6

COMPACT IMBEDDINGS OF SOBOLEV SPACES

The Rellich-Kondrachov Theorem

6.1 (Restricted Imbeddings) Let Ω be a domain in $\mathbb{R}^n$ and let Ω_0 be a subdomain of Ω. Let $X(\Omega)$ denote any of the possible target spaces for imbeddings of $W^{m,p}(\Omega)$, that is, $X(\Omega)$ is a space of the form $C_B^j(\Omega)$, $C^j(\overline{\Omega})$, $C^{j,\lambda}(\overline{\Omega})$, $L^q(\Omega_k)$, or $W^{j,q}(\Omega_k)$, where Ω_k, $1 \le k \le n$, is the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Since the linear restriction operator $i_{\Omega_0} : u \to u\big|_{\Omega_0}$ is bounded from $X(\Omega)$ into $X(\Omega_0)$ (in fact $\|i_{\Omega_0}u; X(\Omega_0)\| \le \|u; X(\Omega)\|$) any imbedding of the form

$$W^{m,p}(\Omega) \to X(\Omega) \tag{1}$$

can be composed with this restriction to yield the imbedding

$$W^{m,p}(\Omega) \to X(\Omega_0) \tag{2}$$

and (2) has imbedding constant no larger than (1).

6.2 (Compact Imbeddings) Recall that a set A in a normed space is precompact if every sequence of points in A has a subsequence converging in norm to an element of the space. An operator between normed spaces is called compact if it maps bounded sets into precompact sets, and is called completely continuous if it is continuous and compact. (See Paragraph 1.24; for linear operators compactness and complete continuity are equivalent.) In this chapter we are concerned with the compactness of imbedding operators which are continuous whenever they exist, and so are completely continuous whenever they are compact.

If Ω satisfies the hypotheses of the Sobolev imbedding Theorem 4.12 and if Ω_0 is a bounded subset of Ω, then, with the exception of certain extreme cases, all the restricted imbeddings (1) corresponding to imbeddings asserted in Theorem 4.12 are compact. The most important of these compact imbedding results originated in a lemma of Rellich [Re] and was proved specifically for Sobolev spaces by Kondrachov [K]. Such compact imbeddings have many important applications in analysis, especially to showing that linear elliptic partial differential equations defined over bounded domains have discrete spectra. See, for example, [EE] and [ET] for such applications and further refinements.

We summarize the various compact imbeddings of $W^{m,p}(\Omega)$ in the following theorem

6.3 THEOREM (The Rellich-Kondrachov Theorem) Let Ω be a domain in $\mathbb{R}^n$, let Ω_0 be a bounded subdomain of Ω, and let Ω_0^k be the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$. Let $j \ge 0$ and $m \ge 1$ be integers, and let $1 \le p < \infty$.

PART I If Ω satisfies the cone condition and $mp \le n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad 0 < n - mp < k \le n \text{ and } 1 \le q < kp/(n-mp), \tag{3}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad n = mp,\ 1 \le k \le n \text{ and } 1 \le q < \infty. \tag{4}$$

PART II If Ω satisfies the cone condition and $mp > n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega_0) \tag{5}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad 1 \le q < \infty. \tag{6}$$

PART III If Ω satisfies the strong local Lipschitz condition, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C^j(\overline{\Omega_0}) \quad \text{if} \quad mp > m, \tag{7}$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}) \quad \text{if} \quad mp > n \ge (m-1)p \text{ and } 0 < \lambda < m - (n/p). \tag{8}$$

PART IV If Ω is an arbitrary domain in $\mathbb{R}^n$, the imbeddings (3)–(8) are compact provided $W^{j+m,p}(\Omega)$ is replaced by $W_0^{j+m,p}(\Omega)$.

6.4 REMARKS

1. Note that if Ω is bounded, we may have $\Omega_0 = \Omega$ in the statement of the theorem.

2. If X, Y, and Z are spaces for which we have the imbeddings $X \to Y$ and $Y \to Z$, and if one of these imbeddings is compact, then the composite imbedding $X \to Z$ is compact. Thus, for example, if $Y \to Z$ is compact, then any sequence $\{u_j\}$ bounded in X will be bounded in Y and will therefore have a subsequence $\{u'_j\}$ convergent in Z.

3. Since the extension operator $u \to \tilde{u}$, where $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and $\tilde{u}(x) = 0$ if $x \notin \Omega$, defines an imbedding $W_0^{j+m,p}(\Omega) \to W^{j+m,p}(\mathbb{R}^n)$ by Lemma 3.27, Part IV of Theorem 6.3 follows from application of Parts I–III to $\mathbb{R}^n$.

4. In proving the compactness of any of the imbeddings (3)–(8) it is sufficient to consider only the case $j = 0$. Suppose, for example, that (3) has been proven compact if $j = 0$. For $j \geq 1$ and $\{u_i\}$ a bounded sequence in $W^{j+m,p}(\Omega)$ it is clear that $\{D^\alpha u_i\}$ is bounded in $W^{m,p}(\Omega)$ for each α such that $|\alpha| \leq j$. Hence $\{D^\alpha u_i|_{\Omega_0^k}\}$ is precompact in $L^q(\Omega_0^k)$ with q specified as in (3). It is possible, therefore, to select (by finite induction) a subsequence $\{u'_i\}$ of $\{u_i\}$ for which $\{D^\alpha u'_i|_{\Omega_0^k}\}$ converges in $L^q(\Omega_0^k)$ for each α such that $|\alpha| \leq j$. Thus $\{u'_i|_{\Omega_0^k}\}$ converges in $W_0^{j,q}(\Omega_0^k)$ and (3) is compact.

5. Since Ω_0 is bounded, $C_B^0(\Omega_0^k) \to L^q(\Omega_0^k)$ for $1 \leq q \leq \infty$; in fact $\|u\|_{0,q,\Omega_0^k} \leq \|u; C_B^0(\Omega_0^k)\|[\mathrm{vol}(\Omega_0^k)]^{1/q}$. Thus the compactness of (6) (for $j = 0$) follows from that of (5).

6. For the purpose of proving Theorem 6.3 the bounded subdomain Ω_0 of Ω may be assumed to satisfy the cone condition in Ω does. If C is a finite cone determining the cone condition for Ω, let $\tilde{\Omega}$ be the union of all finite cones congruent to C, contained in Ω and having nonempty intersection with Ω_0. Then $\Omega_0 \subset \tilde{\Omega} \subset \Omega$ and $\tilde{\Omega}$ is bounded and satisfies the cone condition. If $W^{m,p}(\Omega) \to X(\tilde{\Omega})$ is compact, then so is $W^{m,p}(\Omega) \to X(\Omega_0)$ by restriction.

6.5 (Proof of Theorem 6.3, Part III) If $mp > n \geq (m-1)p$ and if $0 < \lambda < m - (n/p)$, then there exists μ such that $\lambda < \mu < m - (n/p)$. Since Ω_0 is bounded, the imbedding $C^{0,\mu}(\overline{\Omega_0}) \to C^{0,\lambda}(\overline{\Omega_0})$ is compact by Theorem 1.34. Since $W^{m,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega_0})$ by Theorem 4.12 and restriction, imbedding (8) is compact for $j = 0$ by Remark 6.4(2).

If $mp > n$, let j^* be the nonnegative integer satisfying the inequalities $(m - j^*)p > n \geq (m - j^* - 1)p$. Then we have the imbedding chain

$$W^{m,p}(\Omega) \to W^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega_0}) \to C(\overline{\Omega_0}) \qquad (9)$$

where $0 < \mu < m - j^* - (n/p)$. The last imbedding in (9) is compact by Theorem 1.34. Thus (7) is compact for $j = 0$. ∎

6.6 (Proof of Theorem 6.3, Part II) As noted in Remark 6.4(6), Ω_0 may be assumed to satisfy the cone condition. Since Ω_0 is bounded it can, by Lemma 4.22 be written as a finite union, $\Omega_0 = \bigcup_{k=1}^{M} \Omega_k$, where each Ω_k satisfies the strong local Lipschitz condition. If $mp > n$, then

$$W^{m,p}(\Omega) \to W^{m,p}(\Omega_k) \to C(\overline{\Omega_k}),$$

the latter imbedding being compact as proved above. If $\{u_i\}$ is a sequence bounded in $W^{m,p}(\Omega)$, we may select by finite induction on k a subsequence $\{u_i'\}$ whose restriction to Ω_k converges in $C(\overline{\Omega_k})$ for each k, $1 \le k \le M$. But this subsequence then converges in $C_B^0(\Omega_0)$, so proving that (5) is compact for $j = 0$. Therefore (6) is also compact by Remark 6.4(5). ∎

6.7 LEMMA Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a subdomain of Ω, and Ω_0^k the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$ $(1 \le k \le n)$. Let $1 \le q_1 < q_0$ and suppose that

$$W^{m,p}(\Omega) \to L^{q_0}\left(\Omega_0^k\right)$$

and

$$W^{m,p}(\Omega) \to L^{q_1}\left(\Omega_0^k\right) \qquad \text{compactly.}$$

If $q_1 \le q < q_0$, then

$$W^{m,p}(\Omega) \to L^{q}\left(\Omega_0^k\right) \qquad \text{compactly.}$$

Proof. Let $\lambda = q_1(q_0 - q)/q(q_0 - q_1)$ and $\mu = q_0(q - q_1)/q(q_0 - q_1)$. Then $\lambda > 0$ and $\mu \ge 0$. By Hölder's inequality there exists a constant K such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_{0,q,\Omega_0^k} \le \|u\|_{0,q_1,\Omega_0^k}^{\lambda} \|u\|_{0,q_0,\Omega_0^k}^{\mu} \le K \|u\|_{0,q_1,\Omega_0^k}^{\lambda} \|u\|_{m,p,\Omega}^{\mu}.$$

A sequence bounded in $W^{m,p}(\Omega)$ has a subsequence which converges in $L^{q_1}(\Omega_0^k)$ and is therefore a Cauchy sequence in that space. Applying the inequality above to differences between terms of this sequence shows that it is also a Cauchy sequence in $L^q(\Omega_0^k)$, so the imbedding of $W^{m,p}(\Omega)$ into $L^q(\Omega_0^k)$ is compact. ∎

6.8 (Proof of Theorem 6.3, Part I) First we deal with (the case $j = 0$ of) imbedding (3). Assume for the moment that $k = n$ and let $q_0 = np/(n - mp)$. In order to prove that the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega_0), \qquad 1 \le q < q_0, \tag{10}$$

is compact, it sufficed, by Lemma 6.7, to do so only for $q = 1$. For $j = 1, 2, 3, \ldots$ let

$$\Omega_j = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) > 2/j\}.$$

Let S be a set of functions bounded in $W^{m,p}(\Omega)$. We show that S (when restricted to Ω_0) is precompact in $L^1(\Omega_0)$ by showing that S satisfies the conditions of Theorem 2.32. Accordingly, let $\epsilon > 0$ be given and for each $u \in W^{m,p}(\Omega)$ set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

By Hölder's inequality and since $W^{m,p}(\Omega) \to L^{q_0}(\Omega_0)$, we have

$$\begin{aligned} \int_{\Omega_0 - \Omega_j} |u(x)|\, dx &\le \left(\int_{\Omega_0 - \Omega_j} |u(x)|^{q_0}\, dx \right)^{1/q_0} \left(\int_{\Omega_0 - \Omega_j} 1\, dx \right)^{1-1/q_0} \\ &\le K_1 \|u\|_{m,p,\Omega} \left[\operatorname{vol}(\Omega_0 - \Omega_j)\right]^{1-1/q_0}, \end{aligned}$$

with K_1 independent of u. Since $q_0 > 1$ and Ω_0 has finite volume, j may be selected large enough to ensure that for every $u \in S$,

$$\int_{\Omega_0 - \Omega_j} |u(x)|\, dx < \epsilon$$

and also, for every $h \in \mathbb{R}^n$,

$$\int_{\Omega_0 - \Omega_j} |\tilde{u}(x+h) - \tilde{u}(x)|\, dx < \frac{\epsilon}{2}.$$

Now if $|h| < 1/j$, then $x + th \in \Omega_{2j}$ provided $x \in \Omega_j$ and $0 \le t \le 1$. If $u \in C^\infty(\Omega)$, it follows that

$$\begin{aligned} \int_{\Omega_j} |u(x+h) - u(x)|\, dx &\le \int_{\Omega_j} dx \int_0^1 \left| \frac{d}{dt} u(x+th) \right| dt \\ &\le |h| \int_0^1 dt \int_{\Omega_{2j}} |\operatorname{grad} u(y)|\, dy \\ &\le |h|\ \|u\|_{1,1,\Omega_0} \le K_2 |h|\ \|u\|_{m,p,\Omega}, \end{aligned}$$

where K_2 is independent of u. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$, this estimate holds for any $u \in W^{m,p}(\Omega)$. Hence if $|h|$ is sufficiently small, we have

$$\int_{\Omega_0} |\tilde{u}(x+h) - \tilde{u}(x)|\, dx < \epsilon.$$

Hence S is precompact in $L^1(\Omega_0)$ by Theorem 2.32 and imbedding (10) is compact.

