirmative answer by Gleason, Montgomery and Zippin in 1952.

Any locally compact group carries some left (resp. right) invariant measures called left (resp. right) Haar measures. Any two left (resp. right) Haar measures μ_1, μ_2 are related by $\mu_1 = c\mu_2$ for some $c > 0$, so that essentially there is only one left (resp. right) Haar measure. Let μ be a left Haar measure on G. For $g \in G$ and A a Borel subset of G, let $\mu_g(A) = \mu(Ag)$. Then μ_g is another left Haar measure so that there exists $m(g) > 0$ such that $\mu_g = m(g)\mu$. Obviously, $m(gh) = m(g)m(h)$ and $m(\mathrm{id}) = 1$. The function m is called the modular function of G. It can be trivial (i.e., $m \equiv 1$), in which case we say that G is unimodular, or non-trivial. Obviously, G is unimodular if and only if any left Haar measure is also a right Haar measure.

Let T be the tangent space at the identity element id. By the left action of G on itself, any vector ξ in T defines a left-invariant vector field X_ξ on G. Conversely, any left-invariant vector field is determined by its value at id. That is, the space of all invariant vector fields on G (i.e., the Lie algebra of G) is isomorphic, as a vector space, to T. We can now fix a Euclidean metric on T and turn it into a left invariant Riemannian structure on G. If $(\xi_1, \dots, \xi_n)$ is an orthonormal basis of T and $X_i = X_{\xi_i}$ then, at each point $g \in G$, $(X_1(g), \dots, X_n(g))$ is an orthornormal basis of T_g for our Riemannian structure. By construction, the Riemannian measure associated to this structure must be left-invariant: it is a left Haar measure. Call it μ.

Given a function $f \in C_0^\infty(M)$, ∇f is the vector field (not a left-invariant vector field!) defined by

$$df(Z) = \langle \nabla f, Z \rangle$$

for all vector fields Z. Computing in the orthonormal basis $(e_1, \dots, e_n) = (X_1(g), \dots, X_n(g))$ of T_g, $g \in G$, we have

$$df(Z)(g) = \sum_1^n z_i X_i f(g) = \sum_1^n z_i (\nabla f(g))_i$$

where $Z(g) = \sum z_i e_i$. That is, in coordinates,

$$\nabla f(g) = (X_1 f(g), \dots, X_n f(g)). \tag{3.3.6}$$

Hence

$$|\nabla f(g)| = \sqrt{\sum_1^n |X_i f(g)|^2}.$$

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We now want to compute the Laplace–Beltrami operator. Any left-invariant vector field X generates a one-parameter group $\{\phi_X(t) : t \in \mathbb{R}\}$ where $\phi_X : \mathbb{R} \to G$ satisfies $\phi_X'(t) = X(\phi_X(t))$. Moreover, for any function $f \in C_0^\infty(M)$, Xf can be computed by

$$Xf(g) = \lim_{t\to 0} \frac{f(g\phi_X(t)) - f(g)}{t}.$$

Now, observe that if μ_r is a right Haar measure,

$$\int_G f_1(g\phi_X(t)) f_2(g) d\mu_r(g) = \int_G f_1(g) f_2(g\phi_X^{-1}(t)) d\mu_r(g).$$

As $\phi_X^{-1}(t) = \phi_X(-t)$, it follows that

$$\int_G X f_1(g) f_2(g) d\mu_r(g) = -\int_G f_1(g) X f_2(g) d\mu_r(g).$$

In particular this applies to the X_i's, which are thus skew symmetric with respect to μ_r. Let

$$Z(g) = \sum_1^n z_i(g) X_i(g)$$

be any smooth vector field with compact support on G. Then

$$\begin{aligned} \int_G \langle Z, \nabla f\rangle d\mu_r &= \int_G \sum_1^n z_i (X_i f) d\mu_r \\ &= -\int_G \left(\sum_1^n X_i z_i\right) f d\mu_r. \end{aligned}$$

