General Sobolev Inequalities and Embeddings on Riemannian Manifolds

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Introduction

Our goal in this report is to introduce embeddings of various Sobolev spaces into others. The crucial analytic tools here will be certain so-called "Sobolev inequalities".

Though Riemannian manifolds are natural extensions of Euclidean space, several important questions still puzzle mathematicians today.

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Sobolev Spaces on $\mathbf{R}^n$ [1]

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General Sobolev Inequalities (kp<n)

Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$.

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}$$

the constant C depending only on k, p, n and U.

[1] Evans, Lawrence C. *Partial differential equations.* Intersxcience Publishers, 1964.

Sobolev Spaces on $\mathbb{R}^n$ [2]

General Sobolev Inequalities (kp>n)

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$||u||_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{U})} \leq C||u||_{W^{k,p}(U)}$$

the constant C depending only on k, p, n, γ and U.

[2] Evans, Lawrence C. *Partial differential equations.* Intersxcience Publishers, 1964.

Sobolev Spaces on $\mathbf{R}^n$ [3]

Rellich-Kondrachov Compactness Theorem (Compactness)

Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \leq p < n$. Then

$$\boldsymbol{W^{1,p}(U) \subset\subset L^q(U)}$$

for each $1 \leq q < p^*$, where $1 \leq p < n, p^* = \frac{pn}{n-p}$

Before we start on Riemannian manifolds, we may ask:

1. ∂U *is* C^1 is an important assumption, what about manifolds without boundary?
2. can we find better embedded spaces which have more compactness properties?
3. how to generalize the Sobolev inequalities to Riemannian manifolds?

[3] Evans, Lawrence C. *Partial differential equations.* Intersxcience Publishers, 1964.

Sobolev Spaces on $\mathbf{R}^n$

Let Ω be some open subset of $\mathbb{R}^n$, k an integer, $p \geq 1$ real, and $u : \Omega \to \mathbb{R}$ a smooth, real-valued function. We define then the Sobolev spaces

$$H_k^p(\Omega) = \text{ the completion of } \{u \in C^\infty(\Omega)\,,\ ||u||_{k,p} < +\infty\} \text{ for } ||\cdot||_{k,p}$$
$$W^{k,p}(\Omega) = \{u \in L^p(\Omega)\,,\ \forall|\alpha| \leq k, D_\alpha u \text{ exists and belongs to } L^p(\Omega)\}$$

[Meyers-Senin] For any Ω, any k, and any $p \geq 1$, $H_k^p(\Omega) = W^{k,p}(\Omega)$. [4]

[4] Adams, R. A. *Sobolev spaces.* Academic Press, San Diego, 1978.

Sobolev Spaces on (M, g) [5]

Sobolev Spaces on (M, g)

Given (M, g) a smooth Riemannian manifold, k an integer, and $p \geq 1$ real, the Sobolev space $\boldsymbol{H_k^p(M)}$ **is the completion of** $\boldsymbol{\mathcal{C}_k^p(M)}$ **with respect to** $||\cdot||_{\boldsymbol{H_k^p}}$, where

$$\mathcal{C}_k^p(M) = \left\{ u \in C^\infty(M)\,,\, \forall j = 0, \dots, k, \int_M |\nabla^j u|^p \, dv(g) < +\infty \right\}$$

When M is compact, one clearly has that $\mathcal{C}_k^p(M) = \mathcal{C}^\infty(M)$ for any k and any $p \geq 1$. For $u \in \mathcal{C}_k^p(M)$, set also

$$||u||_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p \, dv(g) \right)^{1/p}$$

[5] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities.* 1999.

Compact Manifolds without Boundary [6]

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General Sobolev Inequalities on Compact Manifolds without Boundary

Let (M, g) be a smooth, compact Riemannian n-manifold. Let $p \geq 1$ be real and $0 \leq m < k$ be two integers.

