5

INTERPOLATION, EXTENSION, AND APPROXIMATION THEOREMS

Interpolation on Order of Smoothness

5.1 We consider the problem of determining upper bounds for L^p norms of derivatives $D^\beta u$, $0 < |\beta| < m$, of functions in $W^{m,p}(\Omega)$ in terms of the L^p norms of u and its partial derivatives of order m. Such estimates are conveniently expressed in terms of the seminorms $|\cdot|_{j,p}$ defined in Paragraph 4.29. Theorem 5.2 below provides such an estimate for the seminorm $|u|_{j,p}$ in terms of $|u|_{m,p}$ and $\|u\|_p$, as well as some elementary consequences of this estimate. Such estimates arose in the work of of Ehrling [E], Nirenberg [Nr1, Nr2], Gagliardo [Ga1, Ga2], and Browder [Br1, Br2], and were frequently proved under the assumption that Ω satisfies the uniform cone condition, at least if Ω is unbounded. However, we will prove Theorem 5.2 assuming only the cone condition. In fact, even the weak cone condition is sufficient for the proof, as is shown in [AF1].

5.2 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. For each $\epsilon_0 > 0$ there exist finite constants K and K', each depending on n, m, p, ϵ_0 and the dimensions of the cone C providing the cone condition for Ω such that if $0 < \epsilon \le \epsilon_0$, $0 \le j \le m$, and $u \in W^{m,p}(\Omega)$, then

$$|u|_{j,p} \le K\big(\epsilon\,|u|_{m,p} + \epsilon^{-j/(m-j)}\,\|u\|_p\big), \tag{1}$$

$$\|u\|_{j,p} \le K'\big(\epsilon\,\|u\|_{m,p} + \epsilon^{-j/(m-j)}\,\|u\|_p\big), \tag{2}$$

$$\|u\|_{j,p} \le 2K'\,\|u\|_{m,p}^{j/m}\,\|u\|_p^{(m-j)/m}\,. \tag{3}$$

5.3 Inequality (2) follows from repeated applications of (1), and (3) by setting $\epsilon_0 = 1$ in (2) and choosing ϵ in (2) so that the two terms on the right side are equal. Furthermore, (1) holds when $\epsilon < \epsilon_0$ if it holds for $\epsilon < \epsilon_1$ for any specific positive ϵ_1; to see this just replace ϵ by $\epsilon\epsilon_1/\epsilon_0$ and suitably adjust K. Thus we need only prove (1), and that for just one value of ϵ_0.

We carry out the proof in three lemmas. The first develops a one-dimensional version for the case $m = 2$, $j = 1$. The second establishes (1) for $m = 2$, $j = 1$ for general Ω satisfying the cone condition. The third shows that (1) is valid for general $m \geq 2$ and $1 \leq j \leq m - 1$ whenever the case $m = 2$, $j = 1$ is known to hold.

5.4 LEMMA If $\rho > 0$, $1 \leq p < \infty$, $K_p = 2^{p-1}9^p$, and $g \in C^2([0, \rho])$, then

$$|g'(0)|^p \leq \frac{K_p}{\rho}\left(\rho^p \int_0^\rho |g''(t)|^p\, dt + \rho^{-p} \int_0^\rho |g(t)|^p\, dt\right). \tag{4}$$

Proof. Let $f \in C^2([0, 1])$, let $x \in [0, 1/3]$, and let $y \in [2/3, 1]$. By the mean-value theorem there exists $z \in (x, y)$ such that

$$|f'(z)| = \left|\frac{f(y) - f(x)}{y - x}\right| \leq 3|f(x)| + 3|f(y)|.$$

Thus

$$\begin{aligned} |f'(0)| &= \left|f'(z) - \int_0^z f''(t)\, dt\right| \\ &\leq 3|f(x)| + 3|f(y)| + \int_0^1 |f''(t)|\, dt. \end{aligned}$$

Integration of x over $[0, 1/3]$ and y over $[2/3, 1]$ yields

$$\frac{1}{9}|f'(0)| \leq \int_0^{1/3} |f(x)|\, dx + \int_{2/3}^1 |f(y)|\, dy + \frac{1}{9}\int_0^1 |f''(t)|\, dt.$$

For $p \geq 1$ we therefore have (using Hölder's inequality if $p > 1$)

$$|f'(0)|^p \leq K_p\left(\int_0^1 |f''(t)|^p\, dt + \int_0^1 |f(t)|^p\, dt\right).$$

where $K_p = 2^{p-1}9^p$.

Inequality (4) now follows by substituting $f(t) = g(\rho t)$. ∎

5.5 LEMMA If $1 \leq p < \infty$ and the domain $\Omega \subset \mathbb{R}^n$ satisfies the cone condition, then there exists a constant K depending on n, p, and the height ρ_0 and aperture angle κ of the cone C providing the cone condition for Ω such that for all ϵ, $0 < \epsilon \le \rho_0$ and all $u \in W^{2,p}(\Omega)$ we have

$$|u|_{1,p} \le K\big(\epsilon\,|u|_{2,p} + \epsilon^{-1}\,\|u\|_p\big). \tag{5}$$

Proof. Let $\Sigma = \{\sigma \in \mathbb{R}^n : |\sigma| = 1\}$ be the unit sphere in $\mathbb{R}^n$ with volume element $d\sigma$ and $(n-1)$-volume $K_0 = K_0(n) = \int_\Sigma d\sigma$. If $x \in \Omega$ let σ_x be the unit vector in the direction of the axis of a cone $C_x \subset \Omega$ congruent to C and having vertex at x, and let $\Sigma_x = \{\sigma \in \Sigma : \angle(\sigma, \sigma_x) \le \kappa/2\}$.
Let $u \in C^\infty(\Omega)$. If $x \in \Omega$, $\sigma \in \Sigma_x$, and $0 < \rho \le \rho_0$, then

$$|\sigma \cdot \operatorname{grad} u(x)|^p \le \frac{K_p}{\rho}\, I(\rho, p, u, x, \sigma),$$

where

$$I(\rho, p, u, x, \sigma) = \rho^p \int_0^\rho |D_t^2 u(x + t\sigma)|^p\, dt + \rho^{-p} \int_0^\rho |u(x + t\sigma)|^p\, dt.$$

There exists a constant $K_1 = K_1(n, p, \kappa)$ such that

$$\int_\Sigma |\sigma \cdot \operatorname{grad} u(x)|^p\, d\sigma \ge \int_{\Sigma_x} |\sigma \cdot \operatorname{grad} u(x)|^p\, d\sigma \ge K_1 |\operatorname{grad} u(x)|^p.$$

Accordingly,

$$\int_\Omega |\operatorname{grad} u(x)|^p\, dx \le \frac{K_p}{K_1 \rho} \int_\Sigma d\sigma \int_\Omega I(\rho, p, u, x, \sigma)\, dx.$$

In order to estimate the inner integral on the right, regard u and its derivatives as extended to all of $\mathbb{R}^n$ so as to be identically zero outside Ω. For simplicity, we suppose $\sigma = e_n = (0, \dots, 0, 1)$ and write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. We have

$$\begin{aligned}
&\int_\Omega I(\rho, p, u, x, e_n)\, dx \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^{\infty} dx_n \int_0^\rho \big(\rho^p |D_n^2 u(x', x_n + t)|^p + \rho^{-p} |u(x', x_n + t)|^p\big)\, dt \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_0^\rho dt \int_{-\infty}^{\infty} \big(\rho^p |D_n^2 u(x)|^p + \rho^{-p} |u(x)|^p\big)\, dx_n \\
&\quad \le \rho \int_\Omega \big(\rho^p |D_n^2 u(x)|^p + \rho^{-p} |u(x)|^p\big)\, dx,
\end{aligned}$$

In general, for $\sigma \in \Sigma$

$$\int_\Omega I(\rho, p, u, x, \sigma)\, dx \le \rho \int_\Omega \big(\rho^p\, |u|_{2,p}^p + \rho^{-p}\, \|u\|_p^p\big)\, dx,$$

and since $|D_j(u)| \le |\operatorname{grad} u|$ and the measure of Σ is K_0,

$$|u|_{1,p}^p \le \frac{nK_pK_0}{K_1}\left(\rho^p\,|u|_{2,p}^p + \rho^{-p}\,\|u\|_p^p\right).$$

Inequality (5) now follows by taking pth roots, replacing ρ with ϵ, and noting that $C^\infty(\Omega)$ is dense in $W^{2,p}(\Omega)$. ∎