This shows that, *if G is unimodular*, i.e., $\mu_r = \mu$, then

$$\operatorname{div}(Z)(g) = \sum_1^n X_i z_i(g).$$

In this case, it follows that

$$\Delta f = -\operatorname{div}(\nabla f) = -\sum_1^n X_i^2 f. \tag{3.3.7}$$

If G is not unimodular, we can still compute the divergence by using the following trick. Let m be the modular function on G. Observe that $m^{-1}\mu$ is a right-invariant measure on G. Call it μ_r and write

$$\begin{aligned} \int_G \langle Z, \nabla f\rangle d\mu &= \int_G \langle Z, \nabla f\rangle m d\mu_r = \int_G \sum_1^n z_i X_i f m d\mu_r \\ &= -\int_G \left[\sum_1^n X_i(m z_i)\right] f d\mu_r \\ &= -\int_G f \left(m \sum_1^n X_i z_i + \sum_1^n z_i X_i m\right) m^{-1} d\mu. \end{aligned}$$

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It is not hard to see that $X_i m = \lambda_i m$ for some constant λ_i. This is due to the fact that m is multiplicative. Indeed, for any left-invariant vector field X, we have

$$\begin{aligned} Xm(g) &= \lim_{t\to 0} \frac{m(g\phi_X(t)) - m(g)}{t} \\ &= \lim_{t\to 0} \frac{m(g)m(\phi_X(t)) - m(g)}{t} \\ &= m(g) \lim_{t\to 0} \frac{m(\phi_X(t)) - 1}{t} = \lambda_X m(g) \end{aligned}$$

with $\lambda_X = Xm(\mathrm{id})$. Thus, setting $\lambda_{X_i} = \lambda_i$, we get

$$\int_G \langle Z, \nabla f\rangle d\mu = -\int_G f \left(\sum_1^n X_i z_i + \sum_1^n \lambda_i z_i \right) d\mu$$

which gives

$$\operatorname{div}(Z)(g) = \sum_1^n X_i z_i(g) + \sum_1^n \lambda_i z_i(g)$$

and

$$\Delta f = -\sum_1^n X_i^2 f - \sum_1^n \lambda_i X_i f. \tag{3.3.8}$$

Let us now look at the Riemannian distance function. As the Riemannian structure we consider is left-invariant, the associated distance $(g,h) \mapsto d(g,h)$ is also left-invariant, i.e.,

$$d(g,h) = d(\mathrm{id}, g^{-1}h).$$

For all $g \in G$, we set

$$|g| = d(\mathrm{id}, g)$$

so that $d(g,h) = |g^{-1}h|$.

It follows from left invariance that all balls of a given radius have the same volume, that is,

$$V(g,t) = V(\mathrm{id}, t) = V(t).$$

3.3.4 Pseudo-Poincaré inequalities on Lie groups

This section shows that any unimodular Lie group satisfies the pseudo-Poincaré inequality (3.3.1). As a corollary, the results of Section 3.3.1 apply.

Theorem 3.3.6 *Let G be a unimodular Lie group equipped with a left-invariant Riemannian metric and the associated Haar measure μ. Then*

$$\forall t > 0, \ \ \forall f \in \mathcal{C}_0^\infty(G), \quad \|f - f_t\|_p \le t\|\nabla f\|_p$$

for all $1 \le p \le \infty$.

The proof is simple. Let $h \in G$. Let $\gamma_h : [0,t] \to G$ be a smooth curve in G such that $\gamma_h(0) = \mathrm{id}$, $\gamma_h(t) = h$, and $|\dot{\gamma}_h(s)| \leq 1$. Note that $|h|$ is equal to the infimum of all $t > 0$ such that such a curve exists.

Now, for any $g \in G$ and $f \in \mathcal{C}_0^\infty(M)$,

$$f(gh) - f(g) = \int_0^t \partial_s f(g\gamma_h(s))ds.$$

Thus

$$\begin{aligned}
|f(gh) - f(g)| &\leq \int_0^t |df_{g\gamma_h(s)}(\partial_t g\gamma_h(s))|ds \\
&\leq \int_0^t |\nabla f(g\gamma_h(s))||\dot{\gamma}_h(s)|ds \leq \int_0^t |\nabla f(\gamma_h(s))|ds.
\end{aligned}$$

