# 7

# FRACTIONAL ORDER SPACES

## Introduction

**7.1** This chapter is concerned with extending the notion of the standard Sobolev space $W^{m,p}(\Omega)$ to include spaces where $m$ need not be an integer. There are various ways to define such *fractional order* spaces; many of them depend on using interpolation to construct scales of spaces suitably intermediate between two extreme spaces, say $L^p(\Omega)$ and $W^{m,p}(\Omega)$.

Interpolation methods themselves come in two flavours: real methods and complex methods. We have already seen an example of the real method in the Marcinkiewicz theorem of Paragraph 2.58. Although the details of the real method can be found in several sources, for example, [BB], [BL], and [BSh], we shall provide a treatment here in sufficient detail to make clear its application to the development of the Besov spaces, one of the scales of fractional order Sobolev spaces that particularly lends itself to characterizing the spaces of traces of functions in $W^{m,p}(\Omega)$ on the boundaries of smoothly bounded domains $\Omega$; such characterizations are useful in the study of boundary-value problems. Several older interpolation methods are known [BL, pp. 70–75] to be equivalent to the now-standard real interpolation method that we use here. In the the corresponding chapter of the previous edition [A] of this book, the older method of traces was used rather than the method presented in this edition. Later in this Chapter, we prove a trace theorem (Theorem 7.39) giving an instance of that equivalence.

After that we shall describe more briefly other scales of fractional order Sobolev spaces, some obtained by complex methods and some by Fourier decompositions.

## The Bochner Integral

**7.2** In developing the real interpolation method below we will use the concept of the integral of a Banach-space-valued function defined on an interval on the real line $\mathbb{R}$. (For the complex method we will use the concept of analytic Banach-space-valued functions of a complex variable.) We present here a brief description of the Bochner integral, referring the reader to [Y] or [BB] for more details.

Let $X$ be a Banach space with norm $\|\cdot\|_X$ and let $f$ be a function defined on an interval $(a, b)$ in $\mathbb{R}$ (which may be infinite) and having values in $X$. In addition, let $\mu$ be a measure on $(a, b)$ given by $d\mu(t) = w(t)\,dt$ where $w$ is continuous and positive on $(a, b)$. Of special concern to us later will be the case where $a = 0$, $b = \infty$, and $w(t) = 1/t$. In this case $\mu$ is the Haar measure on $(0, \infty)$, which is invariant under scaling in the multiplicative group $(0, \infty)$: if $(c, d) \subset (0, \infty)$ and $\lambda > 0$, then $\mu(\lambda c, \lambda d) = \mu(c, d)$.

We want to define the integral of $f$ over $(a, b)$.

**7.3 (Definition of the Bochner Integral)** If $\{A_1, \dots, A_k\}$ is a finite collection of mutually disjoint subsets of $(a, b)$ each having finite $\mu$-measure, and if $\{x_1, \dots, x_k\}$ is a corresponding set of elements of $X$, we call the function $f$ defined by

$$f(t) = \sum_{i=1}^{k} \chi_{A_i}(t) x_i, \qquad a < t < b,$$

a *simple function* on $(a, b)$ into $X$. For such simple functions we define, obviously,

$$\int_a^b f(t)\,d\mu(t) = \sum_{i=1}^{k} \mu(A_i) x_i = \sum_{i=1}^{k} \left( \int_{A_i} w(t)\,dt \right) x_i.$$

Of course, a different representation of the simple function $f$ using a different collection of subsets of $(a, b)$ will yield the same value for the integral; the subsets in the collections need not be mutually disjoint, and given two such collections we can always form an equivalent mutually disjoint collection consisting of pairwise intersections of the elements of the two collections.

Now let $f$ an arbitrary function defined on $(a, b)$ into $X$. We say that $f$ is *(strongly) measurable* on $(a, b)$ if there exists a sequence $\{f_j\}$ of simple functions with supports in $(a, b)$ such that

$$\lim_{j \to \infty} \left\| f_j(t) - f(t) \right\|_X \qquad \text{a.e. in } (a, b). \tag{1}$$

It can be shown that $f$ is measurable if its range is separable and if, for each $x'$ in the dual of $X$, the scalar-valued function $x'\big(f(\cdot)\big)$ is measurable on $(a,b)$.

Suppose that a sequence of simple functions $\{f_j\}$ satisfying (1) can be chosen in such a way that

$$\lim_{j\to\infty}\int_a^b \left\|f_j(t)-f(t)\right\|_X \, d\mu(t)=0.$$

Then we say that $f$ is *Bochner integrable* on $(a,b)$ and we define

$$\int_a^b f(t)\, d\mu(t) = \lim_{j\to\infty}\int_a^b f_j(t)\, d\mu(t).$$

Again we observe that the limit does not depend on the choice of the approximating simple functions.

A measurable function $f$ is integrable on $(a,b)$ if and only if the scalar-valued function $\|f(\cdot)\|_X$ is integrable on $(a,b)$. In fact, there holds the "triangle inequality"

$$\left\|\int_a^b f(t)\, d\mu(t)\right\|_X \le \int_a^b \|f(t)\|_X \, d\mu(t).$$

**7.4 (The Spaces $L^q(a,b;d\mu,X)$)** If $1\le q\le\infty$, we say that $f\in L^q(a,b;d\mu,X)$ provided $\|f\,;L^q(a,b;d\mu,X)\|<\infty$, where

$$\left\|f\,;L^q(a,b;d\mu,X)\right\| = \begin{cases} \left(\displaystyle\int_a^b \|f(t)\|_X^q \, d\mu(t)\right)^{1/q} & \text{if } 1\le q<\infty \\ \operatorname{ess\,sup}_{a<t<b}\{\|f(t)\|_X\} & \text{if } q=\infty. \end{cases}$$

In particular, if $X=\mathbb{R}$ or $X=\mathbb{C}$, we will denote $L^q(a,b;d\mu,X)$ simply by $L^q(a,b;d\mu)$.

**7.5 (The spaces $L^q_*$)** Of much importance below is the special case where $X=\mathbb{R}$ or $\mathbb{C}$, $(a,b)=(0,\infty)$, and $d\mu=dt/t$; we will further abbreviate the notation for this , denoting $L^q(a,b;d\mu,X)$ simply $L^q_*$. Note that $L^q_*$ is equivalent to $L^q(\mathbb{R})$ with Lebesgue measure via a change of variable: if $t=e^s$ and $f(t)=f(e^s)=F(s)$, then $\left\|f\,;L^q_*\right\|=\|F\|_{q,\mathbb{R}}$. Most of the properties of $L^q(\mathbb{R})$ transfer to properties of $L^q_*$. In particular Hölder's and Young's inequalities hold; we will need both of them below. It should be noted that the convolution of two functions $f$ and $g$ defined on $(0,\infty)$ and integrated with respect to the Haar measure $dt/t$ is given by

$$f*g(t)=\int_0^\infty f\left(\frac{t}{s}\right) g(s)\,\frac{ds}{s},$$

and Young's inequality proclaims $\|f * g\,; L^r_*\| \le \|f\,; L^p_*\|\,\|g\,; L^q_*\|$ provided $p, q, r \ge 1$ and $1 + (1/r) = (1/p) + (1/q)$.

## Intermediate Spaces and Interpolation — The Real Method

**7.6** In this Section we will be discussing the construction of Banach spaces $X$ that are suitably intermediate between two Banach spaces $X_0$ and $X_1$, each of which is (continuously) imbedded in a Hausdorff topological vector space $\mathcal{X}$, and whose intersection is nontrivial. (Such a pair of spaces $\{X_0, X_1\}$ is called an *interpolation pair* and $X$ is called an intermediate space of the pair. In some of our later applications, we will have $X_1 \to X_0$ (for example, $X_0 = L^p(\Omega)$ and $X_1 = W^{m,p}(\Omega)$), in which case we can clearly take $\mathcal{X} = X_0$. We shall, in fact, be constructing families of such intermediate spaces $X_{\theta,q}$ between $X_0$ and $X_1$, such that if $Y_{\theta,q}$ is the corresponding intermediate space for another such interpolation pair $\{Y_0, Y_1\}$ with $Y_0$ and $Y_1$ imbedded in $\mathcal{Y}$, and if $T$ is a linear operator from $\mathcal{X}$ into $\mathcal{Y}$ (for example an imbedding operator) such that $T$ is bounded from $X_i$ into $Y_i$, $i = 0, 1$, then $T$ will also be bounded from $X_{\theta,q}$ into $Y_{\theta,q}$.

There are many different ways of constructing such intermediate spaces, mostly leading to the same spaces with equivalent norms. We examine here two such methods, the $J$-method and the $K$-method, (together called the real method) due to Lions and Peetre. The theory is developed in several texts, in particular [BB] and [BL]. Our approach follows [BB] and we will omit some aspects of the theory for which we have no future need.

**7.7 (Intermediate Spaces)** Let $\|\cdot\|_{X_i}$ denote the norm in $X_i$, $i = 0, 1$. The intersection $X_0 \cap X_1$ and the algebraic sum $X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0, u_1 \in X_1\}$ are themselves Banach spaces with respect to the norms

$$\begin{aligned}
\|u\|_{X_0 \cap X_1} &= \max\big\{\|u\|_{X_0}, \|u\|_{X_1}\big\} \\
\|u\|_{X_0 + X_1} &= \inf\big\{\|u_0\|_{X_0} + \|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\big\}.
\end{aligned}$$

and $X_0 \cap X_1 \to X_i \to X_0 + X_1$ for $i = 0, 1$.

In general, we say that a Banach space $X$ is *intermediate* between $X_0$ and $X_1$ if there exist the imbeddings

$$X_0 \cap X_1 \to X \to X_0 + X_1.$$

**7.8 (The J and K norms)** For each fixed $t > 0$ the following functionals define norms on $X_0 \cap X_1$ and $X_0 + X_1$ respectively, equivalent to the norms defined above:

$$\begin{aligned}
J(t; u) &= \max\big\{\|u\|_{X_0}, t\,\|u\|_{X_1}\big\} \\
K(t; u) &= \inf\big\{\|u_0\|_{X_0} + t\,\|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\big\}.
\end{aligned}$$

Evidently $J(1;u) = \|u\|_{X_0\cap X_1}$, $K(1;u) = \|u\|_{X_0+X_1}$, and $J(t;u)$ and $K(t;u)$ are continuous and monotonically increasing functions of $t$ on $(0,\infty)$. Moreover

$$\min\{1,t\}\,\|u\|_{X_0\cap X_1} \le J(t;u) \le \max\{1,t\}\,\|u\|_{X_0\cap X_1} \tag{2}$$

$$\min\{1,t\}\,\|u\|_{X_0+X_1} \le K(t;u) \le \max\{1,t\}\,\|u\|_{X_0+X_1}\,. \tag{3}$$

$J(t;u)$ is a convex function of $t$ because, if $0 < a < b$ and $0 < \theta < 1$,

$$\begin{aligned} J\big((1-\theta)a+\theta b;u\big) &= \max\big\{\|u\|_{X_0}, (1-\theta)a\,\|u\|_{X_1} + \theta b\,\|u\|_{X_1}\big\} \\ &\le (1-\theta)\max\big\{\|u\|_{X_0}, a\,\|u\|_{X_1}\big\} + \theta\max\big\{\|u\|_{X_0}, b\,\|u\|_{X_1}\big\} \\ &= (1-\theta)J(a;u) + \theta J(b;u). \end{aligned}$$

Also for such $a, b, \theta$ and any $u_0 \in X_0$ and $u_1 \in X_1$ for which $u = u_0 + u_1$ we have

$$\begin{aligned} &\|u_0\|_{X_0} + \big((1-\theta)a+\theta b\big)\,\|u_1\|_{X_1} \\ &\qquad = (1-\theta)\big(\|u_0\|_{X_0} + a\,\|u_1\|_{X_1}\big) + \theta\big(\|u_0\|_{X_0} + b\,\|u_1\|_{X_1}\big) \\ &\qquad \ge (1-\theta)K(a;u) + \theta K(b;u), \end{aligned}$$

so that $K\big((1-\theta)a+\theta b); u\big) \ge (1-\theta)K(a;u) + \theta K(b;u)$ and $K(t;u)$ is a concave function of $t$.

Finally we observe that if $u \in X_0 \cap X_1$, then for any positive $t$ and $s$ we have $K(t;u) \le \|u\|_{X_0} \le J(s;u)$ and $K(t;u) \le t\,\|u\|_{X_1} = (t/s)s\,\|u\|_{X_1} \le (t/s)J(s;u)$. Accordingly,

$$K(t;u) \le \min\left\{1, \frac{t}{s}\right\} J(s;u). \tag{4}$$

**7.9 (The K-method)** If $0 \le \theta \le 1$ and $1 \le q \le \infty$ we denote by $(X_0, X_1)_{\theta,q;K}$ the space of all $u \in X_0 + X_1$ such that the function $t \to t^{-\theta}K(t;u)$ belongs to $L^q_* = L^q(0,\infty; dt/t)$.

Of course, the zero element $u = 0$ of $X_0 + X_1$ always belongs to $(X_0, X_1)_{\theta,q;K}$. The following theorem shows that if $1 \le q < \infty$ and either $\theta = 0$ or $\theta = 1$, then $(X_0, X_1)_{\theta,q;K}$ contains only this trivial element. Otherwise $(X_0, X_1)_{\theta,q;K}$ is an intermediate space between $X_0$ and $X_1$.

**7.10 THEOREM** If and only if either $1 \le q < \infty$ and $0 < \theta < 1$ or $q = \infty$ and $0 \le \theta \le 1$, then the space $(X_0, X_1)_{\theta,q;K}$ is a nontrivial Banach space with norm

$$\|u\|_{\theta,q;K} = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{-\theta}K(t;u)\big)^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\ \operatorname{ess\,sup}_{0<t<\infty}\{t^{-\theta}K(t;u)\} & \text{if } q = \infty. \end{cases}$$

Furthermore,

$$\|u\|_{X_0+X_1} \le \frac{\|u\|_{\theta,q;K}}{\left\|t^{-\theta}\min\{1,t\}\,;\,L^q_*\right\|} \le \|u\|_{X_0\cap X_1} \tag{5}$$

so there hold the imbeddings

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K} \to X_0 + X_1$$

and $(X_0, X_1)_{\theta,q;K}$ is an intermediate space between $X_0$ and $X_1$.

Otherwise $(X_0, X_1)_{\theta,q;K} = \{0\}$.

**Proof.** It is easily checked that the function $t \to t^{-\theta}\min\{1,t\}$ belongs to $L^q_*$ if and only if $\theta$ and $q$ satisfy the conditions of the theorem. Since (3) shows that

$$\left\|t^{-\theta}\min\{1,t\}\,;\,L^q_*\right\| \, \|u\|_{X_0+X_1} \le \left\|t^{-\theta}K(t;u)\,;\,L^q_*\right\| = \|u\|_{\theta,q;K}\,,$$

there can be no nonzero elements of $(X_0, X_1)_{\theta,q;K}$ unless those conditions are satisfied. If so, then the left inequality in (5) holds and $(X_0, X_1)_{\theta,q;K} \to X_0 + X_1$. Also, by (4) we have $K(t;u) \le \min\{1,t\}J(1;u) = \min\{1,t\}\,\|u\|_{X_0\cap X_1}$ so the right inequality in (5) holds and $X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K}$.

Verification that $\|u\|_{\theta,q;K}$ is a norm and that $(X_0, X_1)_{\theta,q;K}$ is complete under it are left as exercises for the reader. ∎

Note that $u \in X_0$ and $\theta = 0$ implies that $t^{-\theta}K(t;u) \le \|u\|_{X_0}$. Also, $u \in X_1$ and $\theta = 1$ implies that $t^{-\theta}K(t;u) \le \|u\|_{X_1}$. Thus we also have

$$X_0 \to (X_0, X_1)_{0,\infty;K} \qquad \text{and} \qquad X_1 \to (X_0, X_1)_{1,\infty;K}. \tag{6}$$

**7.11 THEOREM (A Discrete Version of the K-method)** For each integer $i$ let $K_i(u) = K(2^i;u)$ . Then $u \in (X_0, X_1)_{\theta,q;K}$ if and only if the sequence $\{2^{-i\theta}K_i(u)\}_{i=-\infty}^{\infty}$ belongs to the space $\ell^q$ (defined in Paragraph 2.27). Moreover, the $\ell^q$-norm of that sequence is equivalent to $\|u\|_{\theta,q;K}$.

**Proof.** First write (for $1 \le q < \infty$)

$$\int_0^\infty \left(t^{-\theta}K(t;u)\right)^q \frac{dt}{t} = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(t^{-\theta}K(t;u)\right)^q \frac{dt}{t}.$$

Since $K(t;u)$ increases and $t^{-\theta}$ decreases as $t$ increases, we have for $2^i \le t \le 2^{i+1}$,

$$2^{-(i+1)\theta}K_i(u) \le t^{-\theta}K(t;u) \le 2^{-i\theta}K_{i+1}(u),$$

so that

$$2^{-\theta q}\ln 2\left[2^{-i\theta}K_i(u)\right]^q \le \int_{2^i}^{2^{i+1}} \left(t^{-\theta}K(t;u)\right)^q \frac{dt}{t} \le 2^{\theta q}\ln 2\left[2^{-(i+1)\theta}K_{i+1}(u)\right]^q.$$

Summing on $i$ and taking $q$th roots then gives

$$2^{-\theta}(\ln 2)^{1/q}\left\|\left\{2^{-i\theta}K_i(u)\right\};\ell^q\right\| \le \|u\|_{\theta,q;K} \le 2^{\theta}(\ln 2)^{1/q}\left\|\left\{2^{-i\theta}K_i(u)\right\};\ell^q\right\|.$$

The proof for $q=\infty$ is easier and left for the reader. ∎

**7.12 (The J-method)** If $0 \le \theta \le 1$ and $1 \le q \le \infty$ we denote by $(X_0, X_1)_{\theta,q;J}$ the space of all $u \in X_0 + X_1$ such that

$$u = \int_0^\infty f(t)\,\frac{dt}{t}$$

(Bochner integral) for some $f \in L^1(0,\infty; dt/t, X_0+X_1)$ having values in $X_0 \cap X_1$ and such that the real-valued function $t \to t^{-\theta}J(t;f)$ belongs to $L^q_*$.

**7.13 THEOREM** If either $1 < q \le \infty$ and $0 < \theta < 1$ or $q = 1$ and $0 \le \theta \le 1$, then $(X_0, X_1)_{\theta,q;J}$ is a nontrivial Banach space with norm

$$\begin{aligned}\|u\|_{\theta,q;J} &= \inf_{f\in S(u)} \left\|t^{-\theta}J\big(t; f(t)\big); L^q_*\right\| \\ &= \inf_{f\in S(u)} \left(\int_0^\infty \left[t^{-\theta}J\big(t;f(t)\big)\right]^q \frac{dt}{t}\right)^{1/q}, \quad (\text{if } q<\infty),\end{aligned}$$

where

$$S(u) = \left\{ f \in L^1(0,\infty; dt/t, X_0+X_1) \;:\; u = \int_0^\infty f(t)\,\frac{dt}{t} \right\}.$$

Furthermore,

$$\|u\|_{X_0+X_1} \le \left(\left\|t^{-\theta}\min\{1,t\}; L^{q'}_*\right\|\right)\|u\|_{\theta,q;J} \le \|u\|_{X_0\cap X_1} \tag{7}$$

so that

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;J} \to X_0 + X_1$$

and $(X_0, X_1)_{\theta,q;J}$ is an intermediate space between $X_0$ and $X_1$.

**Proof.** Again we leave verification of the norm and completeness properties to the reader and we concentrate on the imbeddings.

Let $f \in S(u)$. By (3) and (4) with $t = 1$ and $s = \tau$ we have

$$\|f(\tau)\|_{X_0+X_1} \le K\big(1, f(\tau)\big) \le \min\left\{1, \frac{1}{\tau}\right\} J\big(\tau, f(\tau)\big).$$

Accordingly, If $(1/q) + (1/q') = 1$, then by Hölder's inequality

$$\begin{aligned}\|u\|_{X_0+X_1} &\le \int_0^\infty \|f(\tau)\|_{X_0+X_1} \frac{d\tau}{\tau} \le \int_0^\infty \min\left\{1, \frac{1}{\tau}\right\} J\big(\tau, f(\tau)\big) \frac{d\tau}{\tau} \\ &\le \left\|\tau^\theta \min\left\{1, \frac{1}{\tau}\right\} ; L_*^{q'}\right\| \left\|t^{-\theta} J\big(t; f(t)\big) ; L_*^q\right\|.\end{aligned}$$

The first factor in this product of norms is finite if $\theta$ and $q$ satisfy the conditions of the theorem, and if we replace $\tau$ with $1/t$ in it, we can see that it is equal to $\left\|t^{-\theta} \min\{1, t\} ; L_*^{q'}\right\|$. Since the above inequality holds for all $f \in S(u)$, the left inequality in (7) is established and $(X_0, X_1)_{\theta,q;J} \to X_0 + X_1$.