5.6 LEMMA Let $m \ge 2$, let $0 < \delta_0 < \infty$, and let $\epsilon_0 = \min\{\delta_0, \delta_0^2, \dots, \delta_0^{m-1}\}$. Suppose that for given p, $1 \le p < \infty$, and given $\Omega \subset \mathbb{R}^n$ there exists a constant $K = K(\delta_0, p, \Omega)$ such that for every δ satisfying $0 < \delta \le \delta_0$ and every $u \in W^{2,p}(\Omega)$, we have

$$|u|_{1,p} \le K\delta\,|u|_{2,p} + K\delta^{-1}\,|u|_{0,p}\,. \tag{6}$$

Then there exists a constant $K = K(\epsilon_0, m, p, \Omega)$ such that for every ϵ satisfying $0 < \epsilon \le \epsilon_0$, every integer j satisfying $0 \le j \le m-1$, and every $u \in W^{m,p}(\Omega)$, we have

$$|u|_{j,p} \le K\epsilon\,|u|_{m,p} + K\epsilon^{-j/(m-j)}\,|u|_{0,p}\,. \tag{7}$$

Proof. Since (7) is trivial for $j = 0$, we consider only the case $1 \le j \le m-1$. The proof is accomplished by a double induction on m and j. The constants $K_1, K_2, \dots$ appearing in the argument may depend on δ_0 (or ϵ_0), m, p, and Ω. First we prove (7) for $j = m-1$ by induction on m, so that (6) is the special case $m = 2$. Assume, therefore, that for some k, $2 \le k \le m-1$,

$$|u|_{k-1,p} \le K_1\delta\,|u|_{k,p} + K_1\delta^{-(k-1)}\,|u|_{0,p} \tag{8}$$

holds for all δ, $0 < \delta \le \delta_0$, and all $u \in W^{k,p}(\Omega)$. If $u \in W^{k+1,p}(\Omega)$, we prove (8) with $k+1$ replacing k (and a different constant K_1). If $|\alpha| = k-1$ we obtain from (6)

$$|D^\alpha u|_{1,p} \le K_2\delta\,|D^\alpha u|_{2,p} + K_2\delta^{-1}\,|D^\alpha u|_{0,p}\,.$$

Combining this inequality with (8) we obtain, for $0 < \eta \le \delta_0$,

$$\begin{aligned}
|u|_{k,p} &\le K_3 \sum_{|\alpha|=k-1} |D^\alpha u|_{1,p}\\
&\le K_4\delta\,|u|_{k+1,p} + K_4\delta^{-1}\,|u|_{k-1,p}\\
&\le K_4\delta\,|u|_{k+1,p} + K_4K_1\delta^{-1}\eta\,|u|_{k,p} + K_4K_1\delta^{-1}\eta^{1-k}\,|u|_{0,p}\,.
\end{aligned}$$

We may assume without prejudice that $2K_1K_4 \ge 1$. Therefore, we may take $\eta = \delta/(2K_1K_4)$ and so obtain

$$\begin{aligned}
|u|_{k,p} &\le 2K_4\delta\,|u|_{k+1,p} + \left(\delta/(2K_1K_4)\right)^{-k}\,|u|_{0,p}\\
&\le K_5\delta\,|u|_{k+1,p} + K_5\delta^{-k}\,|u|_{0,p}\,.
\end{aligned}$$

This completes the induction establishing (8) for $0 < \delta \le \delta_0$ and hence (7) for $j = m - 1$ and $0 < \epsilon \le \delta_0$.

We now prove by downward induction on j that

$$|u|_{j,p} \le K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p} \tag{9}$$

holds for $1 \le j \le m - 1$ and $0 < \delta \le \delta_0$. Note that (8) with $k = m$ is the special case $j = m - 1$ of (9). Assume, therefore, that (9) holds for some j, $2 \le j \le m - 1$. We prove that it also holds with j replaced by $j - 1$ (and a different constant K_6). From (8) and (9) we obtain

$$\begin{aligned} |u|_{j-1,p} &\le K_7 \delta |u|_{j,p} + K_7 \delta^{1-j} |u|_{0,p} \\ &\le K_7 \delta \big(K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p} \big) + K_7 \delta^{1-j} |u|_{0,p} \\ &\le K_8 \delta^{m-(j-1)} |u|_{m,p} + K_8 \delta^{1-j} |u|_{0,p} . \end{aligned}$$

Thus (9) holds, and (7) follows by setting $\delta = \epsilon^{1/(m-j)}$ in (7) and noting that $\epsilon \le \epsilon_0$ if $\delta \le \delta_0$. ∎

This completes the proof of Theorem 5.2

5.7 REMARK Careful consideration of the proofs of the previous two lemmas shows that if the height of the cone providing the cone condition for Ω is infinite, then inequalities (5) and (7) (and therefore (1) and (2)) hold for all $\epsilon > 0$, the corresponding constants K being independent of ϵ. This is the case, for example, if $\Omega = \mathbb{R}^n$ or a half-space like $\mathbb{R}^n_+$.

Interpolation on Degree of Summability

The following two interpolation theorems provide sharp estimates for L^q norms of functions in $W^{m,p}(\Omega)$. Some of these estimates follow from Theorem 4.12 while others have traditionally been obtained for regular domains from imbeddings of Sobolev spaces of fractional order. (See Chapter 7.) We obtain them here assuming only that the domain satisfies the cone condition. Again, the weak cone condition would do as well; see [AF1].

5.8 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. If $mp > n$, let $p \le q \le \infty$; if $mp = n$, let $p \le q < \infty$; if $mp < n$, let $p \le q \le p^* = np/(n - mp)$. Then there exists a constant K depending on m, n, p, q and the dimensions of the cone C providing the cone condition for Ω, such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_q \le K \|u\|_{m,p}^{\theta} \|u\|_p^{1-\theta} , \tag{10}$$

where $\theta = (n/mp) - (n/mq)$.

Proof. The case $mp < n$, $p \le q \le p^*$ follows directly from Theorems 2.11 and 4.12:

$$\|u\|_q \le \|u\|_{p^*}^{\theta} \|u\|_p^{1-\theta} \le K \|u\|_{m,p}^{\theta} \|u\|_p^{1-\theta},$$

where $1/q = (\theta/p^*)+(1-\theta)/p$ from which it follows that $\theta = (n/mp)-(n/mq)$. For the cases $mp = n$, $p \le q < \infty$, and $mp > n$, $p \le q \le \infty$ we use the local bound obtained in Lemma 4.15. If $0 < r \le \rho$ (the height of the cone C), then

$$|u(x)| \le K_1 \left(\sum_{|\alpha| \le m-1} r^{|\alpha|-n} \chi_r * |D^\alpha u|(x) + \sum_{|\alpha|=m} (\chi_r \omega_m) * |D^\alpha u|(x) \right), \tag{11}$$

where χ_r is the characteristic function of the ball of radius r centred at the origin in $\mathbb{R}^n$, and $\omega_m(x) = |x|^{m-n}$. We estimate the L^q norms of both terms on the right side of (11) using Young's inequality from Corollary 2.25. If $(1/p)+(1/s) = 1+(1/q)$, then

$$\|\chi_r * |D^\alpha u|\|_q \le \|\chi_r\|_s \|D^\alpha u\|_p = K_2 r^{n-(n/p)+(n/q)} \|D^\alpha u\|_p$$
$$\|(\chi_r \omega_m) * |D^\alpha u|\|_q \le \|\chi_r \omega_m\|_s \|D^\alpha u\|_p = K_3 r^{m-(n/p)+(n/q)} \|D^\alpha u\|_p .$$

(Note that $m - (n/p) + (n/q) > 0$ if q satisfies the above restrictions.) Hence

$$\|u\|_q \le K_4 \left(\sum_{j=0}^{m-1} r^{j-(n/p)+(n/q)} |u|_{j,p} + r^{m-(n/p)+(n/q)} |u|_{m,p} \right).$$

By Theorem 5.2,

$$|u|_{j,p} \le K_5 \big(r^{m-j} |u|_{m,p} + r^{-j} \|u\|_p \big),$$

so

$$\|u\|_q \le K_6 \big(r^{m-(n/p)+(n/q)} \|u\|_{m,p} + r^{-(n/p)+(n/q)} \|u\|_p \big).$$

Adjusting K_6 if necessary, we can assume this inequality holds for all $r \le 1$. Choosing r to make the two terms on the right side equal, we obtain (10). ∎

A special case of the above Theorem asserts that if $mp > n$, then

$$\|u\|_\infty \le K \|u\|_{m,p}^{n/mp} \|u\|_p^{1-(n/mp)}. \tag{12}$$

A similar inequality with $\|u\|_p$ replaced by a more general $\|u\|_q$ is sometimes useful.

5.9 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. Let $p > 1$ and $mp > n$. Suppose that either $1 \le q \le p$ or both $q > p$ and $mp - p < n$. Then there exists a constant K depending on m, n, p, q and the dimensions of the cone C providing the cone condition for Ω, such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_\infty \le K \|u\|_{m,p}^{\theta} \|u\|_q^{1-\theta},$$

where $\theta = np/[np + (mp - n)q]$.

