Relations Between Topological Spaces

Liguang Hou

Abstract

This paper summarized some basic topological spaces and their relationships as below.

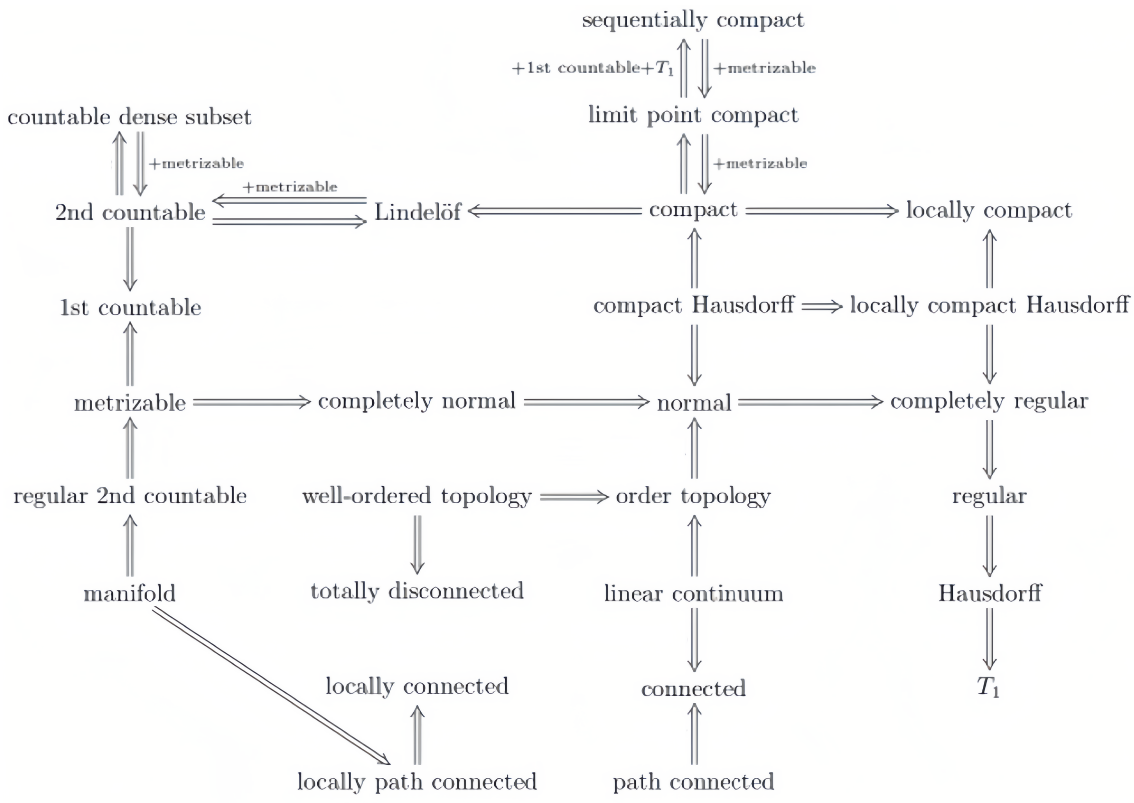

Figure 1: Relations between topological spaces

Introduction

We have learned many different topological spaces that vary in structure, but share some relationships in terms of connectedness, compactness, and separateness. In this paper, we will give some definitions that were not mentioned in the class, and summarize the relations between some common topological spaces. Our work is shown in Figure 1.

Additional Definitions

1. Sets and maps

Def 1.1 Linear order. A linear order on the set A is a relation $<\subset A \times A$ that is

Comparable: If $a \neq b$ then $a < b$ or $b < a$ for all $a, b \in A$

Nonreflexive: $a < a$ for no $a \in A$

Transitive: $a < b$ and $b < c \implies a < c$ for all $a, b, c \in A$

Def 1.2 Dictionary order. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The dictionary order on $A \times B$ is the linear order given by

$$(a_1, b_1) < (a_2, b_2) \iff (a_1 < a_2) \text{ or } (a_1 = a_2 \text{ and } b_1 < b_2)$$

The restriction of a dictionary order to a product subspace is the dictionary order of the restricted linear orders.