To verify the right inequality in (7), let $u \in X_0 \cap X_1$. Let $\phi(t) \ge 0$ satisfy $\left\|t^{-\theta}\phi(t) ; L_*^q\right\| = 1$. Hölder's inequality shows that

$$\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau} < \infty.$$

If

$$f(t) = \frac{\phi(t) \min\{1, 1/t\}}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} u,$$

then $f \in S(u)$ and

$$\begin{aligned}J\big(t; f(t)\big) &= \frac{\phi(t) \min\{1, 1/t\}}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} J(t; u) \\ &\le \frac{\phi(t)}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} \|u\|_{X_0\cap X_1},\end{aligned}$$

the latter inequality following from (2) since $\max\{1, t\} = \big(\min\{1, 1/t\}\big)^{-1}$. Therefore,

$$\begin{aligned}&\left(\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}\right) \|u\|_{\theta,q;J} \\ &\le \left(\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}\right) \left(\int_0^\infty \big(t^{-\theta} J\big(t; f(t)\big)\big)^q \frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_0^\infty \big(t^{-\theta}\phi(t) \|u\|_{X_0\cap X_1}\big)^q \frac{dt}{t}\right)^{1/q} = \|u\|_{X_0\cap X_1}.\end{aligned}$$

By the converse to Hölder's inequality,

$$\sup\left\{\int_0^\infty \phi(\tau)\min\{1,1/\tau\}\,\frac{d\tau}{\tau}\ :\ \left\|\tau^{-\theta}\phi(\tau)\,;L_*^q\right\|=1\right\}$$
$$=\left\|\tau^{\theta}\min\{1,1/\tau\}\,;L_*^{q'}\right\|=\left\|t^{-\theta}\min\{1,t\}\,;L_*^{q'}\right\|.$$

Thus the right inequality in (7) is established and $X_0\cap X_1\to(X_0,X_1)_{\theta,q;J}$. ∎

**7.14** Observe that if $u=\int_0^\infty f(t)\,dt/t$ where $f(t)\in X_0\cap X_1$, then

$$\|u\|_{X_0}\le\int_0^\infty \|f(t)\|_{X_0}\,\frac{dt}{t}\le\int_0^\infty J\big(t,f(t)\big)\,\frac{dt}{t}$$
$$\|u\|_{X_1}\le\int_0^\infty \|f(t)\|_{X_1}\,\frac{dt}{t}\le\int_0^\infty t^{-1}J\big(t,f(t)\big)\,\frac{dt}{t}.$$

Each of these estimates holds for all such representations of $u$, so $\|u\|_{X_0}\le\|u\|_{0,1;J}$ and $\|u\|_{X_1}\le\|u\|_{1,1;J}$. Combining these with (6) we obtain

$$\begin{aligned}(X_0,X_1)_{0,1;J}&\to X_0\to(X_0,X_1)_{0,\infty;K}\\(X_0,X_1)_{1,1;J}&\to X_1\to(X_0,X_1)_{1,\infty;K}.\end{aligned}\tag{8}$$

There is also a discrete version of the J-method leading to an equivalent norm for $(X_0,X_1)_{\theta,q;J}$.

**7.15 THEOREM (A Discrete Version of the J-method)** An element $u$ of $X_0+X_1$ belongs to $(X_0,X_1)_{\theta,q;J}$ if and only if $u=\sum_{i=-\infty}^{\infty}u_i$ where the series converges in $X_0+X_1$ and the sequence $\{2^{-\theta i}J(2^i,u_i)\}$ belongs to $\ell^q$. In this case

$$\inf\left\{\left\|\{2^{-\theta i}J(2^i;u_i)\}\,;\ell^q\right\|\ :\ u=\sum_{i=-\infty}^{\infty}u_i\right\}$$

is a norm on $(X_0,X_1)_{\theta,q;J}$ equivalent to $\|u\|_{\theta,q;J}$.

**Proof.** Again we will show this for $1\le q<\infty$ and leave the easier case $q=\infty$ to the reader.

First suppose that $u\in(X_0,X_1)_{\theta,q;J}$ and let $\epsilon>0$. Then there exists a function $f\in L^1(0,\infty;dt/t,X_0+X_1)$ such that

$$u=\int_0^\infty f(t)\,\frac{dt}{t}$$

and

$$\int_0^\infty \left[t^{-\theta}J\big(t;f(t)\big)\right]^q\frac{dt}{t}\le(1+\epsilon)\,\|u\|_{\theta,q;J}\,.$$

Let the sequence $\{u_i\}_{i=-\infty}^{\infty}$ be defined by

$$u_i = \int_{2^i}^{2^{i+1}} f(t)\,\frac{dt}{t}.$$

then $\sum_{i=-\infty}^{\infty} u_i$ converges to $u$ in $X_0 + X_1$ because the integral representation converges to $u$ there. Moreover,

$$\begin{aligned}
2^{-i\theta} J(2^i; u_i) &\le \int_{2^i}^{2^{i+1}} 2^{-i\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&= 2^{\theta} \int_{2^i}^{2^{i+1}} 2^{-(i+1)\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&\le 2^{\theta} \int_{2^i}^{2^{i+1}} t^{-\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&\le 2^{\theta} \big(\ln 2\big)^{1/q'} \left( \int_{2^i}^{2^{i+1}} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q \,\frac{dt}{t} \right)^{1/q},
\end{aligned}$$

where $q' = q/(q-1)$ and Hölder's inequality was used in the last line. Thus

$$\sum_{i=-\infty}^{\infty} \big[2^{-i\theta} J(2^i; u_i)\big]^q \le 2^{\theta q} \big(\ln 2\big)^{q/q'} \int_0^{\infty} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q \,\frac{dt}{t}$$

and, since $\epsilon$ is arbitrary,

$$\big\| \big\{2^{-i\theta} J(2^i; u_i)\big\} ; \ell^q \big\| \le 2^{\theta} \big(\ln 2\big)^{1/q'} \|u\|_{\theta,q;J}\,.$$

Conversely, if $u = \sum_{i=-\infty}^{\infty} u_i$ where the series converges in $X_0 + X_1$, we can define a function $f \in L^1(0, \infty; dt/t, X_0 + X_1)$ by

$$f(t) = \frac{1}{\ln 2} u_i, \qquad \text{for} \quad 2^i \le t < 2^{i+1}, \quad -\infty < i < \infty,$$

and we will have

$$\int_{2^i}^{2^{i+1}} f(t)\,\frac{dt}{t} = u_i \qquad \text{and} \qquad u = \int_0^{\infty} f(t)\,\frac{dt}{t}.$$

Moreover,

$$\begin{aligned}
\int_{2^i}^{2^{i+1}} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q \,\frac{dt}{t} &\le \int_{2^i}^{2^{i+1}} \big[2^{-i\theta} J\big(2^{i+1}; f(t)\big)\big]^q \,\frac{dt}{t} \\
&\le \left(\frac{2}{\ln 2}\right)^q \int_{2^i}^{2^{i+1}} \big[2^{-i\theta} J(2^i; u_i)\big]^q \,\frac{dt}{t} \\
&= \frac{2^q}{(\ln 2)^{q-1}} \big[2^{-i\theta} J(2^i; u_i)\big]^q.
\end{aligned}$$

Summing on $i$ then gives

$$\|u\|_{\theta,q;J} \le \left(\frac{2}{(\ln 2)^{1/q'}}\right) \left\| \{2^{-i\theta} J(2^i; u_i)\} ; \ell^q \right\| . \blacksquare$$

Next we prove that for $0 < \theta < 1$ the $J$- and $K$-methods generate the same intermediate spaces with equivalent norms.

**7.16 THEOREM (Equivalence Theorem)** If $0 < \theta < 1$ and $1 \le q \le \infty$, then

(a) $(X_0, X_1)_{\theta,q;J} \to (X_0, X_1)_{\theta,q;K}$, and

(b) $(X_0, X_1)_{\theta,q;K} \to (X_0, X_1)_{\theta,q;J}$. Therefore

(c) $(X_0, X_1)_{\theta,q;J} = (X_0, X_1)_{\theta,q;K}$, the two spaces having equivalent norms.

**Proof.** Conclusion (a) is a consequence of the somewhat stronger result

$$(X_0, X_1)_{\theta,p;J} \to (X_0, X_1)_{\theta,q;K}, \qquad \text{if } 1 \le p \le q \tag{9}$$

which we now prove. Let $u = \int_0^\infty f(s)\, ds/s \in (X_0, X_1)_{\theta,p;J}$. Since $K(t; \cdot)$ is a norm on $X_0 + X_1$, we have by the triangle inequality and (4)

$$\begin{aligned} t^{-\theta} K(t; u) &\le t^{-\theta} \int_0^\infty K\big(t; f(s)\big) \frac{ds}{s} \\ &\le \int_0^\infty \left(\frac{t}{s}\right)^{-\theta} \min\left\{1, \frac{t}{s}\right\} s^{-\theta} J\big(s; f(s)\big) \frac{ds}{s} \\ &= \big[t^{-\theta} \min\{1, t\}\big] * \big[t^{-\theta} J\big(t; f(t)\big)\big]. \end{aligned}$$

By Young's inequality with $1 + (1/q) = (1/r) + (1/p)$ (so $r \ge 1$)

$$\begin{aligned} \|u\|_{\theta,q;K} &= \left\| t^{-\theta} K(t; u) ; L^q_* \right\| \\ &\le \left\| t^{-\theta} \min\{1, t\} ; L^r_* \right\| \left\| t^{-\theta} J\big(t; f(t)\big) ; L^p_* \right\| \\ &\le C_{\theta,p,q} \|u\|_{\theta,p;K} , \end{aligned}$$

which confirms (9) and hence (a).

Now we prove (b) by using the discrete versions of the J and K methods. Let $u \in (X_0, X_1)_{\theta,p;K}$. By the definition of $K(t; u)$, for each integer $i$ there exist $v_i \in X_0$ and $w_i \in X_1$ such that

$$u = v_i + w_i \qquad \text{and} \qquad \|v_i\|_{X_0} + 2^i \|w_i\|_{X_1} \le 2K(2^i; u).$$

Then the sequences $\{2^{-i\theta} \|v_i\|_{X_0}\}$ and $\{2^{i(1-\theta)} \|w_i\|_{X_1}\}$ both belong to $\ell^q$ and each has $\ell^q$-norm bounded by a constant times $\|u\|_{\theta,q;K}$. For each index $i$ let $u_i = v_{i+1} - v_i$. Since

$$0 = u - u = \big(v_{i+1} + w_{i+1}\big) - \big(v_i + w_i\big) = \big(v_{i+1} - v_i\big) + \big(w_{i+1} - w_i\big),$$

we have, in fact,

$$u_i = v_{i+1} - v_i = w_i - w_{i+1}.$$

The first of these representations of $u_i$ shows that $\{2^{-i\theta}\,\|u_i\|_{X_0}\}$ belongs to $\ell^q$; the second representations shows that $\{2^{i(1-\theta)}\,\|u_i\|_{X_1}\}$ also belongs to $\ell^q$. Therefore $\{2^{-i\theta}J(2^i; u_i)\} \in \ell^q$ and has $\ell^q$-norm bounded by a constant times $\|u\|_{\theta,q;K}$. Since $\ell^q \subset \ell^\infty$, the sequence $\left\{2^{j(1-\theta)}\left\|w_j\right\|_{X_1}\right\}$ is bounded even though $2^{j(1-\theta)} \to \infty$ as $j \to \infty$. Thus $\left\|w_j\right\|_{X_1} \to 0$ as $j \to \infty$. Since $\sum_{i=0}^{j} u_i = w_0 - w_{j+1}$, the half series $\sum_{i=0}^{\infty}$ converges to $w_0$ in $X_1$ and hence in $X_0 + X_1$. Similarly, the half-series $\sum_{i=-\infty}^{-1} u_i$ converges to $v_0$ in $X_0$, and thus in $X_0 + X_1$. Thus the full series $\sum_{i=-\infty}^{\infty} u_i$ converges to $v_0 + w_0 = u$ in $X_0 + X_1$ and we have

$$\|u\|_{\theta,q;J} \le \text{const.}\, \|u\|_{\theta,q;K}\,.$$

This completes the proof of (b) and hence (c). ∎

**7.17 COROLLARY** If $0 < \theta < 1$ and $1 \le p \le q \le \infty$, then

$$(X_0, X_1)_{\theta,p;K} \to (X_0, X_1)_{\theta,q;K}. \tag{10}$$

**Proof.** $(X_0, X_1)_{\theta,p;K} \to (X_0, X_1)_{\theta,p;J} \to (X_0, X_1)_{\theta,q;K}$ by part (b) and imbedding (9). ∎

**7.18 (Classes of Intermediate Spaces)** We define three classes of intermediate spaces $X$ between $X_0$ and $X_1$ as follows:

(a) $X$ belongs to class $\mathscr{K}(\theta; X_0, X_1)$ if for all $u \in X$

$$K(t; u) \le C_1 t^\theta\,\|u\|_X\,,$$

where $C_1$ is a constant.

(b) $X$ belongs to class $\mathscr{J}(\theta; X_0, X_1)$ if for all $u \in X_0 \cap X_1$

$$\|u\|_X \le C_2 t^{-\theta} J(t; u),$$

where $C_2$ is a constant.

(c) $X$ belongs to class $\mathscr{H}(\theta; X_0, X_1)$ if $X$ belongs to both $\mathscr{K}(\theta; X_0, X_1)$ and $\mathscr{J}(\theta; X_0, X_1)$.

The following lemma gives necessary and sufficient conditions for membership in these classes.

**7.19 LEMMA** Let $0 \le \theta \le 1$ and let $X$ be an intermediate space between $X_0$ and $X_1$.

(a) $X \in \mathscr{K}(\theta; X_0, X_1)$ if and only if $X \to (X_0, X_1)_{\theta,\infty;K}$.

(b) $X \in \mathscr{J}(\theta; X_0, X_1)$ if and only if $(X_0, X_1)_{\theta,1;J} \to X$.

(c) $X \in \mathscr{H}(\theta; X_0, X_1)$ if and only if $(X_0, X_1)_{\theta,1;J} \to X \to (X_0, X_1)_{\theta,\infty;K}$.

**Proof.** Conclusion (a) is immediate since $\|u\|_{\theta,\infty;K} = \sup_{0<t<\infty}\big(t^{-\theta}k(t;u)\big)$. Since (c) follows from (a) and (b), only (b) requires proof.

First suppose $X \in \mathscr{J}(\theta; X_0, X_1)$. Let $u \in (X_0, X_1)_{\theta,1;J}$. If $f(t)$ is any function on $(0,\infty)$ with values in $X_0 \cap X_1$ such that $u = \int_0^\infty f(t)\,dt/t$, then

$$\|u\|_X \le \int_0^\infty \|f(t)\|_X \frac{dt}{t} \le C_2 \int_0^\infty t^{-\theta} J\big(t; f(t)\big)\frac{dt}{t}.$$

Since this holds for all such representations of $u$ we have

$$\|u\|_X \le C_2 (X_0, X_1)_{\theta,1;J}, \tag{11}$$

and so $(X_0, X_1)_{\theta,1;J} \to X$.

Conversely, suppose that $(X_0, X_1)_{\theta,1;J} \to X$; therefore (11) holds with some constant $C_2$. Let $u \in X_0 \cap X_1$, let $\lambda > 0$ and $t > 0$, and let

$$f_\lambda(s) = \begin{cases} (1/\lambda)u & \text{if } te^{-\lambda} \le s \le t \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_0^\infty f_\lambda(s)\frac{ds}{s} = \left(\int_{te^{-\lambda}}^t \frac{ds}{s}\right)\left(\frac{1}{\lambda}\right) u = u.$$

Since $J\big(s; (1/\lambda)u\big) = (1/\lambda)J(s;u)$ we have

$$\|u\|_{\theta,1;J} \le \int_0^\infty s^{-\theta} J\big(s; f_\lambda(s)\big)\frac{ds}{s} = \frac{1}{\lambda}\int_{te^{-\lambda}}^t s^{-\theta} J(s;u)\frac{ds}{s}.$$

Since $s^{-\theta}J(s;u)$ is continuous in $s$ and $\int_{te^{-\lambda}}^t ds/s = \lambda$, we can let $\lambda \to 0+$ in the above inequality and obtain $\|u\|_{\theta,1;J} \le t^{-\theta} J(t;u)$. Hence

$$\|u\|_X \le C_2 (X_0, X_1)_{\theta,1;J} \le C_2 t^{-\theta} J(t;u)$$

and the proof of (b) is complete. ∎

The following corollary follows immediately, using the equivalence theorem, (10), and (8).

**7.20 COROLLARY** If $0 < \theta < 1$ and $1 \le q \le \infty$, then

$$(X_0, X_1)_{\theta,q;J} = (X_0, X_1)_{\theta,q;K} \in \mathscr{H}(\theta; X_0, X_1).$$

Moreover, $X_0 \in \mathscr{H}(0; X_0, X_1)$ and $X_1 \in \mathscr{H}(1; X_0, X_1)$. ∎

Next we examine the result of constructing intermediate spaces between two intermediate spaces.

**7.21 THEOREM (The Reiteration Theorem)** Let $0 \le \theta_0 < \theta_1 \le 1$ and let $X_{\theta_0}$ and $X_{\theta_1}$ be intermediate spaces between $X_0$ and $X_1$. For $0 \le \lambda \le 1$, let $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$.

(a) If $X_{\theta_i} \in \mathscr{K}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 \le q < \infty$ or $0 \le \lambda \le 1$ and $q = \infty$, then

$$\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} \to (X_0, X_1)_{\theta,q;K}.$$

(b) If $X_{\theta_i} \in \mathscr{J}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 < q \le \infty$ or $0 \le \lambda \le 1$ and $q = 1$, then

$$(X_0, X_1)_{\theta,q;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;J}.$$

(c) If $X_{\theta_i} \in \mathscr{H}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if $0 < \lambda < 1$ and $1 \le q \le \infty$, then

$$\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;J} = \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} = (X_0, X_1)_{\theta,q;K} = (X_0, X_1)_{\theta,q;J}.$$

(d) Moreover,

$$(X_0, X_1)_{\theta_0,1;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{0,1;J} \to X_{\theta_0} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{0,\infty;K} \to (X_0, X_1)_{\theta_0,\infty;K}$$
$$(X_0, X_1)_{\theta_1,1;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{1,1;J} \to X_{\theta_1} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{1,\infty;K} \to (X_0, X_1)_{\theta_1,\infty;K}.$$

**Proof.** The important conclusions here are (c) and (d) and these follow from (a) and (b) which we must prove. In both proofs we need to distinguish the function norms $K(t;u)$ and $J(t;u)$ used in the construction of the intermediate spaces between $X_0$ and $X_1$ from those used for the intermediate spaces between $X_{\theta_0}$ and $X_{\theta_1}$. We will use $K^*$ and $J^*$ for the latter.

**Proof of (a)** If $u \in \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K}$, then $u = u_0 + u_1$ where $u_i \in X_{\theta_i}$. Since $X_{\theta_i} \in \mathscr{K}(\theta_i; X_0, X_1)$, we have

$$\begin{aligned} K(t;u) &\le K(t;u_0) + K(t;u_1) \\ &\le C_0 t^{\theta_0} \left\| u_0 ; X_{\theta_0} \right\| + C_1 t^{\theta_1} \left\| u_1 ; X_{\theta_1} \right\| \\ &\le C_0 t^{\theta_0} \left( \left\| u_0 ; X_{\theta_0} \right\| + \frac{C_1}{C_0} t^{\theta_1-\theta_0} \left\| u_1 ; X_{\theta_1} \right\| \right). \end{aligned}$$

Since this estimate holds for all such representations of $u$, we have

$$K(t;u) \le C_0 t^{\theta_0} K^*\left(\frac{C_1}{C_0} t^{\theta_1-\theta_0}; u\right).$$

If $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then $\lambda = (\theta-\theta_0)/(\theta_1-\theta_0)$, and (assuming $q<\infty$)

$$\begin{aligned}
\left\| t^{-\theta} K(t;u)\,;L_*^q \right\| &\le C_0 \left[\int_0^\infty \left(t^{-(\theta-\theta_0)} K^*\left(\frac{C_1}{C_0} t^{\theta_1-\theta_0}; u\right)\right)^q \frac{dt}{t}\right]^{1/q} \\
&= \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\theta_1-\theta_0)^{1/q}} \left[\int_0^\infty \left(s^{-\lambda} K^*(s;u)\right)^q \frac{ds}{s}\right]^{1/q}
\end{aligned}$$

via the transformation $s = (C_1/C_0) t^{\theta_1-\theta_0}$. Hence

$$\|u\|_{\theta,q;K} \le \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\theta_1-\theta_0)^{1/q}} \|u\|_{\lambda,q;K}$$

and so $\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} \to (X_0, X_1)_{\theta,q;K}$.

