# A New Proof of Hartman Theorem

Liguang Hou

**Abstract**

Hartman's theorem is a well-known theorem on the local topological equivalence of hyperbolic nonlinear ordinary differential systems to linear ordinary differential systems. The theorem is proved by geometric qualitative methods with strong condition restrictions. In this paper, we give a new proof of Hartman's theorem by using differential topological geometry under the condition that the second-order differentiable vector field is changed to the first-order differentiable vector field.

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## Introduction

**Definition 1.** [1] [2]. Let the system be

$$\dot{\mathrm{X}} = \mathrm{V}(\mathrm{X}) \quad (\mathrm{X} \in \mathrm{R}^n)\,, \tag{1}$$

where the square vector field $\mathrm{V}(\mathrm{X}) \in \mathrm{R}^n$ is well defined and continuously differentiable in an open neighborhood $W \in \mathrm{R}^n$ of the origin. Besides, the origin is an isolated singularity.

**Definition 2.** [3] Let the linear operator $A : \mathrm{R}^n \to \mathrm{R}^{\mathrm{n}}$ be

$$A = \left( \frac{\partial \mathrm{V}(\mathrm{X})}{\partial \mathrm{X}} \right).$$

Since the system has hyperbolic nonlinearity, the real part of all eigenvalues $\lambda_{\mathrm{i}}$ corresponding to $A$ is nonzero. In a sufficiently small neighborhood of the origin, Eq. (1) can be written as

$$\dot{\mathrm{X}} = A\mathrm{X} + \mathrm{O}(|\mathrm{X}|)$$

$$or \ \left\{ \begin{array}{l} \dot{\mathrm{X}}_1 = A_- \mathrm{X}_1 + \mathrm{O}_1\left(|\mathrm{X}_1| + |\mathrm{X}_2|\right) \overset{def}{=} \mathrm{V}_1 \\ \dot{\mathrm{X}}_2 = A_+ \mathrm{X}_2 + \mathrm{O}_2\left(|\mathrm{X}_1| + |\mathrm{X}_2|\right) \overset{def}{=} \mathrm{V}_2 \end{array} \right. , \tag{2}$$

We specify that $\mathrm{V}_1, \mathrm{V}_2 \in C', \mathrm{X} \in \mathrm{R}^{\mathrm{n}}, \mathrm{X}_1 \in \mathrm{R}^{\mathrm{k}}, \mathrm{X}_2 \in \mathrm{R}^{\mathrm{m}} \quad (k + m = n, k \geq 0, m \geq 0)$, $|\mathrm{X}|^2 = |\mathrm{X}_1|^2 + |\mathrm{X}_2|^2$ ($|\cdot|$ is the Euclidean parameter ), $(A) = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}$.
$A_- : \mathrm{R}^k \to \mathrm{R}^{\mathrm{k}}$. All eigenvalues are negative in real part, i.e. $\mathrm{Re}\lambda_i < 0 (i = 1 \cdots k)$,
$A_+ : \mathrm{R}^m \to \mathrm{R}^{\mathrm{m}}$. All eigenvalues are negative in real part, i.e. $\mathrm{Re}\lambda_i > 0 (i = 1 \cdots k)$.

$O_1(\cdot)$ and $O_2(\cdot)$ are orders of magnitude which satisfy

$$\lim_{|X_1|+|X_2|\to 0} \frac{O_1(|X_1|+|X_2|)}{|X_1|+|X_2|} = O_1 \in R^k,$$
$$\lim_{|X_1|+|X_2|\to 0} \frac{O_2(|X_1|+|X_2|)}{|X_1|+|X_2|} = O_2 \in R^m.$$

**Lemma 1.** First, we consider special systems.

$$\begin{cases} \dot{X}_1 = A_- X_1 + O_1(|X_1|) \overset{\text{def}}{=} V_1' \\ \dot{X}_2 = A_+ X_2 + O_2(|X_2|) \overset{\text{def}}{=} V_2' \end{cases} \tag{3}$$

If the right-hand side of system (3) satisfies $V_1', V_2' \in C'$ in a sufficiently small neighborhood of the origin, noting that $r_i^2 = (X_i, X_i)$ $(i = 1, 2)$, there must be $\sigma_i > 0 (i = 1, 2)$ and positive constants $\alpha_1 > \beta_1 > 0, \beta_2 > \alpha_2 > 0$ such that the directional derivative $L_{v_1'}^{r_i^2}$ of $r_i^2$ along the vector field $V_i' (i = 1, 2)$ satisfy the following inequality.

$$-\alpha_1 r_1^2 < L_{v_1'}^{r_1^2} < -\beta_1 r_1^2 \quad (for\ \forall\,|X_1| < \sigma_1)$$
$$\alpha_2 r_2^2 < L_{v_2'}^{r_2^2} < \beta_2 r_2^2 \qquad (for\ \forall\,|X_2| < \sigma_2)$$