Next suppose that $k < n$ and $p > 1$. The Sobolev Imbedding Theorem 4.12 assures us that $W^{m,p}(\Omega) \to L^{kp/(n-mp)}(\Omega_0^k)$. For any $q < kp/(n-mp)$ we can choose r such that $1 \le r < p, n-mr < k$, and $q \le kr/(n-mr) < kp/(n-mp)$. Since Ω_0 is bounded, the imbeddings

$$W^{m,p}(\Omega) \to W^{m,p}(\Omega_0) \to W^{m,r}(\Omega_0)$$

exist. By Theorem 5.10 we have

$$\begin{aligned}\|u\|_{q,\Omega_0^k} &\le K_1 \|u\|_{kr/(n-mr),\Omega_0^k} \\ &\le K_2 \|u\|_{nr/(n-mr),\Omega_0}^{1-\theta} \|u\|_{m,r,\Omega_0}^{\theta} \\ &\le K_3 \|u\|_{nr/(n-mr),\Omega_0}^{1-\theta} \|u\|_{m,p,\Omega}^{\theta},\end{aligned}$$

where K_j and θ are constants (independent of $u \in W^{m,p}(\Omega)$) and θ satisfies $0 < \theta < 1$. Since $nr/(n-mr) < np/(n-mp)$, a sequence bounded in $W^{m,p}(\Omega)$ must have a subsequence convergent in $L^{nr/(n-mr)}(\Omega_0)$ by the earlier part of this proof. That sequence is therefore a Cauchy sequence in $L^{nr/(n-mr)}(\Omega_0)$, and by the above inequality it is therefore a Cauchy sequence in $L^q(\Omega_0^k)$, so the imbedding $W^{m,p}(\Omega) \to L^q(\Omega_0^k)$ is compact and so is $W^{m,p}(\Omega) \to L^1(\Omega_0^k)$.

If $p = 1$ and $0 \le n - m < k < n$, then necessarily $m \ge 2$. Composing the continuous imbedding $W^{m,1}(\Omega) \to W^{m-1,r}(\Omega)$, where $r = n/(n-1) > 1$, with the compact imbedding $W^{m-1,r}(\Omega) \to L^1(\Omega_0^k)$, (which is compact because $k \ge n-(m-1) > n-(m-1)r$), completes the proof of the compactness of (3).

To prove that imbedding (4) is compact we proceed as follows. If $n = mp$, $p > 1$, and $1 \le q < \infty$, then we may select r so that $1 \le r < p$, $k > n - mr > 0$, and $kr/(n-mr) > q$. Assuming again that Ω_0 satisfies the cone condition, we have

$$W^{m,p}(\Omega) \to W^{m,r}(\Omega_0) \to L^q(\Omega_0^k).$$

The latter imbedding is compact by (3). If $p = 1$ and $n = m \ge 2$, then, setting $r = n/(n-1) > 1$ so that $n = (n-1)r$, we have for $1 \le q < \infty$,

$$W^{n,1}(\Omega) \to W^{n-1,r}(\Omega) \to L^q(\Omega_0^k),$$

the latter imbedding being compact as shown immediately above. Finally, if $n = m = p = 1$, then $k = 1$ also. Letting $q_0 > 1$ be arbitrarily chosen we prove the compactness of $W^{1,1}(\Omega) \to L^1(\Omega_0)$ exactly as in the case $k = n$ considered at the beginning of this proof. Since $W^{1,1}(\Omega) \to L^q(\Omega_0)$ for $1 \le q < \infty$, all these imbeddings are compact by Lemma 6.7. ∎

Two Counterexamples

6.9 (Quasibounded Domains) We say that an unbounded domain $\Omega \subset \mathbb{R}^n$ is *quasibounded* if

$$\lim_{\substack{x\in\Omega \\ |x|\to\infty}} \operatorname{dist}(x, \operatorname{bdry}\Omega) = 0.$$

An unbounded domain is not quasibounded if and only if it contains infinitely many pairwise disjoint congruent balls.

6.10 Two obvious questions arise from consideration of the statement of the Rellich-Kondrachov Theorem 6.3. First, can the theorem be extended to cover unbounded Ω_0? Second, can the *extreme cases*

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right), \qquad 0 < n-, p < k \le n, \quad q = kp/(n-mp)$$

and

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}), \qquad mp > n > (m-1)p, \quad \lambda = m - (n/p)$$

ever be compact? The first of these questions will be investigated later in this chapter. For the moment though we show that the answer is negative if $k = n$ and Ω_0 is not quasibounded. However, the situation changes (see [Lp]) for subspaces of symmetric functions.

6.11 EXAMPLE Let Ω be an unbounded domain in $\mathbb{R}^n$ that is not quasibounded. Then there exists a sequence $\{B_i\}$ of mutually disjoint open balls contained in Ω and all having the same positive radius. Let $\phi_1 \in C_0^\infty(B_1)$ satisfy $\|\phi_1\|_{j,p,B_1} = A_{j,p} > 0$ for each $j = 0, 1, 2, \ldots$ and each $p \ge 1$. Let ϕ_i be a translate of ϕ_1 having support in B_i. Then $\{\phi_i\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$ for any fixed m and p. But for any q,

$$\|\phi_i - \phi_k\|_{j,q,\Omega} = \left(\|\phi_i\|_{j,q,B_i}^q + \|\phi_k\|_{j,q,B_i}^q\right)^{1/q} = 2^{1/q}A_{j,q} > 0$$

so that $\{\phi_i\}$ cannot have a sequence converging in $W^{j,q}((\,)\,\Omega)$ for any $j \ge 0$. Thus no compact imbedding of the form $W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega)$ is possible. The non-compactness of the other imbeddings of Theorem 6.3 is proved similarly. ∎

Now we provide an example showing that the answer to the second question raised in Paragraph 6.10 is always negative.

6.12 EXAMPLE Let integers j, m, n be given with $j \ge 0$ and $m, n \ge 1$. Let $p \ge 1$. If $mp < n$, let k be an integer such that $n - mp < k \le n$ and let $q = kp/(n - mp)$. If $(m-1)p < n < mp$, let $\lambda = m - (n/p)$. Let Ω

be a domain in $\mathbb{R}^n$ and let Ω_0 be a nonempty bounded subdomain of Ω having nonempty intersection Ω_0^k with a k-dimensional plane H in $\mathbb{R}^n$ which, without loss of generality, we can take to be the plane $\mathbb{R}^k$ spanned by the $x_1, x_2, \dots, x_k$ coordinate axes. We show that the imbeddings

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^k) \qquad \text{if} \quad mp < n \tag{11}$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}) \qquad \text{if} \quad (m-1)p \le n < mp \tag{12}$$

cannot be compact.

Let $B_r(x)$ be the open ball of radius r in $\mathbb{R}^n$ centred at x and let ϕ be a nontrivial function in $C_0^\infty\big(B_1(0)\big)$. Let $\{a_i\}$ be a sequence of distinct points in Ω_0^k, and let $B_i = B_{r_i}(a_i)$ where the positive radii r_i satisfy $r_i \le 1$ and are chosen so that the balls B_i are pairwise disjoint and contained in Ω_0. We define a scaled, translated dilation ϕ_i of ϕ with support in B_i by

$$\phi_i(x) = r_i^{j+m-(n/p)}\phi(y), \qquad \text{where} \quad x = a_i + r_i y.$$

The functions ϕ_i have disjoint supports in Ω_0 and, since $D^\alpha\phi_i(x) = r^{-|\alpha|}D^\alpha\phi(y)$ and $dx = r_i^n\, dy$, we have, for $|\alpha| \le j+m$,

$$\int_\Omega |D^\alpha\phi_i(x)|^p\, dx = r_i^{(j+m-|\alpha|)p}\int_\Omega |\mathscr{D}^\alpha\phi(y)|^p\, dy.$$

Therefore, $\{\phi\}$ is bounded in $W^{j+m,p}(\Omega)$.

On the other hand, $dx_1\cdots dx_k = r_i^k\, dy_1\cdots dy_k$, so that if $|\alpha| = j$, then

$$\int_{\Omega_0^k} |D^\alpha\phi_i(x)|^q\, dx_1\cdots dx_k = r_i^{k+q[m-(n/p)]}\int_{\mathbb{R}^k} |D^\alpha\phi(y)|^q\, dy_1\cdots dy_k.$$

Since $k + q\big[m-(n/p)\big] = 0$, this shows that

$$\|\phi_i\|_{j,q,\Omega_0^k} \ge |\phi_i|_{j,q,\Omega_0^k} = C_1\, |\phi|_{j,q,\mathbb{R}^k} > 0$$

for all i, and $\{\phi_i\}$ is bounded away from zero in $W^{j,q}(\Omega_0^k)$. The disjointness of the supports of the functions ϕ_i now implies that $\{\phi\}$ can have no subsequence converging in $W^{j,q}(\Omega_0^k)$, so the imbedding (11) cannot be compact.

Now suppose that $(m-1)p \le n < mp$. Let a be a point in $B_1(0)$ and β be a particular multiindex satisfying $|\beta| = j$ such that $|D^\beta\phi(a)| = C_2 > 0$. Let $b_i = a_i + r_i a$ and let c_i be the point on the boundary of B_i closest to b_i. We have

$$\big|D^\beta\phi_i(b_i)\big| = r_i^{m-(n/p)}C_2 = r_i^\lambda C_2,$$

and, since $D^\beta \phi_i(c_i) = 0$,

$$\left\| \phi_i \,; C^{j,\lambda}(\overline{\Omega_0}) \right\| \ge \frac{\left| D^\beta \phi_i(b_i) - D^\beta \phi_i(c_i) \right|}{|b_i - c_i|^\lambda} = C_2 > 0.$$

Again, this precludes the existence of a subsequence of of $\{\phi_i\}$ convergent in $C^{j,\lambda}(\overline{\Omega_0})$, so the imbedding (12) cannot be compact. ∎

6.13 REMARK Observe that the above examples in fact showed that no imbeddings of $W_0^{j+m,p}(\Omega)$, not just of the larger space $W^{j+m,p}(\Omega)$, into the appropriate target space can be compact. We now examine the possibility of obtaining compact imbeddings of $W_0^{m,p}(\Omega)$ for certain unbounded domains.

Unbounded Domains — Compact Imbeddings of $W_0^{m,p}(\Omega)$

6.14 Let Ω be an unbounded domain in $\mathbb{R}^n$. We shall be concerned below with determining whether the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{13}$$

is compact. If it is, then it will follow by Remark 6.4(4), Lemma 6.7, and the second part of the proof in Paragraph 6.8 that the imbeddings

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad 0 < n - mp < k \le n, \quad p \le q < kp/(n - mp),$$

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad n = mp, \quad 1 \le k \le n, p \le q < \infty$$

are also compact. See Theorem 6.28 for the corresponding compactness of imbeddings into continuous function spaces.

As was shown in Example 6.11, imbedding (13) cannot be compact unless Ω is quasibounded. In Theorem 6.16 we give a geometric condition on Ω that is sufficient to guarantee the compactness of (13), and in Theorem 6.19 we give an analytic condition that is necessary and sufficient for the compactness of (13). Both theorems are from [A2].

6.15 Let Ω_r denote the set $\{x \in \Omega : |x| \ge r\}$. In the following discussion any cube H referred to will have its edges parallel to the coordinate axes. For a domain Ω, a cube H, and an integer ν satisfying $1 \le \nu \le n$, we define the quantity $\mu_{n-\nu}(H, \Omega)$ to be the maximum of the $(n - \nu)$-measure of $P(H - \Omega)$ taken over all projections P onto $(n - \nu)$-dimensional faces of H.

6.16 THEOREM Let ν be an integer such that $1 \le \nu \le n$ and $mp > \nu$ (or $m = p = \nu = 1$). Suppose that for every $\epsilon > 0$ there exist numbers h and r with $0 < h \le 1$ and $r \ge 0$ such that for every cube $H \subset \mathbb{R}^n$ having side h and nonempty intersection with Ω_r we have

$$\frac{\mu_{n-\nu}(H, \Omega)}{h^{n-\nu}} \ge \frac{h^p}{\epsilon}.$$

Then imbedding (13) is compact.