Proof. It is sufficient to show that the inequality

$$|u(x)| \le K \|u\|_{m,p}^{\theta} \|u\|_q^{1-\theta}, \qquad \theta = np/[np + (mp - n)q] \tag{13}$$

holds for all $x \in \Omega$ and all $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$.

First we observe that (13) is a straightforward consequence of Theorems 5.8 and 2.11 if $1 \le q \le p$; since (12) holds we can substitute

$$\|u\|_p \le \|u\|_q^{q/p} \|u\|_\infty^{1-(q/p)}$$

and obtain (13) by cancellation.

Now suppose $q > p$, and, for the moment, that $m = 1$ and $p > n$. We reuse the local bound (11); in this case it says

$$|u(x)| \le K_1\Big(r^{-n}\chi_r * |u|(x) + \sum_{|\alpha|=1} (\chi_r\omega_1) * |D^\alpha u|(x)\Big),$$

for $0 < r \le \rho$, the height of the cone C. By Hölder's inequality,

$$\chi_r * |u|(x) \le K_2 r^{n-(n/q)} \|u\|_q,$$

and, for $|\alpha| = 1$,

$$(\chi_r\omega_1) * |D^\alpha u|(x) \le K_3 r^{1-(n/p)} \|D^\alpha u\|_p. \tag{14}$$

Since $\|u\|_q \le K_5 \|u\|_{1,p}$ (by Part I Case A of Theorem 4.12), and since inequality (14) may be assumed to hold for all r such that $0 < r^{1-(n/p)+(n/q)} \le K_5$ provided K_4 is suitably adjusted, we can choose r to make the two upper bounds above equal. This choice yields (13) with $m = 1$.

For general m, we have $W^{m,p}(\Omega) \to W^{1,r}(\Omega)$, where $r = np/(n - mp + p)$ satisfies $n < r < \infty$ since $(m-1)p < n < mp$. Hence, if $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$, we have

$$|u(x)| \le K_6 \|u\|_{1,r}^{\theta} \|u\|_q^{1-\theta} \le K_7 \|u\|_{m,p}^{\theta} \|u\|_q^{1-\theta},$$

where $\theta = nr/[nr + (r - n)q] = np/[np + (mp - n)q]$. ∎

The following theorem makes use of the above result to provide an alternate direct proof of Part I Case C of the Sobolev imbedding theorem 4.12 as well as a hybrid imbedding inequality that will prove useful for establishing compactness of some of these imbeddings in the next chapter.

5.10 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. Let m and k be positive integers and let $p > 1$. Suppose that $mp < n$ and $n - mp < k \le n$. Let ν be the largest integer less than mp, so that $n - \nu \le k$. Let Ω_k be the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Then there exists a constant K such that the inequality

$$\|u\|_{0,kq/n,\Omega_k} \le K\, \|u\|_{0,q,\Omega}^{1-\theta}\, \|u\|_{m,p,\Omega}^{\theta} \tag{15}$$

holds for all $u \in W^{m,p}(\Omega)$, where

$$q = p^* = \frac{np}{n - mp} \qquad \text{and} \qquad \theta = \frac{\nu p}{\nu p + (mp - \nu)q}.$$

Note that $0 < \theta < 1$.

Proof. Again it is sufficient to establish the inequality for functions in $W^{m,p}(\Omega) \cap C^\infty(\Omega)$. Without loss of generality we assume that H is a coordinate k-plane $\mathbb{R}^k$ in $\mathbb{R}^n$, and, as we did in Lemma 4.24, that Ω is a union of coordinate cubes of fixed edge length, say 2.

Let $\mu = \binom{k}{n-\nu}$, and let E^i, $1 \le i \le \mu$, denote the various coordinate planes in $\mathbb{R}^k$ having dimension $n - \nu$. Let Ω^i be the projection of Ω_k onto E^i, and for each $x \in \Omega^i$ let Ω^i_x denote the intersection of Ω with the ν-dimensional plane through x perpendicular to E^i. Then Ω^i_x contains a ν dimensional cube of unit edge length having a vertex at x, so it satisfies a cone condition with parameters independent of i and x. By Theorem 5.9

$$\|u\|_{0,\infty,\Omega^i_x} \le K_1\, \|u\|_{0,q,\Omega^i_x}^{1-\theta}\, \|u\|_{m,p,\Omega^i_x}^{\theta}.$$

Let $s = (n - \nu)p/(n - mp)$, and let dx^i and dx^i_* denote the volume elements in E^i and its orthogonal complement (in $\mathbb{R}^n$) respectively. Since

$$s(1 - \theta) = \frac{q(mp - \nu)}{mp} \qquad \text{and} \qquad s\theta = \frac{\nu}{m},$$

we have

$$\begin{aligned}
&\int_{\Omega^i} \sup_{y \in \Omega^i_x} |u(y)|^s \, dx^i \\
&\le K_1 \int_{\Omega^i} \left[\int_{\Omega^i_x} |u(x)|^q \, dx^i_* \right]^{(mp-\nu)/mp} \left[\int_{\Omega^i_x} \sum_{|\alpha| \le m} |D^\alpha u(x)|^p \, dx^i_* \right]^{\nu/mp} \\
&\le K_1\, \|u\|_{0,q,\Omega}^{s(1-\theta)}\, \|u\|_{m,p,\Omega}^{s\theta},
\end{aligned}$$

the last line being an application of Hölder's inequality.

Let dx^k denote the k-dimensional volume element in H. We apply the averaging Lemma 4.23 to the family of μ subspaces E^i of $\mathbb{R}^k$. The parameter λ for this application of the lemma is $\lambda = \binom{k-1}{n-\nu-1} = (n-\nu)\mu/k$. Since $(kq/n)(\lambda/\mu) = s$, we obtain

$$\begin{aligned}
\|u\|_{0,kq/n,\Omega_k}^{kq/n} &\le K_2 \int_{\Omega_k} \prod_{i=1}^{\mu} \sup_{y\in\Omega_x^i} |u(y)|^{kq/\mu n}\, dx^k \\
&\le K_2 \prod_{i=1}^{\mu} \left[\int_{\Omega^i} \sup_{y\in\Omega_x^i} |u(y)|^s\, dx^i \right]^{1/\lambda} \\
&\le K_3 \prod_{i=1}^{\mu} \|u\|_{0,q,\Omega}^{kq(1-\theta)/\mu n}\, \|u\|_{m,p,\Omega}^{kq\theta/\mu n},
\end{aligned}$$

so that

$$\|u\|_{0,kq/n,\Omega_k} \le K\, \|u\|_{0,q,\Omega}^{1-\theta}\, \|u\|_{m,p,\Omega}^{\theta}$$

as required. ∎

5.11 REMARK If we take $k = n$ in inequality (15), then the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ follows for $q = np/(n-mp)$ by cancellation. The corresponding imbedding inequality $\|u\|_{0,q,\Omega} \le K\, \|u\|_{m,p,\Omega}$ can then be used to further estimate the right side of (15), yielding the trace imbedding $W^{m,p}(\Omega) \to L^r(\Omega_k)$ for $r = kp/(n-mp)$.

Interpolation Involving Compact Subdomains

Sometimes it is useful to have bounds for intermediate derivatives $D^\beta u$, of a function $u \in W^{m,p}(\Omega)$, where $1 \le |\beta| \le m-1$, in terms of the seminorm $|u|_{m,p,\Omega}$ and the L^p-norm of u over a compact subdomain $\Omega' \Subset \Omega$. Such inequalities are typically not possible unless Ω is bounded, but for bounded Ω they can be established under the assumption that Ω satisfies either the segment condition or the cone condition. (A bounded domain Ω satisfying the cone condition can be decomposed into a finite union of subdomains each of which satisfies the strong local Lipschitz condition, and therefore the segment condition. See Lemma 4.22.) We will prove the following hybrid interpolation theorem. (See Agmon [Ag].)

5.12 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the segment condition. Let $0 < \epsilon_0 < \infty$, let $1 \le p < \infty$, and let j and m be integers with $0 \le j \le m-1$. There exists a constant $K = K(\epsilon_0, m, p, \Omega)$ and for each ϵ satisfying $0 < \epsilon \le \epsilon_0$ a domain $\Omega_\epsilon \Subset \Omega$ such that for every $u \in W^{m,p}(\Omega)$

$$|u|_{j,p,\Omega} \le K\epsilon\, |u|_{m,p,\Omega} + K\epsilon^{-j/(m-j)}\, \|u\|_{p,\Omega_\epsilon}. \tag{16}$$

Note that this theorem implies Theorem 5.2 extends to bounded domains satisfying the segment condition.