**Lemma 2.** Let $\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$ be a nonzero solution of the system (3) with the initial value $|\varphi_i(0)| = \delta_i < \sigma_i (i = 1, 2)$ $(\delta_i > 0)$. Construct $\rho_i(t) = \ln \frac{(\varphi_i(t), \varphi_i(t))}{\delta_i^2}$ of real variable $t$, then we can get a differential homogeneous embryo $\rho_i(t)$ for every $i$.

$$\rho_i : R \cap \{t_: |\varphi_i(t)| < \sigma_i\} \to \rho_i \left(R \cap \{t : |\varphi_i(t)| < \sigma_i\} \; (i = 1, 2)\right.$$

Aditionally, the differentials of $\rho_i(t)$ satisfy the following inequalities.

$$-\alpha_1 < \frac{d\rho_1(t)}{dt} < -\beta_1 \quad (\alpha_1 > \beta_1 > 0)$$
$$\alpha_2 < \frac{d\rho_2(t)}{dt} < \beta_2 \qquad (\beta_2 > \alpha_2 > 0)$$

**Lemma 3.** For every $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \neq 0$, $X_1 \in R^k, X_2 \in R^m, (k + m = n, k \geq 0, m \geq 0)$ $|X_i| < \sigma_i$ $(\sigma_i > 0)$ $(i = 1, 2)$, there is a unique form

$$X = \begin{pmatrix} f_1^t x_{10} \\ f_2^t x_{10} \end{pmatrix},$$

where $X_{10} \in R^k, X_{20} \in R^m$. $\begin{pmatrix} f_1^t \\ f_2^t \end{pmatrix}$ denotes the phase flow of system (3).

*The three lemmas above were well proved in Ref. 3 and here we consider them correct.*

## Proof of Hartman Theorem

**Step 1.** The system (3) is locally topologically equivalent to

$$\begin{cases} \dot{\mathrm{Y}}_1 = -\mathrm{Y}_1 \\ \dot{\mathrm{Y}}_2 = \mathrm{Y}_2\,, \end{cases} \tag{4}$$

$$where\ \mathrm{Y}_1 \in \mathrm{R}^{\mathrm{k}}, \mathrm{Y}_2 \in \mathrm{R}^{\mathrm{m}}, (k+m=n, k \geq 0, m \geq 0).$$

**Proof 1.** Let $\begin{pmatrix} f_1^t \\ f_2^t \end{pmatrix}$ be the phase flow of system ( 3 ) and $\begin{pmatrix} g_1^t \\ g_2^t \end{pmatrix}$ be the phase flow of system (4). Define the sets $S_1$ and $S_2$ as follows.

$$S_1 = \{\mathrm{X}_{10} \mid \mathrm{X}_{10} \in \mathrm{R}^{\mathrm{k}}, (\mathrm{X}_{10}, \mathrm{X}_{10}) = \delta_1^2 < \sigma_1^2\Big\}, S_2 = \big\{\mathrm{X}_{20} \mid \mathrm{X}_{20} \in \mathrm{R}^{\mathrm{m}}, (\mathrm{X}_{20}, \mathrm{X}_{20}) = \delta_2^2 < \sigma_2\big\}$$

Our work is to prove that system (3) is locally topologically equivalent to system (4), which can be divided into three parts.

1. The map $h_1 : \mathrm{R}^{\mathrm{k}} \cap \{\mathrm{X}_1 : \mathrm{X}_1 < \sigma_1\} \to \mathrm{R}^{\mathrm{k}} \cap \{\mathrm{X}_1 : |\mathrm{X}_1| < \sigma_1\}$ is a homomorphism.
2. The map $h_2 : \mathrm{R}^{\mathrm{m}} \cap \{\mathrm{X}_2 : \mathrm{X}_2 < \sigma_1\} \to \mathrm{R}^{\mathrm{m}} \cap \{\mathrm{X}_2 : |\mathrm{X}_2| < \sigma_2\}$ is a homomorphism.
3. The direct product maintains topological equivalence.

The first two parts can be proved by constructing a mapping, and the third part follows from the first two. (See Ref. 3 for details.)