6.17 REMARKS

1. We will deduce this theorem from Theorem 6.19 later in this section.
2. The above theorem shows that for quasibounded Ω the compactness of (13) may depend in an essential way on the dimension of bdry Ω.
3. For $\nu = n$, the condition of Theorem 6.16 places only the minimum restriction of quasiboundedness on Ω; if $mp > n$ then (13) is compact for any quasibounded Ω. It can also be shown that if $p > 1$ and Ω is quasibounded with boundary having no finite accumulation points, then (13) cannot be compact unless $mp > n$.
4. If $\nu = 1$, the condition of Theorem 6.16 places no restrictions on m and p but requires that bdry Ω be "essentially $(n-1)$-dimensional." Any quasibounded domain whose boundary consists of reasonably regular $(n-1)$-dimensional surfaces will satisfy that condition. An example of such a domain is the "spiny urchin" of Figure 4, a domain in $\mathbb{R}^2$ obtained by deleting from the plane the union of the sets S_k, $(k = 1, 2, \ldots)$, specified in polar coordinates by

$$S_k = \left\{(r, \theta) \,:\, r \geq k,\ \theta = n\pi/2^k,\ n = 1, 2, \ldots, 2^{k+1}\right\}.$$

Note that this domain, though quasibounded, is simply connected and has empty exterior.

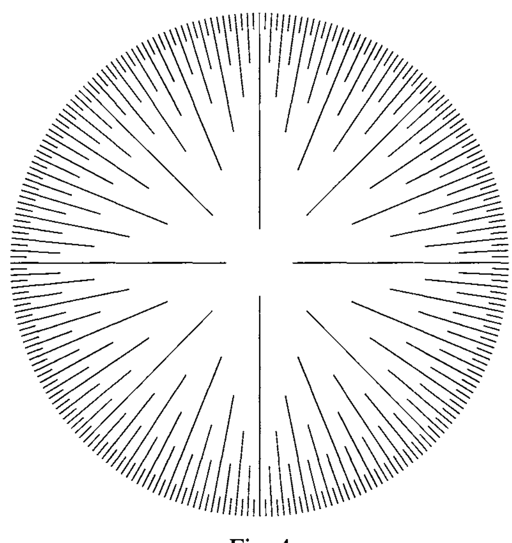

Fig. 4

5. More generally, if ν is the largest integer less than mp, the condition of Theorem 6.16 requires in a certain sense that the part of the boundary of Ω having dimension at least $n - \nu$ should bound a quasibounded domain.

6.18 (A Definition of Capacity) Let H be a cube of edge length h in $\mathbb{R}^n$ and let E be a closed subset of H. Given m and p we define a functional $I_H^{m,p}$ on $C^\infty(H)$ by

$$I_H^{m,p}(u) = \sum_{1\le j\le m} h^{jp}\,|u|_{j,p,H}^p = \sum_{1\le|\alpha|\le m} h^{|\alpha|p}\int_H |D^\alpha u(x)|^p\,dx.$$

Let $C^\infty(H,E)$ denote the set of all nontrivial functions $u \in C^\infty(H)$ that vanish identically in a neighbourhood of E. We define the (m,p)-capacity $Q^{m,p}(H,E)$ of E in H by

$$Q^{m,p}(H,E) = \inf\left\{ \frac{I_H^{m,p}(u)}{\|u\|_{0,p,H}^p} : u \in C^\infty(H,E)\right\}.$$

Clearly $Q^{m,p}(H,E) \le Q^{m+1,p}(H,E)$ and, whenever $E \subset F \subset H$, we have $Q^{m,p}(H,E) \le Q^{m,p}(H,F)$.

The following theorem characterizes those domains for which imbedding (13) is compact in terms of this capacity.

6.19 THEOREM Imbedding (13) is compact if and only if Ω satisfies the following condition: For every $\epsilon > 0$ there exists $h \le 1$ and $r \ge 0$ such that the inequality

$$Q^{m,p}(H, H-\Omega) \ge h^p/\epsilon$$

holds for every n-cube H of edge length h having nonempty intersection with Ω_r. (This condition clearly implies that Ω is quasibounded.)

Prior to proving this theorem we prepare the following lemma.

6.20 LEMMA There exists a constant $K(m,p)$ such that for any n-cube H of edge length h, any measurable subset A of H with positive volume, and any $u \in C^1(H)$, we have

$$\|u\|_{0,p,H}^p \le \frac{2^{p-1}h^n}{\mathrm{vol}(A)}\|u\|_{0,p,A}^p + K\,\frac{h^{n+p}}{\mathrm{vol}(A)}\,\|\mathrm{grad}\,u\|_{0,p,H}^p\,.$$

Proof. Let $y \in A$ and $x = (\rho,\phi) \in H$, where (ρ,ϕ) denote spherical coordinates centred at y, in terms of which the volume element is given by $dx = \omega(\phi)\,\rho^{n-1}\,d\rho\,d\phi$. Let bdry H be specified by $\rho = f(\phi)$, $\phi \in \Sigma$. Clearly $f(\phi) \le \sqrt{n}h$. Since

$$u(x) = u(y) + \int_0^\rho \frac{d}{dr}u(r,\phi)\,dr,$$

we have by Lemma 2.2 and Hölder's inequality

$$\begin{aligned}
&\int_H |u(x)|^p\,dx \\
&\leq 2^{p-1}h^n|u(y)|^p + 2^{p-1}\int_H \left|\int_0^\rho \frac{d}{dr}u(r,\phi)\,dr\right|^p dx \\
&\leq 2^{p-1}h^n|u(y)|^p + 2^{p-1}\int_\Sigma \omega(\phi)\,d\phi \int_0^{f(\phi)} \rho^{n+p-2}\,d\rho \int_0^\rho |\operatorname{grad} u(r,\phi)|^p\,dr \\
&\leq 2^{p-1}h^n|u(y)|^p + \frac{2^{p-1}}{n+p-1}\big(\sqrt{n}h\big)^{n+p-1}\int_H \frac{|\operatorname{grad} u(z)|^p}{|z-y|^{n-1}}\,dz.
\end{aligned}$$

Integrating y over A and using Lemma 4.64 we obtain

$$(\operatorname{vol}(A))\,\|u\|_{0,p,H}^p \leq 2^{p-1}h^n\,\|u\|_{0,p,A}^p + Kh^{n+p}\,\|\operatorname{grad} u\|_{0,p,H}^p,$$

as required. ∎

6.21 (Proof of Theorem 6.19 — Necessity) Suppose that Ω does not satisfy the condition stated in the theorem. Then there exists a finite constant $K_1 = 1/\epsilon$ such that for every h with $0 < h \leq 1$ there exists a sequence $\{H_j\}$ of mutually disjoint cubes of edge length h which intersect Ω and for which

$$Q^{m,p}(H_j, H_j - \Omega) < K_1h^p.$$

By the definition of capacity, for each such cube H_j there exists a function $u_j \in C^\infty(H_j, H_j - \Omega)$ such that $\|u_j\|_{0,p,H_j}^p = h^n$, $\|\operatorname{grad} u_j\|_{0,p,H_j}^p \leq K_1h^n$, and $\|u_j\|_{m,p,H_j}^p \leq K_2(h)$. Let $A_j = \{x \in H_j : |u_j(x)| < \frac{1}{2}\}$. By the previous Lemma we have

$$h^n \leq \frac{2^{p-1}h^n}{\operatorname{vol}(A_j)}\cdot\frac{\operatorname{vol}(A_j)}{2^p} + \frac{KK_1}{\operatorname{vol}(A_j)}h^{2n+p}$$

from which it follows that $\operatorname{vol}(A_j) \leq K_3h^{n+p}$. Let us choose h so small that $K_3h^p \leq \frac{1}{3}$, whence $\operatorname{vol}(A_j) \leq \frac{1}{3}\operatorname{vol}(H_j)$. Choose functions $w_j \in C_0^\infty(H_j)$ such that $w_j(x) = 1$ on a subset S_j of H_j having volume no less than $\frac{2}{3}\operatorname{vol}(H_j)$, and such that

$$\sup_j \max_{|\alpha|\leq m} \sup_{x\in H_j} |D^\alpha w_j(x)| = K_4(h) < \infty.$$

Then $v_j = u_jw_j \in C_0^\infty(H_j \cap \Omega) \subset C_0^\infty(\Omega)$ and $|v_j(x)| \geq \frac{1}{2}$ on $S_j \cap (H_j - A_j)$, a set of volume not less than $h^n/3$. Hence $\|v_j\|_{0,p,H_j}^p \geq h^n/3\cdot 2^p$. On the other hand

$$\int_{H_j} |D^\alpha u_j(x)|^p\cdot|D^\beta w_j(x)|^p\,dx \leq K_4(h)\,K_2(h)$$

provided $|\alpha|, |\beta| \le m$. Hence $\{v_j\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$. Since the supports of the functions v_j are disjoint, $\|v_i - v_j\|_{0,p,\Omega}^p \ge 2h^n/3 \cdot 2^p$ so the imbedding (13) cannot be compact. ∎

6.22 (Proof of Theorem 6.19 — Sufficiency) Suppose Ω satisfies the condition stated in the theorem. Let $\epsilon > 0$ be given and choose $r \ge 0$ and $h \le 1$ such that for every cube H of edge h intersecting Ω_r we have $Q^{m,p}(H, H - \Omega) \ge h^p/\epsilon^p$. Then for every $u \in C_0^\infty(\Omega)$ we obtain

$$\|u\|_{0,p,H}^p \le \frac{\epsilon^p}{h^p} I_H^{m,p}(u) \le \epsilon^p \|u\|_{m,p,H}^p .$$

Since a neighbourhood of Ω_r can be tessellated by such cubes H we have by summation

$$\|u\|_{0,p,\Omega_r} \le \epsilon \|u\|_{m,p,\Omega} .$$

That any bounded set S in $W_0^{m,p}(\Omega)$ is precompact in $L^p(\Omega)$ now follows from Theorems 2.33 and 6.3. ∎

6.23 LEMMA There is a constant K independent of h such that for any cube H in $\mathbb{R}^n$ having edge length h, for every q satisfying $p \le q \le np/(n - mp)$ (or $p \le q < \infty$ if $mp = n$, or $p \le q \le \infty$ if $mp > n$), and for every $u \in C^\infty(H)$ we have

$$\|u\|_{0,q,H} \le K \left(\sum_{|\alpha| \le m} h^{|\alpha|p - n + np/q} \|D^\alpha u\|_{0,p,H}^p \right)^{1/p} .$$

Proof. We may suppose H to be centred at the origin and let $\tilde{H}$ be the cube of unit edge concentric with H and having edges parallel to those of H. The stated inequality holds for $\tilde{u} \in C^\infty(\tilde{H})$ by the Sobolev imbedding theorem. It then follows for H via the dilation $u(x) = \tilde{u}(x/h)$. ∎

6.24 LEMMA If $mp > n$ (or if $m = p = n = 1$), there exists a constant $K = K(m, p, n)$ such that for every cube H of edge length h in $\mathbb{R}^n$ and every $u \in C^\infty(H)$ that vanishes in a neighbourhood of some point $y \in H$, we have

$$\|u\|_{0,p,H}^p \le K\, I_H^{m,p}(u).$$

Proof. Let (ρ, ϕ) be spherical coordinates centred at y. Then

$$u(\rho, \phi) = \int_0^\rho \frac{d}{dt} u(t, \phi)\, dt.$$

If $n > (m-1)p$, then let $q = np/(n - mp + p)$, so that $q > n$. Otherwise let $q > \max\{n, p\}$ be an arbitrary and finite. If $(\rho, \phi) \in H$, then by Hölder's inequality

$$|u(\rho,\phi)|^q\rho^{n-1} \le \left(\sqrt{n}h\right)^{n-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt\left(\int_0^{\sqrt{n}h} t^{-(n-1)/(q-1)}\,dt\right)^{q-1}$$
$$\le K_1h^{q-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt.$$

It follows, using the previous lemma with $m-1$ in place of m, that

$$\begin{aligned}\|u\|_{0,q,H}^q &\le K_2h^q\int_H |\operatorname{grad} u(x)|^q\,dx \\ &\le K_2h^q\sum_{|\alpha|=1}\|D^\alpha u\|_{0,q,H}^q \\ &\le K_3h^q\sum_{|\alpha|=1}\left(\sum_{|\beta|\le m-1} h^{|\beta|p-n+n/q}\left\|D^{\alpha+\beta}u\right\|_{0,p,H}^p\right)^{q/p}.\end{aligned} \tag{14}$$

If $p>n$ (or $p=n=1$) the desired result follows directly from (14) with $q=p$:

$$\|u\|_{0,p,H}^p \le K\,I_H^{1,p}(u) \le K\,I_H^{m,p}(u).$$

Otherwise, a further application of Hölder's inequality yields

$$\begin{aligned}\|u\|_{0,p,H}^p &\le \|u\|_{0,q,H}^p\,(\mathrm{vol}(H))^{(q-p)/q} \\ &\le K_2^{p/q}\sum_{1\le|\gamma|\le m} h^{|\gamma|p}\,\|D^\gamma u\|_{0,p,H}^p = K\,I_H^{m,p}(u). \quad \blacksquare\end{aligned}$$