(A1) If $1/q = 1/p - (k-m)/n > 0$, then $H_k^p(M) \subset H_m^q(M)$.

(A2) If $1/q = 1/p - (k-m)/n < 0$, then $H_k^p(M) \subset C^m(M)$.

where

$$||u||_{H_k^p} = \sum_{j=0}^{k} \left(\int_M \left|\nabla^j u\right|^p dv(g) \right)^{1/p} \text{ and } ||u||_{C^m} = \sum_{j=0}^{m} \max_{x \in M} \left| \left(\nabla^j u\right)(x) \right|$$

[6] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities.* 1999.

Compact Manifolds without Boundary [7]

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General Sobolev Embeddings on Compact Manifolds without Boundary

Let (M, g) be a smooth, compact Riemannian n-manifold. Let $p \geq 1$ be real and $0 \leq m < k$ be two integers.

(A1) For any real q such that $1 \leq q < \frac{np}{(n-(k-m)p)}$, $H_k^p(M) \subset\subset H_m^q(M)$.

Take $k = 1, m = 0$, for any $p < n$ real and any $1 \leq q < \frac{np}{n-p}$, $H_1^p(M) \subset\subset L^q(M)$.

(A2) Take $k = 1, m = 0$, for $p > n$ and for any $\lambda \in (0, 1)$, such that $(1-\lambda)p > n$, $H_1^p(M) \subset\subset C^\lambda(M)$. Particularly, $H_1^p(M) \subset\subset C^0(M)$.

$$||u||_{C^\lambda} = \max_{x\in M} |u(x)| + \max_{x\neq y\in M} \frac{|u(y) - u(x)|}{d_g(x,y)^\lambda}$$

[7] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Complete Manifolds without Boundary [8]

General Sobolev Inequalities on Complete Manifolds without Boundary

Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below. Assume that

$$\inf_{x \in M} \mathrm{Vol}_g\left(B_x(1)\right) > 0$$

where $\mathrm{Vol}_g\left(B_x(1)\right)$ stands for the volume of $B_x(1)$ with respect to g. Then the Sobolev embeddings in their first part (A1) are valid for (M, g). i.e.

(B1) If $\boldsymbol{1/q = 1/p - (k-m)/n > 0}$**, then** $\boldsymbol{H_k^p(M) \subset H_m^q(M)}$.

The assumption of Ricci curvature is satisfactory but not necessary. e.g. $H_1^p \subset L^q$

[8] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities.* 1999.

Complete Manifolds without Boundary [9]

General Sobolev Inequalities on Complete Manifolds without Boundary

Let (M, g) be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below and positive injectivity radius.

For $p \geq 1$ real and $0 \leq m < k$ two integers,

(B2) If $\boldsymbol{1/q = 1/p - (k-m)/n < 0}$, then $\boldsymbol{H_k^p(M) \subset C_B^m(M)}$.

where $C_B^m(\Omega)$ consists of the functions $u \in C^m(\Omega)$ and $\nabla^j u$ is bounded on M for $0 \leq |j| \leq m$. Particularly, for $p \geq 1$ real and $\lambda \in (0, 1)$ real, if $1/p \leq (1-\lambda)/n$, then $\boldsymbol{H_1^p(M) \subset C_B^\lambda(M)}$.

[9] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities.* 1999.

Compact Manifold with Boundary [10]

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Sobolev Inequalities on Compact Manifolds with Boundary

Let (M, g) be a smooth, compact, n-dimensional Riemannian manifold with boundary. For $p < n$ real, set $p^* = np/(n-p)$. Then for any $q \in [1, p^*]$, $\boldsymbol{H_1^p(M) \subset L^q(M)}$

Sobolev Embeddings on Compact Manifolds with Boundary

If $q \in [1, p^*)$, the embedding above is compact, i.e. $\boldsymbol{H_1^p(M) \subset\subset L^q(M)}$.

[10] Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Thank You