**Step 2.** System (2) is locally topologically equivalent to system (3).

**Proof 2.** Transform system (2)

$$\begin{cases} \mathrm{X}_1 = \mathrm{Y}_1 - \varphi\,(\mathrm{Y}_2) \\ \mathrm{X}_2 = \mathrm{Y}_2 - \psi\,(\mathrm{Y}_1) \end{cases}$$

and we can get

$$\begin{cases} \dot{\mathrm{Y}}_1 = A_- \mathrm{Y}_1 + \mathrm{O}(\Delta)\,(|\mathrm{Y}_1| + |\psi\,(\mathrm{Y}_1)|) \\ \dot{\mathrm{Y}}_2 = A_+ \mathrm{Y}_2 + \mathrm{O}_2(\Delta)\,(|\mathrm{Y}_2| + |\varphi\,(\mathrm{Y}_2)|) \\ \dot{\psi}\,(\mathrm{Y}_1) = A_+ \varphi\,(\mathrm{Y}_1) - \mathrm{O}_2(\Delta)\,(|\mathrm{Y}_1| + |\psi\,(\mathrm{Y}_1)|) \\ \dot{\varphi}\,(\mathrm{Y}_2) = A_- \varphi\,(\mathrm{Y}_2) - \mathrm{O}_1(\Delta)\,(|\mathrm{Y}_2| + |\varphi\,(\mathrm{Y}_2)|)\,, \end{cases} \tag{5}$$

$$where\ \Delta = \frac{|\mathrm{Y}_1 - \varphi\,(\mathrm{Y}_2)| + |\mathrm{Y}_2 - \psi\,(\mathrm{Y}_1)|}{|\mathrm{Y}_1| + |\mathrm{Y}_2| + |\varphi\,(\mathrm{Y}_2)| + |\psi\,(\mathrm{Y}_1)|}\ is\ a\ bounded\ quantity.$$

By the existence uniqueness theorem of solutions and the continuous dependence theorem on initial values, it follows that the solution $\mathrm{Y}_1(t), \mathrm{Y}_2(t), \varphi\,(\mathrm{Y}_2)\,, \psi\,(\mathrm{Y}_1)$ of system (5) satisfying $\mathrm{Y}_1 = \mathrm{O}_1, \mathrm{Y}_2 = \mathrm{O}_2, \varphi(O_2) = \mathrm{O}_2, \psi\,(\mathrm{O}_1) = \mathrm{O}_1$ exists uniquely and is continuously differentiable. In a sufficiently small neighborhood of $(O_1, O_2)$, the Jacobian determinant of the transformation made to system (2) $J = \left(\frac{\partial \mathrm{X}_{\mathrm{i}}}{\partial \mathrm{Y}_{\mathrm{i}}}\right) \neq 0\ (i = 1, 2)$. Then by the inverse function existence theorem, the transformation is a differential homomorphism in a sufficiently small neighborhood of $(O_1, O_2)$.

By the differentiability of $\varphi(Y_2), \psi(Y_1)$, in a sufficiently small neighborhood of $(O_1, O_2)$

$$\varphi(Y_2) = \int_0^1 \frac{d\varphi(sY_2)}{ds} ds = \int_0^1 \varphi'(sY_2)\, ds \cdot Y_2$$
$$\psi(Y_1) = \int_1^0 \frac{d\psi(sY_1)}{ds} ds = \int_0^1 \psi'(sY_1)\, ds \cdot Y_1,$$

$$\textit{where} \int_0^1 \varphi'(sY_2)\, ds, \int_0^1 \psi'(sY_1)\, ds \in C'.$$

Substitute the two equations into the first two equations of the system (5) and we can get

$$\begin{cases} \dot{Y}_1 = A_- Y_1 + O_1(\Delta)\left(1 + \left|\int_0^1 \psi(sY_1)\, ds\right|\right)|Y_1| \\ \dot{Y}_2 = A_+ Y_2 + O_2(\Delta)\left(1 + \left|\int_0^1 \varphi(sY_2)\, ds\right|\right)|Y_2|, \end{cases} \tag{6}$$

where $O_1(\Delta)\left(1 + \mid \int_0^1 \psi'(sY_1)\, ds \mid\right), O_2(\Delta)\left(1 + \mid \int_0^1 \varphi'(sY_2)\, ds \mid\right)$ are both bounded and infinitesimal. Therefore, system (2) and system (6) are locally topologically equivalent. Since system (6) is of the same form as system (3), system (2) and system (3) are also locally topologically equivalent.

**Step 3.** (*Hartman theorem*) In a sufficiently small neighborhood of the origin, system (1) is locally equivalent to system (7).

$$\begin{cases} \dot{X}_1 = A_- X_1 \\ \dot{X}_2 = A_+ X_2 \end{cases} \tag{7}$$

**Proof 3.** Since $V(X) \in C'$, system (1) can be written in the form of system (2) in a sufficiently small neighborhood of the origin. From step 2, system (2) is locally topologically equivalent to system (3), while from step 1, system (3) is locally topologically equivalent to system (4). Since system (7) is a special case of system (3) ($O_1 |X_1| = O_1, O_2 |X_2| = O_2$), according to the above Lemmas 1-3 and the argument of Step 1, it follows that the local topology of system (4) is equivalent to system (7), and then the local topology of system (1) is equivalent to system (7). Above all, Hartman theorem is proved.

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