6.25 (Proof of Theorem 6.16) Let $mp>\nu$ (or $m=p=\nu=1$) and let H be a cube in $\mathbb{R}^n$ for which $\mu_{n-\nu}(H,\Omega)\ge h^p/\epsilon$. Let P be the maximal projection of $H-\Omega$ onto an $(n-\nu)$-dimensional face of H and let $E=P(H-\Omega)$. Without loss of generality we may assume that the face F of H containing E is parallel to the $x_{\nu+1},\dots,x_n$ coordinate plane. For each point $x=(x',x'')$ in E, where $x'=(x_1,\dots,x_\nu)$ and $x''=(x_{\nu+1},\dots,x_n)$ let $H_{x''}$ be the ν-dimensional cube of edge length h in which H intersects the ν-plane through x normal to F. By the definition of P there exists $y\in H_{x''}-\Omega$. If $u\in C^\infty(H,H-\Omega)$, then $u(\cdot,x'')\in C^\infty(H_{x''},y)$. Applying the previous lemma to $u(\cdot,x'')$ we obtain

$$\int_{H_{x''}}|u(x',x'')|^p\,dx' \le K_1\sum_{1\le|\alpha|\le m} h^{|\alpha|p}\int_{H_{x''}}|D^\alpha u(x',x'')|^p\,dx',$$

where K_1 is independent of H, x'', and u. Integrating this inequality over E and denoting $H' = \{x' : x = (x', x'') \in H \text{ for some } x''\}$, we obtain

$$\|u\|_{0,p,H'\times E}^p \le K_1 I_{H'\times E}^{m,p}(u) \le K_1 I_H^{m,p}(u).$$

Now we apply Lemma 6.20 with $A = H' \times E$ so that $\mathrm{vol}(A) = h^\nu \mu_{n-\nu}(H, \Omega)$. This yields

$$\|u\|_{0,p,H}^p \le K_2 \frac{h^{n-\nu}}{\mu_{n-\nu}(H,\Omega)} I_H^{m,p}(u),$$

where K_2 is independent of H. It follows that

$$Q^{m,p}(H, H - \Omega) \ge \frac{\mu_{n-\nu}(H,\Omega)}{K_2 h^{n-\nu}} \ge \frac{h^p}{\epsilon K_2}.$$

Hence Ω satisfies the hypothesis of Theorem 6.19 if it satisfies that of Theorem 6.16. ∎

The following two interpolation lemmas enable us to extend Theorem 6.16 to cover imbeddings into spaces of continuous functions.

6.26 LEMMA Let $1 \le p < \infty$ and $0 < \mu \le 1$. There exists a constant $K = K(n, p, \mu)$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le K \|u\|_{0,p,\mathbb{R}^n}^\lambda \left(\sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} \right)^{1-\lambda}, \tag{15}$$

where $\lambda = p\mu/(n + p\mu)$.

Proof. We may assume

$$\sup_{x\in\mathbb{R}^n} |u(x)| = N > 0 \qquad \text{and} \qquad \sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} = M < \infty.$$

Let ϵ satisfy $0 < \epsilon \le N/2$. Then there exists a point x_0 in $\mathbb{R}^n$ such that we have $|u(x_0)| \ge N - \epsilon \ge N/2$. Now $|u(x_0 - u(x)|/|x_0 - x|^\mu \le M$ for all x, so

$$|u(x)| \ge |u(x_0)| - M|x_0 - x|^\mu \ge \frac{1}{2}|u(x_0)|$$

provided $|x - x_0| \le (N/4M)^{1/\mu} = r$. Hence

$$\int_{\mathbb{R}^n} |u(x)|^p\,dx \ge \int_{B_r(x_0)} \left(\frac{|u(x_0)|}{2}\right)^p dx \ge K_1 \left(\frac{N-\epsilon}{2}\right)^p \left(\frac{N}{4M}\right)^{n/\mu}.$$

Since this holds for arbitrarily small ϵ we have

$$\|u\|_{0,p,\mathbb{R}^n} \geq \left(\frac{K_1^{1/p}}{2\cdot 4^{n/\mu p}}\right) N^{1+(n/\mu p)} M^{-n/\mu p}$$

from which (15) follows at once. ∎

6.27 LEMMA Let Ω be an arbitrary domain in $\mathbb{R}^n$, and let $0 < \lambda < \mu \leq 1$. For every function $u \in C^{0,\mu}(\overline{\Omega})$ we have

$$\left\|u;\, C^{0,\lambda}(\overline{\Omega})\right\| \leq 3^{1-\lambda/\mu} \left\|u;\, C(\overline{\Omega})\right\|^{1-\lambda/\mu} \left\|u;\, C^{0,\mu}(\overline{\Omega})\right\|^{\lambda/\mu}. \tag{16}$$

Proof. Let $p = \mu/\lambda$ and $p' = p/(p-1)$. Let

$$A_1 = \left\|u;\, C(\overline{\Omega})\right\|^{1/p}, \qquad B_1 = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \left(\frac{|u(x)-u(y)|}{|x-y|^{\mu}}\right)^{1/p},$$

$$A_2 = \left\|u;\, C(\overline{\Omega})\right\|^{1/p'}, \qquad B_2 = \sup_{\substack{x,y\in\Omega\\ x\neq y}} |u(x)-u(y)|^{1/p'}.$$

Clearly $A_1^p + B_1^p = \left\|u;\, C^{0,\mu}(\overline{\Omega})\right\|$ and $B_2^{p'} \leq 2\left\|u;\, C(\overline{\Omega})\right\|$. By Hölder's inequality for sums we have

$$\begin{aligned}
\left\|u;\, C^{0,\lambda}(\overline{\Omega})\right\| &= \left\|u;\, C(\overline{\Omega})\right\| + \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\lambda}} \\
&\leq A_1A_2 + B_1B_2 \\
&\leq \left(A_1^p + B_1^p\right)^{1/p}\left(A_2^{p'} + B_2^{p'}\right)^{1/p'} \\
&\leq \left\|u;\, C^{0,\mu}(\overline{\Omega})\right\|^{\lambda/\mu}\left(3\left\|u;\, C(\overline{\Omega})\right\|\right)^{1-\lambda/\mu}
\end{aligned}$$

as required. ∎

6.28 THEOREM Let Ω satisfy the hypotheses of Theorem 6.16. Then the following imbeddings are compact:

$$W_0^{j+m,p}(\Omega) \to C^j(\overline{\Omega}) \qquad \text{if} \quad mp > n \tag{17}$$

$$W_0^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}) \qquad \text{if} \quad mp > n \geq (m-1)p \quad \text{and} \quad 0 < \lambda < m - (n/p). \tag{18}$$

Proof. It is sufficient to deal with the case $j = 0$. If $mp > n$, let j^* be the nonnegative integer satisfying $(m - j^*)p > n \geq (m - j^* - 1)p$. Then we have the chain of imbeddings

$$W_0^{m,p}(\Omega) \to W_0^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C(\overline{\Omega}),$$

where $0 < \mu < m - j^* - (n/p)$. If $\{u_i\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$, then it is also bounded in $C^{0,\mu}(\overline{\Omega})$. By Theorem 6.16, $\{u_i\}$ has a subsequence $\{u_i'\}$ converging in $L^p(\Omega)$. By (15), which applies by completion to the functions u_i, this subsequence is a Cauchy sequence in $C(\overline{\Omega})$ and so converges there. Hence (17) is compact for $j = 0$. Furthermore, if $mp > n \geq (m-1)p$ (that is, if $j^* = 0$) and $0 < \lambda < \mu$, then by (16) $\{u_i'\}$ is also a Cauchy sequence in $C^{0,\lambda}(\overline{\Omega})$ whence (18) is also compact. ∎

An Equivalent Norm for $W_0^{m,p}(\Omega)$

6.29 (Domains of Finite Width) Consider the problem of determining for what domains Ω in $\mathbb{R}^n$ is the seminorm

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{0,p,\Omega}^p \right)^{1/p}$$

actually a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{0,p,\Omega}^p \right)^{1/p}.$$

This problem is closely related to the problem of determining for which unbounded domains Ω the imbedding $W_0^{m,p}(\Omega) \to L^p(\Omega)$ is compact because both problems depend on estimates for the L^p norm of a function in terms of L^p estimates for its derivatives.

We can easily show the equivalence of the above seminorm and norm for a domain of *finite width*, that is, a domain in $\mathbb{R}^n$ that lies between two parallel planes of dimension $(n-1)$. In particular, this is true for any bounded domain.

6.30 THEOREM (Poincaré's Inequality) If domain $\Omega \subset \mathbb{R}^n$ has finite width, then there exists a constant $K = K(p)$ such that for all $\phi \in C_0^\infty(\Omega)$

$$\|\phi\|_{0,p,\Omega} \leq K\, |\phi|_{1,p,\Omega}. \tag{19}$$

This inequality is known as *Poincare's Inequality.*

Proof. Without loss of generality we can assume that Ω lies between the hyperplanes $x_n = 0$ and $x_n = c > 0$. Denoting $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$, we have for any $\phi \in C_0^\infty(\Omega)$,

$$\phi(x) = \int_0^{x_n} \frac{d}{dt}\phi(x', t)\, dt$$

so that, by Hölder's inequality,

$$\begin{aligned}\|\phi\|_{0,p,\Omega}^p &= \int_{\mathbb{R}^{n-1}} dx' \int_0^c |\phi(x)|^p \, dx_n \\ &\le \int_{\mathbb{R}^{n-1}} dx' \int_0^c x_n^{p-1} \, dx_n \int_0^c |D_n\phi(x', t)|^p \, dt \\ &\le \frac{c^p}{p} \, |\phi|_{1,p,\Omega}^p .\end{aligned}$$

Inequality (19) follows with $K = c/p^{1/p}$. ∎

6.31 COROLLARY If Ω has finite width, $|\cdot|_{m,p,\Omega}$ is a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm $\|\cdot\|_{m,p,\Omega}$.

Proof. If $\phi \in C_0^\infty(\Omega)$ then any derivative of ϕ also belongs to $C_0^\infty(\Omega)$. Now (19) implies

$$|\phi|_{1,p,\Omega}^p \le \|\phi\|_{1,p,\Omega}^p = \|\phi\|_{0,p,\Omega}^p + |\phi|_{1,p,\Omega}^p \le (1 + K^p) \, |\phi|_{1,p,\Omega}^p ,$$

and successive iterations of this inequality to derivatives $D^\alpha \phi$, ($|\alpha| \le m-1$) leads to

$$|\phi|_{m,p,\Omega}^p \le \|\phi\|_{m,p,\Omega}^p \le K_1 \, |\phi|_{m,p,\Omega}^p .$$

By completion, this holds for all u in $W_0^{m,p}(\Omega)$. ∎

6.32 (Quasicylindrical Domains) An unbounded domain Ω in $\mathbb{R}^n$ is called *quasicylindrical* if

$$\limsup_{x\in\Omega,\ |x|\to\infty} \operatorname{dist}(x, \operatorname{bdry}\Omega) < \infty.$$

Every quasibounded domain is quasicylindrical, as is every (unbounded) domain of finite width. The seminorm $|\cdot|_{m,p,\Omega}$ is not equivalent to the norm $\|\cdot\|_{m,p,\Omega}$ on $W_0^{m,p}(\Omega)$ for unbounded Ω if Ω is not quasicylindrical. We leave it to the reader to construct a suitable counterexample.

The following theorem is clearly analogous to Theorem 6.16.

6.33 THEOREM Suppose there exist an integer ν and constants K, R, and h such that $1 \le \nu \le n$, $0 < K \le 1$, $0 \le R < \infty$, and $0 < h < \infty$. Suppose also that either $\nu < p$ or $\nu = p = 1$, and that for every cube H in $\mathbb{R}^n$ having edge length h and nonempty intersection with $\Omega_R = \{x \in \Omega : |x| \ge R\}$ we have

$$\frac{\mu_{n-\nu}(H, \Omega)}{h^{n-\nu}} \ge K,$$

where $\mu_{n-\nu}(H, \Omega)$ is as defined prior to the statement of Theorem 6.16. Then $|\cdot|_{m,p,\Omega}$ is a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm $\|\cdot\|_{m,p,\Omega}$.