Def 1.3 Linear continuum. An ordered set $(A, <)$ has the least upper bound property if any nonempty subset of A that has an upper bound has a least upper bound. If also $(x, y) \neq \emptyset$ for all $x < y$, then $(A, <)$ is a linear continuum.

Def 1.4 Well-ordered sets. A set A with a linear order $<$ is well-ordered if any nonempty subset has a smallest element.

2. Topological spaces and continuous maps

Def 2.1 Order topology. Let $(X, <)$ be a linearly ordered set containing at least two points. The open rays in X are the subsets

$$(-\infty, b) = \{x \in X \mid x < b\}\ ,\ (a, +\infty) = \{x \in X \mid a < x\}$$

of X. The order topology $T_<$ on the linearly ordered set X is the topology generated by all open rays. A linearly ordered space is a linearly ordered set with the order topology. The open intervals in X are the subsets of the form

$$(a, b) = (-\infty, b) \cap (a, +\infty) = \{x \in X \mid a < x < b\},\ a, b \in X,\ a < b.$$

Def 2.2 Separation axioms T_0, T_1, and T_2.

T_0**-space**: A topological space X is a T_0-space (or a Kolmogorov space) if for any two distinct points $x_1 \neq x_2$ in X there exists an open set U containing one but not both points.

T_1**-space**: A topological space is T_1-space if points are closed: For any two distinct points $x_1 \neq x_2$ in X there exists an open set U such that $x_1 \in U$ and $x_2 \notin U$.

T_2**-space**: A topological space X is a T_2-space (or a Hausdorff space) if there are enough open sets to separate points: For any two distinct points $x_1 \neq x_2$ in X there exist disjoint open sets, U_1 and U_2, such that $x_1 \in U_1$ and $x_2 \in U_2$.

Def 2.3 Limit point compactness and sequential compactness. A space X is

(1) limit point compact if any infinite subset of X has a limit point.

(2) sequentially compact if any sequence in X has a convergent subsequence.

Def 2.4 Locally compact. A space X is locally compact at the point $x \in X$ if x lies in the interior of some compact subset of X. A space is locally compact if it is locally compact at each of its points.

3. Regular and normal spaces

Def 3.1 Lindelöf. A topological space in which every open covering has a countable subcovering is called a Lindelöf Space.

Def 3.2 Completely regular. A space X is completely regular if points are closed in X and for any closed subset C of X and any point $x \notin C$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$.

Def 3.3 Completely normal. A space X is completely normal if every subspace of X is normal, equivalent to the fact that for any two disjoint closed subsets of X there exist disjoint open sets containing them.

Proof of Relations

[Deduction 1]

$$connected \overset{1}{\Longleftarrow} linear\ continuum \overset{2}{\Longrightarrow} order\ topology \overset{3}{\Longrightarrow} normal \tag{1}$$

Proof 1. [Lemma 1] Suppose that X is a linear continuum. Then [1](2.118)

$$\{connected\ subsets\ of\ X\} = \{convex\ subsets\ of\ X\}$$

Assume X is a linear continuum. From [Lemma 1] we know that the the connected and the convex subsets of X are the same. In particular, the linear continuum X, certainly convex in itself, is connected in the order topology. Let C be a nonempty convex subset of X. We look at two cases:

- **Case 1.** C is neither bounded from above nor below. Let x be any point of X. Since x is neither a lower nor an upper bound for C there exist $a, b \in C$ so that $a < x < b$. Then $x \in C$ by convexity. Thus $C = X$.
- **Case 2.** C is bounded from above but not from below. Let $c = sup\ C$ be its least upper bound. Then $C \subset (-\infty, c]$. Let $x < c$ be any point. Since x is neither a lower nor an upper bound for C there exist $a, b \in C$ so that $a < x < b$. Then $x \in C$ by convexity. Thus $(-\infty, c) \subset C \subset (-\infty, c]$ and C is either $(-\infty, c)\ or\ (-\infty, c]$.
- **Case 3.** The arguments are similar for the other cases, since X also has the greatest lower bound property [2] (Ex 3.13).