As in the proof of Theorem 5.12, we begin with a one-dimensional inequality.

5.13 LEMMA Let $1 \le p < \infty$ and let $0 < l_1 < l_2 < \infty$. Then there exists a constant $K = K(p, l_1, l_2)$ and, for every $\epsilon > 0$, a number $\delta = \delta(\epsilon, l_1, l_2)$ satisfying $0 < 2\delta < l_1$ such that if (a, b) is a finite open interval in $\mathbb{R}$ whose length $b - a$ satisfies $l_1 \le b - a \le l_2$, and $g \in C^1(a, b)$, then

$$\int_a^b |g(t)|^p\,dt \le K\epsilon \int_a^b |g'(t)|^p\,dt + K \int_{a+\delta}^{b-\delta} |g(t)|^p\,dt. \tag{17}$$

Proof. If $f \in C^1(0, 1)$, $0 < t < 1$, and $1/3 < \tau < 2/3$, then

$$|f(s)| = \left| f(\tau) + \int_\tau^s f'(\xi)\,d\xi \right| \le |f(\tau| + \int_0^1 |f'(\xi)|\,d\xi.$$

Integrating τ over $(1/3, 2/3)$, applying Hölder's inequality if $p > 1$, and finally integrating s over $(0, 1)$ gives

$$\int_0^1 |f(s)|^p \le K_p \int_{1/3}^{2/3} |f(s)|^p\,ds + K_p \int_0^1 |f'(s)|^p\,ds,$$

where $K_p = 3 \cdot 2^{p-1}$. Now substitute $f(s) = g\big(a + s(b - a)\big) = g(t)$ to obtain

$$\int_a^b |g(t)|^p \le K_p(b-a)^p \int_a^b |g'(t)|^p\,dt + K_p \int_{(2a+b)/3}^{(a+2b)/3} |g(t)|^p\,dt.$$

For given $\epsilon > 0$ pick a positive integer k such that $k^{-p} \le \epsilon$. Let $a_j = a + (b-a)j/k$ for $j = 0, 1, \ldots, k$ and pick δ so that $0 < \delta \le (b - a)/3k$. Then

$$\begin{aligned}
\int_a^b |g(t)|^p\,dt &= \sum_{j=1}^k \int_{a_{j-1}}^{a_j} |g(t)|^p\,dt \\
&\le K_p \sum_{j=1}^k \left[\left(\frac{b-a}{k}\right)^p \int_{a_{j-1}}^{a_j} |g'(t)|^p\,dt + \int_{a_{j-1}+\delta}^{a_j-\delta} |g(t)|^p\,dt \right] \\
&\le K_p \max\{1, (b-a)^p\} \left[\epsilon \int_a^b |g'(t)|^p\,dt + \int_{a+\delta}^{b-\delta} |g(t)|^p\,dt \right]
\end{aligned}$$

which is the desired inequality (17). ∎

5.14 LEMMA Let Ω be a bounded domain in $\mathbb{R}^n$ that satisfies the segment condition. Then there exists a constant $K = K(p, \Omega)$ and, for any positive number ϵ, a domain $\Omega_\epsilon \Subset \Omega$, such that

$$|u|_{0,p,\Omega} \leq K\epsilon\, |u|_{1,p,\Omega} + K\, |u|_{0,p,\Omega_\epsilon} \tag{18}$$

holds for every $u \in W^{1,p}(\Omega)$.

Proof. Since Ω is bounded, and its boundary is therefore compact, the open cover $\{U_j\}$ of bdry Ω and corresponding set $\{y_j\}$ of nonzero vectors referred to in the definition of the segment condition (Paragraph 3.21) are both finite sets. Therefore open sets $V_j \Subset U_j$ can be found such that bdry $\Omega \subset \bigcup_j V_j$ and even, for sufficiently small δ, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\} \subset \bigcup_j V_j$. Thus $\Omega = \bigcup_j (V_j \cap \Omega) \cup \tilde{\Omega}$, where $\tilde{\Omega} \Subset \Omega$. It is thus sufficient to prove that for each j

$$|u|_{0,p,V_j\cap\Omega^p} \leq K_1\epsilon^p\, |u|^p_{1,p,\Omega} + K_1\, |u|^p_{0,p,\Omega_{\epsilon,j}}$$

for some $\Omega_{\epsilon,j} \Subset \Omega$. For simplicity, we now drop the subscripts j.

Consider the sets Q, Q_η, $0 \leq \eta < 1$, defined by

$$\begin{aligned} Q &= \{x + ty : x \in U \cap \Omega, 0 < t < 1\}, \\ Q_\eta &= \{x + ty : x \in V \cap \Omega, \eta < t < 1\}. \end{aligned}$$

If $\eta > 0$, then $Q_\eta \Subset Q$, and by the segment condition, $Q \subset \Omega$. Any line ℓ parallel to y and passing through a point in $V \cap \Omega$ intersects Q_0 in one or more intervals each having length between $|y|$ and diam Ω. By 5.13 there exists $\eta > 0$ and a constant K_1 such that for every $u \in C^\infty(\Omega)$ and any such line ℓ

$$\int_{\ell\cap Q_0} |u(x)|^p\, ds \leq K_1\epsilon^p \int_{\ell\cap Q_0} |D_y u(x)|^p\, ds + K_1 \int_{\ell\cap Q_\eta} |u(x)|^p\, ds,$$

D_y denoting differentiation in the direction of y and ds being the length element in that direction. We integrate this inequality over the projection of Q_0 on a hyperplane perpendicular to y and so obtain

$$\begin{aligned} |u|^p_{0,p,V\cap\Omega} \leq |u|^p_{0,p,Q_0} &\leq K_1\epsilon^p\, |u|^p_{1,p,Q_0} + K_1\, |u|^p_{0,p,Q_\eta} \\ &\leq K_1\epsilon^p\, |u|^p_{1,p,\Omega} + K_1\, |u|^p_{0,p,Q_\epsilon}, \end{aligned}$$

where $\Omega_\epsilon = \Omega_\eta \Subset \Omega$. By density, this inequality holds for every $u \in W^{1,p}(\Omega)$.

5.15 (Completion of the Proof of Theorem 5.12) We apply Lemma 5.14 to derivatives $D^\beta u$, $|\beta| = m - 1$ to obtain

$$|u|_{m-1,p,\Omega} \leq K\epsilon\, |u|_{m,p,\Omega} + K_1\, |u|_{m-1,p,\Omega_\epsilon}, \tag{19}$$

where $\Omega_\epsilon \Subset \Omega$. Since $\overline{\Omega_\epsilon}$ is a compact subset of Ω, there exists a constant $\delta > 0$ such that $\operatorname{dist}(\overline{\Omega_\epsilon}, \operatorname{bdry}\Omega) > \delta$. The union Ω' of open balls of radius δ about points in $\overline{\Omega_\epsilon}$ clearly satisfies the cone condition and also $\Omega' \Subset \Omega$. We can use Ω' in place of Ω_ϵ in (19), and so we can assume Ω_ϵ satisfies the cone condition. By Theorem 5.2, for given $\epsilon_0 > 0$ the inequality

$$|u|_{m-1,p,\Omega_\epsilon} \le K_2\epsilon\, |u|_{m,p,\Omega_\epsilon} + K_2\epsilon^{-(m-1)}\, |u|_{0,p,\Omega_\epsilon}\,.$$

Combining this with inequality (19) we obtain the case $j = m - 1$ of (16).

The rest of the proof is by downward induction on j. Assuming that (16) holds for some j satisfying $1 \le j \le m-1$, and replacing ϵ with ϵ^{m-j} (with consequent alterations to K and Ω_ϵ), we obtain

$$|u|_{j,p,\Omega} \le K_3\epsilon^{m-j}\, |u|_{m,p,\Omega} + K_3\epsilon^{-j}\, |u|_{0,p,\Omega_{\epsilon,1}}\,.$$

Also, by the case already proved,

$$|u|_{j-1,p,\Omega} \le K_4\epsilon\, |u|_{j,p,\Omega} + K_4\epsilon^{-(j-1)}\, |u|_{0,p,\Omega_{\epsilon,2}}\,.$$

Combining these we get

$$|u|_{j-1,p,\Omega} \le K_5\epsilon^{m-(j-1)}\, |u|_{m,p,\Omega} + K_5\epsilon^{-(j-1)}\, |u|_{0,p,\Omega_\epsilon}\,,$$

where $K_5 = K_4(K_3 + 1)$ and $\Omega_\epsilon = \Omega_{\epsilon,1} \cup \Omega_{\epsilon,2}$. Replacing ϵ by $\epsilon^{1/(m-j+1)}$ we complete the induction. ∎

5.16 REMARK The conclusion of Theorem 5.12 is also valid for bounded domains satisfying the cone condition. Although the cone condition does not imply the segment condition, the decomposition of a domain Ω satisfying the cone condition into a finite union of subdomains each of which is a union of parallel translates of a parallelepiped (see Lemma 4.22) can be refined, for bounded Ω, so that each of the subdomains satisfies a strong local Lipschitz condition and therefore also the segment condition.