Proof. As observed in the previous Corollary, it is again sufficient to prove that $\|u\|_{0,p,\Omega} \le K_1\,|u|_{1,p,\Omega}$ holds for all $u \in C_0^\infty(\Omega)$. Let H be a cube of edge length h having nonempty intersection with Ω_R. Since $\nu < p$ (or $\nu = p = 1$) the proof of Theorem 6.16 shows that

$$Q^{1,p}(H, H - \Omega) \ge \frac{\mu_{n-\nu}(H,\Omega)}{K_2 h^{n-\nu}} \ge \frac{K}{K_2}$$

for all $u \in C_0^\infty(\Omega)$, K_2 being independent of u. Hence

$$\|u\|_{0,p,H}^p \le (K_2/K) I_H^{1,p} = K_3\,|u|_{1,p,H}^p\,.$$

By summing this inequality over the cubes comprising a tessellation of some neighbourhood of Ω_R, we obtain

$$\|u\|_{0,p,\Omega_R}^p \le K_3\,|u|_{1,p,\Omega}^p\,. \tag{20}$$

It remains to be proven that

$$\|u\|_{0,p,B_R}^p \le K_3\,|u|_{1,p,\Omega}^p\,,$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Let (ρ, ϕ) denote the spherical coordinates of the point $x \in \mathbb{R}^n$ ($\rho \ge 0$, $\phi \in \Sigma$) so that $dx = \rho^{n-1}\omega(\phi)\,d\rho\,d\phi$. For any $u \in C^\infty(\mathbb{R}^n)$ we have

$$u(\rho,\phi) = u(\rho + R, \phi) - \int_\rho^{R+\rho} \frac{d}{dt} u(t,\phi)\,dt$$

so that (by Lemma 2.2)

$$|u(\rho,\phi)|^p \le 2^{p-1}|u(\rho+R,\phi)|^p + 2^{p-1}R^{p-1}\rho^{1-n}\int_\rho^{R+\rho} |\operatorname{grad} u(t,\phi)|^p t^{n-1}\,dt.$$

Hence

$$\begin{aligned}
\|u\|_{0,p,B_R}^p &= \int_\Sigma \omega(\phi)\,d\phi \int_0^R |u(\rho,\phi)|^p \rho^{n-1}\,d\rho \\
&\le 2^{p-1}\int_\Sigma \omega(\phi)\,d\phi \int_0^R |u(\rho+R,\phi)|^p (\rho+R)^{n-1}\,d\rho \\
&\quad + 2^{p-1}R^p \int_\Sigma \omega(\phi)\,d\phi \int_0^{2R} |\operatorname{grad} u(t,\phi)|^p t^{n-1}\,dt.
\end{aligned}$$

Therefore, we have for $u \in C_0^\infty(\Omega)$

$$\begin{aligned}\|u\|_{0,p,B_R}^p &\le 2^{p-1}\|u\|_{0,p,B_{2R}-B_R}^p + 2^{p-1}R^p\,|u|_{1,p,B_{2R}}^p \\ &\le 2^{p-1}\|u\|_{0,p,\Omega_R}^p + 2^{p-1}R^p\,|u|_{1,p,\Omega}^p \le K_4\,|u|_{1,p,\Omega}^p\end{aligned}$$

by (20). ∎

Unbounded Domains — Decay at Infinity

6.34 The fact that elements of $W_0^{m,p}(\Omega)$ vanish in a generalized sense on the boundary of Ω played a critical role in our showing that the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{21}$$

is compact for certain unbounded domains Ω. Since elements of $W^{m,p}(\Omega)$ do not have this property, there remains a question of whether an imbedding of the form

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{22}$$

can ever be compact for unbounded Ω, or even for bounded Ω which are sufficiently irregular that no imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{23}$$

can exist for any $q > p$. Note that if Ω has finite volume, the existence of imbedding (23) for some $q > p$ implies the compactness of imbedding (22) by the method of the first part of the proof in Paragraph 6.8. By Theorem 4.46 imbedding (23) cannot, however, exist if $q > p$ and Ω is unbounded but has finite volume.

6.35 EXAMPLE For $j = 1, 2, \dots$ let B_j be an open ball in $\mathbb{R}^n$ having radius r_j, and suppose that $\overline{B_j} \cap \overline{B_i}$ is empty whenever $j \ne i$. Let $\Omega = \bigcup_{j=1}^\infty B_j$. Note that Ω may be bounded or unbounded. The sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} (\mathrm{vol}(B_j))^{-1/p} & \text{if } x \in \overline{B_j} \\ 0 & \text{if } x \notin \overline{B_j} \end{cases}$$

is bounded in $W^{m,p}(\Omega)$ for every integer $m \ge 0$, but is not precompact in $L^p(\Omega)$ no matter how fast $r_j \to 0$ as $j \to \infty$. (Of course, imbedding (21) is compact by Theorem 6.16 provided $\lim_{j\to\infty} r_j = 0$.) Even if Ω is bounded, imbedding (23) cannot exist for any $q > p$.

6.36 Let us state at once that there do exist unbounded domains Ω for which the imbedding (22) is compact. See Example 6.53. An example of such a domain was given by the authors in [AF2] and it provided a basis for an investigation of the general problem in [AF3]. The approach of this latter paper is used in the following discussion.

First we concern ourselves with necessary conditions for the compactness of (23) for $q \geq p$. These conditions involve rapid decay at infinity for any unbounded domain (see Theorem 6.45). The techniques involved in the proof also yield a strengthened version of Theorem 4.46, namely Theorem 6.41, and a converse of the assertion [see Remark 4.13(3)] that $W^{m,p}(\Omega) \to L^q(\Omega)$ for $1 \leq q < p$ if Ω has finite volume.

A sufficient condition for the compactness of (22) is given in Theorem 6.52. It applies to many domains, bounded and unbounded, to which neither the Rellich-Kondrachov theorem nor any generalization of that theorem obtained by the same methods can be applied. (e.g. exponential cusps — see Example 6.54).

6.37 (Tessellations and λ-fat Cubes) Let T be a tessellation of $\mathbb{R}^n$ by closed n-cubes of edge length h. If H is one of the cubes in T, let $N(H)$ denote the cube of edge length $3h$ concentric with H and therefore consisting of the 3^n elements of T that intersect H. We call $N(H)$ the *neighbourhood* of H. By the *fringe* of H we shall mean the shell $F(H) = N(H) - H$.

Let Ω be a given domain in $\mathbb{R}^n$ and T a given tessellation as above. Let $\lambda > 0$. A cube $H \in T$ will be called *λ-fat* (with respect to Ω) if

$$\mu(H \cap \Omega) > \lambda\, \mu(F(H) \cap \Omega),$$

where μ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. (We use μ instead of "vol" for notational simplicity in the following discussion where the symbol must be used many times.) If H is not λ-fat then we will say it is *λ-thin.*

6.38 THEOREM Suppose there exists a compact imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for some $q \geq p$. Then for every $\lambda > 0$ and every tessellation T of $\mathbb{R}^n$ by cubes of fixed size, T can have only finitely many λ-fat cubes.

Proof. Suppose, to the contrary, that for some $\lambda > 0$ there exists a tessellation T of $\mathbb{R}^n$ by cubes of edge length h containing a sequence $\{H_j\}_{j=1}^{\infty}$ of λ-fat cubes. Passing to a subsequence if necessary we may assume that $N(H_j) \cap N(H_i)$ is empty whenever $j \neq i$. For each j there exists $\phi_j \in C_0^{\infty}(N(H_j))$ such that

(i) $|\phi_j(x)| \leq 1$ for all $x \in \mathbb{R}^n$,

(ii) $\phi_j(x) = 1$ for $x \in H_j$, and

(iii) $|D^{\alpha}\phi_j(x)| \leq M$ for all j, all $x \in \mathbb{R}^n$, and all α satisfying $0 \leq |\alpha| \leq m$.

In fact, all the ϕ_j can be taken to be translates of one of them. Let $\psi_j = c_j\phi_j$, where the positive constants c_j are chosen so that

$$\|\psi_j\|_{0,q,\Omega}^q \geq c_j^q \int_{H_j\cap\Omega} |\phi_j(x)|^q\,dx = c_j^q \mu(H_j\cap\Omega) = 1.$$

But then

$$\begin{aligned}\|\psi_j\|_{m,p,\Omega}^p &= c_j^p \sum_{0\leq|\alpha|\leq m} \int_{N(H_j)\cap\Omega} |D^\alpha\phi_j(x)|^p\,dx \\ &\leq M^p c_j^p\,\mu\big(N(H_j)\cap\Omega\big) \\ &< M^p c_j^p\,\mu\big(H_j\cap\Omega\big)\left(1+\frac{1}{\lambda}\right) = M^p\left(1+\frac{1}{\lambda}\right)c_j^{p-q},\end{aligned}$$

since H_j is λ-fat. Now $\mu(H_j\cap\Omega) \leq \mu(H_j) = h^n$ so $c_j \geq h^{-n/q}$. Since $p-q \leq 0$, $\{\psi_j\}$ is bounded in $W^{m,p}(\Omega)$. But the functions ψ_j have disjoint supports, so $\{\psi_j\}$ cannot be precompact in $L^q(\Omega)$, contradicting the assumption that $W^{m,p}(\Omega) \to L^q(\Omega)$ is compact. Thus every T can possess at most finitely many λ-fat cubes. ∎

6.39 COROLLARY Suppose that $W^{m,p}(\Omega) \to L^q(\Omega)$ for some $q > p$. If T is a tessellation of $\mathbb{R}^n$ by cubes of fixed edge-length, and if $\lambda > 0$ is given, then there exists $\epsilon > 0$ such that $\mu(H\cap\Omega) \geq \epsilon$ for every λ-fat $H \in T$.

Proof. Suppose, to the contrary, that there exists a sequence $\{H_j\}$ of λ-fat cubes with $\lim_{j\to\infty}\mu(H_j\cap\Omega) = 0$. If c_j is defined as in the above proof, we have $\lim_{j\to\infty} c_j = \infty$. But then $\lim_{j\to\infty}\|\psi_j\|_{m,p,\Omega} = 0$ since $p < q$. Since $\{\psi_j\}$ is bounded away from 0 in $L^q(\Omega)$, we have contradicted the continuity of the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$. ∎

6.40 REMARK It follows from the above corollary that if there exists an imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{24}$$

for some $q > p$ then one of the following alternatives must hold:

(a) There exists $\epsilon > 0$ and a tessellation T of $\mathbb{R}^n$ consisting of cubes of fixed size such that $\mu(H\cap\Omega) \geq \epsilon$ for infinitely many cubes $H \in T$.

(b) For every $\lambda > 0$, every tessellation T of $\mathbb{R}^n$ consisting of cubes of fixed size contains only finitely many λ-fat cubes.

We will show in Theorem 6.42 that (b) implies that Ω has finite volume. By Theorem 4.46, (b) is therefore inconsistent with the existence of (24) for $q > p$. On the other hand, (a) implies that $\mu\big(\{x \in \Omega : N \leq |x| \leq N+1\}\big)$ does not approach zero as N tends to infinity. We have therefore proved the following strengthening of Theorem 4.46.

6.41 THEOREM Let Ω be an unbounded domain in $\mathbb{R}^n$ satisfying

$$\limsup_{N\to\infty} \mathrm{vol}(\{x \in \Omega : N \le |x| \le N+1\}) = 0.$$

Then there can be no imbedding of the form (24) for any $q > p$.

6.42 THEOREM Suppose that imbedding (24) is compact for some $q \ge p$. Then Ω has finite volume.

Proof. Let T be a tessellation of $\mathbb{R}^n$ by cubes of unit edge length, and let $\lambda = 1/[2(3^n - 1)]$. Let P be the union of the finitely many λ-fat cubes in T. Clearly $\mu(P \cap \Omega) \le \mu(P) < \infty$.

Let H be a λ-thin cube in T. Let H_1 be one of the $3^n - 1$ cubes in T constituting the fringe of H selected so that $\mu(H_1 \cap \Omega)$ is maximal. Thus

$$\mu(H \cap \Omega) \le \lambda\, \mu\big(F(H) \cap \Omega\big) \le \lambda(3^n - 1)\mu(H_1 \cap \Omega) = \frac{1}{2}\mu(H_1 \cap \Omega).$$

If H_1 is also λ-thin, then we may select a cube $H_2 \in T$ with $H_2 \subset F(H_1)$ such that $\mu(H_1 \cap \Omega) \le \frac{1}{2}\mu(H_2 \cap \Omega)$.

Suppose an infinite chain $\{H_1, H_2, \dots\}$ of λ-thin cubes can be constructed in the above manner. Then

$$\mu(H \cap \Omega) \le \frac{1}{2}\mu(H_1 \cap \Omega) \le \cdots \le \frac{1}{2^j}\mu(H_j \cap \Omega) \le \frac{1}{2^j}$$

for each j since $\mu(H_j \cap \Omega) \le \mu(H_j) = 1$. Hence $\mu(H \cap \Omega) = 0$. Denoting by P_∞ the union of λ-thin cubes $H \in T$ for which such an infinite chain can be constructed, we have $\mu(P_\infty \cap \Omega) = 0$.