**Proof of (b)** Let $u \in (X_0, X_1)_{\theta,q;J}$. Then $u = \int_0^\infty f(s)\,ds/s$ for some $f$ taking values in $X_0 \cap X_1$ satisfying $s^{-\theta} J\big((s; f(s)\big) \in L_*^q$. Clearly $f(s) \in X_{\theta_0} \cap X_{\theta_1}$. Since $X_{\theta_i} \in \mathscr{J}(\theta_i; X_0, X_1)$ we have

$$\begin{aligned}
J^*\big(s; f(s)\big) &= \max\left\{ \left\| f(s)\,; X_{\theta_0}\right\|, s\left\| f(s)\,; X_{\theta_1}\right\| \right\} \\
&\le \max\left\{ C_0 t^{-\theta_0} J\big(t; f(s)\big), C_1 t^{-\theta_1} s J\big(t; f(s)\big) \right\} \\
&= C_0 t^{-\theta_0} \max\left\{ 1, \frac{C_1}{C_0} t^{-(\theta_1-\theta_0)} s \right\} J\big(t; f(s)\big).
\end{aligned}$$

This estimate holds for all $t>0$ so we can choose $t$ so that $t^{-(\theta_1-\theta_0)} s = C_0/C_1$ and obtain

$$J^*\big(s; f(s)\big) \le C_0 \left(\frac{C_1}{C_0} s\right)^{-\theta_0/(\theta_1-\theta_0)} J\left(\left(\frac{C_1}{C_0} s\right)^{1/(\theta_1-\theta_0)}; f(s)\right).$$

If $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then

$$\begin{aligned}
&\left\| s^{-\lambda} J^*\big(s; f(s)\big)\,; L_*^q \right\| \\
&\quad \le C_0^{1-\lambda} C_1^{\lambda} \left(\int_0^\infty \left[\left(\frac{C_1}{C_0} s\right)^{-\theta/(\theta_1-\theta_0)} J\left(\left(\frac{C_1}{C_0} s\right)^{1/(\theta_1-\theta_0)}; f(s)\right)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad \le C_0^{1-\lambda} C_1^{\lambda} (\theta_1-\theta_0)^{1/q} \left(\int_0^\infty \left[t^{-\theta} J\big(t; g(t)\big)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = C_0^{1-\lambda} C_1^{\lambda} (\theta_1-\theta_0)^{1/q} \left\| t^{-\theta} J\big(t; g(t)\big)\,; L_*^q \right\|,
\end{aligned}$$

where $g(t) = f\big((C_0/C_1)t^{\theta_1-\theta_0}\big) = f(s) \in X_0 \cap X_1$. Since

$$\int_0^\infty g(t)\,\frac{dt}{t} = \frac{1}{\theta_1-\theta_0}\int_0^\infty f(s)\,\frac{ds}{s} = \frac{1}{\theta_1-\theta_0}u,$$

we have

$$\|u\|_{\lambda,q;J} \le \frac{C_0^{1-\lambda}C_1^{\lambda}}{(\theta_1-\theta_0)^{(q-1)/q}}\,\|u\|_{\theta.q;J}$$

and so $(X_0, X_1)_{\theta,q;J} \to \big(X_{\theta_0}, X_{\theta_1}\big)_{\lambda,q;J}$. ∎

**7.22 (Interpolation Spaces)** Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces, and let $T$ be a bounded linear operator from $X_0 + X_1$ into $Y_0 + Y_1$ having the property that $T$ is bounded from $X_i$ into $Y_i$, with norm at most $M_i$, $i = 0, 1$; that is,

$$\|Tu_i\|_{Y_i} \le M_i\,\|u_i\|_{X_i}, \quad \text{for all } u_i \in X_i), \qquad (i = 1, 2).$$

If $X$ and $Y$ are intermediate spaces for $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, respectively, we call $X$ and $Y$ *interpolation spaces of type* $\theta$ for $P$ and $Q$, where $0 \le \theta \le 1$, if every such linear operator $T$ maps $X$ into $Y$ with norm $M$ satisfying

$$M \le CM_0^{1-\theta}M_1^{\theta}, \tag{12}$$

where constant $C \ge 1$ is independent of $T$. We say that the interpolation spaces $X$ and $Y$ are *exact* if inequality (12) holds with $C = 1$. If $X_0 = Y_0$, $X_1 = Y_1$, $X = Y$ and $T = I$, the identity operator on $X_0 + X_1$, then $C = 1$ for all $0 \le \theta \le 1$, so no smaller $C$ is possible in (12).

**7.23 THEOREM (An Exact Interpolation Theorem)** Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs.

(a) If either $0 < \theta < 1$ and $1 \le q \le \infty$ or $0 \le \theta \le 1$ and $q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;K}$ and $(Y_0, Y_1)_{\theta,q;K}$ are exact interpolation spaces of type $\theta$ for $P$ and $Q$

(b) If either $0 < \theta < 1$ and $1 < q \le \infty$ or $0 \le \theta \le 1$ and $q = 1$, then the intermediate spaces $(X_0, X_1)_{\theta,q;J}$ and $(Y_0, Y_1)_{\theta,q;J}$ are exact interpolation spaces of type $\theta$ for $P$ and $Q$.

**Proof.** Let $T : X_0 + X_1 \to Y_0 + Y_1$ satisfy $\|Tu_i\|_{Y_i} \le M_i\,\|u_i\|_{X_i}$, $i = 0, 1$. If $u \in X_0 + X_1$, then

$$\begin{aligned} K(t; Tu) &= \inf\Big\{\|Tu_0\|_{Y_0} + t\,\|Tu_1\|_{Y_1} : u = u_0 + u_1, u_i \in X_i\Big\} \\ &\le M_0 \inf_{\substack{u=u_0+u_1\\ u_i\in X_i}} \left(\|u_0\|_{X_0} + \frac{M_1}{M_0}t\,\|u_1\|_{X_1}\right) = M_0K\big((M_1/M_0)t; u\big). \end{aligned}$$

If $u \in (X_0, X_1)_{\theta,q;K}$, then

$$\begin{aligned}\|Tu\|_{\theta,q;K} &= \left\|t^{-\theta}K(t;Tu)\,;\,L^q_*\right\| \le M_0\left\|t^{-\theta}K\big((M_1/M_0)t;u\big)\,;\,L^q_*\right\|\\ &= M_0\left(\frac{M_0}{M_1}\right)^{-\theta}\left\|s^{-\theta}K(s;u)\,;\,L^q_*\right\| = M_0^{1-\theta}M_1^{\theta}\|u\|_{\theta,q;K},\end{aligned}$$

which proves (a).

If $u \in X_0 \cap X_1$, then

$$\begin{aligned}J(t;Tu) &= \max\left\{\|Tu\|_{Y_0}, t\,\|Tu\|_{Y_1}\right\}\\ &\le M_0\max\left\{\|u\|_{X_0}, (M_1/M_0)t\,\|u\|_{X_1}\right\} = M_0 J\big((M_1/M_0)t;u\big).\end{aligned}$$

If $u = \int_0^\infty f(t)\,dt/t$, where $f(t) \in X_0 \cap X_1$ and $t^{-\theta}J\big(t;f(t)\big) \in L^q_*$, then

$$\begin{aligned}&\|Tu\|_{\theta,q;J} = \left\|t^{-\theta}J\big(t;Tf(t)\big)\,;\,L^q_*\right\|\\ &\le M_0\left\|t^{-\theta}J\big((M_1/M_0)t;f(t)\,;\,L^q_*\right\| = M_0\left(\frac{M_0}{M_1}\right)^{-\theta}\left\|s^{-\theta}J\big(s;g(s)\big)\,;\,L^q_*\right\|,\end{aligned}$$

where $g(s) = f\big((M_0/M_1)s\big) = f(t)$. Since this estimate holds for all representations of $u = \int_0^\infty g(s)\,ds/s$, we have

$$\|Tu\|_{\theta,q;J} \le M_0^{1-\theta}M_1^{\theta}\|u\|_{\theta,q;J}$$

and the proof of (b) is complete. ∎

## The Lorentz Spaces

**7.24 (Equimeasurable Decreasing Rearrangement)** Recall that, as defined in Paragraph 2.53, the distribution function $\delta_u$ corresponding to a measurable function $u$ finite a.e. in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$\delta_u(t) = \mu\{x \in \Omega : |u(x)| > t\}$$

and is nonincreasing on $[0,\infty)$. (It is also right continuous there, but that is of no relevance for integrals involving the distribution function since a nonincreasing function can have at most countably many points of discontinuity.) Moreover, if $u \in L^p(\Omega)$, then

$$\|u\|_p = \begin{cases}\left(p\displaystyle\int_0^\infty t^p\delta_u(y)\,\frac{dt}{t}\right)^{1/p} & \text{if } 1 \le p < \infty,\\ \inf\{t : \delta_u(t) = 0\} & \text{if } p = \infty.\end{cases}$$

The *equimeasurable decreasing rearrangement of u* is the function $u^*$ defined by

$$u^*(s) = \inf\{t : \delta_u(t) \le s\}.$$

This definition and the fact that $\delta_u$ is nonincreasing imply that $u*$ is nonincreasing too. Moreover, $u^*(s) > t$ if and only if $\delta_u(t) > s$, and this latter condition is trivially equivalent to $s < \delta_u(t)$. Therefore,

$$\delta_{u^*}(t) = \mu\left\{s : u^*(s) > t\right\} = \mu\{s : 0 \le s < \delta_u(t)\} = \mu\left\{\left[0, \delta_u(t)\right)\right\} = \delta_u(t).$$

This justifies our calling $u^*$ and $u$ equimeasurable; the size of both functions exceeds any number $s$ on sets having the same measure. Also,

$$\delta_{u^*}(t) = \mu\{s : u^*(s) > t\} = \inf\{s : u^*(s) \le t\}$$

so that

$$\delta_u(t) = \inf\{s : u^*(s) \le t\}.$$

This further illustrates the symmetry between $\delta_u$ and $u^*$.

Note also that

$$u^*\big(\delta_u(t)\big) = \inf\big\{s : \delta_u(s) \le \delta_u(t)\big\} \le t.$$

If $u^*\big(\delta_u(t)\big) = s < t$, then $\delta_u$ is constant on the interval $(s, t)$ in which case $u^*$ has a jump discontinuity of magnitude at least $t - s$ at $\delta_u(t)$.

Similarly, $\delta_u\big(u^*(s)\big) \le s$, with equality if $\delta_u$ is continuous at $t = u^*(s)$. The relationship between $\delta_u$ and $u^*$ is illustrated in Figure 8. Except at points where either function is discontinuous (and the other is constant on an interval), each is the inverse of the other.

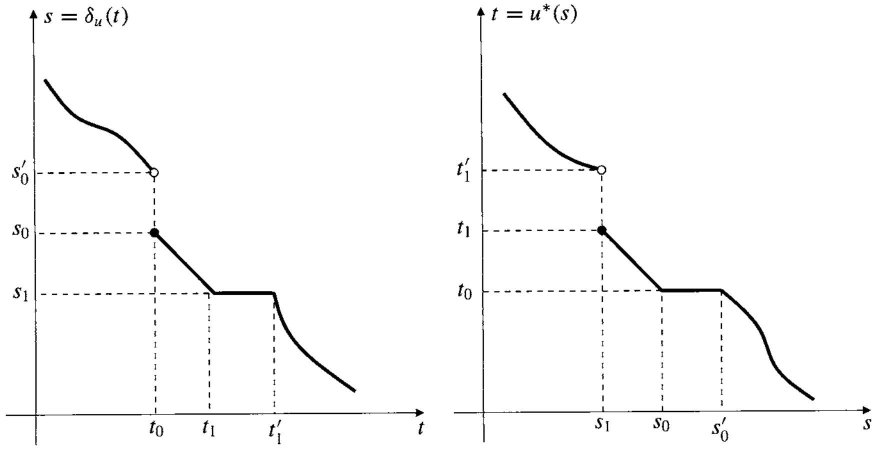


Fig. 8

If $S_t = \{x \in \Omega : |u(x)| > t\}$, then

$$\int_{S_t} |u(x)|\, dx = \int_0^{\delta_u(t)} u^*(s)\, ds, \tag{13}$$

and if $u \in L^p(\Omega)$, then

$$\|u\|_p = \begin{cases} \left( \displaystyle\int_0^\infty \big(u^*(s)\big)^p\, dx \right)^{1/p} & \text{if } 1 \le p < \infty \\ \displaystyle\sup_{0<s<\infty} u^*(s) & \text{if } p = \infty. \end{cases}$$

**7.25 (The Lorentz Spaces)** For $u$ measurable on $\Omega$ let

$$u^{**}(t) = \frac{1}{t} \int_0^t u^*(s)\, ds,$$

that is, the average value of $u^*$ over $[0, t]$. Since $u^*$ is nonincreasing, we have $u^*(t) \le u^{**}(t)$.

For $1 \le p \le \infty$ we define the functional

$$\big\| u\, ;\, L^{p,q}(\Omega) \big\| = \begin{cases} \left( \displaystyle\int_0^\infty \big(t^{1/p} u^{**}(t)\big)^q\, \frac{dt}{t} \right)^{1/q} & \text{if } 1 \le q < \infty \\ \displaystyle\sup_{t>0} t^{1/p} u^{**}(t) & \text{if } q = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega)$ consists of those measurable functions $u$ on $\Omega$ for which $\|u\, ;\, L^{p,q}(\Omega)\| < \infty$. Theorem 7.26 below shows that if $1 < p < \infty$, then $L^{p,q}(\Omega)$ is, in fact, identical to the intermediate space $\big(L^1(\Omega), L^\infty(\Omega)\big)_{(p-1)/p,q;K}$ and $\|u\, ;\, L^{p,q}(\Omega)\| = \|u\|_{(p-1)/p,q;K}$. Thus $L^{p,q}(\Omega)$ is a Banach space under the norm $\|u\, ;\, L^{p,q}(\Omega)\|$. It is also a Banach space if $p = 1$ or $p = \infty$.

The second corollary to Theorem 7.26 shows that if $1 < p < \infty$, then $L^{p,q}(\Omega)$ coincides with the set of measurable $u$ for which $\big[u;\, L^{p,q}(\Omega)\big] < \infty$, where

$$\big[u;\, L^{p,q}(\Omega)\big] = \begin{cases} \left( \displaystyle\int_0^\infty \big(t^{1/p} u^{*}(t)\big)^q\, \frac{dt}{t} \right)^{1/q} & \text{if } 1 \le q < \infty \\ \displaystyle\sup_{t>0} t^{1/p} u^{*}(t) & \text{if } q = \infty, \end{cases}$$

and that

$$\big[u;\, L^{p,q}(\Omega)\big] \le \big\| u\, ;\, L^{p,q}(\Omega) \big\| \le \frac{p}{p-1}\big[u;\, L^{p,q}(\Omega)\big].$$

The index $p$ in $L^{p,q}(\Omega)$ is called the principal index; $q$ is the secondary index. Unless $q = p$, the functional $\left[\,\cdot\,; L^{p,q}(\Omega)\right]$ is not a norm since it does not satisfy the triangle inequality; it, however, is a quasi-norm since

$$\left[u+v; L^{p,q}(\Omega)\right] \le 2\left(\left[u; L^{p,q}(\Omega)\right] + \left[v; L^{p,q}(\Omega)\right]\right).$$

For $1 < p < \infty$ it is evident that $\left[\,\cdot\,; L^{p,p}(\Omega)\right] = \|\cdot\|_{p,\Omega}$, and therefore $L^{p,p}(\Omega) = L^p(\Omega)$. Moreover, if we recall the definition of the space weak-$L^p(\Omega)$ given in Paragraph 2.55 and having quasi-norm given (for $p < \infty$) by

$$[u]_p = [u]_{p,\Omega} = \left(\sup_{t>0} t^p \delta_u(t)\right)^{1/p},$$

we can show that $L^{p,\infty}(\Omega) = \text{weak-}L^p(\Omega)$. This is also clear for $p = \infty$. If $1 < p < \infty$ and $K > 0$, then for all $t > 0$ we have, putting $s = K^p t^{-p}$,

$$\delta_u(t) \le K^p t^{-p} = s \qquad \Longleftrightarrow \qquad u^*(s) \le t = K s^{-1/p}.$$

Hence $[u]_p \le K$ if and only if $\left[u; L^{p,\infty}(\Omega)\right] \le K$, and these two quasi-norms are, in fact, equal.

For $p = 1$ the situation is a little different. Observe that

$$\left\|u\,; L^{1,\infty}(\Omega)\right\| = \sup_{t>0} t u^{**}(t) = \sup_{t>0} \int_0^t u^*(s)\,ds = \int_0^\infty u^*(s)\,ds = \|u\|_1$$

so $L^1(\Omega) = L^{1,\infty}(\Omega)$ (*not* $L^{1,1}(\Omega)$ which contains only the zero function).

For $p = \infty$ we have $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$ since

$$\left\|u\,; L^{\infty,\infty}(\Omega)\right\| = \sup_{t>0} u^{**}(t) = \sup_{t>0} \frac{1}{t}\int_0^t u^*(s)\,ds = u^*(0) = \|u\|_\infty .$$

**7.26 THEOREM** If $u \in L^1(\Omega) + L^\infty(\Omega)$, then for $t > 0$ we have

$$K(t; u) = \int_0^t u^*(s)\,ds = t u^{**}(t). \tag{14}$$

Therefore, if $1 < p < \infty$, $1 \le q \le \infty$, and $\theta = 1 - (1/p)$,

$$L^{p,q}(\Omega) = \left(L^1(\Omega), L^\infty(\Omega)\right)_{\theta,q;K}$$

with equality of norms: $\|u\,; L^{p,q}(\Omega)\| = \|u\|_{\theta,q;K}$.

**Proof.** The second conclusion follows immediately from the representation (14) which we prove as follows.

Since $K(t;u) = K(t;|u|)$ we can assume that $u$ is real-valued and nonnegative. Let $u = v + w$ where $v \in L^1(\Omega)$ and $w \in L^\infty(\Omega)$. In order to calculate

$$K(t;u) = \inf_{u=v+w} \left(\|v\|_1 + t\,\|w\|_\infty\right) \tag{15}$$

we can also assume that $v$ and $w$ are real-valued functions since, in any event, $u = \operatorname{Re} v + \operatorname{Re} w$ and $\|\operatorname{Re} v\|_1 \le \|v\|_1$ and $\|\operatorname{Re} w\|_\infty \le \|w\|_\infty$. We can also assume that $v$ and $w$ are nonnegative, for if

$$v_1(x) = \begin{cases} \min\{v(x), u(x)\} & \text{if } v(x) \ge 0 \\ 0 & \text{if } v(x) < 0 \end{cases} \qquad \text{and} \qquad w_1(x) = u(x) - v_1(x),$$

then $0 \le v_1(x) \le |v(x)|$ and $0 \le w_1(x) \le |w(x)|$. Thus the infimum in (15) does not change if we restrict to nonnegative functions $v$ and $w$.

Thus we consider $u = v + w$, where $v \ge 0$, $v \in L^1(\Omega)$, $w \ge 0$, and $w \in L^\infty(\Omega)$. Let $\lambda = \|w\|_\infty$ and define $u_\lambda(x) = \min\{\lambda, u(x)\}$. Evidently $w(x) \le u_\lambda(x)$ and $u(x) - u_\lambda(x) \le u(x) - w(x) = v(x)$. Now let

$$g(t,\lambda) = \|u - u_\lambda\|_1 + t\lambda \le \|v\|_1 + t\,\|w\|_\infty .$$

Then $K(t;u) = \inf_{0<\lambda<\infty} g(t,\lambda)$. We want to show that this infimum is, in fact, a minimum and is assumed at $\lambda = \lambda_t = \inf\{\tau : \delta_u(\tau) < t\}$.

If $\lambda > \lambda_t$, then $u_\lambda(x) - u_{\lambda_t}(x) \le \lambda - \lambda_t$ if $u(x) > \lambda_t$, and $u_\lambda(x) - u_{\lambda_t}(x) = 0$ if $u(x) \le \lambda_t$. Since $\delta_u(\lambda_t) \le t$, we have

$$\begin{aligned} g(t,\lambda) - g(t,\lambda_t) &= -\int_\Omega \left(u_\lambda(x) - u_{\lambda_t}(x)\right) dx + t(\lambda - \lambda_t) \\ &\ge (\lambda - \lambda_t)\left(t - \delta_u(\lambda_t)\right) \ge 0. \end{aligned}$$

Thus $K(t;u) \le g(t,\lambda_t)$.

On the other hand, if $g(t,\lambda^*) < \infty$ for some $\lambda^* < \lambda_t$, then $g(t,\lambda)$ is a continuous function of $\lambda$ for $\lambda \ge \lambda^*$ and so for any $\epsilon > 0$ there exists $\lambda$ such that $\lambda^* \le \lambda < \lambda_t$ and

$$|g(t,\lambda) - g(t,\lambda_t)| < \epsilon.$$

Now $u_\lambda(x) - u_{\lambda^*}(x) = \lambda - \lambda^*$ if $u(x) > \lambda$, and since $\delta_u(\lambda) \ge t$ we have

$$\begin{aligned} g(t,\lambda^*) - g(t,\lambda) &= \int_\Omega \left(u_\lambda(x) - u_{\lambda^*}(x)\right) dx - t(\lambda - \lambda^*) \\ &\ge (\lambda - \lambda^*)\left(\delta_u(\lambda) - t\right) \ge 0. \end{aligned}$$

Thus

$$g(t,\lambda^*) - g(t,\lambda_t) \ge g(t,\lambda^*) - g(t,\lambda) - |g(t,\lambda) - g(t,\lambda_t)| \ge -\epsilon.$$

Since $\epsilon$ is arbitrary, $g(t,\lambda^*) \ge g(t,\lambda_t)$ and $K(t;u) \ge g(t,\lambda_t)$. Thus

$$K(t;u) = g(t,\lambda_t) = \left\|u - u_{\lambda_t}\right\|_1 + t\lambda_t.$$

Now $u(x)-u_{\lambda_t}(x) = 0$ except where $u(x) > \lambda_t$ and $\lambda_t = u^*(s)$ for $\delta_u(\lambda_t) \le s \le t$. Therefore, by (13),

$$\begin{aligned} K(t;u) &= \int_0^{\delta_u(\lambda_t)} \big(u^*(s) - \lambda_t\big)\, ds + t\lambda_t = \int_0^{\delta_u(\lambda_t)} u^*(s)\, ds - \lambda_t \delta_u(\lambda_t) + t\lambda_t \\ &= \int_0^{\delta_u(\lambda_t)} u^*(s)\, ds + \int_{\delta_u(\lambda_t)}^t u^*(s)\, ds = \int_0^t u^*(s)\, ds \end{aligned}$$

which completes the proof. ∎

**7.27 COROLLARY** If $1 \le p_1 < p < p_2 \le \infty$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, then by the Reiteration Theorem 7.21, up to equivalence of norms,

$$L^{p,q}(\Omega) = \big(L^{p_1}(\Omega), L^{p_2}(\Omega)\big)_{\theta,q;K}.$$

**7.28 COROLLARY** For $1 < p < \infty$, $1 \le q \le \infty$, and $\theta = 1 - (1/p)$, we have

$$\big[u; L^{p,q}(\Omega)\big] \le \left\|u\,; L^{p,q}(\Omega)\right\| \le \frac{p}{p-1}\big[u; L^{p,q}(\Omega)\big].$$

**Proof.** Since $u^*$ is decreasing, (14) implies that $tu^*(t) \le K(t;u)$. Thus

$$\begin{aligned} \big[u; L^{p,q}(\Omega)\big] &= \left(\int_0^\infty \big(t^{1/p}u^*(t)\big)^q \,\frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_0^\infty \big(t^{-\theta}K(t;u)\big)^q \,\frac{dt}{t}\right)^{1/q} = \|u\|_{\theta,q;K} = \left\|u\,; L^{p,q}(\Omega)\right\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} t^{-\theta}K(t;u) &= \int_0^t t^{-\theta}u^*(s)\, ds \\ &= \int_1^\infty \sigma^{-\theta}\left(\frac{t}{\sigma}\right)^{1-\theta} u^*\left(\frac{t}{\sigma}\right)\frac{d\sigma}{\sigma} = f * g(t), \end{aligned}$$

where

$$f(t) = t^{1-\theta}u^*(t) = t^{1/p}u^*(t), \qquad \text{and} \qquad g(t) = \begin{cases} t^{-\theta} & \text{if } t \geq 1 \\ 0 & \text{if } 0 \leq 1, \end{cases}$$

and the convolution is with respect to the measure $dt/t$. Since we have $\|f\,;L^q_*\| = [u;\,L^{p,q}(\Omega)]$ and $\|g\,;L^1_*\| = 1/\theta = p/(p-1)$, Young's inequality (see Paragraph 7.5) gives

$$\|u\,;L^{p,q}(\Omega)\| = \|u\|_{\theta,q;K} = \|f * g\,;L^q_*\| \leq \frac{p}{p-1}[u;\,L^{p,q}(\Omega)]. \quad \blacksquare$$

**7.29 REMARK** Working with Lorentz spaces and using the real interpolation method allows us to sharpen the cases of the Sobolev imbedding theorem where $p > 1$ and $mp < n$. In those cases, the proof in Chapter IV used Lemma 4.18, where convolution with the kernel $\omega_m$ was first shown to be of weak type $(p, p^*)$ (where $p* = np/(n-mp)$) for all such indices $p$. Then other such indices $p_1$ and $p_2$ were chosen with $p_1 < p < p_2$, and Marcinkiewicz interpolation implied that this linear convolution operator must be of strong type $(p, p^*)$.