Proof 2. From Proof 1. we know that

$$X\ is\ a\ linear\ continuum \implies X\ is\ connected\ in\ the\ order\ topology$$

Proof 3. Linearly ordered spaces are normal.

General proofs can be referred to [3] (Problem 1.7.4). We shall only prove the special case that every well-ordered space is normal.

- **Step 1.** Let X be a well-ordered set. In the order topology, sets of the form $(a, +\infty) = [a^+, +\infty) = X - (-\infty, a]$, $(-\infty, b] = (-\infty, b+) = X - (b, +\infty)$ and $(a, b] = (a, +\infty) \cap (-\infty, b]$ are closed and open.
- **Step 2.** Let A and B be two disjoint closed subsets and let a_0 denote the smallest element of X. Suppose that neither A nor B contain a_0. For any point $a \in A$ there exists a point $x_a < a$ such that $(x_a, a]$ is disjoint from B. Similarly, for any point $b \in B$ there exists a point $x_b < b$ such that $(x_b, b]$ is disjoint from A. The proof now proceeds as the proof for normality of R_l. Suppose next that $a_0 \in A \cup B$, say $a_0 \in A$. The one-point set $a_0 = [a_0, a_0^+)$ is open and closed (as X is Hausdorff). By the above, we can find disjoint open sets U, V such that $A - a_0 \subset U$ and $B \subset V$. Then $A \subset U \cup a_0$ and $B \subset V - a_0$ where the open sets $U \cup a_0$ and $V - a_0$ are disjoint.

[Deduction 2]

$$well-ordered\ topology \overset{4}{\Longrightarrow} totally\ disconnected \tag{2}$$

Proof 4. A space is totally disconnected if its only connected subspaces are one-point sets. Any well-ordered set X containing at least two points is totally disconnected in the order topology. For if $C \subset X$ contains $a < b$, then $a \notin C \cap (a, b] \ni b$ is closed and open in C, since $(a, b]$ is closed and open in X.

[Deduction 3]

$$compact \overset{5}{\Longrightarrow} limit\ point\ compact \overset{6\ (+1st\ countable + T_1)}{\Longrightarrow} sequentially\ compact \tag{3}$$

Proof 5. [Lemma 2] Let A be a subset of X and A' the set of limit points of A. Then $A \cup A' = A$ and $A \cap A' = \{a \in A \mid a\ is\ not\ an\ isolated\ point\ of\ A\}$ so that

$$\begin{aligned} A \supset A' &\iff A\ is\ closed \\ A \subset A' &\iff A\ has\ no\ isolated\ points \\ A \cap A' = \emptyset &\iff A\ is\ discrete \\ A' = \emptyset &\iff A\ is\ closed\ and\ discrete \end{aligned}$$

For any compact topological space X, a subset with no limit points is closed and discrete [Lemma 2], hence finite (Any closed subspace of a compact space is compact).

Proof 6. Let X be a limit point compact space, 1st countable and T_1. Let (x_n) be a sequence in X. Consider the set $A = \{x_n \mid n \in Z_+\}$ of points in the sequence.

- **Case 1.** If A is finite, there is a constant subsequence.
- **Case 2.** If A is infinite, A has a limit point x by hypothesis. As X is 1st countable, there is a countable nested countable basis $U_1 \supset U_2 \supset \cdots$ at x just as in the proof of [1](2.100). Since x is a limit point, there is a sequence element $x_{n_1} \in U_1$ and we can find $n_1 < \cdots < n_k$ such that $x_{n_i} \in U_i$. Since x is a limit point and X is T_1, there are infinitely many points from A in U_{k+1}. In particular, there is $n_{k+1} < n_k$ such that $x_{n_{k+1}}$ is in U_{k+1}. The subsequence x_{n_k} converges to x.

[Deduction 4]

$$compact \overset{7\ (+metrizable)}{\Longleftarrow} limit\ point\ compact \overset{8\ (+metrizable)}{\Longleftarrow} sequentially\ compact \tag{4}$$

Proof 7. 8. All three forms of compactness are equivalent for a metrizable space. It is well-known from our experience with metric spaces that

$$X\ is\ sequentially\ compact \iff X\ is\ compact.$$

Hence X is sequentially compact and metrizable $\Rightarrow$ X compact, and due to Proof 5. the three forms of compactness are equivalent.