This yields

$$|f(gh) - f(g)|^p \leq t^{p-1} \int_0^t |\nabla f(g\gamma_h(s))|^p ds$$

and

$$\begin{aligned}
\int_G |f(gh) - f(g)|^p dg &\leq t^{p-1} \int_0^t \int_G |\nabla f(g\gamma_h(s))|^p dgds \\
&= t^{p-1} \int_0^t \int_G |\nabla f(g)|^p dgds \\
&= t^p \int_G |\nabla f(g)|^p dg.
\end{aligned}$$

Note that here we have used the *right invariance* of the left Haar measure on G, i.e., the fact that G is unimodular. We can optimize over all curves joining the identity to h. This yields

$$\int_G |f(gh) - f(g)|^p dg \leq |h|^p \int_G |\nabla f(g)|^p dg.$$

Integrating this inequality in h over all $h \in B(t) = B(\mathrm{id}, t)$ yields

$$\begin{aligned}
\|f - f_t\|_p^p &= \int_G \left| f(g) - \frac{1}{V(t)} \int_{B(t)} f(gh)dh \right|^p dg \\
&\leq \frac{1}{V(t)} \int_B \int_G |f(g) - f(gh)|^p dgdh \\
&\leq \frac{1}{V(t)} \int_B |h| \|\nabla f\|_p^p dh \\
&\leq t^p \|\nabla f\|_p^p.
\end{aligned}$$

This is the desired inequality.

Theorem 3.3.6 has the following corollary.

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Corollary 3.3.7 *Let G be a unimodular Lie group equipped with a left-invariant Riemannian metric and the associated Haar measure μ. Assume that the volume growth function $V(t)$ satisfies*

$$\inf_{t>0} t^{-\nu} V(t) > 0$$

for some $\nu > 0$. Then G satisfies the isoperimetric inequality

$$\mu_n(\Omega)^{1-1/\nu} \leq C_1(\nu)\mu_{n-1}(\partial\Omega)$$

for all bounded sets $\Omega \subset G$ with smooth boundary.

For each $1 \leq p < \nu$, G sastisfies the (L^p, ν)-Sobolev inequality

$$\forall f \in C_0^\infty(G), \quad \|f\|_{p\nu/(\nu-p)} \leq C(p,\nu)\|\nabla f\|_p.$$

More generally, for any $1 \leq p < \infty$ define $q(p)$ by $1/q(p) = 1/p - 1/\nu$. Then G satisfies all the inequalities $(S^\theta_{r,s}(p))$ for $1 \leq p < \infty$, $0 < s, r \leq \infty$ and $0 < \theta \leq 1$ such that

$$\frac{1}{r} = \frac{\theta}{q(p)} + \frac{1-\theta}{s}.$$

In particular, G sastisfies the Nash inequality

$$\forall f \in C_0^\infty(G), \quad \|f\|_2^{2(1+2/\nu)} \leq C_2(\nu)\|\nabla f\|_2^2 \|f\|_1^{4/\nu}.$$

This corollary is a useful result because the volume growth of unimodular Lie groups is well understood. First, for $0 < t \leq 1$ we have

$$c_0 \leq t^{-n}V(t) \leq C_0$$

where n is the topological dimension of G. Second, by the work of Guivarc'h [36] (see also [45]), the volume growth function of any unimodular Lie group G satisfies the following alternative: either there exist $c, \alpha > 0$ such that for all $t \geq 1$,

$$V(t) \geq c\exp(\alpha t),$$

or there exist $0 < c \leq C < \infty$ and $d = 0, 1, 2, \ldots$ such that for all $t \geq 1$

$$c \leq t^{-d}V(t) \leq C.$$

A typical case where the volume of G has a polynomial behavior is when G is a simply connected nilpotent Lie group. In this case, the volume growth function V satisfies

$$\forall t > 1, \quad ct^d \leq V(t) \leq Ct^d$$

where d is an integer given by

$$d = \sum_1^k i \dim(\mathcal{G}_i/\mathcal{G}_{i+1}).$$

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Here, the $\mathcal{G}_i$'s are subalgebras of the Lie algebra $\mathcal{G}$ of G defined inductively by $\mathcal{G}_1 = \mathcal{G}$, $\mathcal{G}_i = [\mathcal{G}, \mathcal{G}_{i-1}]$, $i = 2, \ldots$. The integer k is the smallest integer such that $\mathcal{G}_{k+1} = \{0\}$. That such a k exists is essentially the definition of G (i.e., $\mathcal{G}$) being nilpotent. As the topological dimension n of G is $n = \sum_1^k \dim(\mathcal{G}_i/\mathcal{G}_{i+1})$, it follows that any simply connected nilpotent Lie group G satisfies $n \leq d$. In particular, such a group has volume growth bounded below by

$$\forall t > 0, \quad V(t) \geq c_m t^m$$

for all $m \in [n, d]$.