We can instead apply the the Exact Interpolation Theorem 7.23 and Lorentz interpolation as in Corollary 7.27, to deduce, from the weak-type estimates above, that convolution with $\omega_m$ maps $L^p(\Omega)$ into $L^{p^*,p}(\Omega)$; this target space is strictly smaller than $L^{p^*}(\Omega)$, since $p < p^*$. It follows that $W^{m,p}(\Omega)$ imbeds in the smaller spaces $L^{p^*,p}(\Omega)$ when $p > 1$ and $mp < n$.

Recall too that convolution with $\omega_m$ is *not* of strong type $(1, 1^*)$ when $m < n$, but an averaging argument, in Lemma 4.24, showed that $W^{m,1}(\Omega) \subset L^{1^*}(\Omega)$ in that case. That argument can be refined as in Fournier [F] to show that in fact $W^{m,1}(\Omega) \subset L^{1^*,1}(\Omega)$ in these cases. This sharper endpoint imbedding had been proved earlier by Poornima [Po] using another method, and also in a dual form in Faris [Fa].

An ideal context for applying interpolation is one where there are apt endpoint estimates from which everything else follows. We illustrate that idea for convolution with $\omega_m$. It is easy, via Fubini's theorem, to verify that if $f \in L^1(\Omega)$ then $\|f * g_0\|_\infty \leq \|f\|_1\,\|g_0\|_\infty$ and $\|f * g_1\|_1 \leq \|f\|_1\,\|g_1\|_1$ for all functions $g_0$ in $L^\infty(\Omega)$ and $g_1$ in $L^1(\Omega)$. Fixing $f$ and interpolating between the endpoint conditions on the functions $g$ gives that $\|f * g\,;L^{p,q}(\Omega)\| \leq C_p\,\|f\|_1\,\|g\,;L^{p,q}(\Omega)\|$ for all indices $p$ and $q$ in the intervals $(1, \infty)$ and $[1, \infty]$ respectively. Apply this with $g = \omega_m$, which belongs to $L^{n/(n-m),\infty}(\Omega) = L^{1^*,\infty}(\Omega) = \text{weak-}L^{1^*}(\Omega)$ to deduce that convolution with $\omega_m$ maps $L^1(\Omega)$ into $L^{1^*,\infty}(\Omega)$. On the other hand, if $f \in L^{(1^*)',1}(\Omega) = L^{n/m,1}(\Omega)$, then

$$\begin{aligned}|\omega_m * f(x)| &\le \int_{R^n} |\omega_m(x-y)f(y)|\,dy \\ &\le \int_0^\infty (\omega_m)^*(t) f^*(t)\,dt = \int_0^\infty [t^{1/1^*}(\omega_m)^*(t)][t^{1/(1^*)'} f^*(t)]\,\frac{dt}{t} \\ &\le \left\|\omega_m\,;L^{1^*,\infty}(\Omega)\right\| \int_0^\infty \left[t^{1/(1^*)'} f^*(t)\right]\frac{dt}{t} \le C_m \left\|f\,;L^{(1^*)',1}(\Omega)\right\|.\end{aligned}$$

That is, convolution with $\omega_m$ maps $L^1(\Omega)$ into $L^{1^*,\infty}(\Omega)$ and $L^{(1^*)',1}(\Omega)$ into $L^\infty(\Omega)$. Real interpolation then makes this convolution a bounded mapping of $L^{p,q}(\Omega)$ into $L^{p^*,q}(\Omega)$ for all indices $p$ in the interval $(1,(1^*)') = (1,n/m)$ and all indices $q$ in $[1,\infty]$.

These conclusions are sharper than those coming from Marcinkiewicz interpolation. On the other hand, the latter applies to mappings of weak-type $(1,1)$, a case not covered by the $K$ and $J$ methods for Banach spaces, since weak $L^1$ is not a Banach space. The statement of the Marcinkiewicz Theorem 2.58 also applies to sublinear operators of weak-type $(p,q)$ rather than just linear operators. It is easy, however, to extend the $J$ and $K$ machinery to cover sublinear operators between $L^p$ spaces and Lorentz spaces. As above, this gives target spaces $L^{q,p}$ that are strictly smaller than $L^q$ when $p < q$. Marcinkiewicz does not apply when $p > q$, but the $J$ and $K$ methods still apply, with target spaces $L^{q,p}$ that are larger than $L^q$ in these cases.

## Besov Spaces

**7.30** The real interpolation method also applies to scales of spaces based on smoothness. For Sobolev spaces on sufficiently smooth domains the resulting intermediate spaces are called Besov spaces. Before defining them, we first establish the following theorem which shows that if $0 < k < m$, then $W^{k,p}(\Omega)$ is suitably intermediate between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ provided $\Omega$ is sufficiently regular. Since the proof requires both Theorem 5.2, for which the cone condition suffices, and the approximation property of Paragraph 5.31 which we know holds for $\mathbb{R}^n$ and by extension for any domain satisfying the strong local Lipschitz condition, which implies the cone condition, we state the theorem for domains satisfying the strong local Lipschitz condition even though it holds for some domains which do not satisfy this condition. (See Paragraph 5.31.)

**7.31 THEOREM** If $\Omega \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition and if $0 < k < m$ and $1 \le p < \infty$, then

$$W^{k,p}(\Omega) \in \mathscr{H}\big(k/m;\, L^p(\Omega),\, W^{m,p}(\Omega)\big).$$

**Proof.** In this context we deal with the function norms

$$J(t; u) = \max\{\|u\|_p \,, t\,\|u\|_{m,p}\}$$
$$K(t; u) = \inf\{\|u_0\|_p + t\,\|u_1\|_{m,p} \;:\; u = u_0 + u_1, u \in L^p(\Omega), u_1 \in W^{m,p}(\Omega)\}.$$

We must show that

$$\|u\|_{k,p} \le C\,t^{-(k/m)}\,J(t; u) \tag{16}$$
$$K(t; u) \le C\,t^{k/m}\,\|u\|_{k,p}\,. \tag{17}$$

Now Theorem 5.2 asserts that for some constant $C$ and all $u \in W^{m,p}(\Omega)$

$$\|u\|_{k,p} \le C\,\|u\|_p^{1-(k/m)}\,\|u\|_{m,p}^{k/m}\,.$$

The expression on the right side is $C$ times the minimum value of

$$t^{-k/m}\,J(t; u) = \max\{t^{-k/m}\,\|u\|_p \,, t^{1-(k/m)}\,\|u\|_{m,p}\},$$

which occurs for $t = \|u\|_p \,/\, \|u\|_{m,p}$, the value of $t$ making both terms in the maximum equal. This proves (16).

We show that (17) is equivalent to the approximation property. If $u \in W^{k,p}(\Omega)$, then

$$K(t; u) \le \|u\|_p + t\,\|0\|_{m,p} = \|u\|_p \le \|u\|_{k,p}\,.$$

Thus $t^{-k/m}K(t, u) \le \|u\|_p$ when $t \ge 1$, and inequality (17) holds in that case. If $t^{-(k/m)}K(t; u) \le C\,\|u\|_{k,p}$ also holds for $0 < t \le 1$, then since we can choose $u_0 \in L^p(\Omega)$ and $u_1 \in W^{m,p}(\Omega)$ with $u = u_0 + u_1$ and $\|u_0\|_p + t\,\|u_1\|_{m,p} \le 2K(t; u)$, we must have

$$\|u - u_1\|_p = \|u_0\|_p \le 2Ct^{k/m}\,\|u\|_{k,p} \qquad \text{and} \qquad \|u_1\|_{m,p} \le 2Ct^{(k/m)-1}\,\|u\|_{k,p}\,,$$

so that with $t = \epsilon^m$, $u_\epsilon = u_1$ is a solution of the approximation problem of Paragraph 5.31. Conversely, if the approximation problem has a solution, that is, if for each $\epsilon \le 1$ there exists $u_\epsilon \in W^{m,p}(\Omega)$ satisfying

$$\|u - u_\epsilon\|_p \le C\epsilon^k\,\|u\|_{k,p} \qquad \text{and} \qquad \|u_\epsilon\|_{m,p} \le C\epsilon^{k-m}\,\|u\|_{k,p}\,,$$

then, with $\epsilon = t^{1/m}$, we will have

$$t^{-(k/m)}K(t; u) \le t^{-(k/m)}\big(\|u - u_\epsilon\|_p + t\,\|u_\epsilon\|_{m,p}\big) \le C\,\|u\|_{k,p}$$

and (17) holds. This completes the proof. ∎

**7.32 (The Besov Spaces)** We begin with a definition of Besov spaces on general domains by interpolation.

Let $0 < s < \infty$, $1 \le p < \infty$ , and $1 \le q \le \infty$. Also let $m$ be the smallest integer larger than $s$. We define the *Besov space* $B^{s;p,q}(\Omega)$ to be the intermediate space between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ corresponding to $\theta = s/m$, specifically:

$$B^{s;p,q}(\Omega) = \big(L^p(\Omega), W^{m,p}(\Omega)\big)_{s/m,q;J}.$$

It is a Banach space with norm $\|u\,;B^{s;p,q}(\Omega)\| = \Big\|u\,;\big(L^p(\Omega), W^{m,p}(\Omega)\big)_{s/m,q;J}\Big\|$ and enjoys many other properties inherited from $L^p(\Omega)$ and $W^{m,p}(\Omega)$, for example the density of the subspace $\{\phi \in C^\infty(\Omega) : \|u\|_{m,p} < \infty\}$. Also, imposing the strong local Lipschitz property on $\Omega$ guarantees the existence of an extension operator from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^n)$ and so from $B^{s;p,q}(\Omega)$ to $B^{s;p,q}(\mathbb{R}^n)$. On $\mathbb{R}^n$, there are many equivalent definitions $B^{s;p,q}$ (see [J]), each leading to a definition of $B^{s;,p,q}(\Omega)$ by restriction. For domains with good enough extension properties, these definitions by restriction are equivalent to the definition by real interpolation. Although somewhat indirect, that definition is intrinsic. As in Remark 6.47(1), the definitions by restriction can give smaller spaces for domains without extension properties.

For domains for which the conclusion of Theorem 7.31 holds, that theorem and the Reiteration Theorem 7.21 show that, up to equivalence of norms, we get the same space $B^{s;p,q}(\Omega)$ if we use any integer $m > s$ in the definition above. In fact, if $s_1 > s$ and $1 \le q_1 \le \infty$, then

$$B^{s;p,q}(\Omega) = \big(L^p(\Omega), B^{s_1;p,q_1}(\Omega)\big)_{s/s_1,q;J}.$$

More generally, if $0 \le k < s < m$ and $s = (1-\theta)k + \theta m$, then

$$B^{s;p,q}(\Omega) = \big(W^{k,p}(\Omega), W^{m,p}(\Omega)\big)_{\theta,q;J},$$

and if $0 < s_1 < s < s_2$, $s = (1-\theta)s_1 + \theta s_2$, and $1 \le q_1, q_2 \le \infty$, then

$$B^{s;p,q}(\Omega) = \big(B^{s_1;p,q_1}(\Omega), B^{s_2;p,q_2}(\Omega)\big)_{\theta,q;J}.$$

**7.33** Theorem 7.31 also implies that for integer $m$,

$$B^{m;p,1}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,\infty}(\Omega).$$

In Paragraph 7.67 we will see that

$$\begin{aligned} &B^{m;p,p}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,2}(\Omega) && \text{for} \quad 1 < p \le 2, \\ &B^{m;p,2}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,p}(\Omega) && \text{for} \quad 2 \le p < \infty. \end{aligned}$$

The indices here are best possible; even in the case $\Omega = \mathbb{R}^n$ it is not true that $B^{m;p,q}(\Omega) = W^{m,p}(\Omega)$ for any $q$ unless $p = q = 2$.

The following imbedding theorem for Besov spaces requires only that $\Omega$ satisfy the cone condition (or even the weak cone condition) since it makes no use of Theorem 7.31.

**7.34 THEOREM (An Imbedding Theorem for Besov Spaces)** Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the cone condition, and let $1 \le p < \infty$ and $1 \le q \le \infty$.

(a) If $sp < n$, then $B^{s;p,q}(\Omega) \to L^{r,q}(\Omega)$ for $r = np/(n - sp)$.

(b) If $sp = n$, then $B^{s;p,1}(\Omega) \to C_B^0(\Omega) \to L^\infty(\Omega)$.

(c) If $sp > n$, then $B^{s;p,q}(\Omega) \to C_B^0(\Omega)$.

**Proof.** Observe that part (a) follows from part (b) and the Exact Interpolation Theorem 7.23 since if $0 < s < s_1$ and $s_1 p = n$, then (b) implies

$$B^{s;p,q}(\Omega) = \left(L^p(\Omega), B^{s_1;p,1}(\Omega)\right)_{s/s_1,q;J} \to \left(L^p(\Omega), L^\infty(\Omega)\right)_{s/s_1,q;J} = L^{r,q}(\Omega),$$

where $r = [1 - (s/s_1)]/p = np/(n - sp)$.

To prove (b) let $m$ be the smallest integer greater than $s = n/p$. Let $u \in B^{n/p;p,1}(\Omega) = \left(L^p(\Omega), W^{m,p}(\Omega)\right)_{n/(mp),1;J}$. By the discrete version of the J-method, there exist functions $u_i$ in $W^{m,p}(\Omega)$ such that the series $\sum_{i=-\infty}^{\infty} u_i$ converges to $u$ in $B^{n/p;p,1}(\Omega)$ and such that the sequence $\left\{2^{-in/mp} J(2^i; u_i)\right\}_{i=-\infty}^{\infty}$ belongs to $\ell^1$ and has $\ell^1$ norm no larger than $C \left\| u \,;\, B^{n/p;p,1}(\Omega) \right\|$. Since $mp > n$ and $\Omega$ satisfies the cone condition, Theorem 5.8 shows that

$$\|v\|_\infty \le C_1 \|v\|_p^{1-(n/mp)} \|v\|_{m,p}^{n/mp}$$

for all $v \in W^{m,p}(\Omega)$. Thus

$$\begin{aligned}
\|u\|_\infty &\le \sum_{i=-\infty}^{\infty} \|u_i\|_\infty \\
&\le C_1 \sum_{i=-\infty}^{\infty} \|u_i\|_p^{1-(n/mp)} \|u_i\|_{m,p}^{n/mp} \\
&\le C_1 \sum_{i=-\infty}^{\infty} 2^{-in/mp} J\left(2^i; u_i\right) \le C_2 \left\| u \,;\, B^{n/p;p,1}(\Omega) \right\|.
\end{aligned}$$

Thus $B^{n/p;p,1}(\Omega) \to L^\infty(\Omega)$. The continuity of $u$ follows as in the proof of Part I, Case A of Theorem 4.12 given in Paragraph 4.16.

Part (c) follows from part (b) since $B^{s;p,q}(\Omega) \to B^{s_1;p,1}(\Omega)$ if $s > s_1$. This imbedding holds because $W^{m,p}(\Omega) \to L^p(\Omega)$. ∎

## Generalized Spaces of Hölder Continuous Functions

**7.35 (The Spaces $C^{j,\lambda,q}(\overline{\Omega})$)** If $\Omega$ satisfies the strong local Lipschitz condition and $sp > n$, the Besov space $B^{s;p,q}(\Omega)$ also imbeds into an appropriate space of Hölder continuous functions. To formulate that imbedding we begin by generalizing the Hölder space $C^{j,\lambda}(\overline{\Omega})$ to allow for a third parameter. For this purpose we consider the *modulus of continuity* of a function $u$ defined on $\Omega$ given by

$$\omega(u;t) = \sup\big\{|u(x)-u(y)| \,:\, x,y\in\Omega,\ |x-y|\le t\big\}, \qquad (t>0).$$

Observe that $\omega(u;t) = \omega_\infty^*(u;t)$ in the notation of Paragraph 7.46. Also observe that if $0<\lambda\le 1$ and $t^{-\lambda}\omega(t,u)\le k<\infty$ for all $t>0$, then $u$ is uniformly continuous on $\Omega$. Since $C^j(\overline{\Omega})$ is a subspace of $W^{j,\infty}(\Omega)$ with the same norm, $C^{j,\lambda}(\overline{\Omega})$ consists of those $u\in W^{j,\infty}(\Omega)$ for which $t^{-\lambda}\omega(t,D^\alpha u)$ is bounded for all $0<t<\infty$ and all $\alpha$ with $|\alpha|=j$.

We now define the generalized spaces $C^{j,\lambda,q}(\overline{\Omega})$ as follows. If $j\ge 0$, $0<\lambda\le 1$, and $q=\infty$, then $C^{j,\lambda,\infty}(\overline{\Omega}) = C^{j,\lambda}(\overline{\Omega})$ with norm

$$\left\|u\,;C^{j,\lambda,\infty}(\overline{\Omega})\right\| = \left\|u\,;C^{j,\lambda}(\overline{\Omega})\right\| = \|u\|_{j,\infty} + \max_{|\alpha|=j}\sup_{t>0}\frac{\omega(D^\alpha u;t)}{t^\lambda}.$$

For $j\ge 0$, $0<\lambda\le 1$, and $1\le q<\infty$, the space $C^{j,\lambda,q}(\overline{\Omega})$ consists of those functions $u\in W^{j,\infty}(\Omega)$ for which $\left\|u\,;C^{j,\lambda,q}(\overline{\Omega})\right\|<\infty$, where

$$\left\|u\,;C^{j,\lambda,q}(\overline{\Omega})\right\| = \left\|u\,;C^{j}(\overline{\Omega})\right\| + \max_{|\alpha|=j}\left(\int_0^\infty \left(t^{-\lambda}\omega(D^\alpha u;t)\right)^q\frac{dt}{t}\right)^{1/q}.$$

$C^{j,\lambda,q}(\overline{\Omega})$ is a Banach space under the norm $\left\|\cdot\,;C^{j,\lambda,q}(\overline{\Omega})\right\|$.