Extension Theorems

5.17 (Extension Operators) Let Ω be a domain in $\mathbb{R}^n$. For given m and p a linear operator E mapping $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$ is called a *simple* (m, p)-*extension operator for* Ω if there exists a constant $K = K(m, p)$ such that for every $u \in W^{m,p}(\Omega)$ the following conditions hold:

(i) $Eu(x) = u(x)$ a.e. in Ω,

(ii) $\|Eu\|_{m,p,\mathbb{R}^n} \le K\, \|u\|_{m,p,\Omega}$.

E is called a *strong m-extension operator for* Ω if E is a linear operator mapping functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ and if, for every p, $1 \le p < \infty$, and every integer k, $0 \le k \le m$, the restriction of E to $W^{k,p}(\Omega)$ is a simple (k, p)-extension operator for Ω.

Finally, E is called a *total extension operator for* Ω if E is a strong m-extension operator for Ω for every m. Such a total extension operator necessarily extends functions in $C^m(\overline{\Omega})$ to lie in $C^m(\mathbb{R}^n)$.

5.18 The existence of even a simple (m, p)-extension operator for Ω guarantees that $W^{m,p}(\Omega)$ inherits many properties possessed by $W^{m,p}(\mathbb{R}^n)$. For instance, if an imbedding $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is known to hold, so that

$$\|u\|_{q,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\mathbb{R}^n},$$

then the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ must also hold, for if $u \in W^{m,p}(\Omega)$, then

$$\|u\|_{0,q,\Omega} \le \|Eu\|_{0,q,\mathbb{R}^n} \le K_1 \|Eu\|_{m,p,\mathbb{R}^n} \le K_1 K \|u\|_{m,p,\Omega}.$$

The reason we did not use this technique to prove the Sobolev imbedding theorem 4.12 is that extension theorems cannot be obtained for some domains satisfying such weak conditions as the cone condition or even the weak cone condition.

We will construct extension operators of each of the three types defined above. First we will use successive reflections in smooth boundaries to construct strong and total extension operators for half spaces, and strong extension operators for domains with suitably smooth boundaries. The method is attributed to Whitney [W] and later Hestenes [He] and Seeley [Se]. Stein [St] obtained a total extension operator under the minimal assumption that Ω satisfies the strong local Lipschitz condition. He used integral averaging instead of reflections. We will give only an outline of his proof here, leaving the interested reader to consult [St] for the details. See also [Ry]. The third construction, due to Calderón [Ca1] involves the use of the Calderón-Zygmund theory of singular integrals. It is less transparent than the reflection or averaging methods, and only works when $1 < p < \infty$, but requires only that the domain Ω satisfies the uniform cone condition. Unlike the other methods, it has the property that if the trivial extension $\tilde{u}$ belongs to $W^{m,p}(\mathbb{R}^n)$, then $\tilde{u}$ is the extension produced by the method. By Theorem 5.29 below, this happens if and only if $u \in W_0^{m,p}(\Omega)$. The paper [Jn] provides an extension method that works under a geometric hypothesis that is necessary and sufficient in $\mathbb{R}^2$, and is nearly optimal in higher dimensions.

Except for very simple domains all of our constructions require the use of partitions of unity subordinate to open covers of bdry Ω chosen in such a way that the functions in the partition have uniformly bounded derivatives.

To illustrate the reflection technique we begin by constructing a strong m-extension operator and a total extension operator for a half-space. Then we extend these to apply to domains that satisfy the uniform C^m-regularity condition and also have a bounded boundary.

5.19 THEOREM Let Ω be the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Then there exists a strong m-extension operator E for Ω. Moreover, for every multi-index α satisfying $|\alpha| \le m$ there exists a strong $(m - |\alpha|)$-extension operator E_α for Ω, such that

$$D^\alpha Eu(x) = E_\alpha D^\alpha u(x).$$

Proof. For functions u defined a.e. on $\mathbb{R}^n_+$ we define Eu and $E_\alpha u$, $|\alpha| \le m$ a.e. on $\mathbb{R}^n$ via

$$Eu(x) = \begin{cases} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{m+1} \lambda_j u(x_1, \ldots, x_{n-1}, -jx_n) & \text{if } x_n < 0, \end{cases}$$

$$E_\alpha u(x) = \begin{cases} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x_1, \ldots, x_{n-1}, -jx_n) & \text{if } x_n < 0, \end{cases}$$

where the coefficients $\lambda_1, \ldots, \lambda_{m+1}$ are the unique solutions of the $(m+1)\times(m+1)$ system of linear equations

$$\sum_{j=1}^{m+1} (-j)^k \lambda_j = 1, \qquad k = 0, \ldots, m.$$

If $u \in C^m(\overline{\mathbb{R}^n_+})$, it is readily checked that $Eu \in C^m(\mathbb{R}^n)$ and

$$D^\alpha Eu(x) = E_\alpha D^\alpha u(x), \qquad |\alpha| \le m.$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^n} |D^\alpha Eu(x)|^p \, dx \\ &= \int_{\mathbb{R}^n_+} |D^\alpha u(x)|^p \, dx + \int_{\mathbb{R}^n_-} \left| \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x_1, \ldots, x_{n-1}, -jx_n) \right|^p dx \\ &\le K(m, p, \alpha) \int_{\mathbb{R}^n_+} |D^\alpha u(x)|^p \, dx. \end{aligned}$$

By Theorem 3.22, the above inequality extends to functions $u \in W^{k,p}\left(\mathbb{R}^n_+\right)$, $m \ge k \ge |\alpha|$. Hence, E is a strong m-extension operator for $\mathbb{R}^n_+$. Since $D^\beta E_\alpha u(x) = E_{\alpha+\beta} u(x)$, a similar calculations shows that E_α is a strong $(m - |\alpha|)$-extension. ∎

The reflection technique used in the above proof can be modified to yield a total extension operator. The proof, due to Seeley [Se], is based on the following lemma.

5.20 LEMMA There exists a sequence $\{a_k\}_{k=0}^{\infty}$ such that for every nonnegative integer n we have

$$\sum_{k=0}^{\infty} 2^{nk} a_k = (-1)^n, \tag{20}$$

and

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty. \tag{21}$$

Proof. For fixed N, let $a_{k,N}$, $k = 0, 1, \dots, N$ be the solution of the system of linear equations

$$\sum_{k=0}^{N} 2^{nk} a_{k,N} = (-1)^n, \qquad n = 0, 1, \dots, N. \tag{22}$$

In terms of the Vandermonde determinant

$$V(x_0, x_1, \dots, x_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ x_0^2 & x_1^2 & \cdots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_0^N & x_1^N & \cdots & x_N^N \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{N} (x_j - x_i),$$

$a_{k,N}$ as given by Cramer's rule is

$$\begin{aligned} a_{k,N} &= \frac{V(1, 2, \dots, 2^{k-1}, -1, 2^{k+1}, \dots, 2^N)}{V(1, 2, \dots, 2^N)} \\ &= \left[\prod_{\substack{i,j=0 \\ i,j \neq k \\ i<j}} (2^j - 2^i) \prod_{i=0}^{k-1} (-1 - 2^i) \prod_{j=k+1}^{N} (2^j + 1) \right] \cdot \left[\prod_{\substack{i,j=0 \\ i<j}}^{N} (2^j - 2^i) \right]^{-1} \\ &= A_k B_{k,N} \end{aligned}$$

where

$$A_k = \prod_{i=1}^{k-1} \frac{1 + 2^i}{2^i - 2^k}, \qquad B_{k,N} = \prod_{j=k+1}^{N} \frac{1 + 2^j}{2^j - 2^k},$$

it being understood that $\prod_{i=l}^{m} P_i = 1$ if $l > m$. Now

$$|A_k| \le \prod_{i=1}^{k-1} \frac{2^{i+1}}{2^{k-1}} \le 2^{(5k-k^2)/2}.$$

Also

$$\begin{aligned}\log B_{k,N} &= \sum_{j=k+1}^{N} \log\left(1 + \frac{1+2^k}{2^j - 2^k}\right)\\ &< \sum_{j=k+1}^{N} \frac{1+2^k}{2^j - 2^k} < (1+2^k)\sum_{j=k+1}^{N} \frac{1}{2^{j-1}} < 4,\end{aligned}$$

where we have used the inequality $\log(1+x) < x$ valid for $x > 0$. It follows that the increasing sequence $\{B_{k,N}\}_{N=0}^{\infty}$ converges to a limit $B_k \le e^4$. Let $a_k = A_k B_k$, so that

$$|a_k| \le e^4 \cdot 2^{(5k-k^2)/2}.$$

Then for any n

$$\sum_{k=0}^{\infty} 2^{nk}|a_k| \le e^4 \sum_{k=0}^{\infty} 2^{(2nk+5k-k^2)/2} < \infty.$$