For instance, the group of three by three upper-triangular matrices with diagonal entries equal to 1 (see (5.6.1)) is a simply connected nilpotent group known as the Heisenberg group and having $k = 2$ and $d = 4$. Of course, its topological dimension is $n = 3$. See [87] for details and references.

3.3.5 Ricci ≥ 0 and maximal volume growth

Let (M, g) be a complete Riemannian manifold. The Ricci curvature tensor $\mathcal{R}$ is a symmetric two-tensor obtained by contraction of the full curvature tensor. See, e.g., [13, 29]. Thus, it can be compared with the metric tensor g. Hypotheses of the type $\mathcal{R} \geq kg$, for some $k \in \mathbb{R}$, turn out to be sufficient to derive important analytic and geometric results. For instance, if $\mathcal{R} \geq kg$ with $k > 0$, then M must be compact. See, e.g., [13, Theorem 2.12]. If $k = 0$, the volume growth on (M, g) is at most Euclidean, that is, $\forall r > 0, \quad V(x, r) \leq \Omega_n r^n$. See, e.g., [13, Theorem 3.9].

We want to show that complete manifolds of dimension n having non-negative Ricci curvature and maximal volume growth, that is, for which there exists $c > 0$ such that

$$\forall r > 0, \quad V(x, r) \geq cr^n,$$

satisfy the pseudo-Poincaré inequality (3.3.1).

Theorem 3.3.8 *Let (M, g) be a complete manifold of dimension n having non-negative Ricci curvature. Assume that there exists $c > 0$ such that*

$$\forall r > 0, \quad V(x, r) \geq cr^n.$$

Then

$$\forall r > 0, \ \forall f \in C_0^\infty(M), \quad \|f - f_r\|_1 \leq (\Omega_n/c)\, r\, \|\nabla f\|_1.$$

Moreover, the Sobolev-type inequalities $(S^\theta_{r,s}(p))$ with

$$\frac{1}{r} = \theta\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-\theta}{s}, \quad 0 < r, s \leq \infty, \ \ 0 < \theta \leq 1$$

are all satisfied on M. In particular,

$$\forall f \in C_0^\infty(M), \ \ \|f\|_{np/(n-p)} \leq C_{n,p} \|\nabla f\|_p$$

for all $1 \leq p < n$.

Note that the hypothesis $V(x,r) \geq cr^n$ is also necessary for the last conclusion of this theroem to hold. By the results developed in this chapter, it suffices to prove the first assertion of the theorem. Observe that

$$\begin{aligned}\|f - f_r\|_1 &= \int_M \left| f(x) - \frac{1}{V(x,r)} \int_{B(x,r)} f(y)dy \right| dx \\ &\leq \int_M \int_M |f(x) - f(y)| \frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)} dydx.\end{aligned}$$

Now, consider the integral

$$\int_{B(x,r)} |f(x) - f(y)|dy.$$

To estimate this integral, we use polar (exponential) coordinates around x. See [13, Proposition 3.1]. This gives (in somewhat abusive notation)

$$\begin{aligned}\int_{B(x,r)} |f(x) - f(y)|dy &= \int_0^r |f(x) - f(\rho,\theta)|\sqrt{\mathbf{g}}(\rho,\theta)d\rho d\theta \\ &\leq \int_0^r \int_0^\rho |\partial_t f(t,\theta)|dt\sqrt{\mathbf{g}}(\rho,\theta)d\rho d\theta \\ &\leq \int_0^r \int_0^\rho |\nabla f(t,\theta)|dt\sqrt{\mathbf{g}}(\rho,\theta)d\rho d\theta.\end{aligned}$$

Here, we have simply used the usual trick to control $f(x) - f(y)$ by integrating along the geodesic segment from x to y and used polar exponential coordinates $y = (\rho,\theta)$ around x. In particular, $\sqrt{\mathbf{g}}(\rho,\theta)d\rho d\theta = dy$ is by definition the Riemannian volume element in polar coordinates. Strictly speaking, one should avoid the cut locus $C(x)$ of x in this computation. This however is not a problem because $C(x)$ has measure zero and $M \setminus C(x)$ is star shaped with respect to x. We refer the reader to [13] for a detailed treatment.