**7.36 LEMMA** If $0<\lambda\le 1$ and $0<\theta<1$, then

$$\left(L^\infty(\Omega),C^{0,\lambda}(\overline{\Omega})\right)_{\theta,q;K}\to C^{0,\theta\lambda,q}(\overline{\Omega}).$$

**Proof.** Let $u\in C^{0,\lambda}(\overline{\Omega})_{\theta,q;K}$. Then there exists $v\in L^\infty(\Omega)$ and $w\in C^{0,\lambda}(\overline{\Omega})$ such that $u=v+w$ and

$$\|v\|_\infty + t^\lambda\left\|w\,;C^{0,\lambda}(\overline{\Omega})\right\| \le 2K(t^\lambda;u) \qquad \text{for } t>0.$$

If $|h|\le t$, then

$$\begin{aligned}|u(x+h)-u(x)| &\le |v(x+h)|+|v(x)| + \frac{|w(x+h)-w(x)|}{|h|^\lambda}\,|h|^\lambda\\ &\le 2\|v\|_\infty + \left\|w\,;C^{0,\lambda}(\overline{\Omega})\right\|t^\lambda \le 4K(t^\lambda;u).\end{aligned}$$

Thus $\omega(u;t) \le 4K(t^\lambda;u)$.
Since $\|u\|_\infty \le \left\|u\,;C^{0,\lambda}(\overline{\Omega})\right\|$, we have $\|u\|_\infty \le \|u\|_{\theta,q;K}$. Thus, if $1 \le q < \infty$,

$$\begin{aligned}
\left\|u\,;C^{0,\lambda\theta,q}(\overline{\Omega})\right\| &= \|u\|_\infty + \left(\int_0^\infty \left(t^{-\lambda\theta}\omega(u;t)\right)^q \frac{dt}{t}\right)^{1/q} \\
&\le \|u\|_{\theta,q;K} + 4\left(\int_0^\infty \left(t^{-\lambda\theta}K(t^\lambda;u)\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \|u\|_{\theta,q;K} + 4\lambda^{-1/q}\left(\int_0^\infty \left(\tau^{-\theta}K(\tau;u)\right)^q \frac{d\tau}{\tau}\right)^{1/q} \\
&\le \left(1+4\lambda^{-1/q}\right)\|u\|_{\theta,q;K}\,.
\end{aligned}$$

Similarly, for $q = \infty$, we obtain

$$\left\|u\,;C^{0,\lambda\theta,\infty}(\overline{\Omega})\right\| \le \|u\|_{\theta,\infty;K} + 4\sup_t t^{-\lambda\theta}K(t^\lambda;u) \le 5\,\|u\|_{\theta,\infty;K}\,.$$

This completes the proof. ∎

**7.37 THEOREM** Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition. Let $m-1-j \le n/p < s \le m-j$ and $1 \le q \le \infty$. If $\mu = s - n/p$, then

$$B^{s;p,q}(\Omega) \to C^{j,\mu,q}(\overline{\Omega}).$$

**Proof.** It is sufficient to prove this for $j = 0$. By Theorem 7.34(b),

$$B^{n/p;p,1}(\Omega) \to C_B^0(\Omega) \to L^\infty(\Omega).$$

By Part II of Theorem 4.12,

$$W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega}), \qquad \text{where} \quad \lambda = m - \frac{n}{p}.$$

Now $B^{s;p,q}(\Omega) = \left(B^{n/p;p,1}(\Omega),\, W^{m,p}(\Omega)\right)_{\theta,q;K}$, where

$$(1-\theta)\frac{n}{p} + \theta m = s.$$

Since $\lambda\theta = \mu$, we have by the Exact Interpolation Theorem and the previous Lemma,

$$B^{s;p,q}(\Omega) \to \left(L^\infty(\Omega),\, C^{0,\lambda}(\overline{\Omega})\right)_{\theta,q;K} \to C^{j,\mu,q}(\overline{\Omega}). \quad ∎$$

## Characterization of Traces

**7.38** As shown in the Sobolev imbedding theorem (Theorem 4.12) functions in $W^{m,p}(\mathbb{R}^{n+1})$ (where $mp < n+1$) have traces on $\mathbb{R}^n$ that belong to $L^q(\mathbb{R}^n)$ for $p \le q \le np/(n+1-mp)$. The following theorem asserts that these traces are exactly the functions that belong to $B^{m-(1/p);p,p}(\mathbb{R}^n)$. This is an instance of the phenomenon that passing from functions in $W^{m,p}(\Omega)$ to their traces on surfaces of codimension 1 results in a loss of smoothness corresponding to $1/p$ of a derivative.

In the following we denote points in $\mathbb{R}^{n+1}$ by $(x,t)$ where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The trace $u(x)$ of a smooth function $U(x,t)$ defined on $\mathbb{R}^{n+1}$ is therefore given by $u(x) = U(x,0)$.

**7.39 THEOREM (The Trace Theorem)** If $1 < p < \infty$, the following conditions on a measurable function $u$ on $\mathbb{R}^n$ are equivalent.

(a) There is a function $U$ in $W^{m,p}(\mathbb{R}^{n+1})$ so that $u$ is the trace of $U$.

(b) $u \in B^{m-(1/p);p,p}(\mathbb{R}^n)$. ∎

As the proof of this theorem is rather lengthy, we split it into two lemmas; (a) implies (b) and (b) implies (a).

**7.40 LEMMA** Let $1 < p < \infty$. If $U \in W^{m,p}(\mathbb{R}^{n+1})$, then its trace $u$ belongs to the space $B = B^{m-(1/p);p,p}(\mathbb{R}^n)$ and

$$\|u\|_B \le K\,\|U\|_{m,p,\mathbb{R}^{n+1}}, \tag{18}$$

for some constant $K$ independent of $U$.

**Proof.** We represent

$$B \equiv B^{m-(1/p);p,p}(\mathbb{R}^n) = \big(W^{m-1,p}(\mathbb{R}^n),\, W^{m,p}(\mathbb{R}^n)\big)_{\theta,p;J},$$

where

$$\theta = 1 - \frac{1}{p} = \frac{1}{p'}$$

and use the discrete version of the J-method; we have $u \in B^{m-(1/p);p,p}(\mathbb{R}^n)$ if and only if there exist functions $u_i$ in $W^{m-1,p}(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ for $-\infty < i < \infty$ such that the series $\sum_{i=-\infty}^{\infty} u_i$ converges to $u$ in norm in the space $W^{m-1,p}(\mathbb{R}^n) + W^{m,p}(\mathbb{R}^n) = W^{m-1,p}(\mathbb{R}^n)$, and such that the sequences $\{2^{-i/p'}\|u_i\|_{m-1,p}\}$ and $\{2^{i/p}\|u_i\|_{m,p}\}$ both belong to $\ell^p$. We verify (18) by splitting $U$ into pieces $U_i$ with traces $u_i$ that satisfy these conditions.

Let $\Phi$ be an even function on the real line satisfying the following conditions:

(i) $\Phi(t) = 1$ if $-1 \le t \le 1$,

(ii) $\Phi(t) = 0$ if $|t| \ge 2$,

(iii) $|\Phi(t)| \le 1$ for all $t$,

(iv) $|\Phi^{(j)}(t)| \le C_j < \infty$ for all $j \ge 1$ and all $t$.

For each integer $i$ let $\Phi_i(t) = \Phi(t/2^i)$; then $\Phi_i$ takes the value 1 on the interval $[-2^i, 2^i]$ and takes the value 0 on the intervals $[2^{i+1}, \infty)$ and $(-\infty, -2^{i+1}]$. Also, $|\Phi(t)| \le 1$ and $|\Phi_i'(t)| \le 2^{-i}C_1$ for all $t$.

Let $\phi_i = \Phi_{i+1} - \Phi_i$. Then $\phi_i(\tau)$ vanishes outside the open intervals $(2^i, 2^{i+2})$ and $(-2^{i+2}, -2^i)$; in particular it vanishes at the endpoints of these intervals. Also $\|\phi_i\|_\infty = 1$ and $\left\|\phi_i'\right\|_\infty \le 2^{-i}C_1$.

Now suppose that $U \in C_0^\infty(\mathbb{R}^{n+1})$. Then for each $t$ we have

$$U(x,t) = -\int_t^\infty \frac{\partial U}{\partial \tau}(x,\tau)\,d\tau = -\int_t^\infty D^{(0,1)}U(x,\tau)\,d\tau.$$

Let

$$U_i(x,t) = -\int_t^\infty \phi_i(\tau)D^{(0,1)}U(x,\tau)\,d\tau.$$

Let $u(x) = U(x,0)$ be the trace of $U$ on $\mathbb{R}^n$, and let $u_i$ be the corresponding trace of $U_i$. Since $U$ has compact support, the functions $U_i$ and $u_i$ vanish when $i$ is sufficiently large. Moreover, $U_i(x,t) = 0$ for all $i$ when $|x|$ is sufficiently large. Therefore the trace $u$ vanishes except on a compact set, on which the series $\sum_{i=\infty}^{\infty} u_i(x)$ converges uniformly to $u(x)$. The terms in this series also vanish off that compact set and taking any partial derivative term-by-term gives a series that converges uniformly on that compact set to the corresponding partial derivative of $u$.

We use two representations of $u_i(x) = U_i(x,0)$, namely

$$u_i(x) = -\int_{2^i}^{2^{i+2}} \phi_i(\tau)D^{(0,1)}U(x,\tau)\,d\tau = \int_{2^i}^{2^{i+2}} \phi_i'(\tau)U(x,\tau)\,d\tau, \tag{19}$$

where the second expression follows from the first by integration by parts. If $|\alpha| \le m-1$ we obtain from the first representation a corresponding representations of $D^\alpha u_i(x)$:

$$D^\alpha u_i(x) = -\int_{2^i}^{2^{i+2}} \phi_i(\tau)D^{(\alpha,1)}U(x,\tau)\,d\tau,$$

so that, by Hölder's inequality,

$$|D^\alpha u_i(x)| \le (2^{i+2})^{1/p'}\left(\int_{2^i}^{2^{i+2}} \left|D^{(\alpha,1)}U(x,\tau)\right|^p d\tau\right)^{1/p}.$$

Each positive number $\tau$ lies in exactly two of the intervals $[2^i, 2^{i+1})$ over which the integrals above run. Multiplying by $2^{-i/p'}$, taking $p$-th powers on both sides, summing with respect to $i$, and integrating $x$ over $\mathbb{R}^n$ shows that the $p$-th power of the $\ell^p$ norm of the sequence $\left\{2^{-i/p'}\|D^\alpha u_i\|_p\right\}_{i=-\infty}^{\infty}$ is no larger than

$$2^{1+2p/p'}\int_{R_+^{n+1}}\left|D^{(\alpha,1)}U(x,\tau)\right|^p\,d\tau\,dx.$$

Thus that $\ell^p$ norm is bounded by a constant times $\|U\|_{m,p,\mathbb{R}^{n+1}}$.

Using the second representation of $u_i$ in (19), our bound on $\|\phi_i'\|_\infty$, and Hölder's inequality gives us a second estimate

$$|D^\alpha u_i(x)| \le 2^{-i}C_1(2^{i+2})^{1/p'}\left(\int_{2^i}^{2^{i+2}}|D^{(\alpha,0)}U(x,\tau)|^p\,d\tau\right)^{1/p},$$

this one valid for any $\alpha$ with $|\alpha| \le m$. Multiplying by $2^{i/p}$, taking $p$-th powers on both sides, and summing with respect to $i$ shows that the $p$-th power of the $\ell^p$ norm of the sequence $\left\{2^{i/p}\|D^\alpha u_i\|_p\right\}_{i=-\infty}^{\infty}$ is no larger than

$$2^{1+2p/p'}C_1^p\int_{R_+^{n+1}}|D^{(\alpha,0)}U(x,\tau)|^p\,d\tau\,dx.$$

Thus that $\ell^p$ norm is also bounded by a constant times $\|U\|_{m,p,\mathbb{R}^{n+1}}$.

Together, these estimates show that the norm of $u$ in $B^{m-(1/p);p,p}(\mathbb{R}^n)$ is bounded by a constant times the norm of $U$ in $W^{m,p}(\mathbb{R}^{n+1})$ whenever $U \in C_0^\infty(\mathbb{R}^{n+1})$. Since the latter space is dense in $W^{m,p}(\mathbb{R}^{n+1})$, the proof is complete. ∎

**7.41 LEMMA** Let $1 < p < \infty$ and $B = B^{m-(1/p);p,p}(\mathbb{R}^n)$. If $u \in B$, then $u$ is the trace of a function $U \in W^{m,p}(\mathbb{R}^{n+1})$ satisfying

$$\|U\|_{m,p,\mathbb{R}^{n+1}} \le K\,\|u\|_B \tag{20}$$

for some constant $K$ independent of $u$.

**Proof.** In this proof it is convenient to use a characterization of $B$ different (if $m > 1$) from the one used in the previous lemma, namely

$$B = B^{m-(1/p);p,p}(\mathbb{R}^n) = \left(L^p(\mathbb{R}^n),\,W^{m,p}(\mathbb{R}^n)\right)_{\theta,p;J},$$

where $\theta = 1-(1/mp)$. Again we use the discrete version of the J-method. For $u \in B$ we can find $u_i \in L^p(\mathbb{R}^n)\cap W^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ (for $-\infty < i < \infty$)

such that $\sum_{i=-\infty}^{\infty} u_i$ converges to $u$ in $L^p(\mathbb{R}^n) + W^{m,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and such that

$$\left\| \left\{2^{-\theta i} \|u_i\|_p\right\} ; \ell^p \right\| \le K_1 \|u\|_B ,$$
$$\left\| \left\{2^{(1-\theta)i} \|u_i\|_{m,p}\right\} ; \ell^p \right\| \le K_1 \|u\|_B .$$

These estimates imply that $\sum_{i=-\infty}^{\infty} u_i$ converges to $u$ in $B$. We will construct an extension $U(x,t)$ of $u(x)$ defined on $\mathbb{R}^{n+1}$ such that (20) holds.

It is sufficient to extend the partial sums $s_k = \sum_{i=-k}^{k} u_i$ to $S_k$ on $\mathbb{R}^{n+1}$ with control of the norms:

$$\|S_k\|_{m,p,\mathbb{R}^{n+1}} \le K_1 \left\| s_k \,;\, B^{m-(1/p);p,p}(\mathbb{R}^n)\right\| ,$$

since $\{S_k\}$ will then be a Cauchy sequence in $W^{m,p}(\mathbb{R}^{n+1})$ and so will converge there. Furthermore, we can assume that the functions $u$ and $u_i$ are smooth since the mollifiers $J_\epsilon * u$ and $J_\epsilon * u_i$ (as considered in Paragraphs 2.28 and 3.16) converge to $u$ and $u_i$ in norm in $W^{m,p}(\mathbb{R}^n)$ as $\epsilon \to 0+$. Accordingly, therefore, in the following construction we assume that the functions $u$ and $u_i$ are smooth and that all but finitely many of the $u_i$ vanish identically on $\mathbb{R}^n$.

Let $\Phi(t)$ be as defined in the previous lemma. Here, however, we redefine $\Phi_i$ as follows:

$$\Phi_i(t) = \Phi\left(\frac{t}{2^{i/m}}\right), \qquad -\infty < i < \infty.$$

The derivatives of $\Phi_i$ then satisfy $|\Phi_i^{(j)}(t)| \le 2^{-ij/m} C_j$. Also, note that for $j \ge 1$, $\Phi_i^{(j)}$ is zero outside the two intervals $(-2^{(i+1)/m}, -2^{i/m})$ and $(2^{i/m}, 2^{(i+1)/m})$, which have total length not exceeding $2^{1+(i/m)}$.

We define the extension of $u$ as follows:

$$U(x,t) = \sum_{i=-\infty}^{\infty} U_i(x,t), \qquad \text{where} \qquad U_i(x,t) = \Phi_i(t) u_i(x).$$

Note that the sum is actually a finite one under the current assumptions. In order to verify (20) it is sufficient to bound by multiples of $\|u\|_B$ the $L^p$-norms of $U$ and all its $m$th order derivatives; the Ehrling-Nirenberg-Gagliardo interpolation theorem 5.2 then supplies similar bounds for intermediate derivatives. The $m$th order derivatives are of three types: $D^{(0,m)}U$, $D^{(\alpha,j)}U$ for $1 \le j \le m-1$ and $|\alpha| + j = m$, and $D^{(\alpha,0)}U$ for $|\alpha| = m$. We examine each in turn.

Since $D^{(0,m)}U_i(x,t) = 2^{-i}\Phi^{(m)}(t/2^{i/m})u_i(x)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |D^{(0,m)}U_i(x,t)|^p \, dx\, dt \\
&\le \left(\int_{-2^{(i+1)/m}}^{-2^{i/m}} dt + \int_{2^{i/m}}^{2^{(i+1)/m}} dt\right) \int_{\mathbb{R}^n} |D^{(0,m)}U_i(x,t)|^p \, dx \\
&\le 2^{1+(i/m)} 2^{-ip} C_m^p \|u_i\|_p^p = 2C_m^p 2^{-\theta i p} \|u_i\|_p^p .
\end{aligned}$$

Since the functions $\Phi_i^{(m)}$ have non-overlapping supports, we can sum the above inequality on $i$ to obtain

$$\begin{aligned}\left\| D^{(0,m)}U \right\|_{p,\mathbb{R}^{n+1}}^p &\le 2C_m^p \sum_{i=-\infty}^{\infty} \left(2^{-\theta i} \|u_i\|_p\right)^p \\ &= 2C_m^p \left\| \left\{2^{-\theta i} \|u_i\|_p\right\} ; \ell^p \right\|^p \le 2C_m^p \|u\|_B^p\end{aligned}$$

and the required estimate for $D^{(0,m)}U$ is proved.

Now consider $D^{(\alpha,j)}U_i(x,t) = 2^{-ij/m}\Phi^{(j)}(t/2^{i/m})D^\alpha u_i(x)$ for which we obtain similarly

$$\int_{\mathbb{R}^{n+1}} |D^{(\alpha,j)}U_i(x,t)|^p \, dx\, dt \le C_j^p 2^{-i(jp-1)/m} \|D^\alpha u_i\|_p^p .$$

Since $|\alpha| = m - j$, we can replace the $L^p$-norm of $D^\alpha u_i$ with the seminorm $|u_i|_{m-j,p}$, and again using the non-overlapping of the supports of the $\Phi_i^{(j)}$ (since $j \ge 1$) to get

$$\left\| D^{(\alpha,j)}U \right\|_{p,\mathbb{R}^{n+1}}^p \le C_j^p \sum_{i=-\infty}^{\infty} 2^{-i(jp-1)/m} |u_i|_{m-jp}^p .$$

As remarked in Paragraph 5.7, for $1 \le j \le m-1$ Theorem 5.2 assures us that there exists a constant $K_2$ such that for any $\epsilon > 0$ and any $i$

$$|u_i|_{m-j,p}^p \le K_2\left(\epsilon^p |u_i|_{m,p}^p + \epsilon^{-(m-j)p/j} \|u\|_p^p\right).$$

Let $\epsilon = 2^{ij/m}$. Then we have

$$\begin{aligned}\left\| D^{(\alpha,j)}U \right\|_{p,\mathbb{R}^{n+1}}^p &\le C_j^p K_2 \sum_{i=-\infty}^{\infty} \left(2^{i/m} |u_i|_{m,p}^p + 2^{-ip(1-(1/mp))} \|u_i\|_p^p\right) \\ &= C_j^p K_2 \sum_{i=-\infty}^{\infty} \left(2^{(1-\theta)ip} |u_i|_{m,p}^p + 2^{-\theta ip} \|u_i\|_p^p\right) \\ &\le C_j^p K_2 \left( \left\| \left\{2^{(1-\theta)i} \|u_i\|_{m,p}\right\} ; \ell^p \right\|^p + \left\| \left\{2^{-\theta i} \|u_i\|_p\right\} ; \ell^p \right\| \right) \\ &\le 2K_1^p C_j^p K_2 \|u\|_B\end{aligned}$$

and the bound for $D^{(\alpha,j)}U$ is proved.

Finally, we consider $U$ and $D^{(\alpha,0)}U$ together. (We allow $0 \le |\alpha| \le m$.) Unlike their derivatives, the functions $\Phi_i$ have nested rather than non-overlapping supports. We must proceed differently than in the previous cases. Consider

$D^{(\alpha,0)}U(x,t)$ on the strip $2^{j/m} < t \le 2^{(j+1)/m}$ in $\mathbb{R}^{n+1}$. Since $|\Phi_i(t)| \le 1$ and since $U_i(x,t) = 0$ on this strip if $i < j-1$, we have

$$|D^{(\alpha,0)}U(x,t)| \le \sum_{i=j-1}^{\infty} |D^{(\alpha,0)}U_i(x,t)| = \sum_{i=j-1}^{\infty} 2^{-i/mp}a_i,$$

where $a_i = 2^{i/mp}|D^\alpha u_i(x)|$. Thus,

$$\begin{aligned} b_j \equiv \left(\int_{2^{j/m}}^{2^{(j+1)/m}} |D^{(\alpha,0)}U(x,t)|^p\,dt\right)^{1/p} &\le \sum_{i=j-1}^{\infty} 2^{j/mp}2^{-i/mp}\,a_i \\ &= \sum_{i=j-1}^{\infty} 2^{(j-i)/mp}a_i = (c*a)_j, \end{aligned}$$

where $c_j = 2^{j/mp}$ when $-\infty < j \le 1$ and $c_j = 0$ otherwise. Observe that $c \in \ell^1$ (say, $\|c\,;\ell^1\| = K_3$), and so by Young's inequality for sequences

$$\|b\,;\ell^p\| \le K_3\,\|a\,;\ell^p\|.$$

Taking $p$th powers and summing on $j$ now leads to

$$\int_0^\infty \left|D^{(\alpha,0)}U(x,t)\right|^p dt \le K_3^p\,\left\|\left\{2^{i/mp}|D^\alpha u_i(x)|\right\};\ell^p\right\|^p.$$

Integrating $x$ over $\mathbb{R}^n$ and taking $p$th roots then gives

$$\begin{aligned} \left\|D^{(\alpha,0)}U\right\|_{0,p,\mathbb{R}^{n+1}_+} &\le K_3\left\|\left\{2^{i/mp}\,\|D^\alpha u_i\|_p\right\};\ell^p\right\| \\ &\le K_3\left\|\left\{2^{(1-\theta)i}\,\|u_i\|_{m,p}\right\};\ell^p\right\| \le K_1K_3\,\|u\|_B. \end{aligned}$$

A similar estimate holds for $\left\|D^{(\alpha,0)}U\right\|_{0,p,\mathbb{R}^{n+1}_-}$, so the proof is complete. ∎

**7.42** We can now complete the imbedding picture for Besov spaces by proving an analog of the trace imbedding part of the Sobolev Imbedding Theorem 4.12 for Besov spaces. We will show in Lemma 7.44 below that the trace operator $T$ defined for smooth functions $U$ on $\mathbb{R}^{n+1}$ by

$$(TU)(x) = U(x,0)$$

is linear and bounded from $B^{1/p;p,1}(\mathbb{R}^{n+1})$ into $L^p(\mathbb{R}^n)$. Since Theorem 7.39 assures us that $T$ is also bounded from $W^{m,p}(\mathbb{R}^{n+1})$ onto $B^{m-1/p;p;p}(\mathbb{R}^n)$ for every $m \geq 1$, by the exact interpolation theorem (Theorem 7.23), it is bounded from $B^{s;p,q}(\mathbb{R}^{n+1})$ into $B^{s-1/p;p;q}(\mathbb{R}^n)$, that is,

$$B^{s;p,q}(\mathbb{R}^{n+1}) \to B^{s-1/p;p;q}(\mathbb{R}^n),$$

for every $s > 1/p$ and $1 \leq q \leq \infty$. (Although Theorem 7.39 does not apply if $p = 1$, we already know from the Sobolev Theorem 4.12 that traces of functions in $W^{m,1}(\mathbb{R}^{n+1})$ belong to $W^{m-1,1}(\mathbb{R}^n)$.)