Let P_j denote the union of λ-thin cubes $H \in T$ for which some such chain ends on the jth step; that is, H_j is λ-fat. Any particular λ-fat cube H' can occur as the end H_j of a chain beginning at H only if H is contained in the cube of edge $2j+1$ centred on H'. Hence there are at most $(2j+1)^n$ such cubes $H \subset P_j$ having H' as chain endpoint. Thus

$$\begin{aligned}
\mu(P_j \cap \Omega) &= \sum_{H\subset P_j} \mu(H \cap \Omega) \\
&\le \frac{1}{2^j} \sum_{H\subset P_j} \mu(H_j \cap \Omega) \\
&\le \frac{(2j+1)^n}{2^j} \sum_{H'\subset P} \mu(H' \cap \Omega) = \frac{(2j+1)^n}{2^j}\mu(P \cap \Omega),
\end{aligned}$$

so that $\mu(\Omega) = \mu(P \cap \Omega) + \mu(P_\infty \cap \Omega) + \sum_{j=1}^{\infty} \mu(P_j \cap \Omega) < \infty$. ∎

Suppose $1 \le q < p$. If $\mathrm{vol}(\Omega) < \infty$, then the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

exists because $W^{m,p}(\Omega) \to L^p(\Omega)$ trivially and $L^p(\Omega) \to L^q(\Omega)$ by Theorem 2.14.

We are now in a position to prove the converse.

6.43 THEOREM If the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ exists for some p and q satisfying $1 \le q < p$, then Ω has finite volume.

Proof. Let T, λ, and again let P denote the union of the λ-fat cubes in T. If we can show that $\mu(P \cap \Omega)$ is finite, it will follow by the same argument used in that theorem that $\mu(\Omega)$ is finite.

Accordingly, suppose that $\mu(P \cap \Omega)$ is not finite. Then there exists a sequence $\{H_j\}$ of λ-fat cubes in T such that $\sum_{j=1}^{\infty} \mu(H_j \cap \Omega) = \infty$. If L is the lattice of centres of cubes in T, we may break up L into 3^n mutually disjoint sublattices $\{L_i\}_{i=1}^{3^n}$ each having period 3 in each coordinate direction. For each i let T_i be the set of all cubes in T that have centres in L_i. For some i we must have $\sum_{\lambda-\mathrm{fat} H \in T_i} \mu(H \cap \Omega) = \infty$. Thus we may assume that the cubes of the sequence $\{H_j\}$ all belong to T_i for some i, so that $N(H_j) \cap N(H_k)$ is empty if $j \ne k$.

Choose integer j_1 so that

$$2 \le \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 4.$$

Let ϕ_j be as in the proof of Theorem 6.38, and let

$$\psi_1(x) = 2^{-1/p} \sum_{j=1}^{j_1} \phi_j(x).$$

Since the supports of the functions ϕ_j are mutually disjoint and since the cubes H_j are λ-fat, for $|\alpha| \le m$ we have

$$\begin{aligned} \|D^\alpha \psi_1\|_{0,p,\Omega}^p &= \frac{1}{2} \sum_{j=1}^{j_1} \int_\Omega |D^\alpha \phi_j(x)|^p \, dx \\ &\le \frac{1}{2} M^p \sum_{j=1}^{j_1} \mu\big(N(H_j) \cap \Omega\big) \\ &< \frac{1}{2} M^p \left(1 + \frac{1}{\lambda}\right) \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 2M^p \left(1 + \frac{1}{\lambda}\right). \end{aligned}$$

On the other hand,

$$\|\psi_1\|_{0,q,\Omega}^q \geq 2^{-q/p} \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) \geq 2^{1-(q/p)}.$$

Having so defined j_1 and ψ_1, we can now define $j_2, j_3, \dots$ and $\psi_2, \psi_3, \dots$ inductively so that

$$2^k \leq \sum_{j=j_{k-1}+1}^{j_k} \mu(H_j \cap \Omega) < 2^{k+1}$$

and

$$\psi_k(x) = 2^{-k/p} k^{-2/p} \sum_{j=j_{k-1}+1}^{j_k} \phi_j(x).$$

As above, we have for $|\alpha| \leq m$,

$$\|D^\alpha \psi_k\|_{0,p,\Omega}^p \leq \frac{2}{k^2} M^p \left(1 + \frac{1}{\lambda}\right)$$

and

$$\|\psi_k\|_{0,q,\Omega}^q \geq 2^{k(1-q/p)} M^p \left(\frac{1}{k}\right)^{2q/p}.$$

Thus $\psi = \sum_{k=1}^{\infty} \psi_k$ belongs to $W^{m,p}(\Omega)$ but not to $L^q(\Omega)$ contradicting the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$. Hence $\mu(P \cap \Omega) < \infty$ as required. ∎

6.44 If there exists a compact imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ for some $q \geq p$, then, as we have shown, Ω has finite volume. In fact, considerably more is true; $\mu(\{x \in \Omega : |x| \geq R\})$ must approach zero very rapidly as $R \to \infty$, as we show in Theorem 6.45 below.

If Q is a union of cubes H in some tessellation T of $\mathbb{R}^n$ by cubes of fixed edge length, we extend the notions of neighbourhood and fringe to Q in an obvious manner:

$$N(Q) = \bigcup_{\substack{H \in T \\ H \subset Q}} N(H), \qquad F(Q) = N(Q) - Q.$$

Given $\delta > 0$, let $\lambda = \delta/[3^n(1+\delta)]$. If all the cubes $H \in T$ satisfying $H \subset Q$ are λ-thin, then Q is itself δ-thin in the sense that

$$\mu(Q \cap \Omega) \leq \delta \mu\big(F(Q) \cap \Omega\big).$$

To see this note that as H runs through the cubes comprising Q, $F(H)$ covers $N(Q)$ at most 3^n times. Hence

$$\begin{aligned}\mu(Q \cap \Omega) &= \sum_{H \subset Q} \mu(H \cap \Omega) \leq \lambda \sum_{H \subset Q} \mu\big(F(H) \cap \Omega\big) \\ &\leq 3^n \lambda \mu\big(N(Q) \cap \Omega\big) = 3^n \lambda \big[\mu(Q \cap \Omega) + \mu\big(F(Q) \cap \Omega\big)\big]\end{aligned}$$

and the fact that Q is δ-thin follows by transposition (permissible since $\mu(\Omega) < \infty$) and since $3^n\lambda/(1-3^n\lambda) = \delta$.

For any measurable set $S \subset \mathbb{R}^n$ let Q be the union of all cubes in T whose interiors intersect S, and define $F(S) = f(Q)$. If S is at a positive distance from the finitely many λ-fat cubes in T, then Q consists of λ-thin cubes and we obtain

$$\mu(S \cap \Omega) \le \mu(Q \cap \Omega) \le \delta\mu\big(F(S) \cap \Omega\big). \tag{25}$$

6.45 THEOREM (Rapid Decay) Suppose there exists a compact imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{26}$$

for some $q \ge p$. For each $r \ge 0$ let $\Omega_r = \{x \in \Omega : |x| > r\}$, let $S_r = \{x \in \Omega : |x| = r\}$, and let A_r denote the surface area (Lebesgue $(n-1)$-measure) of S_r. Then

(a) For given $\epsilon, \delta > 0$ there exists R such that if $r \ge R$, then

$$\mu(\Omega_r) \le \delta\, \mu\big(\{x \in \Omega : r - \epsilon \le |x| \le r\}\big).$$

(b) If A_r is positive and ultimately nonincreasing as $r \to \infty$, then for each $\epsilon > 0$

$$\lim_{r\to\infty} \frac{A_{r+\epsilon}}{A_r} = 0.$$

Proof. Given $\epsilon > 0$ and $\delta > 0$, let $\lambda = \delta/[3^n(1+\delta)]$ and let T be a tessellation of $\mathbb{R}^n$ by cubes of edge length $\epsilon/(2\sqrt{n})$. Let R be large enough that the finitely many λ-fat cubes in T lie in the ball of radius $R - \epsilon/2$ about the origin. If $r \ge R$ and $S = \Omega_r$, then any $H \in T$ whose interior intersects S is λ-thin. Moreover, any cube in the fringe of S can only intersect Ω at points x satisfying $r - \epsilon/2 \le |x| \le r$. By (25),

$$\mu(\Omega_r) = \mu(S \cap \Omega) \le \delta\, \mu\big(F(S) \cap \Omega\big) = \delta\, \mu\big(\{x \in \Omega : r - \epsilon \le |x| \le r\}\big),$$

which proves (a).

For (b) choose R_0 so that A_r is nonincreasing for $r \ge R_0$. Fix $\epsilon', \delta > 0$ and let $\epsilon = \epsilon'/2$. Let R be as in (a). If $r \ge \max\{R, R_0 + \epsilon'\}$, then

$$\begin{aligned} A_{r+\epsilon'} &\le \frac{1}{\epsilon}\int_{r+\epsilon}^{r+2\epsilon} A_s\, ds \le \frac{1}{\epsilon}\mu(\Omega_{r+\epsilon}) \\ &\le \frac{\delta}{\epsilon}\mu\left(\{x \in \Omega : r \le |x| \le r+\epsilon\}\right) = \frac{\delta}{\epsilon}\int_r^{r+\epsilon} A_s\, ds \le \delta A_r. \end{aligned}$$

Since ϵ' and δ are arbitrary, (b) follows. ∎

6.46 COROLLARY If there exists a compact imbedding of the form (26) for some $q \geq p$, then for every $k > 0$ we have

$$\lim_{r \to \infty} e^{kr} \mu(\Omega_r) = 0. \tag{27}$$

Proof. Fix k and let $\delta = e^{-(k+1)}$. From conclusion (a) of Theorem 6.45 there exists R such that $r \geq R$ implies $\mu(\Omega_{r+1}) \leq \delta \mu(\Omega_r)$. Thus if j is a positive integer and $0 \leq t < 1$, we have

$$\begin{aligned} e^{k(R+j+t)} \mu(\Omega_{R+j+t}) &\leq e^{k(R+j+1)} \mu(\Omega_{R+j}) \\ &\leq e^{k(R+1)}\, e^{kj}\, \delta^j\, \mu(\Omega_R) = e^{k(R+1)}\, \mu(\Omega_R)\, e^{-j}. \end{aligned}$$

The last term approaches zero as j tends to infinity. ∎

6.47 REMARKS

1. We work with Sobolev spaces defined intrinsically in domains. If instead, we had defined $W^{m,p}(\Omega)$ to consist of all restrictions to Ω of functions in $W^{m,p}(R^n)$, then the outcome for Corollary 6.46 would have been different. With that definition, it is shown in [BSc] that $W^{m,p}(\Omega)$ imbeds compactly in $L^p(\Omega)$ if and only if the volume of the intersection of Ω with cubes of fixed edge-length tends to 0 as the centres of those cubes tend to ∞. There are many domains Ω satisfying the latter condition but not satisfying (27). None of these domains can have any Sobolev extension property.
2. The argument used in the proof of Theorem 6.45(a) works for any norm ρ on $\mathbb{R}^n$ in place of the usual Euclidean norm $\rho(x) = |x|$. The same holds for Theorem 6.45(b) provided A_r is well defined (with respect to the norm ρ) and provided

$$\mu(\{x \in \Omega : r \leq \rho(x) \leq r + \epsilon\}) = \int_r^{r+\epsilon} A_s\, ds.$$

   This is true, for example, if $\rho(x) = \max_{1 \leq i \leq n} |x_i|$.
3. For the proof of Theorem 6.45(b) it is sufficient that A_r have an equivalent nonincreasing majorant, that is, there should exist a positive, nonincreasing function $f(r)$ and a constant $M > 0$ such that for sufficiently large r

$$A_r \leq f(r) \leq M\, A_r.$$

4. Theorem 6.38 is sharper than Theorem 6.45, because the conclusions of the latter theorem are global whereas the compactness of (26) depends on local properties of Ω. We illustrate this by means of two examples.

6.48 EXAMPLE Let $f \in C^1([0,\infty))$ be positive and nonincreasing with bounded derivative f'. We consider the planar domain (Figure 5)

$$\Omega = \{(x,y) \in \mathbb{R}^2 \, : \, x > 0, \, 0 < y < f(x)\}.$$

With respect to the supremum norm on $\mathbb{R}^2$, that is $\rho(x,y) = \max\{|x|,|y|\}$, we have $A_s = f(s)$ for sufficiently large s. Hence Ω satisfies conclusion (b) of Theorem 6.45 (and, since f is monotonic, conclusion (a) as well) if and only if

$$\lim_{s\to\infty} \frac{f(s+\epsilon)}{f(s)} = 0 \tag{28}$$

holds for every $\epsilon > 0$. For example, $f(x) = \exp(-x^2)$ satisfies this condition but $f(x) = \exp(-x)$ does not. We shall show in Example 6.53 that the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{29}$$

is compact if (28) holds. Thus (28) is necessary and sufficient for compactness of the above imbedding for domains of this type. ∎

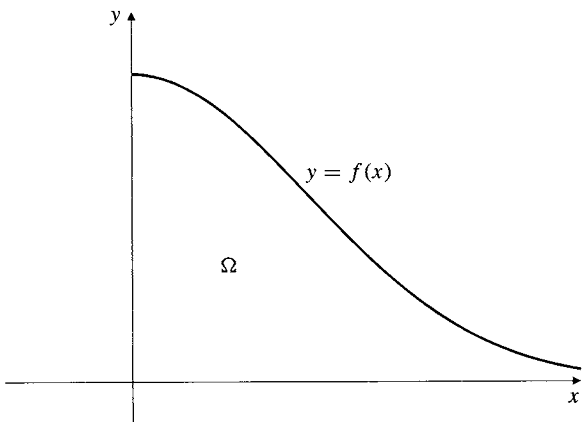

Fig. 5

6.49 EXAMPLE Let f be as in the previous example, and assume also that $f'(0) = 0$. Let g be a positive, nonincreasing function in $C^1([0,\infty))$ satisfying

(i) $g(0) = \frac{1}{2}f(0)$, and $g'(0) = 0$,

(ii) $g(x) < f(x)$ for all $x \geq 0$,

(iii) $g(x)$ is constant on infinitely many disjoint intervals of unit length.