We now use the hypothesis that (M, g) has non-negative Ricci curvature. By Bishop's theorem [13, Theorem 3.8], the function $s \mapsto \sqrt{\mathbf{g}}(s,\theta)/s^{n-1}$ is non-increasing. It follows that

$$\begin{aligned}\int_{B(x,r)} |f(x) - f(y)|dy &\leq \int_0^r \int_0^\rho |\nabla f(t,\theta)|t^{1-n}\sqrt{\mathbf{g}}(t,\theta)dt\rho^{n-1}d\rho d\theta \\ &\leq \frac{r^n}{n}\int_0^r |\nabla f(t,\theta)|t^{1-n}\sqrt{\mathbf{g}}(t,\theta)dtd\theta \\ &= \frac{r^n}{n}\int_{B(x,r)} \frac{|\nabla f(y)|}{d(x,y)^{n-1}}dy.\end{aligned}$$

Using the hypothesis of maximal volume growth, we get

$$\int_M \int_M |f(x) - f(y)| \frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)} dydx$$

$$\begin{aligned}
&\le \frac{1}{cn}\int_M\int_{B(x,r)} \frac{|\nabla f(y)|}{d(x,y)^{n-1}}dydx \\
&\le \frac{1}{cn}\int_M |\nabla f(y)| \int_{B(y,r)} \frac{1}{d(x,y)^{n-1}}dxdy.
\end{aligned}$$

Finally, by Bishop's theorem again, $\sqrt{\mathrm{g}}(t,\theta) \le t^{n-1}$. It follows that

$$\int_{B(y,r)} \frac{1}{d(x,y)^{n-1}}dx \le \omega_{n-1}r.$$

This yields

$$\|f - f_r\|_1 \le \frac{\omega_{n-1}}{cn} r\|\nabla\|_1.$$

Recalling that $\Omega_n = \omega_{n-1}/n$, we see that this is the desired inequality. A variation on this proof yields a similar inequality in L^p norms.

The above proof of the pseudo-Poincaré inequality (3.3.1) for manifolds having non-negative Ricci curvature uses the additional hypothesis that the volume growth of M is maximal. One may wonder whether (3.3.1) holds without this additional hypothesis. The answer is yes.

Theorem 3.3.9 *Let (M,g) be a complete manifold of dimension n having non-negative Ricci curvature. Then*

$$\forall r > 0,\ \ \forall f \in C_0^\infty(M),\ \ \ \|f - f_r\|_1 \le 4^n r\,\|\nabla f\|_1.$$

To prove the desired inequality it suffices to bound

$$\int_M\int_M |f(x) - f(y)|\,\frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)}dydx$$

by $C\,r\,\|\nabla f\|_1$. To make things more symmetric in x and y, note that

$$\frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)} \le 2^n \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}}.$$

This uses the fact that $V(y,r) \le V(x,2r) \le 2^n V(x,r)$ if $d(x,y) \le r$. The inequality $V(x,2r) \le 2^n V(x,r)$ follows from the celebrated Bishop–Gromov comparison theorem (see, e.g., [13, 29]). Hence it suffices to bound

$$\int_M\int_M |f(x) - f(y)|\,\frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}}dydx.$$

Now, let $\gamma_{x,y}(t)$ be the geodesic from x to y parametrized by arc length (as in the proof of Theorem 3.3.8, we can ignore the cut locus). Then,

$$\begin{aligned}
&\int_M\int_M |f(x) - f(y)|\,\frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}}dydx \\
&\le \int_M\int_M \left[\int_0^{d(x,y)} |\nabla f(\gamma_{x,y}(t))|dt\right] \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}}dydx.
\end{aligned}$$