[Deduction 5]

$$countable\ dense\ subset \overset{9}{\Longleftarrow} 2nd\ countable \overset{10}{\Longrightarrow} Lindelöf \tag{5}$$

Proof 9. Suppose that X is second countable and let $\mathcal{B}$ be a countable basis for the topology. X has a countable dense subset. Pick a point $b_B \in B$ in each basis set $B \in \mathcal{B}$. Then $\{b_B \mid B \in \mathcal{B}\}$ is countable and dense.

Proof 10. Let $\mathcal{U}$ be an open covering of X. For each basis set $B \in \mathcal{B}$ which is contained in some open set from the collection $\mathcal{U}$, pick any $\mathrm{U}_B \in \mathcal{U}$ such that $B \subset \mathrm{U}_B$. Then the at most countable collection $\{\mathrm{U}_B\}$ of these open sets from $\mathcal{U}$ is an open covering. Let x be any point in X. Since x is contained in a member U of $\mathcal{U}$ and every open set is a union of basis sets, we have $x \in B \subset \mathrm{U}$ for some basis set $B \in \mathcal{B}$. But then $x \in B \subset \mathrm{U}_B$.

[Deduction 6]

$$countable\ dense\ subset \overset{11\ (+metrizable)}{\Longrightarrow} 2nd\ countable \overset{12\ (+metrizable)}{\Longleftarrow} Lindelöf \tag{6}$$

Proof 11. Let X be a metric space with a countable dense subset $A \subset X$. Then the collection $\{B(a,r) \mid a \in A, r \in Q_+\}$ of balls centered at points in A and with a rational radius is a countable basis for the topology. It suffices to show that for any open ball $B(x,\epsilon)$ in X and any $y \in B(x,\epsilon)$ there are $a \in A$ and $r \in Q_+$ such that $y \in B(a,r) \subset B(x,\epsilon)$. Let r be a positive rational number such that $2r + d(x,y) < \epsilon$ and let $a \in A \cap B(y,r)$. Then $y \in B(a,r)$, of course, and $B(a,r) \subset B(x,\epsilon)$ for if $d(a,z) < r$ then $d(x,z) \leq d(x,y) + d(y,z) \leq d(x,y) + d(y,a) + d(a,z) < d(x,y) + 2r < \epsilon$.

Proof 12. Let X be a metric Lindelöf space. For each positive rational number r, let A_r be a countable subset of X such that $X = \cup_{a\in A_r} B(a,r)$. Then $A = \cup_{r\in Q_+} A_r$ is a dense countable subset. For any open ball $B(x,\epsilon)$ and any positive rational $r < \epsilon$ there is an $a \in A_r$ such that $x \in B(a,r)$. Then $a \in B(x,r) \subset B(x,\epsilon)$.

[Deduction 7]

$$compact\ Hausdorff \overset{13}{\Longrightarrow} normal \tag{7}$$

Proof 13. Let X be a compact Hausdorff space. We claim that

$$\{compact\ subspaces\ of\ X\} = \{closed\ subspaces\ of\ X\}$$

Compact subspaces of Hausdorff spaces are closed, hence $\subset$, and closed subspaces of compact spaces are compact, hence $\supset$. Let A and B be disjoint closed subsets of X, then A and B are compact as just shown. Since any two disjoint compact subspaces of a Hausdorff space can be separated by disjoint open sets, there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

[Deduction 8]

$$completely\ regular \overset{14}{\Longleftarrow} locally\ compact\ Hausdorff \tag{8}$$

Proof 14. We will prove that locally compact Hausdorff spaces are open subspaces of compact spaces, and completely regular spaces are subspaces of compact spaces.