Let $h(x) = f(x) - g(x)$ and consider the domain (Figure 6)

$$\tilde{\Omega} = \{(x,y) \in \mathbb{R}^2 \, : \, 0 < y < g(x) \text{ if } x \geq 0, \, 0 < y < h(-x) \text{ if } x < 0\}.$$

Again we have $A_s = f(s)$ for sufficiently large s, so $\tilde{\Omega}$ satisfies the conclusions of Theorem 6.45 if (28) holds.

If, however, T is a tessellation of $\mathbb{R}^2$ by squares of edge length $\frac{1}{4}$ having edges parallel to the coordinate axes, and if one of the squares in T has centre at the origin, then T has infinitely many $\frac{1}{3}$-fat squares with centres on the positive x-axis. By Theorem 6.38 the imbedding (29) cannot be compact for the domain $\tilde{\Omega}$. ∎

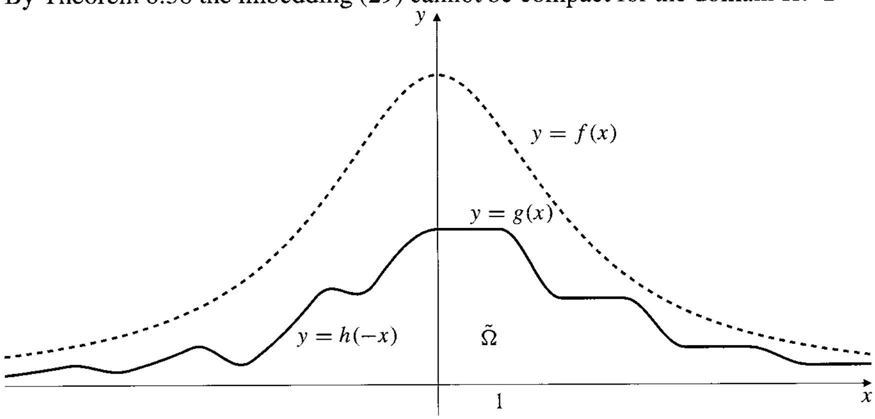

Fig. 6

Unbounded Domains — Compact Imbeddings of $W^{m,p}(\Omega)$

6.50 (Flows) The above examples suggest that any sufficient condition for the compactness of the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega)$$

for unbounded domains Ω must involve the rapid decay of volume locally in each branch of Ω_r as r tends to infinity. A convenient way to express such local decay is in terms of flows on Ω.

By a *flow* on Ω we mean a continuously differentiable map $\Phi : U \to \Omega$ where U is an open set in $\Omega \times \mathbb{R}$ containing $\Omega \times \{0\}$, and where $\Phi(x, 0) = x$ for every $x \in \Omega$.

For fixed $x \in \Omega$ the curve $t \to \Phi(x, t)$ is called a *streamline* of the flow. For fixed t the map $\Phi_t : x \to \Phi(x, t)$ sends a subset of Ω into Ω. We shall be concerned with the Jacobian of this map:

$$\det \Phi_t'(x) = \left. \frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(x_1, \ldots, x_n)} \right|_{(x,t)} .$$

It is sometimes required of a flow Φ that $\Phi_{s+t} = \Phi_s \circ \Phi_t$ but we do not need this property and so do not assume it.

6.51 EXAMPLE Let Ω be the domain of Example 6.48. Define the flow

$$\Phi(x, y, t) = \left(x - t, \frac{f(x-t)}{f(x)} y \right), \qquad 0 < t < x.$$

The direction of the flow is towards the line $x = 0$ and the streamlines (some of which are shown in Figure 7) diverge as the domain widens. Φ_t is a local magnification for $t > 0$:

$$\det \Phi_t'(x, y) = \frac{f(x-t)}{f(x)}.$$

Note that $\lim_{x\to\infty} \det \Phi_t'(x, y) = \infty$ if f satisfies (28).
For $N = 1, 2, \ldots$ let $\Omega_N^* = \{(x, y) \in \Omega : 0 < x < N\}$. Since Ω_N^* is bounded and satisfies the cone condition, the imbedding

$$W^{1,p}\left(\Omega_N^*\right) \to L^p(\Omega_N^*)$$

is compact. This compactness, together with properties of the flow Φ are sufficient to force the compactness of $W^{m,p}(\Omega) \to L^p(\Omega)$ as we now show.

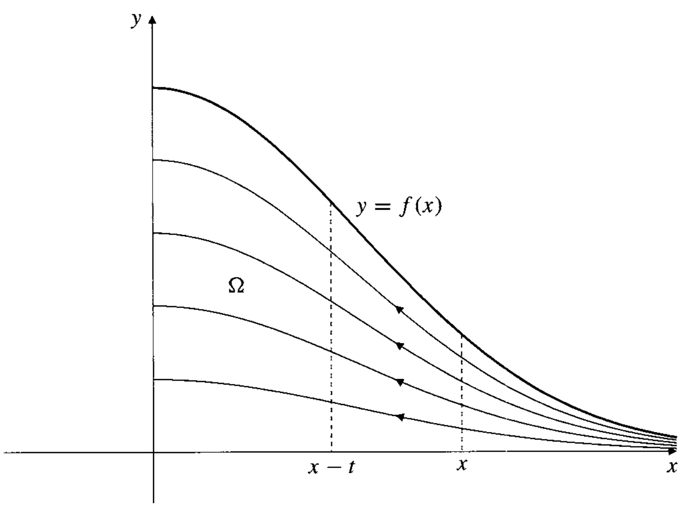

Fig. 7

6.52 THEOREM Let Ω be an open set in $\mathbb{R}^n$ having the following properties:

(a) There exists an infinite sequence $\{\Omega_N^*\}_{N=1}^{\infty}$ of open subsets of Ω such that $\Omega_N^* \subset \Omega_{N+1}^*$ and such that for each N the imbedding

$$W^{1,p}\left(\Omega_N^*\right) \to L^p(\Omega_N^*)$$

is compact.

(b) There exists a flow $\Phi : U \to \Omega$ such that if $\Omega_N = \Omega - \Omega_N^*$, then

(i) $\Omega_N \times [0,1] \subset U$ for each N,

(ii) Φ_t is one-to-one for all t,

(iii) $|(\partial/\partial t)\Phi(x,t)| \le M$ (constant) for all $(x,t) \in U$.

(c) The functions $d_N(t) = \sup_{x\in\Omega_N} |\det \Phi_t'(x)|^{-1}$ satisfy

(i) $\lim_{N\to\infty} d_N(1) = 0$,

(ii) $\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0$.

Then the imbedding $W^{m,p}(\Omega) \to L^p(\Omega)$ is compact.

Proof. Since we have $W^{m,p}(\Omega) \to W^{1,p}(\Omega) \to L^p(\Omega)$ it is sufficient to prove that the latter imbedding is compact. Let $u \in C^1(\Omega)$. For each $x \in \Omega_N$ we have

$$u(x) = u\big(\Phi_1(x)\big) - \int_0^1 \frac{\partial}{\partial t} u\big(\Phi_t(x)\big)\,dt.$$

Now

$$\begin{aligned}\int_{\Omega_N} \left|u\big(\Phi_1(x)\big)\right| dx &\le d_N(1) \int_{\Omega_N} \left|u\big(\Phi_1(x)\big)\right| \left|\det \Phi_1'(x)\right| dx \\ &= d_N(1) \int_{\Phi_1(\Omega_N)} |u(y)|\,dy \\ &\le d_N(1) \int_{\Omega} |u(y)|\,dy.\end{aligned}$$

Also

$$\begin{aligned}\int_{\Omega_N} \left| \int_0^1 \frac{\partial}{\partial t} u\big(\Phi_t(x)\big)\,dt \right| dx &\le \int_{\Omega_N} dx \int_0^1 \left|\operatorname{grad} u\big(\Phi_t(x)\big)\right| \left|\frac{\partial}{\partial t}\Phi_t(x)\right| dt \\ &\le M \int_0^1 d_N(t)\,dt \int_{\Omega_N} \left|\operatorname{grad} u\big(\Phi_t(x)\big)\right| \left|\det \Phi_t'(x)\right| dx \\ &\le M \left(\int_0^1 d_N(t)\,dt\right) \left(\int_{\Omega} |\operatorname{grad} u(y)|\,dy\right).\end{aligned}$$

Putting $\delta_N = \max\left\{d_N(1), M\int_0^1 d_N(t)\,dt\right\}$, we have

$$\int_{\Omega_N} |u(x)|\,dx \le \delta_N \int_{\Omega} \big(|u(y)| + |\operatorname{grad} u(y)|\big)\,dy \le \delta_N \|u\|_{1,1,\Omega} \tag{30}$$

and $\lim_{N\to\infty} \delta_N = 0$.

Now suppose u is real-valued and belongs to $C^1(\Omega) \cap W^{1,p}(\Omega)$. By Hölder's inequality, the distributional derivatives of $|u|^p$

$$D_j|u|^p = p \cdot |u|^{p-1} \cdot \operatorname{sgn} u \cdot D_j u,$$

satisfy

$$\int_\Omega \left|D_j|u(x)|^p\right| dx \le p \left\|D_j u\right\|_{0,p,\Omega} \|u\|_{0,p,\Omega}^{p-1} \le p \|u\|_{1,p,\Omega}^p .$$

Thus $|u|^p \in W^{1,1}(\Omega)$ and by Theorem 3.17 there is a sequence $\{\phi_j\}$ of functions in $C^1(\Omega) \cap W^{1,1}(\Omega)$ such that $\lim_{j\to\infty} \left\|\phi_j - |u|^p\right\|_{1,1,\Omega} = 0$. Thus, by (30)

$$\begin{aligned}\int_{\Omega_N} |u(x)|^p\,dx &= \lim_{j\to\infty} \int_{\Omega_N} \phi_j(x)\,dx \le \limsup_{j\to\infty} \delta_N \left\|\phi_j\right\|_{1,1,\Omega} \\ &\le \delta_N \left\| |u|^p \right\|_{1,1,\Omega} \le K\delta_N \|u\|_{1,p,\Omega}^p ,\end{aligned}$$

where $K = K(n, p)$. This inequality holds for arbitrary complex-valued function $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ by virtue of its separate applications to the real and imaginary parts of u.