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By symmetry with respect to x, y, we can restrict integration in t to the interval $(d(x,y)/2, d(x,y))$ so that we obtain

$$\begin{aligned}
&\int_M \int_M |f(x) - f(y)| \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}} dydx \\
&\leq 2 \int_M \int_M \int_{d(x,y)/2}^{d(x,y)} \frac{|\nabla f(\gamma_{x,y}(t))| \mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}} dtdydx.
\end{aligned}$$

Now, by Bishop's theorem [13, Theorem 3.8], the Jacobian of the map $y \mapsto z = \gamma_{x,y}(t)$ is larger than 2^{-n+1} (see Lemma 5.6.7 for details) and

$$\frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}} \leq 2^n \frac{\mathbf{1}_{B(x,r)}(\gamma_{x,y}(t))}{V(x,r)}.$$

Hence

$$\begin{aligned}
&\int_M \int_M |f(x) - f(y)| \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(y,r)V(x,r)}} dydx \\
&\leq 2^{2n} \int_M \int_M \int_0^r \frac{|\nabla f(z)| \mathbf{1}_{B(x,r)}(z)}{V(x,r)} dtdzdx \\
&\leq 2^{2n} r \int_M |\nabla f(z)| dz.
\end{aligned}$$

This yields the desired pseudo-Poincaré inequality. See also Theorem 5.6.6.

3.3.6 Sobolev inequality in precompact regions

Let (M, g) be a complete, non-compact, manifold of dimension n. It is often useful to invoke the fact that, if Ω is an open precompact subset of M, the usual $\mathbb{R}^n$ (local) Sobolev inequality $\|f\|_{pn/(n-p)} \leq C(\|\nabla f\|_p + \|f\|_p)$ holds for smooth functions with compact support in Ω, with a constant depending on Ω. Here we outline a direct proof of this fact.

Theorem 3.3.10 *Let (M, g) be a complete manifold of dimension n. For any open precompact region $\Omega \subset M$ and any $1 \leq p < n$, there exists a constant $C(\Omega, p)$ such that*

$$\forall f \in C_0^\infty(\Omega), \quad \|f\|_{pn/(n-p)} \leq C(\Omega, p)(\|\nabla f\|_p + \|f\|_p).$$

We only need to prove the case $p = 1$. Now, the neigborhood $U = \{x \in M : d(x, \Omega) \leq 1\}$ of Ω is also precompact. Clearly, the volume function $V(x,t)$ satisfies

$$\inf_{\substack{x \in U \\ t \in (0,1)}} t^{-n} V(x,t) > 0.$$

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Moreover, the volume element in polar coordinates around x, which we denote by $\sqrt{\mathbf{g}}_x(t,\theta)$ as in the previous section, satisfies

$$\forall x \in U,\ \ \forall t \in (0,1), \quad c \le t^{1-n}\sqrt{\mathbf{g}}_x(t,\theta) \le C.$$

(This must be understood outside the cut locus of x. In fact, one can avoid the cut locus completely in this argument by considering only $t \in (0,\epsilon)$ with ϵ small enough). See [13]. With these observations we can use the proof technique of Section 3.3.5 to show that, for all $0 < t < 1$ and all $f \in \mathcal{C}_0^\infty(\Omega)$,

$$\|f - f_t\|_1 \le C(U)\, t\, \|\nabla f\|_1.$$

Applying Theorem 3.3.5 yields the desired result. In general, one cannot dispense with the $\|f\|_p$ term appearing on the right-hand side in Theorem 3.3.10 (to see this consider constant functions on a compact manifold). However, if Ω is a relatively compact set in a complete *non-compact* manifold then the technique of Section 3.3.5 can be used to show that, for all $f \in \mathcal{C}_0^\infty(\Omega)$,

$$\|f\|_p \le C'(\Omega,p)\|\nabla f\|_p$$

(a Poincaré inequality). Thus, one has the following result.

Theorem 3.3.11 *Let (M,g) be a complete non-compact manifold of dimension n. For any open precompact region $\Omega \subset M$ and any $1 \le p < n$, there exists a constant $C(\Omega,p)$ such that*

$$\forall f \in \mathcal{C}_0^\infty(\Omega), \quad \|f\|_{pn/(n-p)} \le C(\Omega,p)\|\nabla f\|_p.$$