Letting $n \to \infty$ in (22) completes the proof. ∎

5.21 THEOREM Let Ω be a half-space in $\mathbb{R}^n$. Then there exists a total extension operator E for Ω.

Proof. The restrictions to $\mathbb{R}^n_+$ of functions $\phi \in C_0^\infty(\mathbb{R}^n)$ being dense in $W^{m,p}\left(\mathbb{R}^n_+\right)$ for any m and p, we need only define the extension operator for such functions. Let f be a real-valued function, infinitely differentiable on $[0,\infty)$ and satisfying $f(t) = 1$ if $0 \le t \le 1/2$ and $f(t) = 0$ if $t \ge 1$, If $\phi \in C_0^\infty(\mathbb{R}^n)$, let

$$E\phi(x) = E\phi(x', x_n) = \begin{cases} \phi(x) & \text{if } x_n \ge 0,\\ \sum_{k=0}^{\infty} a_k f(-2^k x_n)\phi(x', -2^k x_n) & \text{if } x_n < 0,\end{cases}$$

where $\{a_k\}$ is the sequence constructed in the previous lemma. $E\phi$ is well-defined on $\mathbb{R}^n$ since the sum above has only finitely many nonvanishing terms for any particular $x \in \mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$. Moreover, $E\phi$ has compact support and belongs to $C^\infty(\overline{\mathbb{R}^n_+}) \cap C^\infty(\overline{\mathbb{R}^n_-})$. If $x \in \mathbb{R}^n_-$, we have

$$\begin{aligned}D^\alpha E\phi(x) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\alpha_n} \binom{\alpha_n}{j} (-2^k)^{\alpha_n} f^{(\alpha_n - j)}(-2^k x_n) D_n^j D^{\alpha'} \phi(x', -2^k x_n)\\ &= \sum_{k=0}^{\infty} \psi_k(x).\end{aligned}$$

Since $\psi_k(x) = 0$ when $-x_n > 1/2^{k-1}$ it follows from (21) that the above series converges absolutely and uniformly as $x_n \to 0-$. Hence by (20)

$$\begin{aligned}\lim_{x_n \to 0-} D^\alpha E\phi(x) &= \sum_{k=0}^{\infty} (-2^k)^{\alpha_n} a_k D^\alpha \phi(x', 0+) \\ &= D^\alpha \phi(x', 0+) = \lim_{x_n \to 0+} D^\alpha E\phi(x) = D^\alpha E\phi(0).\end{aligned}$$

Thus $E\phi \in C_0^\infty(\mathbb{R}^n)$. Moreover, if $|\alpha| \le m$,

$$|\psi_k(x)|^p \le K_1^p |a_k|^p 2^{kmp} \sum_{|\beta| \le m} |D^\beta \phi(x', -2^k x_n)|^p,$$

where K_1 depends only on m, p, n, and f. Thus

$$\begin{aligned}\|\psi_k\|_{0,p,\mathbb{R}^n_-} &\le K_1 |a_k| 2^{km} \left(\sum_{|\beta| \le m} \int_{\mathbb{R}^n_-} |D^\beta \phi(x', -2^k x_n)|^p \, dx \right)^{1/p} \\ &= K_1 |a_k| 2^{km} \left(\frac{1}{2^k} \sum_{|\beta| \le m} \int_{\mathbb{R}^n_+} |D^\beta \phi(y)|^p \, dy \right)^{1/p} \\ &\le K_1 |a_k| 2^{km} \, \|\phi\|_{m,p,\mathbb{R}^n_+} .\end{aligned}$$

It follows from (21) that

$$\|D^\alpha E\phi\|_{0,p,\mathbb{R}^n_-} \le K_1 \, \|\phi\|_{m,p,\mathbb{R}^n_+} \sum_{k=0}^{\infty} 2^{km} |a_k| \le K_2 \, \|\phi\|_{m,p,\mathbb{R}^n_+} .$$

Combining this with a similar (trivial) estimate for $\|D^\alpha E\phi\|_{0,p,\mathbb{R}^n_+}$, we obtain

$$\|E\phi\|_{m,p,\mathbb{R}^n} \le K_3 \, \|\phi\|_{m,p,\mathbb{R}^n_+}$$

with $K_3 = K_3(m, p, n)$. This completes the proof. ▮

5.22 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition and having a bounded boundary. Then there exists a strong m-extension operator E for Ω. Moreover, if α and γ are multi-indices with $|\gamma| \le |\alpha| \le m$, then there exists a linear operator $E_{\alpha\gamma}$ continuous from $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$, $1 \le p < \infty$, such that if $u \in W^{|\alpha|,p}(\Omega)$, then

$$D^\alpha (Eu)(x) = \sum_{|\gamma| \le |\alpha|} E_{\alpha\gamma} D^\gamma u(x). \tag{23}$$

Proof. Since Ω is uniformly C^m-regular and has a bounded boundary the open cover $\{U_j\}$ of bdry Ω and the corresponding m-smooth maps Φ_j from U_j onto B referred to in Paragraph 4.10 are finite collections, say $1 \le j \le N$. Let $Q = \{y = (y', y_n) \in \mathbb{R}^n : |y'| < 1/2, \ |y_n| < \sqrt{3}/2\}$. Then

$$\{y \in \mathbb{R}^n : |y| < 1/2\} \subset Q \subset B = \{y \in \mathbb{R}^n : |y| < 1\}.$$

By condition (ii) of Paragraph 4.10 the open sets $V_j = \Psi_j(Q)$, $1 \le j \le N$, form an open cover of $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\}$ for some $\delta > 0$. There exists an open set $V_0 \subset \Omega$, bounded away from bdry Ω, such that $\Omega \subset \bigcup_{j=0}^{N} V_j$. By Theorem 3.15 we can find infinitely differentiable functions $\omega_0, \omega_1, \dots, \omega_N$ such that the support of ω_j is a subset of V_j and $\sum_{j=0}^{N} \omega_j(x) = 1$ for all $x \in \Omega$. (Note that the support of ω_0 need not be compact if Ω is unbounded.)

Since Ω is uniformly C^m-regular it satisfies the segment condition and so restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ are dense in $W^{k,p}(\Omega)$. If $\phi \in C_0^\infty(\mathbb{R}^n)$, then for $x \in \Omega$, $\phi(x) = \sum_{j=0}^{N} \phi_j(x)$, where $\phi_j = \omega_j \cdot \phi$.

For $j \ge 1$ and $y \in B$ let $\psi_j(y) = \phi_j\big(\Psi_j(y)\big)$. Then $\psi_j \in C_0^\infty(Q)$. We extend ψ_j to be identically zero outside Q. With E and E_α defined as in Theorem 5.19, we have $E\psi_j \in C_0^m(Q)$, $E\psi_j = \psi_j$ on $Q_+ = \{y \in Q : y_n > 0\}$, and

$$\left\| E\psi_j \right\|_{k,p,Q} \le K_1 \left\| \psi_j \right\|_{k,p,Q_+}, \qquad 0 \le k \le m,$$

where K_1 depends on k, m, and p. If $\theta_j(x) = E\psi_j\big(\Phi_j(x)\big)$, then $\theta_j \in C_0^\infty(V_j)$ and $\theta_j(x) = \phi_j(x)$ if $x \in \Omega$. It may be checked by induction that if $|\alpha| \le m$, then

$$D^\alpha \theta_j(x) = \sum_{|\beta| \le |\alpha|} \sum_{|\gamma| \le |\alpha|} a_{j;\alpha\beta}(x) \big[E_\beta \big(b_{j;\beta\gamma} \cdot (D^\gamma \phi_j \circ \Psi_j) \big) \big] \big(\Phi_j(x) \big),$$

where $a_{j;\alpha\beta} \in C^{m-|\alpha|}(\overline{U_j})$ and $b_{j;\beta\gamma} \in C^{m-|\beta|}(\overline{B})$ depend on the transformations Φ_j and $\Psi_j = \Phi_j^{-1}$ and satisfy

$$\sum_{|\beta| \le |\alpha|} a_{j;\alpha\beta}(x) b_{j;\beta\gamma}\big(\Phi_j(x)\big) = \begin{cases} 1 & \text{if } \gamma = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.41 we have for $k \le m$,

$$\left\| \theta_j \right\|_{k,p,\mathbb{R}^n} \le K_2 \left\| E\psi_j \right\|_{k,p,Q} \le K_1 K_2 \left\| \psi_j \right\|_{k,p,Q_+} \le K_3 \left\| \psi_j \right\|_{k,p,\Omega},$$

where K_3 may be chosen to be independent of j. The operator $\tilde{E}$ defined by

$$\tilde{E}\phi(x) = \phi_0(x) + \sum_{j=1}^{N} \theta_j(x)$$

clearly satisfies $\tilde{E}\phi(x) = \phi(x)$ if $x \in \Omega$, and

$$\left\|\tilde{E}\phi\right\|_{k,p,\mathbb{R}^n} \le \|\phi_0\|_{k,p,\Omega} + K_3 \sum_{j=1}^{N} \|\phi_j\|_{k,p,\Omega} \le K_4(1 + NK_3)\,\|\phi\|_{k,p,\Omega}\,, \tag{24}$$

where

$$K_4 = \max_{0\le j\le N}\ \max_{|\alpha|\le m}\ \sup |D^\alpha \omega_j(x)| < \infty.$$