If S is a bounded set in $W^{1,p}(\Omega)$ and $\epsilon > 0$, we may, by the above inequality, select N so that for all $u \in S$

$$\int_{\Omega_N} |u(x)|^p\,dx < \epsilon.$$

Since $W^{1,p}(\Omega - \Omega_N) \to L^p(\Omega - \Omega_N)$ is compact, the precompactness of S in $L^p(\Omega)$ follows by Theorem 2.33. Hence $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. ∎

6.53 EXAMPLE Consider again the domain of Examples 6.48 and 6.51 and the flow Φ given in the latter example. We have

$$d_N(t) = \sup_{x\ge N} \frac{f(x)}{f(x-t)} \le 1 \qquad \text{if} \quad 0 \le t \le 1$$

and by (28)

$$\lim_{N\to\infty} d_N(t) = 0 \qquad \text{if} \quad t > 0.$$

Thus by dominated convergence

$$\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0.$$

The assumption that f' is bounded guarantees that the speed $|(\partial/\partial t)\Phi(x,y,t)|$ is bounded on U. Thus Ω satisfies the hypotheses of Theorem 6.52 and the imbedding $W^{m,p}(\Omega) \to L^p(\Omega)$ is compact for this domain. ∎

6.54 EXAMPLE Theorem 6.52 can also be used to show the compactness of $W^{m,p}(\Omega) \to L^p(\Omega)$ for some bounded domains to which neither the Rellich-Kondrachov theorem nor the techniques used in its proof can be applied. For example, we consider

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \; : \; 0 < x < 2,\; 0 < y < f(x)\right\},$$

where $f \in C^1([0,2])$ is positive, nondecreasing, has bounded derivative f', and satisfies $\lim_{x\to 0+} f(x) = 0$. Let

$$U = \left\{(x,y,t) \in \mathbb{R}^3 \; : \; (x,y) \in \Omega,\; -x < t < 2 - x\right\}$$

and define the flow $\Phi : U \to \Omega$ by

$$\Phi(x,y,t) = \left(x + t,\; \frac{f(x+t)}{f(x)}\, y\right).$$

Then $\det \Phi_t'(x,y) = f(x+t)/f(x)$. If $\Omega_N^* = \{(x,y) \in \Omega : x > 1/N\}$, then

$$d_N(t) = \sup_{0<x\le 1/N} \left| \frac{f(x)}{f(x+t)} \right|$$

satisfies $d_N(t) \le 1$ for $0 \le t \le 1$, and $\lim_{N\to\infty} d_N(t) = 0$ if $t > 0$. Hence also $\lim_{N\to\infty} \int_0^1 d_N(t)\, dt = 0$ by dominated convergence. Since Ω_N^* is bounded and satisfies the cone condition, and since the boundedness of $\partial\Phi/\partial t$ is assured by that of f', we have, by Theorem 6.52 the compactness of

$$W^{m,p}(\Omega) \to L^p(\Omega). \tag{31}$$

However, suppose that $\lim_{x\to 0+} f(x)/x^k = 0$ for every k. (For example, this is true if $f(x) = e^{-1/x}$.) Then Ω has an exponential cusp at the origin and by Theorem 4.48 there exists no imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ for any $q > p$ so the method of proof of the Rellich-Kondrachov theorem cannot be used to show the compactness of (31).

6.55 REMARKS

1. It is easy to imagine domains more general than those in the above examples to which Theorem 6.52 applies, although it may be difficult to specify an appropriate flow. A domain with many (perhaps infinitely many) unbounded branches can, if connected, admit a suitable flow provided volume decays sufficiently rapidly in each branch, a condition not fulfilled by the domain $\tilde{\Omega}$ in Example 6.49. For unbounded domains in which volume

decays monotonically in each branch Theorem 6.45 is essentially a converse of Theorem 6.52 in that the proof of Theorem 6.45 can be applied separately to show that the volume decays in each branch in the required way.

2. Since the only unbounded domains for which $W^{m,p}(\Omega)$ imbeds compactly into $L^p(\Omega)$ have finite volume there can be no extensions of Theorem 6.52 to give compact imbeddings into $L^q(\Omega)$ (where $q > p$), or $C_B(\Omega)$ etc.; there do not exist such imbeddings.

Hilbert-Schmidt Imbeddings

6.56 (Complete Orthonormal Systems) A *complete orthonormal system* in a separable Hilbert space X is a sequence $\{e_i\}_{i=1}^{\infty}$ of elements of X satisfying

$$(e_i, e_j)_X = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

(where $(\cdot, \cdot)_X$ is the inner product on X), and such that for each $x \in X$ we have

$$\lim_{k\to\infty} \left\| x - \sum_{i=1}^{k} (x, e_i)_X e_i \, ; X \right\| = 0. \tag{32}$$

Thus $x = \sum_{i=1}^{\infty} (x, e_i) e_i$, the series converging with respect to the norm in X. It is well known that every separable Hilbert space possesses such a complete orthonormal system. There follows from (32) the Parseval identity

$$\|x \, ; X\|^2 = \sum_{i=1}^{\infty} |(x, e_i)_X|^2.$$

6.57 (Hilbert-Schmidt Operators) Let X and Y be two separable Hilbert spaces and let $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be given complete orthornomal systems in X and Y respectively. Let A be a bounded linear operator with domain X taking values in Y, and let A^* be the adjoint of A taking Y into X and defined by

$$(x, A^* y)_X = (Ax, y)_Y, \qquad x \in X, \quad y \in Y.$$

Define

$$|||A|||^2 = \sum_{i=1}^{\infty} \|Ae_i \, ; Y\|^2, \qquad |||A^*|||^2 = \sum_{i=1}^{\infty} \left\|A^* f_i \, ; X\right\|^2.$$

If $|||A|||$ is finite, A is called a *Hilbert-Schmidt operator* and we call $|||A|||$ its *Hilbert-Schmidt norm*. Recall that the operator norm of A is given by

$$\|A\| = \sup\big\{\|Ax\,;Y\| \,:\, \|x\,;X\| \le 1\big\}.$$

We must justify the definition of the Hilbert-Schmidt norm.

6.58 LEMMA The norms $|||A|||$ and $|||A^*|||$ are independent of the particular orthonormal systems $\{e_i\}$ and $\{f_i\}$ used to define them. Moreover

$$|||A||| = |||A^*||| \ge \|A\|.$$

Proof. By Parseval's identity

$$\begin{aligned}|||A|||^2 &= \sum_{i=1}^{\infty} \|Ae_i\,;Y\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |(Ae_i, f_j)_Y|^2 \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |(e_i, A^*f_j)_X|^2 = \sum_{j=1}^{\infty} \big\|A^*f_j\,;X\big\|^2 = |||A^*|||^2.\end{aligned}$$

Hence each expression is independent of $\{e_i\}$ and $\{f_j\}$. For any $x \in X$ we have

$$\begin{aligned}\|Ax\,;Y\|^2 &= \left\|\sum_{i=1}^{\infty}(x, e_i)_X Ae_i\,;Y\right\|^2 \le \left(\sum_{i=1}^{\infty} |(x, e_i)_X|\,\|Ae_i\,;Y\|\right)^2 \\ &\le \left(\sum_{i=1}^{\infty} |(x, e_i)_X|^2\right)\left(\sum_{j=1}^{\infty} \big\|Ae_j\,;Y\big\|^2\right) = \|x\,;X\|^2\,|||A|||^2.\end{aligned}$$

Hence $\|A\| \le |||A|||$ as required. ∎

6.59 REMARK Consider the scalars (Ae_i, f_j) for $1 \le i, j < \infty$; they are the entries in an infinite matrix representing the operator A. The lemma above shows that the Hilbert-Schmidt norm of A is the sum of the squares of the absolute values of the elements of this matrix. Similarly, the numbers (A^*f_j, e_i) are the entries in a matrix representing A^*. Since these matrices are adjoints of each other, the equality of the corresponding Hilbert-Schmidt norms of the operators is assured.

6.60 We leave to the reader the task of verifying the following assertions.

(a) If X, Y, and Z are separable Hilbert spaces and A and B are bounded linear operators from X into Y and Y into Z, respectively, then $B \circ A$, which maps X into Z, is a Hilbert-Schmidt operator if either A or B is. If A is Hilbert-Schmidt, then $|||B \circ A||| \le \|B\|\,|||A|||$.

(b) Every Hilbert-Schmidt operator is compact.

The following Theorem, due to Maurin [Mr] has far-reaching implications for eigenfunction expansions corresponding to differential operators.

6.61 THEOREM (Maurin's Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let m and k be nonnegative integers with $k > n/2$. Then the imbedding map

$$W^{m+k,2}(\Omega) \to W^{m,2}(\Omega) \tag{33}$$

is a Hilbert-Schmidt operator. Similarly the imbedding map

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2}(\Omega) \tag{34}$$

is a Hilbert-Schmidt operator for any bounded domain Ω.

Proof. Given $y \in \Omega$ and α with $|\alpha| \le m$ we define a linear functional T_y^α on $W^{m+k,2}(\Omega)$ by

$$T_y^\alpha(u) = D^\alpha u(y).$$

Since $2k > m$, the Sobolev Imbedding Theorem 4.12 implies that T_y^α is bounded on $W^{m+k,2}(\Omega)$ and has norm bounded by a constant K independent of y and α:

$$|T_y^\alpha(u)| \le \max_{0\le|\alpha|\le m} \sup_{x\in\Omega} |D^\alpha u(x)| \le K \, \|u\|_{m+k,2,\Omega} \, .$$

By the Riesz representation theorem for Hilbert spaces there exists $v_y^\alpha \in W^{m+k,2}(\Omega)$ such that

$$D^\alpha u(y) = T_y^\alpha(u) = \left(u, v_y^\alpha\right)_{m+k},$$

where $(\cdot, \cdot)_{m+k}$ is the inner product on $W^{m+k,2}(\Omega)$. Moreover

$$\left\| v_y^\alpha \right\|_{m+k,2,\Omega}^2 = \left\| T_y^\alpha \, ; \, \left[W^{m+k,2}(\Omega)\right]' \right\| \le K.$$

If $\{e_i\}_{i=1}^\infty$ is a complete orthonormal system in $W^{m+k,2}(\Omega)$, then

$$\left\| v_y^\alpha \right\|_{m+k,2,\Omega}^2 = \sum_{i=1}^\infty \left| \left(e_i, v_y^\alpha\right)_{m+k} \right|^2 = \sum_{i=1}^\infty |D^\alpha e_i(y)|^2.$$

Consequently,

$$\sum_{i=1}^\infty \|e_i\|_{m,2,\Omega}^2 \le \sum_{|\alpha|\le m} \int_\Omega \left\| v_y^\alpha \right\|_{m+k,2,\Omega}^2 \, dy \le \sum_{|\alpha|\le m} K \operatorname{vol}(\Omega) < \infty.$$

Hence imbedding (33) is Hilbert-Schmidt. The corresponding imbedding (34) is also Hilbert-Schmidt without the cone-condition requirement as it is not needed for the application of Theorem 4.12 in this case. ∎

The following generalization of Maurin's theorem is due to Clark [Ck].

6.62 THEOREM Let μ be a nonnegative, measurable function defined on the domain Ω in $\mathbb{R}^n$. Let $W_0^{m,2;\mu}(\Omega)$ be the Hilbert space obtained by completing $C_0^\infty(\Omega)$ with respect to the weighted norm

$$\|u\|_{m,2;\mu} = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2 \,\mu(x)\,dx\right)^{1/2}.$$

For $y \in \Omega$ let $\tau(y) = \operatorname{dist}(y, \operatorname{bdry}\Omega)$. Suppose that

$$\int_\Omega (\tau(y))^{2\nu}\,\mu(y)\,dy < \infty \tag{35}$$

for some nonnegative integer ν. If $k > \nu + n/2$, then the imbedding

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2;\mu}(\Omega) \tag{36}$$

(exists and) is Hilbert-Schmidt.

Proof. The argument is parallel to that given in the proof of Maurin's theorem above. Let $\{e_i\}$, T_y^α, and v_y^α be defined as there. If $y \in \Omega$, let y_0 be chosen in $\operatorname{bdry}\Omega$ such that $\tau(y) = |y - y_0|$. If ν is a positive integer and $u \in C_0^\infty(\Omega)$, we have by Taylor's formula with remainder

$$D^\alpha u(y) = \sum_{|\beta|=\nu} \frac{1}{\beta!} D^{\alpha+\beta} u(y_\beta)(y - y_\beta)^\beta$$

for some points y_β satisfying $|y - y_\beta| \le \tau(y)$. If $|\alpha| \le m$ and $k > \nu + n/2$, we obtain from Theorem 4.12

$$|D^\alpha u(y)| \le K\,\|u\|_{m+k,2,\Omega}\,(\tau(y))^\nu.$$

By completion this inequality holds for any $u \in W_0^{m+k,2}(\Omega)$. As in the proof of Maurin's theorem, it follows that

$$\left\|v_y^\alpha\right\|_{m+k,2,\Omega} = \sup_{\|u\|_{m+k,2,\Omega}=1} |D^\alpha u(y)| \le K(\tau(y))^\nu,$$

and hence also that

$$\sum_{i=1}^{\infty} \|e_i\|_{m,2;\mu}^2 \le \sum_{|\alpha|\le m} \int_\Omega \|v_y^\alpha\|_{m+k,2,\Omega}^2 \, \mu(y)\, dy$$
$$\le K^2 \sum_{|\alpha|\le m} \int_\Omega \big(\tau(y)\big)^{2\nu} \mu(y)\, dy < \infty$$

by (35). Hence imbedding (36) is Hilbert-Schmidt. ∎

6.63 REMARK Various choices of μ and ν lead to generalizations of Maurin's theorem for imbeddings of the sort (34). If $\mu(x) = 1$ and $\nu = 0$ we obtain the obvious generalization to unbounded domains of finite volume. If $\mu(x) = 1$ and $\nu > 0$, Ω may be unbounded and even have infinite volume, but it must be quasibounded by (35). Of course quasiboundedness may not be sufficient to guarantee (35). If μ is the characteristic function of a bounded subdomain Ω_0 of Ω, and $\nu = 0$, we obtain the Hilbert-Schmidt imbedding

$$W_0^{m+k,2}(\Omega) \to W^{m,2}(\Omega_0), \qquad k > n/2.$$