We can now take traces of traces. If $n - k < sp < n$ (so that $s - (n-k)/p > 0$), then

$$B^{s;p,q}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;q}(\mathbb{R}^k),$$

We can combine this imbedding with Theorem 7.34 to obtain for $n - k < sp < n$ and $r = kp/(n - sp)$,

$$B^{s;p,p}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;p}(\mathbb{R}^k) \to L^{r,p}(\mathbb{R}^k) \to L^r(\mathbb{R}^k).$$

More generally:

**7.43 THEOREM (Trace Imbeddings for Besov Spaces on $\mathbb{R}^n$)** If $k$ is an integer satisfying $1 < k < n$, $n - k < sp < n$, and $r = kp/(n - sp)$, then

$$\begin{aligned} &B^{s;p,q}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;q}(\mathbb{R}^k) \to L^{r,q}(\mathbb{R}^k), \quad \text{and} \\ &B^{s;p,q}(\mathbb{R}^n) \to L^r(\mathbb{R}^k) \quad \text{for } q \leq r. \end{aligned}$$

To establish this theorem, we need only prove the following lemma.

**7.44 LEMMA** The trace operator $T$ defined by $(TU)(x) = U(x, 0)$ imbeds $B^{1/p;p,1}(\mathbb{R}^{n+1})$ into $L^p(\mathbb{R}^n)$.

**Proof.** Suppose that $U$ belongs to $B \equiv B^{1/p;p,1}(\mathbb{R}^{n+1})$ and, without loss of generality, that $\|U\|_B \leq 1$. Then there exist functions $U_i$ for $-\infty < i < \infty$ such that $U = \sum_i U_i$ and

$$\sum_i 2^{-i/p} \|U_i\|_{p,\mathbb{R}^{n+1}} \leq C \qquad \text{and} \qquad \sum_i 2^{i/p'} \|U_i\|_{1,p,\mathbb{R}^{n+1}} \leq C$$

for some constant $C$. As in the proof of Lemma 7.40, we can assume that only finitely many of the functions $U_i$ have nonzero values and that they are smooth functions. For any of these functions we have, for $2^i \leq h \leq 2^{i+1}$,

$$\begin{aligned} |U_i(x, 0)| &\leq \int_0^h \left|D^{(0,1)} U_i(x, t)\right| dt + |U_i(x, h)| \\ &\leq \int_0^{2^{i+1}} \left|D^{(0,1)} U_i(x, t)\right| dt + |U_i(x, h)|. \end{aligned}$$

Averaging $h$ over $[2^i, 2^{i+1}]$ then gives the estimate

$$|U_i(x,0)| \le \int_0^{2^{i+1}} \left|D^{(0,1)}U_i(x,t)\right| dt + \frac{1}{2^i}\int_{2^i}^{2^{i+1}} |U_i(x,t)|\, dt.$$

By Hölder's inequality,

$$\begin{aligned}
|U_i(x,0)| &\le 2^{(i+1)/p'}\left(\int_0^{2^{i+1}} \left|D^{(0,1)}U_i(x,t)\right|^p dt\right)^{1/p} \\
&\quad + \frac{2^{i/p'}}{2^i}\left(\int_{2^i}^{2^{i+1}} |U_i(x,t)|^p\, dt\right)^{1/p} \\
&= a_i(x) + b_i(x), \quad \text{say.}
\end{aligned}$$

Then $\|a_i\|_{p,\mathbb{R}^n} \le 2(2^{i/p'})\left\|U_j\right\|_{1,p,\mathbb{R}^{n+1}}$ and $\|b_i\|_{p,\mathbb{R}^n} \le 2^{-i/p}\left\|U_j\right\|_{p,\mathbb{R}^{n+1}}$. We now have

$$\begin{aligned}
\|U(\cdot,0)\|_{p,\mathbb{R}^n} &\le \sum_i \|U_i(\cdot,0)\|_{p,\mathbb{R}^n} \\
&\le 2\left(\sum_i 2^{i/p'}\left\|U_j\right\|_{1,p,\mathbb{R}^{n+1}} + \sum_i 2^{-i/p}\left\|U_j\right\|_{p,\mathbb{R}^{n+1}}\right) \le 4C.
\end{aligned}$$

This completes the proof. ∎

**7.45 REMARKS**

1. Theorems 7.39 and 7.43 extend to traces on arbitrary planes of sufficiently high dimension, and, as a consequence of Theorem 3.41, to traces on sufficiently smooth surfaces of sufficiently high dimension.
2. Both theorems also extend to traces of functions in $B^{s;p,q}(\Omega)$ on the intersection of the domain $\Omega$ in $\mathbb{R}^n$ with planes or smooth surfaces of dimension $k$ satisfying $k > n - sp$, provided there exists a suitable extension operator for $\Omega$. This will be the case if, for example, $\Omega$ satisfies a strong local Lipschitz condition. (See Theorem 5.21.)
3. Before Besov spaces were fully developed, Gagliardo [Ga3] identified the trace space as a space defined by a version of the intrinsic condition (c) in the characterization of Besov spaces in Theorem 7.47 below, where $q = p$ and $s = m - (1/p)$.

## Direct Characterizations of Besov Spaces

**7.46** The $K$ functional for the pair $\left(L^p(\Omega), W^{m,p}(\Omega)\right)$ measures how closely a given function $u$ can be approximated in $L^p$ norm by functions whose $W^{m,p}$ norm are not too large. For instance, a splitting $u = u_0 + u_1$ with $\|u_0\|_p + t\,\|u_1\|_{m,p} \le 2K(t;u)$ provides such an approximation $u_1$ to $u$; then the error $u - u_1 = u_0$ has $L^p(\Omega)$ norm at most $2K(t;u)$ and the approximation $u_1$ has $W^{m,p}(\Omega)$ norm at most $(2/t)K(t;u)$. So, in principle, the definition of $B^{s;p,q}(\Omega)$ by real interpolation characterizes functions in $B^{s;p,q}(\Omega)$ by the way in which they can be approximated in $L^p(\Omega)$ norm by functions in $W^{m,p}(\Omega)$.

Like many other descriptions of Besov spaces, the one above seems indirect, but it can yield useful upper bounds for Besov norms. On $\mathbb{R}^n$, more direct characterizations come from considering the $L^p$*-modulus of continuity* and higher-order versions of that modulus. Given a point $h$ in $\mathbb{R}^n$ and a function $u$ in $L^p(\mathbb{R}^n)$, let $u_h$ be the function mapping $x$ to $u(x-h)$, let $\Delta_h u = u - u_h$, let $\omega_p(u;h) = \|\Delta_h u\|_p$, and for positive integers $m$, let $\omega_p^{(m)}(u;h) = \|(\Delta_h)^m u\|_p$. When $1 \le p < \infty$, mollification shows that $\omega_p(u;h)$ tends to 0 as $h \to 0$, and the same is true for $\omega_p^{(m)}(u;h)$; as stated below, when $m > s$, the rate of the latter convergence to 0 determines whether $u \in B^{s;p,q}(\mathbb{R}^n)$. We also define functions on $\mathbb{R}_+$ by letting $\omega_p^*(u;t) = \sup\{\omega_p(u;h);\, |h| \le t\}$ and letting $\omega_p^{(m)*}(u;t) = \sup\{\omega_p^{(m)}(u;h);\, |h| \le t\}$.

**7.47 THEOREM (Intrinsic Characterization of $B^{s;p,q}(\mathbb{R}^n)$)** Whenever $m > s > 0$, $1 < p < \infty$, and $1 \le q < \infty$, the following conditions on a function $u$ in $L^p(\mathbb{R}^n)$ are equivalent. If $q = \infty$ condition (a) is equivalent to the versions of conditions (b) and (c) with the integrals replaced by the suprema of the quantities inside the square brackets.

(a) $u \in B^{s;p,q}(\mathbb{R}^n)$.

(b) $\displaystyle\int_0^\infty \left[t^{-s}\omega_p^{(m)*}(u;t)\right]^q \frac{dt}{t} < \infty.$

(c) $\displaystyle\int_{R^n} \left[|h|^{-s}\omega_p^{m}(u;h)\right]^q \frac{dh}{|h|^n} < \infty.$ ∎

Before proving this theorem, we observe a few things. First, the moduli of continuity in parts (b) and (c) are never larger than $2^m\,\|u\|_p$; so we get conditions equivalent to (b) and (c) respectively if we use integrals with $t \le 1$ and $|h| \le 1$. Next, the equivalence of conditions (b) and (c) with condition (a), where $m$ does not appear, means that if (b) or (c) holds for some $m > s$, then both conditions hold for all $m > s$.

It follows from our later discussion of Fourier decompositions that if $1 < p < \infty$, then these conditions are equivalent to requiring that the derivatives of $u$ of order $k$, where $k$ is the largest integer less than $s$, belong to $L^p(\mathbb{R}^n)$ and satisfy the versions of condition (b) or (c) with $m = 1$ and $s$ replaced with $s - k$.

While we assumed $1 < p < \infty$ in the statement of the theorem, the only part of the proof that requires this is the part showing that (c) $\Rightarrow$ (a) when $m > 1$. The rest of the proof is valid for $1 \le p \le \infty$.

**7.48 (The Proof of Theorem 7.47 for $m$ = 1)** We assume, for the moment, that $m = 1$ and $s < 1$; in the next Paragraph we will outline with rather less detail how to modify the argument for the case $m > 1$. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

The first part is similar to the proof of Lemma 7.36. Suppose first that condition (a) holds and consider condition (b) with $m = 1$. Fix a positive value of the parameter $t$ and split a nontrivial function $u$ as $v + w$ with $\|v\|_p + t\|w\|_{1,p} \le 2K(t; u)$. Then $\Delta_h u = \Delta_h v + \Delta_h w$, and it suffices to control the $L^p$ norms of the these two differences. For the first term, just use the fact that $\|\Delta_h v\|_p \le 2\|u\|_p$.

For the second term, we use mollification to replace $v$ and $w$ with smooth functions satisfying the same estimate on their $L^p$ and $W^{1,p}$ norms respectively. We majorize $|w(x-h) - w(x)|$ by the integral of $|\operatorname{grad} w|$ along the line segment joining $x - h$ to $x$, and use Hölder's inequality to majorize that by $|h|^{1/p'}$ times the one-dimensional $L^p$ norm of the restriction of $|\operatorname{grad} w|$ to that segment. Finally, we take $p$-th powers, integrate with respect to $x$, and take a $p$-th root to get that $\|\Delta_h w\|_p \le |h||w|_{1,p}$. When $|h| \le t$ we then obtain

$$\|\Delta_h u\|_p \le \|\Delta_h v\|_p + \|\Delta_h w\|_p \le 2\|v\|_p + t|w|_{1,p} \le 4K(t; u),$$

so condition (a) implies condition (b).

Since $t^{-s}$ decreases and $\omega_p(u; t)$ increases as $t$ increases, condition (b) holds with $m = 1$ if and only if the sequence $\{2^{-is}\omega_p^*(u; 2^i)\}_{i=-\infty}^{\infty}$ belongs to $\ell^q$. To deduce condition (c) with $m = 1$, we split the integral in (c) into dyadic pieces with $2^i < |h| \le 2^{i+1}$. The integral of the measure $dh/|h|^n$ over each such piece is the same. In the $i$-th piece, $\omega_p(u; h) \le \omega_p^*(u; 2^{i+1})$ by the definition of the latter quantity. And in that piece, $|h|^{-s} \le 2^s 2^{-s(i+1)}$. So the integral in (c) is majorized by a constant time the $q$-th power of the $\ell^q$ norm of the sequence $\{2^{-(i+1)s}\omega_p^*(f; 2^{(i+1)})\}_{i=-\infty}^{\infty}$, and (c) follows from (b).

We now show that (c) $\Rightarrow$ (a) when $m = 1 > s > 0$. Choose a nonnegative smooth function $\Phi$ vanishing outside the ball of radius 2 centred at 0 and inside the ball of radius 1, and satisfying

$$\int_{R^n} \Phi(x)\, dx = 1.$$

For fixed $t > 0$ let $\Phi_t(x) = t^{-n}\Phi(x/t)$; this nonnegative function also integrates to 1, and it vanishes outside the ball of radius $2t$ centred at 0 and inside the ball of radius $t$.

For $u$ satisfying condition (c), split $u = v + w$ where $w = u * \Phi_t$ and $v = u - w$.

The fact that the density $\Phi_t$ has mass 1 ensures that

$$v(x) = \int_{\mathbb{R}^n} \Phi_t(h)[u(x) - u(x-h)]\,dh = \int_{R^n} \Phi_t(h)\Delta_h u(x)\,dh$$
$$= \int_{t<|h|<2t} \Phi_t(h)\Delta_h u(x)\,dh.$$

The function $v$ belongs to $L^p(\mathbb{R}^n)$, being the difference of two functions in that space. To estimate its norm, we use the converse of Hölder's inequality to linearize that norm as the supremum of $\int_{\mathbb{R}^n} |v(x)|g(x)\,dx$ over all nonnegative functions $g$ in the unit ball of $L^{p'}(\mathbb{R}^n)$. For each such function $g$, we find that

$$\int_{\mathbb{R}^n} |v(x)|g(x)\,dx \le \int_{t<|h|<2t} \Phi_t(h)\left[\int_{\mathbb{R}^n} g(x)|\Delta_h u(x)|\,dx\right] dh$$
$$\le \int_{t<|h|<2t} \Phi_t(h)\|g\|_{p'}\|\Delta_h u\|_p\,dh$$
$$= \int_{t<|h|<2t} \Phi_t(h)\|\Delta_h u\|_p\,dh.$$

Since $\|\Phi_t\|_\infty \le C/t^n$, the last integral above is in turn bounded above by

$$\frac{C}{t^n}\int_{t<|h|<2t} \|\Delta_h u\|_p\,dh \le C\int_{t<|h|<2t} \|\Delta_h u\|_p \frac{dh}{|h|^n}$$
$$\le C_q\left(\int_{t<|h|<2t} [\|\Delta_h u\|_p]^q \frac{dh}{|h|^n}\right)^{1/q},$$

where the last step uses Hölder's inequality and the fact that the coronas $\{h \in R^n : t < |h| < 2t\}$ all have the same measure. Thus we have shown that

$$\|v\|_p \le C_q\left(\int_{t<|h|<2t} [\|\Delta_h u\|_p]^q \frac{dh}{|h|^n}\right)^{1/q}. \tag{21}$$

To bound $K(t;u)$ for the interpolation pair $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))$, we also require a bound for $\|w\|_{1,p} = \|u * \Phi_t\|_{1,p}$. Note that $\|w\|_p \le \|u\|_p\|\Phi_t\|_1 = \|u\|_p$. Moreover,

$$\operatorname{grad} w(x) = [u * \operatorname{grad}(\Phi_t)](x) = \int_{t<|h|<2t} u(x-h)\operatorname{grad}(\Phi_t)(h)\,dh$$
$$= \int_{t<|h|<2t} \big[u(x-h) - u(x)\big]\operatorname{grad}(\Phi_t)(h)\,dh$$
$$= \int_{t<|h|<2t} \Delta_h u(x)\operatorname{grad}(\Phi_t)(h)\,dh,$$

where we used the fact that the average value of $\nabla(\Phi_t)(h)$ is $\mathbf{0}$ to pass from the first line above to the second line. Linearizing as we did for $v$ leads to an upper bound like (21) for $\|\operatorname{grad} w\|_p$, except that $\|\Phi_t\|_\infty$ is replaced by $\|\operatorname{grad}\Phi_t\|_\infty$, which is bounded by $\check{C}/t^{n+1}$ rather than by $C/t^n$. This division by an extra factor of $t$ leads to the estimate

$$|w|_{1,p} \le \frac{C_q^*}{t}\left(\int_{t<|h|<2t}\left[\|(\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q}.$$

Therefore

$$\begin{aligned} K(t;u) &\le \|v\|_p + t\|w\|_{1,p} \\ &\le \text{const.}\left(\int_{t<|h|<2t}\left[\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q} + t\|u\|_p. \end{aligned} \tag{22}$$

We also have the cheap estimate $K(t;u) \le \|u\|_p$ from the splitting $u = u + 0$.

We use the discrete version of the $K$ method to describe $B^{s;p,q}(\mathbb{R}^n)$. The cheap estimate suffices to make $\sum_{i=1}^{\infty}[2^{-is}K(2^i;u)]^q$ finite. When $i \le 0$ we use inequality (22) with $t = 2^i$, and we find that distinct indices $i$ lead to disjoint coronas for the integral appearing in (22). It follows that the part of the $\ell^q$ norm with $i \le 0$ is bounded above by a constant times $\|u\|_p$ plus a constant times the quantity

$$\left(\int_{|h|\le 2}\left[|h|^{-s}\omega_p(u;h)\right]^q \frac{dh}{|h|^n}\right)^{1/q}.$$

This completes the proof when $m = 1$ and $1 \le q < \infty$. The proof when $m = 1$ and $q = \infty$ is similar. ∎

**7.49 (The Proof of Theorem 7.47 for $m > 1$)** We can easily modify some parts of the above proof for the case where $m = 1$ to work when $m > 1$. In particular, to prove that condition (b) implies condition (c) when $m > 1$, simply take the argument for $m = 1$ and replace $\omega_p^*$ by $\omega_p^{(m)*}$ and $\omega_p$ by $\omega_p^{(m)}$.

To get from (a) to (b) when $m > 1$, consider $B^{s;p,q}(\mathbb{R}^n)$ as a real interpolation space between $X_0 = L^p(\mathbb{R}^n)$ and $X_1 = W^{m,p}(\mathbb{R}^n)$ with $\theta = s/m$; since $m > s$, we have $\theta < 1$. Given a value of $t$, split $u$ as $v + w$ with $\|v\|_p + t^m\|w\|_{1,p} \le 2K(t^m;u)$. Then $\|\Delta_h^m v\|_p \le 2^m\|v\|_p \le 2^{m+1}\|v\|_p$.

Again we can mollify $w$ and then write differences of $w$ as integrals of derivatives of $w$. When $m = 1$ we found that $\Delta_h w$ was an integral of a first directional derivative of $w$ with respect to path length along the line segment from $x - h$ to $x$. Denote that directional derivative by $D_h w$. Then $\Delta_h^2$ is equal to the integral along the same line segment of $\Delta_h(D_h w)$. That integrand is itself equal to an integral along a line segment of length $|h|$ with integrand $D_h^2 w$. This represents $\Delta_h^2 w(x)$ as an iterated double integral of $D_h^2 w$, with both integrations running over intervals of length $|h|$. Iteration then represents $\Delta_h^m w$ as an $m$-fold iterated integral of $D_h^m w$ over intervals of length $|h|$. Applying Hölder's inequality to that integral and then integrating $p$-th powers over $\mathbb{R}^n$ yields the estimate $\|\Delta_h^m w\|_p \le C|h|^m |w|_{m,p}$. It follows that $\omega_p^{(m)*}(u;t) \le \hat{C} K(t^m; u)$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \left[t^{-s}\omega_p^{(m)*}(u;t)\right]^q \frac{dt}{t} &\le \hat{C}^q \int_{-\infty}^{\infty} \left[t^{-s} K(t^m; u)\right]^q \frac{dt}{t} \\ &= \hat{C}^q \int_{-\infty}^{\infty} \left[(t^m)^{-s/m} K(t^m; u)\right]^q \frac{dt}{t} \\ &= \check{C} \int_{-\infty}^{\infty} [\tau^{-\theta} K(\tau; u) \frac{d\tau}{\tau},\end{aligned}$$

after the change of variable $\tau = t^m$. So condition (a) still implies condition (b) when $m > 1$.

We now give an outline of the proof that (c) implies (a). See [BB, pp. 192–194] for more details on some of what we do. Since condition (c) for any value of $m$ implies the corresponding condition for larger values of $m$, we free to assume that $m$ is even, and we do so.

Given a function $u$ satisfying condition (c) for an even index $m > \max\{1, s\}$, and given an integer $i \le 0$, we can split $u = v_i + w_i$, where $v_i$ is an averaged $m$-fold integral of $\Delta_h^m u$; each single integral in this nest runs over an interval of length comparable to $t = 2^i$, and the averaging involves dividing by a multiple of $t^m$. The outcome is that we can estimate $\|v_i\|_p$ by the average of $\|\Delta_h^m u\|_p$ over a suitable $h$-corona. As in the case where $m = 1$, this leads to an estimate for the $\ell^q$ norm of the sequence $\left\{2^{-is}\|v_i\|_p\right\}_{i=-\infty}^{\infty}$ in terms of the integral in condition (c).

There is still a cheap estimate to guarantee for the pair $X_0 = L^p(\mathbb{R}^n)$ and $X_1 = W^{m,p}(\mathbb{R}^n)$ that the half-sequence $\left\{2^{-is}K(2^{im}; u)\right\}_{i=1}^{\infty}$ belongs to $\ell^q$. This leaves the problem of suitably controlling the $\ell^q$ norm of the half-sequence $\left\{2^{i(m-s)}\|w_i\|_{1,p}\right\}_{i=-\infty}^{0}$. We can represent $w_i$ as a sum of $m$ terms, each involving an average, with an $m$-fold iterated integral, of translates of $u$ in a fixed direction. We can use this representation to estimate the norms in $L^p(\mathbb{R}^n)$ of $m$-fold directional derivatives of $w_i$ in any fixed direction. In particular, we can do this for the unmixed partial derivatives $D_j^m w_i$, in each case getting an $L^p$ norm that we can control with the part of (c) corresponding to a suitable corona. It is known that $L^p$ estimates for all unmixed derivatives of even order $m$ imply similar estimate for all mixed $m$th-order derivatives derivatives, and thus for $|w_i|_{m,p}$. (See [St, p. 77]; this is the place where we need $m$ to be even and $1 < p < \infty$.)