Thus $\tilde{E}$ is a strong m-extension operator for Ω. Also

$$D^\alpha \tilde{E}\phi(x) = \sum_{|\gamma|\le|\alpha|} (E_{\alpha\gamma} D^\gamma \phi)(x),$$

where

$$E_{\alpha\gamma} v(x) = \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)\big[E_\beta\big(b_{j;\beta\gamma}\cdot(v\cdot\omega_j)\circ\Psi_j\big)\big]\big(\Phi_j(x)\big)$$

if $\alpha \ne \gamma$, and

$$E_{\alpha\alpha} v(x) = (v\cdot\omega_0)(x) + \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)\big[E_\beta\big(b_{j;\beta\gamma}\cdot(v\cdot\omega_j)\circ\Psi_j\big)\big]\big(\Phi_j(x)\big).$$

We note that if $x \in \Omega$, then $E_{\alpha\gamma} v(x) = 0$ for $\alpha \ne \gamma$ and $E_{\alpha\alpha} v(x) = v(x)$. Clearly $E_{\alpha\gamma}$ is a linear operator. By the differentiability properties of $a_{j;\alpha\beta}$ and $b_{j;\beta\gamma}$, $E_{\alpha\gamma}$ is continuous on $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$. This completes the proof. ∎

5.23 REMARKS

1. If Ω is uniformly C^m-regular for all m, and has a bounded boundary, then we can use the total extension operator of Theorem 5.21 in place of that of Theorem 5.19 in the above proof to obtain a total extension operator for Ω.
2. The restriction that bdry Ω be bounded was imposed in Theorem 5.22 so that the open cover $\{V_j\}$ would be finite. This finiteness was used in two places in the proof, first in asserting the existence of the constant K_4, and secondly in obtaining the last inequality in (24). This latter use is, however, not essential for the proof because (24) could still be obtained from the finite intersection property (condition (i) in Paragraph 4.10) even if the cover $\{V_j\}$ were not finite. Theorem 5.22 extends to any suitably regular domain for which there exists a partition of unity $\{\omega_j\}$ subordinate to $\{V_j\}$ with $D^\alpha\omega_j$ bounded on $\mathbb{R}^n$ uniformly in j for any given α. The reader may find it

interesting to construct, by the above techniques, extension operators for domains not covered by the above theorems, for example, quadrants, strips, rectangular boxes, and smooth images of these.

3. The previous remark also applies to the Calderón Extension Theorem 5.28 given below. Although it is proved by methods quite different from the reflection methods used above, the proof still makes use of a partition of unity in the same way as does that of Theorem 5.22. Accordingly, the above considerations also apply to it. The theorem is proved under a strengthened uniform cone condition that reduces to the uniform cone condition of Paragraph 4.8 if Ω has a bounded boundary.

Clearly subsuming the extension theorems obtained above is the following theorem of Stein [St].

5.24 THEOREM (The Stein Extension Theorem) If Ω is a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition, then there exists a total extension operator for Ω.

We will provide here only an outline of the proof. The details can be found in Chapter 6 of [St].

5.25 (Outline of the Proof of the Stein Extension Theorem) .

1. Let $\Omega_e = \mathbb{R}^n - \overline{\Omega}$ be the open exterior of Ω. The function $\delta(x) = \operatorname{dist}(x, \overline{\Omega})$ is Lipschitz continuous on Ω_e since
$$|\delta(x) - \delta(y)| \le |x - y| \qquad \text{for} \quad x, y \in \Omega_e,$$
but might not be smooth there. However, there exists a function Δ in $C^\infty(\Omega_e)$ and positive constants c_1, c_2, and C_α for all multiindices α such that for all $x \in \Omega_e$,
$$\begin{aligned} &c_1\delta(x) \le \Delta(x) \le c_2\delta(x), \qquad \text{and} \\ &|D^\alpha \Delta(x)| \le C_\alpha\big(\delta(x)\big)^{1-|\alpha|}. \end{aligned}$$

2. There exists a continuous function ϕ on $[1, \infty)$ for which
   (a) $\lim_{t\to\infty} t^k\phi(t) = 0$ for $k = 0, 1, 2, \ldots,$
   (b) $\int_1^\infty \phi(t)\,dt = 1$
   (c) $\int_1^\infty t^k\phi(t) = 0$ for $k = 1, 2, \ldots.$

   In fact, $\phi(t) = \dfrac{e}{\pi t}\operatorname{Im}\left(e^{-w(t-1)^{1/4}}\right)$, where $w = e^{-i\pi/4}$, is such a function.

3. For the special case $\Omega = \{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, y > f(x)$ where f satisfies a Lipschitz condition $|\phi(x) - \phi(x')| \le M|x - x'|$, there exists a constant c such that if $(x, y) \in \Omega_e$, then $\phi(x) - y \le c\Delta(x, y)$.
4. For Ω as specified in 3, $\Delta^*(x, y) = 2c\Delta(x, y)$, and $u \in C_0^\infty(\mathbb{R}^n)$, the operator E defined by

$$E(u)(x, y) = \begin{cases} u(x, y) & \text{if } y > f(x) \\ \int_1^\infty u\big(x, y + t\Delta^*(x, y)\big)\phi(t)\, dt & \text{if } y < f(x) \end{cases}$$

satisfies, for every $m \ge 0$ and $1 \le p \le \infty$,

$$\|E(u)\|_{m,p,\mathbb{R}^n} \le K\, \|u\|_{m,p,\Omega}\,, \tag{25}$$

where $K = K(m, p, n, M)$. Since Ω satisfies the strong local Lipschitz condition it also satisfies the segment condition and so, by Theorem 3.22 the restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ are dense in $W^{m,p}(\Omega)$ and so (25) holds for all $u \in W^{m,p}(\Omega)$. Thus Stein's theorem holds for this Ω.
5. The case of general Ω satisfying the strong local Lipschitz condition now follows via a partition of unity subordinate to an open cover of bdry Ω by open sets in each of which (a rotated version of) the special case 4 can be applied. ∎

5.26 The proof of the Calderón extension theorem is based on a special case, suitable for our purposes, of a well-known inequality of Calderón and Zygmund [CZ] for convolutions involving kernels with nonintegrable singularities. The proof of this inequality is rather lengthy and can be found in many sources (e.g. Stein and Weiss [SW]). It will be omitted here. Neither the inequality nor the extension theorem itself will be required hereafter in this monograph.

Let $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$, let $\Sigma_R = \{x \in \mathbb{R}^n : |x| = R\}$, and let $d\sigma_R$ be the area element (Lebesgue $(n-1)$-volume element) on Σ_R. A function g is said to be *homogeneous of degree* μ on $B_R - \{0\}$ if $g(tx) = t^\mu g(x)$ for all $x \in B_R - \{0\}$ and $0 < t \le 1$.

5.27 THEOREM (The Calderón Zygmund Inequality) Let

$$g(x) = G(x)|x|^{-n},$$

where

(i) G is bounded on $\mathbb{R}^n - \{0\}$ and has compact support,

(ii) G is homogeneous of degree 0 on $B_R - \{0\}$ for some $R > 0$, and

(iii) $\int_{\Sigma_R} G(x)\, d\sigma_R = 0$.

If $1 < p < \infty$ and $u \in L^p(\mathbb{R}^n)$, then the principal-value convolution integral

$$u * g(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^n - B_\epsilon} u(x-y)g(y)\,dy$$

exists for almost all $x \in \mathbb{R}^n$, and there exists a constant $K = K(G, p)$ such that for all such u

$$\|u * g\|_p \leq K \|u\|_p .$$

Conversely, if G satisfies (i) and (ii) and if $u * g$ exists for all $u \in C_0^\infty(\mathbb{R}^n)$, then G satisfies (iii).