- **Step 1.** For a Hausdorff space X,

  $$X\ is\ locally\ compact \iff X\ is\ homeomorphic\ to\ an\ open\ subset\ of\ a\ compact\ space$$

  The locally compact Hausdorff space X is homeomorphic to the open subspace $\omega X - \{\omega\}$ of the compact Hausdorff space ωX, where $\omega X = X \cup \{\omega\}$ denote the union of X with a set consisting of a single point ω. Hence locally compact Hausdorff spaces are open subspaces of compact spaces.
- **Step 2.** For a topological space X,

  $$X\ is\ completely\ regular \iff X \cong a\ subspace\ of\ a\ compact\ Hausdorff\ space$$

  If X is completely regular then the set $C(X)$ of continuous maps $X \to [0,1]$ separates points and closed sets. The evaluation map

  $$\Delta : X \to \textstyle\prod_{j \in C(X)} [0,1],\ \pi_j(\Delta(x)) = j(x),\ j \in C(X),\ x \in X$$

  is therefore an embedding. By the Tychonoff theorem[1](2.149), $[0,1]^J$ is compact Hausdorff. A compact Hausdorff space is normal, hence completely regular and subspaces of completely regular spaces are completely regular.

[Deduction 9]

$$manifold \overset{15}{\Longrightarrow} regular\ 2nd\ countable \overset{16}{\Longrightarrow} metrizable \tag{9}$$

Proof 15. According to the definition of manifolds, every point has a neighbourhood that is homeomorphic to some R^n. So this means (as R^n is locally compact) that every manifold is locally compact and a locally compact Hausdorff space is (completely) regular.

Proof 16. [Urysohn metrization theorem] The following conditions are equivalent for a second countable space X: (1) X is regular; (2) X is normal; (3) X is homeomorphic to a subspace of $[0,1]^\omega$, where the Hilbert cube $[0,1]^\omega$ is a universal second countable metrizable (or normal or regular) space; (4) X is metrizable.[1](3.27)

Summary

In [Deduction 1-9], we have proved that

$$
\begin{aligned}
&connected \overset{1}{\Longleftarrow} linear\ continuum \overset{2}{\Longrightarrow} order\ topology \overset{3}{\Longrightarrow} normal\\
&well-ordered\ topology \overset{4}{\Longrightarrow} totally\ disconnected\\
&compact \overset{5}{\Longrightarrow} limit\ point\ compact \overset{6\ (+1st\ countable+T_1)}{\Longrightarrow} sequentially\ compact\\
&compact \overset{7\ (+metrizable)}{\Longleftarrow} limit\ point\ compact \overset{8\ (+metrizable)}{\Longleftarrow} sequentially\ compact\\
&countable\ dense\ subset \overset{9}{\Longleftarrow} 2nd\ countable \overset{10}{\Longrightarrow} Lindel\ddot{o}f\\
&countable\ dense\ subset \overset{11\ (+metrizable)}{\Longrightarrow} 2nd\ countable \overset{12\ (+metrizable)}{\Longleftarrow} Lindel\ddot{o}f\\
&compact\ Hausdorff \overset{13}{\Longrightarrow} normal\\
&completely\ regular \overset{14}{\Longleftarrow} locally\ compact\ Hausdorff\\
&manifold \overset{15}{\Longrightarrow} regular\ 2nd\ countable \overset{16}{\Longrightarrow} metrizable.
\end{aligned}
$$

The following deductions can be easily proved by the definitions.

$$
\begin{aligned}
&well-ordered\ topology \Longrightarrow ordered\ topology\\
&Lindel\ddot{o}f \Longleftarrow compact \Longrightarrow locally\ compact \Longrightarrow locally\ compact\\
&compact \Longleftarrow compact\ Hausdorff \Longrightarrow locally\ compact\ Hausdorff\\
&locally\ compact\ Hausdorff \Longrightarrow locally\ compact\\
&completely\ normal \Longrightarrow normal \Longrightarrow completely\ regular \Longrightarrow regular\\
&rugular \Longrightarrow Hausdorff \Longrightarrow T_1\\
&2nd\ countable \Longrightarrow 1st\ countable \Longleftarrow metrizable\\
&manifold \Longrightarrow locally\ path\ connected \Longrightarrow locally\ connected
\end{aligned}
$$

Above all, we have proved the relations between topological spaces shown in Figure 1.

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