Finally, for $K(2^{im}; u)$ we also need estimates for $\|w_i\|_p$. Since $w_i$ comes from averages of translates of $u$, these estimates take the form $\|w_i\|_p \le C\|u\|_p$. For the half sequence $\{2^{-is}K(2^{im};u)\}_{i=-\infty}^{0}$ we then need to multiply by $2^{im}$ and $2^{-is}$; again the outcome is a finite $\ell^q$ norm, since $i \le 0$ and $m > s$. ∎

## Other Scales of Intermediate Spaces

**7.50** The Besov spaces are not the only scale of intermediate spaces that can fill the gap between Sobolev spaces of integer order. Several other such scales have been constructed, each slightly different from the others and each having properties making it useful in certain contexts. As we have seen, the Besov spaces are particularly useful for characterizing traces of functions in Sobolev spaces. However, except when $p = 2$, the Sobolev spaces do not actually belong to the scale of Besov spaces.

Two other scales we will introduce below are:

(a) the scale of fractional order Sobolev spaces (also called spaces of Bessel potentials), denoted $W^{s,p}(\Omega)$, which we will define for positive, real $s$ by a complex interpolation method introduced below. It will turn out that if $s = m$, a positive integer and $\Omega$ is reasonable, then the space obtained coincides with the usual Sobolev space $W^{m,p}(\Omega)$.

(b) the scale of Triebel-Lizorkin spaces, $F^{s;p,q}(\mathbb{R}^n)$, which we will define only on $\mathbb{R}^n$ but which will provide a link between the Sobolev, Bessel potential, and Besov spaces, containing members of each of those scales for appropriate choices of the parameters $s$, $p$, and $q$.

We will use Fourier transforms to characterize both of the scales listed above, and will therefore normally work on the whole of $\mathbb{R}^n$. Some results can be extended to more general domains for which suitable extension operators exist.

For the rest of this chapter we will present only descriptive introductions to the topics considered and will eschew formal proofs, choosing to refer the reader to the available literature, e.g., [Tr1, Tr2, Tr3, Tr4], for more information. We particularly recommend the first chapter of [Tr4].

We begin by describing another interpolation method for Banach spaces; this one is based on properties of analytic functions in the complex plane.

**7.51 (The Complex Interpolation Method)** Let $\{X_0, X_1\}$ be an interpolation pair of complex Banach spaces defined as in Paragraph 7.7 so that $X_0 + X_1$ is a Banach space with norm

$$\|u\|_{X_0+X_1} = \inf\Big\{\|u_0\|_{X_0} + \|u_1\|_{X_1} \ : \ u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\Big\}.$$

Let $\mathscr{F} = \mathscr{F}(X_0, X_1)$ be the space of all functions $f$ of the complex variable $\zeta = \theta + i\tau$ with values in $X_0 + X_1$ that satisfy the following conditions:

(a) $f$ is continuous and bounded on the strip $0 \le \theta \le 1$ into $X_0 + X_1$.

(b) $f$ is analytic from $0 < \theta < 1$ into $X_0 + X_1$ (i.e., the derivative $f'(\zeta)$ exists in $X_0 + X_1$ if $0 < \theta = \operatorname{Re}\zeta < 1$).

(c) $f$ is continuous on the line $\theta = 0$ into $X_0$ and

$$\|f(i\tau)\|_{X_0} \to 0 \qquad \text{as} \qquad |\tau| \to \infty.$$

(d) $f$ is continuous on the line $\theta = 1$ into $X_1$ and

$$\|f(1+i\tau)\|_{X_1} \to 0 \qquad \text{as} \qquad |\tau| \to \infty.$$

**7.52** $\mathscr{F}$ is a Banach space with norm

$$\|f\,;\mathscr{F}\| = \max\left\{\sup_\tau \|f(i\tau)\|_{X_0},\ \sup_\tau \|f(1+i\tau)\|_{X_1}\right\}.$$

Given a real number $\theta$ in the interval (0, 1), we define

$$X_\theta = [X_0, X_1]_\theta = \left\{u \in X_0 + X_1 \ :\ u = f(\theta) \quad \text{for some} \quad f \in \mathscr{F}\right\}.$$

$X_\theta$ is called a *complex interpolation space* between $X_0$ and $X_1$; it is a Banach space with norm

$$\|u\|_{X_\theta} = \|u\|_{[X_0,X_1]_\theta} = \inf\left\{\|f\,;\mathscr{F}\| \ :\ f(\theta) = u\right\}.$$

It follows from the above definitions that an analog of the Exact Interpolation Theorem (Theorem 7.23) holds for the complex interpolation method too. (See Calderón [Ca2, p. 115] and [BL, chapter 4].) If $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ are two interpolation pairs and $T$ is a bounded linear operator from $X_0 + X_1$ into $Y_0 + Y_1$ such that $T$ is bounded from $X_0$ into $Y_0$ with norm $M_0$ and from $X_1$ into $Y_1$ with norm $M_1$, then $T$ is also bounded from $X_\theta$ into $Y_\theta$ with norm $M \le M_0^{1-\theta} M_1^\theta$ for each $\theta$ in the interval [0, 1].

There is also a version of the Reiteration Theorem 7.21 for complex interpolation; if $0 < \theta_0 < \theta_1 < 1$, $0 < \lambda < 1$, and $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then

$$\left[[X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1}\right]_\lambda = [X_0, X_1]_\theta$$

with equivalent norms. This was originally proved under the assumption that $X_0 \cap X_1$ is dense in $[X_0, X_1]_{\theta_0} \cap [X_0, X_1]_{\theta_1}$, but this restriction was removed by Cwikel [Cw].

**7.53 (Banach Lattices on $\Omega$)** Most of the Banach spaces considered in this book are spaces of (equivalence classes of almost everywhere equal) real-valued or complex-valued functions defined in a domain $\Omega$ in $\mathbb{R}^n$. Such a Banach space

$B$ is called a *Banach lattice on* $\Omega$ if, whenever $u \in B$ and $v$ is a measurable, real- or complex-valued function on $\Omega$ satisfying $|v(x)| \le |u(x)|$ a.e. on $\Omega$, then $v \in B$ and $\|v\|_B \le \|u\|_B$. Evidently only function spaces whose norms depend only on the size of the function involved can be Banach lattices. The Lebesgue spaces $L^p(\Omega)$ and Lorentz spaces $L^{p,q}(\Omega)$ are Banach lattices on $\Omega$, but Sobolev spaces $W^{m,p}(\Omega)$ (where $m \ge 1$) are not, since their norms also depend on the size of derivatives of their member functions.

We say that a Banach lattice $B$ on $\Omega$ has the *dominated convergence property* if, whenever $u \in B$, $u_j \in B$ for $1 \le j < \infty$, and $|u_j(x)| \le |u(x)|$ a.e. in $\Omega$, then

$$\lim_{j\to\infty} u_j(x) = 0 \text{ a.e.} \quad \Longrightarrow \quad \lim_{j\to\infty} \|u_j\|_B = 0.$$

The Lebesgue spaces $L^p(\Omega)$ and Lorentz spaces $L^{p,q}(\Omega)$ have this property provided both $p$ and $q$ are finite, but $L^\infty(\Omega)$, $L^{p,\infty}(\Omega)$, and $L^{\infty,q}(\Omega)$ do not. (As a counterexample for $L^\infty$, consider a sequence of translates with non-overlapping supports of dilates of a nontrivial bounded function with bounded support.)

**7.54 (The spaces $X_0^{1-\theta}X_1^\theta$)** Now suppose that $X_0$ and $X_1$ are two Banach lattices on $\Omega$ and let $0 < \theta < 1$. We denote by $X_0^{1-\theta}X_1^\theta$ the collection of measurable functions $u$ on $\Omega$ for each of which there exists a positive number $\lambda$ and non-negative real-valued functions $u_0 \in X_0$ and $u_1 \in X_1$ such that $\|u_0\|_{X_0} = 1$, $\|u_1\|_{X_1} = 1$ and

$$|u(x)| \le \lambda u_0(x)^{1-\theta} u_1(x)^\theta. \tag{23}$$

Then $X_0^{1-\theta}X_1^\theta$ is a Banach lattice on $\Omega$ with respect to the norm

$$\|u\,;X_0^{1-\theta}X_1^\theta\| = \inf\{\lambda : \text{inequality (23) holds}\}.$$

The key result concerning the complex interpolation of Banach lattices is the following theorem of Calderón [Ca2, p.125] which characterizes the intermediate spaces.

**7.55 THEOREM** Let $X_0$ and $X_1$ be Banach lattices at least one of which has the dominated convergence property. If $0 < \theta < 1$, then

$$[X_0, X_1]_\theta = X_0^{1-\theta}X_1^\theta$$

with equality of norms. ∎

**7.56 EXAMPLE** It follows by factorization and Hölder's inequality that if $1 \le p_i \le \infty$ for $i = 0, 1$, $p_1 \ne p_2$, and $0 < \theta < 1$, then

$$\big[L^{p_0}(\Omega), L^{p_1}(\Omega)\big]_\theta = L^p(\Omega),$$

with equality of norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Moreover, if also $1 \le q_i \le \infty$ and at least one of the pairs $(p_0, q_0)$ and $(p_1, q_1)$ has finite components, then

$$\left[L^{p_0,q_0}(\Omega), L^{p_1,q_1}(\Omega)\right]_\theta = L^{p,q}(\Omega),$$

with equivalence of norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

**7.57 (Fractional Order Sobolev Spaces)** We can define a scale of fractional order spaces by complex interpolation between $L^p$ and Sobolev spaces. Specifically, if $s > 0$ and $m$ is the smallest integer greater than $s$ and $\Omega$ is a domain in $\mathbb{R}^n$, we define the space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \left[L^p(\Omega), W^{m,p}(\Omega)\right]_{s/m}.$$

Again, as for Besov spaces, we can use the Reiteration Theorem to replace $m$ with a larger integer and also observe that $W^{s,p}(\Omega)$ is an appropriate complex interpolation space between $W^{s_0,p}(\Omega)$ and $W^{s_1,p}(\Omega)$ if $0 < s_0 < s < s_1$. We will see later that if $s$ is a positive integer and $\Omega$ has a suitable extension property, then $W^{s,p}(\Omega)$ coincides with the usual Sobolev space with the same name.

Because $W^{m,p}(\Omega)$ is not a Banach lattice on $\Omega$ we cannot use Theorem 7.55 to characterize $W^{s,p}(\Omega)$. Instead we will use properties of the Fourier transform on $\mathbb{R}^n$ for this purpose. Therefore, as we did for Besov spaces, we will normally work only with $W^{s,p}(\mathbb{R}^n)$, and rely on extension theorems to supply results for domains $\Omega \subset \mathbb{R}^n$.

We begin by reviewing some basic aspects of the Fourier transform.

**7.58 (The Fourier Transform)** The *Fourier transform* of a function $u$ belonging to $L^1(\mathbb{R}^n)$ is the function $\hat{u}$ defined on $\mathbb{R}^n$ by

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx.$$

By dominated convergence the function $\hat{u}$ is continuous; moreover, we have $\|\hat{u}\|_\infty \le (2\pi)^{-n/2} \|u\|_1$. If $u \in C^1(\mathbb{R}^n)$ and both $u$ and $D_j u$ belong to $L^1(\mathbb{R}^n)$, then $\widehat{D_j u}(y) = i y_j \hat{u}(y)$ by integration by parts. Similarly, if both $u$ and the function mapping $x$ to $|x|u(x)$ belong to $L^1(\mathbb{R}^n)$, then $\hat{u} \in C^1(\mathbb{R}^n)$; in this case $D_j\hat{u}(y)$ is the value at $y$ of the Fourier transform of the function mapping $x$ to $-ix_ju(x)$.

**7.59 (The Space of Rapidly Decreasing Functions)** Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denote the space of all functions $u$ in $C^\infty(\mathbb{R}^n)$ such that for all multi-indices $\alpha \geq 0$ and $\beta \geq 0$ the function mapping $x$ to $x^\alpha D^\beta u(x)$ is bounded on $\mathbb{R}^n$. Unlike functions in $\mathscr{D}(\mathbb{R}^n)$, functions in $\mathscr{S}$ need not have compact support; nevertheless, they must approach 0 at infinity faster than any rational function of $x$. For this reason the elements of $\mathscr{S}$ are usually called *rapidly decreasing functions.*

The properties of the Fourier transform mentioned above extend to verify the assertion that the Fourier transform of an element of $\mathscr{S}$ also belongs to $\mathscr{S}$.

The *inverse Fourier transform* $\check{u}$ of an element $u$ of $L^1(\mathbb{R}^n)$ is defined for $x \in \mathbb{R}^n$ by

$$\check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(y)\, dy.$$

The *Fourier inversion theorem* [RS, chapter 9] asserts that if $u \in \mathscr{S}$, then the inverse Fourier transform of $\hat{u}$ is $u$ ($\check{\hat{u}} = u$), and, moreover, that the same conclusion holds under the weaker assumptions that $u \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\hat{u} \in L^1(\mathbb{R}^n)$. One advantage of considering the Fourier transform on $\mathscr{S}$ is that $u \in \mathscr{S}$ guarantees that $\hat{u} \in L^1(\mathbb{R}^n)$, and the same is true for the function mapping $y$ to $y^\alpha \hat{u}(y)$ for any multi-index $\alpha \geq 0$. In fact, the inverse Fourier transforms of functions in $\mathscr{S}$ also belong to $\mathscr{S}$ and the transform of the inverse transform also returns the original function. Thus the Fourier transform is a one-to-one mapping of $\mathscr{S}$ onto itself.

**7.60 (The Space of Tempered Distributions)** Given a linear functional $F$ on the space $\mathscr{S}$, we can define another such functional $\hat{F}$ by requiring $\hat{F}(u) = F(\hat{u})$ for all $u \in \mathscr{S}$. Fubini's theorem shows that if $F$ operates by integrating functions in $\mathscr{S}$ against a fixed integrable function $f$, then $\hat{F}$ operates by integrating against the transform $\hat{f}$:

$$\begin{aligned} & F(u) = \int_{\mathbb{R}^n} f(x)u(x)\, dx, \qquad f \in L^1(\mathbb{R}^n), \\ \Longrightarrow \quad & \hat{F}(v) = \int_{\mathbb{R}^n} \hat{f}(y)v(y)\, dy. \end{aligned} \tag{24}$$

There exists a locally convex topology on $\mathscr{S}$ such that the mapping $F \to \hat{F}$ maps the dual space $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ in a one-to-one way onto itself. The elements of this dual space $\mathscr{S}'$ are called *tempered distributions.* As was the case for $\mathscr{D}'(\Omega)$, not all tempered distributions can be represented by integration against functions.

**7.61 (The Plancherel Theorem)** An easy calculation shows that if $u$ and $v$ belong to $L^1(\mathbb{R}^n)$, then $\widehat{u * v} = (2\pi)^{n/2}\hat{u}\hat{v}$; Fourier transformation converts convolution products into pointwise products. If $u \in L^1(\mathbb{R}^n)$, let $\tilde{u}(x) = \overline{u(-x)}$. Then $\widehat{\tilde{u}} = \overline{\hat{u}}$, and $\widehat{u * \tilde{u}} = (2\pi)^{n/2}|\hat{u}|^2$. If $u \in \mathscr{S}$, then both $u * \tilde{u}$ and $|\hat{u}|^2$ also belong to $\mathscr{S}$. Applying the Fourier inversion theorem to $u * \tilde{u}$ at $x = 0$ then gives the following result, known as *Plancherel's Theorem.*

$$u \in \mathscr{S} \qquad \Longrightarrow \qquad \|\hat{u}\|_2^2 = \|u\|_2^2.$$

That is, the Fourier transform maps the space $\mathscr{S}$ equipped with the $L^2$-norm isometrically onto itself. Since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, the isometry extends to one mapping $L^2(\mathbb{R}^n)$ onto itself. Also, $L^2(\mathbb{R}^n) \subset \mathscr{S}'$ and the distributional Fourier transforms of an $L^2$ function is the same $L^2$ function as defined by the above isometry. (That is, the Fourier transform of an element of $\mathscr{S}'$ that operates by integration against $L^2$ functions as in (24) does itself operate in that way.)

**7.62 (Characterization of $W^{s,2}(\mathbb{R}^n)$)** Given $u \in L^2(\mathbb{R}^n)$ and any positive integer $m$, let

$$u_m(y) = (1 + |y|^2)^{m/2}\hat{u}(y). \tag{25}$$

It is easy to verify that $u \in W^{m,2}(\mathbb{R}^n)$ if and only if $u_m$ belongs to $L^2(\mathbb{R}^n)$, and the $L^2$-norm of $u_m$ is equivalent to the $W^{m,2}$-norm of $u$. So the Fourier transform identifies $W^{m,2}(\mathbb{R}^n)$ with the Banach lattice of functions $w$ for which $(1 + |\cdot|^2)^{m/2}\hat{w}(\cdot)$ belongs to $L^2(\mathbb{R}^n)$. For each positive integer $m$ that lattice has the dominated convergence property. It follows that $u \in W^{s,2}(\mathbb{R}^n)$ if and only if $(1 + |\cdot|^2)^{s/2}\hat{u}(\cdot)$ belongs to $L^2(\mathbb{R}^n)$.

**7.63 (Characterization of $W^{s,p}(\mathbb{R}^n)$)** The description of $W^{s,p}(\mathbb{R}^n)$ when $1 < p < 2$ or $2 < p < \infty$ is more complicated. If $u \in L^p(\mathbb{R}^n)$ with $1 < p < 2$, then $u \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$; this guarantees that $\hat{u} \in L^\infty(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, and in particular that the distribution $\hat{u}$ is a function. Moreover, it follows by complex interpolation that $\hat{u} \in L^{p'}(\mathbb{R}^n)$ and by real interpolation that $\hat{u} \in L^{p',p}(\mathbb{R}^n)$. But the set of such transforms of $L^p$ functions is not a lattice when $1 < p < 2$. This follows from the fact (see [FG]) that every set of positive measure contains a subset $E$ of positive measure so that if the Fourier transform of an $L^p$ function, where $1 < p < 2$, vanishes off $E$, then the function must be 0. If $u \in L^p(\mathbb{R}^n)$ and $E$ is such a subset on which $\hat{u}(y) \neq 0$, then the function that equals $\hat{u}$ on $E$ and 0 off $E$ is not trivial but would have to be trivial if the set of Fourier transforms of $L^p$ functions were a lattice.

A duality argument shows that the set of (distributional) Fourier transforms of functions in $L^p(\mathbb{R}^n)$ for $p > 2$ cannot be a Banach lattice either. Moreover (see [Sz]), there are functions in $L^p(\mathbb{R}^n)$ whose transforms are not even functions.

Nevertheless, the product of any tempered distribution and any sufficiently smooth function that has at most polynomial growth is always defined. For any distribution $u \in \mathscr{S}'$ we can then define the distribution $u_m$ by analogy with formula (25); we multiply the tempered distribution $\hat{u}$ by the smooth function $(1 + |\cdot|^2)^{m/2}$. When $1 < p < 2$ or $2 < p < \infty$, the theory of singular integrals [St, p. 138] then shows that $u \in W^{m,p}(\mathbb{R}^n)$ if and only if the function $u_m$ is the Fourier transform of some function in $L^p(\mathbb{R}^n)$. Again the space $W^{s,p}(\mathbb{R}^n)$ is characterized by the version of this condition with $m$ replaced by $s$. In particular, if $s = m$ we obtain the usual Sobolev space $W^{m,p}(\mathbb{R}^n)$ up to equivalence of norms, when $1 < p < \infty$. The fractional order Sobolev spaces spaces are natural generalizations of the Sobolev spaces that allow for fractional orders of smoothness.

One can pass between spaces $W^{s,p}(\mathbb{R}^n)$ having the same index $p$ but different orders of smoothness $s$ by multiplying or dividing Fourier transforms by factors of the form $(1 + |\cdot|^2)^{-r/2}$. When $r > 0$ these radial factors are constant multiples of Fourier transforms of certain Bessel functions; for this reason the spaces $W^{s,p}(\mathbb{R}^n)$ are often called *spaces of Bessel potentials.* (See [AMS].)

In order to show the relationship between the fractional order Sobolev spaces and the Besov spaces, it is, however, more useful to refine the scale of spaces $W^{s,p}(\mathbb{R}^n)$ using a dyadic splitting of the Fourier transform.

**7.64 (An Alternate Characterization of $W^{s,p}(\mathbb{R}^n)$)** In proving the Trace Theorem 7.39 we used a splitting of a function in $W^{m,p}(\mathbb{R}^{n+1})$ into dyadic pieces supported in slabs parallel to the subspace $\mathbb{R}^n$ of the traces. Here we are going to use a similar splitting of the Fourier transform of an $L^p$ function into dyadic pieces supported between concentric spheres.

Recall the $C^\infty$ function $\phi_i$ defined in the proof of Lemma 7.40 and having support in the interval $(2^i, 2^{i+2})$. For each integer $i$ and $y$ in $\mathbb{R}^n$, let $\psi_i(y) = \phi_i(|y|)$. Each of these radially symmetric functions belongs to $\mathscr{S}$ and so has an inverse transform, $\Psi_i$ say, that also belongs to $\mathscr{S}$.

Fix an index $p$ in the interval $(1, \infty)$ and let $u \in L^p(\mathbb{R}^n)$. For each integer $i$ let $T_i u$ be the convolution of $u$ with $(2\pi)^{-n/2}\Psi_i$; thus $\widehat{T_i u}(y) = \psi_i(y) \cdot \hat{u}(y)$. One can regard the functions $T_i u$ as dyadic parts of $u$ with nearly disjoint frequencies. Littlewood-Paley theory [FJW] shows that the $L^p$-norm of $u$ is equivalent to the $L^p$-norm of the function mapping $x$ to $[\sum_{i=-\infty}^{\infty} |T_i u(x)|^2]^{1/2}$. That is

$$\|u\|_p \approx \left( \int_{R^n} \left[ \sum_{i=-\infty}^{\infty} |T_i u(x)|^2 \right]^{p/2} dx \right)^{1/p}.$$

To estimate the norm of $u$ in $W^{s,p}(\mathbb{R}^n)$ we should replace each term $T_i u$ by the function obtained by not only multiplying $\hat{u}$ by $\psi_i$, but also multiplying the transform by the function mapping $y$ to $(1 + |y|^2)^{s/2}$.

On the support of $\psi_i$ the values of that second Fourier multiplier are all roughly equal to $1 + 2^{si}$. It turns out that $u \in W^{s,p}(\mathbb{R}^n)$ if and only if

$$\|u\|_{s,p} = \|u\,;\,W^{s,p}(\mathbb{R}^n)\| \approx \left(\int_{R^n}\left[\sum_{i=-\infty}^{\infty}(1+2^{si})^2|T_iu(x)|^2\right]^{p/2} dx\right)^{1/p} < \infty.$$

This is a complicated but intrinsic characterization of the space $W^{s,p}(\mathbb{R}^n)$. That is, the following steps provide a recipe for determining whether an $L^p$ function $u$ belongs to $W^{s,p}(\mathbb{R}^n)$:

(a) Split $u$ into the the pieces $T_iu$ by convolving with the functions $\Psi_i$ or by multiplying the distribution $\hat{u}$ by $\psi_i$ and then taking the inverse transform. For each point $x$ in $\mathbb{R}^n$ this gives a sequence $\{T_iu(x)\}$.

(b) Multiply the $i$-th term in that sequence by $(1+2^{si})$ and compute the $\ell^2$-norm of the result. This gives a function of $x$.

(c) Compute the $L^p$-norm of that function.

The steps in this recipe can be modified to produce other scales of spaces.

**7.65 (The Triebel-Lizorkin Spaces)** Define $F^{s;p,q}(\mathbb{R}^n)$ to be the space obtained by using steps (a) to (c) above but taking an $\ell^q$-norm rather than an $\ell^2$-norm in step (b). This gives the family of Triebel-Lizorkin spaces; if $1 \le q < \infty$,

$$\|u\,;\,F^{s;p,q}(\mathbb{R}^n)\| \approx \left(\int_{R^n}\left[\sum_{i=-\infty}^{\infty}(1+2^{si})^q|T_iu(x)|^q\right]^{p/q} dx\right)^{1/p} < \infty.$$

Note that $F^{m;p,2}(\mathbb{R}^n)$ coincides with $W^{m,p}(\mathbb{R}^n)$ when $m$ is a positive integer, and $F^{s;p,2}(\mathbb{R}^n)$ coincides with $W^{s,p}(\mathbb{R}^n)$ when $s$ is positive.

**7.66 REMARKS**

1. The space $F^{0;p,2}(\mathbb{R}^n)$ coincides with $L^p(\mathbb{R}^n)$ when $1 < p < \infty$.
2. The definitions of $W^{s,p}(\mathbb{R}^n)$ and $F^{s;p,q}(\mathbb{R}^n)$ also make sense if $s < 0$, and even if $0 < p, q < 1$. However they may contain distributions that are not functions if $s < 0$, and they will not be Banach spaces unless $p \ge 1$ and $1 \le q < \infty$.
3. If $s > 0$, the recipes for characterizing $W^{s,p}(\mathbb{R}^n)$ and $F^{s;p,q}(\mathbb{R}^n)$ given above can be modified to replace the multiplier $(1+2^{si})$ by $2^{si}$ and restricting the summations in the $\ell^2$ or $\ell^p$ norm expressions to $i \ge 0$, provided we also explicitly require $u \in L^p(\mathbb{R}^n)$. Thus, for example, $u \in F^{s;p,q}(\mathbb{R}^n)$ if and only if

$$\|u\|_p + \left(\int_{R^n}\left[\sum_{i=0}^{\infty}2^{siq}|T_iu(x)|^q\right]^{p/q} dx\right)^{1/p} < \infty.$$

4. If $s > 0$ and we modify the recipe for $F^{s;p,q}(\mathbb{R}^n)$ by replacing the multiplier $(1+2^{si})$ by $2^{si}$ but continuing to take the summation over all integers $i$, then we obtain the so-called *homogeneous Triebel-Lizorkin space* $\dot{F}^{s;p,q}(\mathbb{R}^n)$ which contain equivalence classes of distributions modulo polynomials of low enough degree. Only smoothness and not size determines whether a function belongs to this homogeneous space.

**7.67 (An Alternate Definition of the Besov Spaces)** It turns out that the Besov spaces $B^{s;p,q}(\mathbb{R}^n)$ arises from the variant of the recipe given in Paragraph 7.64 where the last two steps are modified as follows.

(b') Multiply the $i$-th term in the sequence $\{T_i u(x)\}$ by $(1 + 2^{si})$ and compute the $L^p$-norm of the result. This gives a sequence of nonnegative numbers.

(c') Compute the $\ell^q$-norm of that sequence.

$$\left\| u\, ; B^{s;p,q}(\mathbb{R}^n) \right\| \approx \left[ \sum_{i=-\infty}^{\infty} \left( \int_{R^n} (1 + 2^{si})^p |T_i u(x)|^p \, dx \right)^{q/p} \right]^{1/q} .$$

This amounts to reversing the order in which the two norms are computed. That order does not matter when $q = p$; thus $B^{s;p,p}(\mathbb{R}^n) = F^{s;p,p}(\mathbb{R}^n)$ with equivalent norms. When $q \neq p$, Minkowski's inequality for sums and integrals reveals that in comparing the outcomes of steps (b) and (c), the larger norm and the smaller space of functions arises when the larger of the indices $p$ and $q$ is used first. That is,

$$\begin{cases} F^{s;p,q}(\mathbb{R}^n) \subset B^{s;p,q}(\mathbb{R}^n) & \text{if } q < p \\ B^{s;p,q}(\mathbb{R}^n) \subset F^{s;p,q}(\mathbb{R}^n) & \text{if } q > p. \end{cases}$$

For fixed $s$ and $p$ the inclusions between the Besov spaces $B^{s;p,q}(\mathbb{R}^n)$ are the same as those between $\ell^q$ spaces, and the same is true for the Triebel-Lizorkin spaces $F^{s;p,q}(\mathbb{R}^n)$.

Finally, the only link with the scale of fractional order Sobolev spaces and in particular with the Sobolev spaces occurs through the Triebel-Lizorkin scale with $q = 2$. We have

$$\begin{cases} W^{s,p}(\mathbb{R}^n) = F^{s;p,2}(\mathbb{R}^n) \subset F^{s;p,q}(\mathbb{R}^n) & \text{if } q \geq 2 \\ F^{s;p,q}(\mathbb{R}^n) \subset F^{s;p,2}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n) & \text{if } q \leq 2. \end{cases}$$

As another example, the trace of $W^{m,p}(\mathbb{R}^{n+1})$ on $\mathbb{R}^n$ is exactly the space $B^{m-1/p;p,p}(\mathbb{R}^n) = F^{m-1/p;p,p}(\mathbb{R}^n)$. When $p \leq 2$, this trace space is included in the space $F^{m-1/p;p,2}(\mathbb{R}^n)$ and thus in the space $W^{m-1/p,p}(\mathbb{R}^n)$. When $p \geq 2$, this inclusion is reversed.

**7.68 REMARKS**

1. Appropriate versions of Remarks 7.66 for the Triebel-Lizorkin spaces apply to the above characterization of the Besov spaces too. In particular, modifying recipe item (b') to use the multiplier $2^{si}$ instead of $1 + 2^{si}$ results in a *homogeneous Besov space* $\dot{B}^{s;p,q}(\mathbb{R}^n)$ of equivalence classes of distributions modulo certain polynomials. Again membership in this space depends only on smoothness and not on size.
2. The $K$-version of the definition of $B^{s;p,q}(\mathbb{R}^n)$ as an intermediate space obtained by the real method is a condition on how well $u \in L^p(\mathbb{R}^n)$ can be approximated by functions in $W^{m,p}(\mathbb{R}^n)$ for some integer $m > s$. But the $J$-form of the definition requires a splitting of $u$ into pieces $u_i$ with suitable control on the norms of the functions $u_i$ in the spaces $L^p(\mathbb{R}^n)$ and $W^{m,p}(\mathbb{R}^n)$. The Fourier splitting also gives us pieces $T_i u$ for which we can control those two norms, and these can serve as the pieces $u_i$. Conversely, if we have pieces $u_i$ with suitable control on appropriate norms, and it we apply Fourier decomposition to each piece, we would find that the norms $\|T_j u_i\|_p$ are negligible when $|j - i|$ is large, leading to appropriate estimates for the norms $\|T_j u\|_p$.

**7.69 (Extensions for General Domains)** Many of the properties of the scales of Besov spaces, spaces of Bessel potentials, and Triebel-Lizorkin spaces on $\mathbb{R}^n$ can be extended to more general domains $\Omega$ via the use of an extension operator. Rychkov [Ry] has constructed a linear total extension operator $\mathscr{E}$ that simultaneously and boundedly extends functions in $F^{s;p,q}(\Omega)$ to $F^{s;p,q}(\mathbb{R}^n)$ and functions in $B^{s;p,q}(\Omega)$ to $B^{s;p,q}(\mathbb{R}^n)$ provided $\Omega$ satisfies a strong local Lipschitz condition. The same operator $\mathscr{E}$ works for both scales, all real $s$, and all $p > 0$, $q > 0$; it is an extension operator in the sense that $\mathscr{E}u|_\Omega = u$ in $\mathscr{D}'(\Omega)$ for every $u$ in any of the Besov or Triebel-Lizorkin spaces defined on $\Omega$ as restrictions in the sense of $\mathscr{D}'(\Omega)$ of functions in the corresponding spaces on $\mathbb{R}^n$.

The existence of this operator provides, for example, an intrinsic characterization of $B^{s;p,q}(\Omega)$ in terms of that for $B^{s;p,q}(\mathbb{R}^n)$ obtained in Theorem 7.47.

## Wavelet Characterizations

**7.70** We have seen above how membership of a function $u$ in a space $B^{s;p,q}(\mathbb{R}^n)$ can be determined by the size of the sequence of norms $\|T_i u\|_p$, while its membership in the space $F^{s;p,q}(\mathbb{R}^n)$ requires pointwise information about the sizes of the functions $T_i u$ on $\mathbb{R}^n$. Both characterizations use the functions $T_i u$ of a dyadic decomposition of $u$ defined as inverse Fourier transforms of products of $\hat{u}$ with dilates of a suitable smooth function $\phi$. We conclude this chapter by describing how further refining these decompositions to the level of wavelets reduces questions about membership of $u$ in these smoothness classes to questions about the sizes of the (scalar) coefficients of $u$ in such decompositions. These coefficients do form a Banach lattice.

This contrasts dramatically with the situation for Fourier transforms of $L^p$ functions with $1 < p < 2$, where these transforms fail to form a lattice.

**7.71 (Wavelet Analysis)** An *analyzing wavelet* is a nontrivial function on $\mathbb{R}^n$ satisfying some decay conditions, some cancellation conditions, and some smoothness conditions. Different versions of these conditions are appropriate in different contexts. Two classical examples of wavelets on $\mathbb{R}$ are the following:

(a) The basic *Haar* function $h$ given by

$$h(x) = \begin{cases} 1 & \text{if} \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) A basic Shannon wavelet $S$ defined as the inverse Fourier transform of the function $\hat{S}$ satisfying

$$\hat{S}(y) = \begin{cases} 1 & \text{if } \pi \le |y| < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

The Haar wavelet has compact support, and *a fortiori* decays extremely rapidly. The only cancellation condition it satisfies is that $\int_{\mathbb{R}} ch(x)\,dx = 0$ for all constants $c$. It fails to be smooth, but compensates for that by taking only two nonzero values and thus being simple to use numerically.

The Shannon wavelet does *not* have compact support; instead it decays like $1/|x|$, that is, at a fairly slow rate. However, it has very good cancellation properties, since

$$\int_{\mathbb{R}} x^m S(x)\,dx = 0 \quad \text{for all nonnegative integers } m.$$

(These integrals are equal to constants times the values at $y = 0$ of derivatives of $\hat{S}(y)$. Since $\hat{S}$ vanishes in a neighbourhood of 0, those derivatives all vanish at 0.) Also, $S \in C^\infty(\mathbb{R})$ and even extends to an entire function on the complex plane.

To get a better balance between these conditions, we will invert the roles of function and Fourier transform from the previous section, and use below a wavelet $\phi$ defined on $\mathbb{R}^n$ as the inverse Fourier transform of a nontrivial smooth function that vanishes outside the annulus where $1/2 < |y| < 2$. Then $\phi$ has all the cancellation properties of the Shannon wavelet, for the same reasons. Also $\phi$ decays very rapidly because $\hat{\phi}$ is smooth, and $\phi$ is smooth because $\hat{\phi}$ decays rapidly. Again the compact support of $\hat{\phi}$ makes $\phi$ the restriction of an entire function.

Given an analyzing wavelet, $w$ say, we consider some or all of its translates mapping $x$ to $w(x - h)$ and some or all of its (translated) dilates, mapping $x$ to

$w(2^r x - h)$. These too are often called wavelets. Translation preserves $L^p$ norms; dilation does *not* do so, except when $p = \infty$; however, we will use the multiple $2^{rn/2} w(2^r x - h)$ to preserve $L^2$ norms.

If we apply the same operations to the complex exponential that maps $x$ to $e^{ixy}$ on $\mathbb{R}$, we find that dilation produces other such exponentials, but that translation just multiplies the exponential by a complex constant and so does not produce anything really new. In contrast, the translates of the basic Haar wavelet by integer amounts have disjoint supports and so are orthogonal in $L^2(\mathbb{R})$. A less obvious fact is that translating the Shannon wavelet by integers yields orthogonal functions, this time without disjoint supports.

In both cases, dilating by factors $2^i$, where $i$ is an integer, yields other wavelets that are orthogonal to their translates by $2^i$ times integers, and these wavelets are orthogonal to those in the same family at other dyadic scales. Moreover, in both examples, this gives an orthogonal basis for $L^2(\mathbb{R})$.

Less of this orthogonality persists for wavelets like the one we called $\phi$ above. But it can still pay to consider the *wavelet transform* of a given function $u$ which maps positive numbers $a$ and vectors $h$ in $\mathbb{R}^n$ to

$$\frac{1}{\sqrt{a^n}} \int_{\mathbb{R}^n} u(x) \phi\left(\frac{x-h}{a}\right) dx.$$

For our purposes it will suffice to consider only those dilations and translates mapping $x$ to $\phi_{i,k}(x) = 2^{in/2}\phi(2^i x - k)$, where $i$ runs through the set of integers, and $k$ runs through the integer lattice in $\mathbb{R}^n$. Integrating $u$ against such wavelets yields *wavelet coefficients* that we can index by the pairs $(i, k)$ and use to characterize membership of $u$ in various spaces.

For much more on wavelets, see [Db].

**7.72 (Wavelet Characterization of Besov Spaces)** Let $\phi$ be a function in $\mathscr{S}$ whose Fourier transform $\hat{\phi}$ satisfies the following two conditions:

(i) $\hat{\phi}(y) = 0$ if $|y| < 1/2$ or $|y| > 2$.

(ii) $|\hat{\phi}(y)| > c > 0$ if $3/5 < |y| < 5/3$.

Note that the conditions on $\hat{\phi}$ imply that

$$\int_{\mathbb{R}^n} P(x)\phi(x)\, dx = 0$$

for any polynomial $P$.

Also, it can be shown (see [FJW, p. 54]) that there exists a dual function $\psi \in \mathscr{S}$ satisfying the same conditions (i) and (ii) and such that

$$\sum_{i=-\infty}^{\infty} \overline{\hat{\phi}(2^{-i}y)}\, \hat{\psi}(2^{-i}y)\, dy = 1 \quad \text{for all } y \neq 0.$$

Let $\mathbb{Z}$ denote the set of all integers. For each $i \in \mathbb{Z}$ and each $n$-tuple $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we define two wavelet families by using dyadic dilates and translates of $\phi$ and $\psi$:

$$\phi_{i,k}(x) = 2^{-in/2}\phi(2^{-i}x - k) \qquad \text{and} \qquad \psi_{i,k}(x) = 2^{in/2}\psi(2^i x - k).$$

Note that the dilations in these two families are in opposite directions and that $\phi_{i,k}$ and $\psi_{i,k}$ have the same $L^2$ norms as do $\phi$ and $\psi$ respectively. Moreover, for any polynomial $P$,

$$\int_{\mathbb{R}^n} P(x)\phi_{i,k}(x)\, dx = 0.$$

Let $I$ denote the set of all indices $(i, k)$ such that $i \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, and let $\mathscr{F}$ denote the wavelet family $\{\phi_{i,k} : (i, k) \in I\}$. Given a locally integrable function $u$, we define its wavelet coefficients $c_{i,k}(u)$ with respect to the family $\mathscr{F}$ by

$$c_{i,k}(u) = \int_{\mathbb{R}^n} u(x)\overline{\phi_{i,k}(x)}\, dx,$$

and consider the wavelet series representation

$$u = \sum_{(i,k)\in I} c_{i,k}(u)\psi_{i,k}. \tag{26}$$

The series represents $u$ modulo polynomials as all its terms vanish if $u$ is a polynomial.

It turns out that $u$ belongs to the homogeneous Besov space $\dot{B}^{s;p,q}(\mathbb{R}^n)$ if and only if its coefficients $\{c_{i,k}(u) : (i, k) \in I\}$ belong to the Banach lattice on $I$ having norm

$$\left( \sum_{i=-\infty}^{\infty} \left[ 2^{i(s+n[1/2-1/p])} \sum_{k\in\mathbb{Z}^n} |c_{i,k}|^p \right]^{q/p} \right)^{1/q}. \tag{27}$$

The condition for membership in the ordinary Besov space $B^{s;p,q}(\mathbb{R}^n)$ is a bit more complicated. We use only the part of the wavelet series (26) with $i \geq 0$ and replace the rest with a new series

$$\sum_{k\in\mathbb{Z}^n} c_k(u)\Psi_k,$$

where $\Psi_k(x) = \Psi(x - k)$ and $\Psi$ is a function in $\mathscr{S}$ satisfying the conditions $\hat{\Psi}(y) = 0$ if $|y| \geq 1$ and $|\hat{\Psi}(y)| > c > 0$ if $|y| \leq 5/6$. Again there is a dual such function $\Phi$ with the same properties such that the coefficients $c_k(u)$ are given by

$$c_k(u) = \int_{\mathbb{R}^n} u(x)\overline{\Phi_k(x)}\, dx.$$

We have $u \in B^{s;p,q}(\mathbb{R}^n)$ if and only if the expression

$$\left(\sum_{k\in\mathbb{Z}^n} |c_k|^p\right)^{1/p} + \left(\sum_{i=0}^{\infty}\left[2^{i(s+n[1/2-1/p])}\sum_{k\in\mathbb{Z}^n}|c_{i,k}|^p\right]^{q/p}\right)^{1/q} \tag{28}$$

is finite, and this expression provides an equivalent norm for $B^{s;p,q}(\mathbb{R}^n)$.

Note that, in expressions (27) and (28) the part of the recipe in item 7.66 involving the computation of an $L^p$-norm seems to have disappeared. In fact, however, for any fixed value of the index $i$, the wavelet "coefficients" $c_{i,k}$ are actually values of the convolution $u * \phi_{i,0}$ taken at points in the discrete lattice $\{2^i k\}$, where the index $i$ is fixed but $k$ varies. This lattice turns out to be fine enough that the $L^p$-norm of $u * \phi_{i,0}$ is equivalent to the $\ell^p$-norm over this lattice of the values of $u * \phi_{i,0}$.

**7.73 (Wavelet Characterization of Triebel-Lizorkin Spaces)** Membership in the homogeneous Triebel-Lizorkin space $\dot{F}^{s;p,q}(\mathbb{R}^n)$ is also characterized by a condition where only the sizes of the coefficients $c_{i,k}$ matter, namely the finiteness of

$$\left\|\left(\sum_{i=-\infty}^{\infty}\left[2^{i(s+n/2)}\sum_{k\in\mathbb{Z}}|c_{i,k}|\chi_{i,k}\right]^q\right)^{1/q}\right\|_{p,\mathbb{R}^n}.$$

where $\chi_{i,k}$ is the characteristic function of the cube $2^i k_j \le x_j < 2^i(k_j+)$, $(1 \le j \le n)$. At any point $x$ in $\mathbb{R}^n$ the inner sum above collapses as follows. For each value of the index $i$ the point $x$ belongs to the cube corresponding to $i$ and $k$ for only one value of $k$, say $k_i(x)$. This reduces matters to the finiteness of

$$\left\|\left(\sum_{i=-\infty}^{\infty}\left[2^{i(s+n/2)}\,|c_{i,k_i(\cdot)}|\right]^q\right)^{1/q}\right\|_{p,\mathbb{R}^n}.$$

We refer to section 12 in [FJ] for information on how to deal in a similar way with the inhomogeneous space $F^{s;p,q}(\mathbb{R}^n)$.

Recall that in the discrete version of the $J$-method, the pieces $u_i$ in suitable splittings of $u$ are not unique. This flexibility sometimes simplified our analysis, for instance in the proofs of (trace) imbeddings for Besov spaces. The same is true for the related idea of *atomic decomposition*, for which we refer to [FJW] and [FJ] for sharper results and much more information.