5.28 THEOREM (The Calderón Extension Theorem) Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform cone condition (Paragraph 4.8) modified as follows:

(i) the open cover $\{U_j\}$ of bdry Ω is required to be finite, and

(ii) the sets U_j are not required to be bounded.

Then for any $m \in \{1, 2, \ldots\}$ and any p satisfying $1 < p < \infty$, there exists a simple (m, p)-extension operator $E = E(m, p)$ for Ω.

Proof. Let $\{U_1, \ldots, U_N\}$ be the open cover of bdry Ω given by the uniform cone condition, and let U_0 be an open subset of Ω bounded away from bdry Ω such that $\Omega \subset \bigcup_{j=0}^N U_j$. (Such a U_0 exists by condition (ii) of Paragraph 4.8.) Let $\omega_0, \omega_1, \ldots, \omega_N$ be a C^∞ partition of unity for Ω with $\operatorname{supp}(\omega)_j \subset U_j$. For $1 \leq j \leq N$ we shall define operators E_j so that if $u \in W^{m,p}(\Omega)$, then $E_j u \in W^{m,p}(\mathbb{R}^n)$ and satisfies

$$E_j u = u \qquad \text{in} \quad U_j \cap \Omega,$$
$$\left\| E_j u \right\|_{m,p,\mathbb{R}^n} \leq K_{m,p,j} \|u\|_{m,p,\Omega} .$$

The desired extension operator is then clearly given by

$$Eu = \omega_0 u + \sum_{j=1}^N \omega_j E_j u.$$

We shall write $x \in \mathbb{R}^n$ in the polar coordinate form $x = \rho\sigma$ where $\rho \geq 0$ and σ is a unit vector. Let C_j, the the cone associated with U_j in the description of the uniform cone condition, have vertex at 0. Let ϕ_j be a nontrivial function defined in $\mathbb{R}^n - \{0\}$ satisfying

(i) $\phi_j(x) \geq 0$ for all $x \neq 0$,

(ii) $\operatorname{supp}(\phi_j) \subset -C_j \cup \{0\}$,

(iii) $\phi_j \in C^\infty(\mathbb{R}^n - \{0\})$, and

(iv) for some $\epsilon > 0$, ϕ_j is homogeneous of degree $m - n$ in $B_\epsilon - \{0\}$.

Now $\rho^{n-1}\phi_j$ is homogeneous of degree $m - 1 \geq 0$ on $B_\epsilon - \{0\}$ and so the function $\psi_j(x) = (\partial/\partial\rho)^m\big(\rho^{n-1}\phi_j(x)\big)$ vanishes on $B_\epsilon - \{0\}$. Hence ψ_j, extended to be zero at $x = 0$, belongs to $C_0^\infty(-C_j)$. Define

$$E_j u(y) = K_j\left((-1)^m \int_\Sigma \int_0^\infty \phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m \tilde{u}(y - \rho\sigma)\,d\rho\,d\sigma \right.$$
$$\left. - \int_\Sigma \int_0^\infty \psi_j(\rho\sigma)\tilde{u}(y - \rho\sigma)\,d\rho\,d\sigma\right) \tag{26}$$

where $\tilde{u}$ is the zero extension of u outside Ω and where the constant K_j will be determined shortly. If $y \in U_j \cap \Omega$, then, assuming for the moment that $u \in C^\infty(\Omega)$, we have, for $\rho\sigma \in \operatorname{supp}(\phi_j)$, by condition (iii) of Paragraph 4.8, that $\tilde{u}(y - \rho\sigma) = u(y - \rho\sigma)$ is infinitely differentiable. Now integration by parts m times yields

$$\begin{aligned}
&(-1)^m \int_0^\infty \rho^{n-1}\phi_j(\rho\sigma)\left(\frac{\partial}{\partial\rho}\right)^m u(y - \rho\sigma)\,d\rho \\
&= \sum_{k=0}^{m-1}(-1)^{m-k}\left(\frac{\partial}{\partial\rho}\right)^k \big(\rho^{n-1}\phi_j(\rho\sigma)\big)\left(\frac{\partial}{\partial\rho}\right)^{m-k-1} u(y - \rho\sigma)\Bigg|_{\rho=0}^{\rho=\infty} \\
&\quad + \int_0^\infty \left(\frac{\partial}{\partial\rho}\right)^m \big(\rho^{n-1}\phi_j(\rho\sigma)\big)u(y - \rho\sigma)\,d\rho \\
&= \left(\frac{\partial}{\partial\rho}\right)^{m-1} \big(\rho^{n-1}\phi_j(\rho\sigma)\big)\Big|_{\rho=0} u(y) + \int_0^\infty \psi_j(\rho\sigma)u(y - \rho\sigma)\,d\rho.
\end{aligned}$$

Hence

$$E_j u(y) = K_j u(y)\int_\Sigma \left(\frac{\partial}{\partial\rho}\right)^{m-1}\big(\rho^{n-1}\phi_j(\rho\sigma)\big)\Big|_{\rho=0} d\sigma.$$

Since $(\partial/\partial\rho)^{m-1}\big(\rho^{n-1}\phi_j(\rho\sigma)\big)$ is homogeneous of degree zero near 0, the above integral does not vanish if ϕ_j is not identically zero. Hence K_j can be chosen so that $E_j u(y) = u(y)$ for $y \in U_j \cap \Omega$ and all $u \in C^\infty(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ we have $E_j u(y) = u(y)$ a.e. in $U_j \cap \Omega$ for every $u \in W^{m,p}(\Omega)$. The same argument shows that if $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$, then $E_j u(y) = \tilde{u}(y)$ a.e. in $\mathbb{R}^n$.

It remains, therefore, to show that

$$\big\| D^\alpha E_j u \big\|_{0,p,\mathbb{R}^n} \leq K_\alpha \, \|u\|_{m,p,\Omega}$$

holds for any α with $|\alpha| \leq m$ and all $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$. The last integral in (26) is of the form $\theta_j * \tilde{u}(y)$, where $\theta_j(x) = \psi_j(x)|x|^{1-n}$. Since $\theta_j \in L^1(\mathbb{R}^n)$ and has compact support, we obtain via Young's inequality for convolution (Corollary 2.25),

$$\left\| D^\alpha(\theta_j * \tilde{u}) \right\|_{0,p,\mathbb{R}^n} = \left\| \theta_j * (\widetilde{D^\alpha u}) \right\|_{0,p,\mathbb{R}^n} \le \left\| \theta_j \right\|_{0,1,\mathbb{R}^n} \| D^\alpha u \|_{0,p,\Omega} .$$

It now remains to be shown that the first integral in (26) defines a bounded map from $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$. Since $(\partial/\partial\rho)^m = \sum_{|\alpha|=m}(m!/\alpha!)\sigma^\alpha D^\alpha$ we obtain

$$\begin{aligned}
\int_\Sigma \int_0^\infty & \phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m \tilde{u}(y-\rho\sigma)\, d\rho\, d\sigma \\
&= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} \phi_j(x) \widetilde{D_x^\alpha u}(y-x)\sigma^\alpha\, dx \\
&= \sum_{|\alpha|=m} \xi_\alpha * \widetilde{D^\alpha u},
\end{aligned}$$

where $\xi_\alpha = (-1)^{|\alpha|}(m!/\alpha!)\sigma^\alpha\phi_j$ is homogeneous of degree $m-n$ in $B_\epsilon - \{0\}$ and belongs to $C^\infty(\mathbb{R}^n - \{0\})$. It is now clearly sufficient to show that for any β satisfying $|\beta| \le m$

$$\left\| D^\beta(\xi_\alpha * v) \right\|_{0,p,\mathbb{R}^n} \le K_{\alpha,\beta} \|v\|_{0,p,\mathbb{R}^n} . \tag{27}$$

If $|\beta| \le m-1$, then $D^\beta\xi_\alpha$ is homogeneous of degree not exceeding $1-n$ in $B_\epsilon - \{0\}$ and so belongs to $L^1(\mathbb{R}^n)$. Inequality (27) now follows by Young's inequality for convolution. Thus we need consider only the case $|\beta| = m$, in which we write $D^\beta = (\partial/\partial x_i)D^\gamma$ for some i, $1 \le i \le n$, and some γ with $|\gamma| = m-1$. Suppose, for the moment, that $v \in C_0^\infty(\mathbb{R}^n)$. Then we may write

$$\begin{aligned}
D^\beta(\xi_\alpha * v)(x) &= (D^\gamma\xi_\alpha) * \left[\left(\frac{\partial}{\partial x_i}\right) v\right](x) = \int_{\mathbb{R}^n} D_i v(x-y) D^\gamma \xi_\alpha(y)\, dy \\
&= \lim_{\delta\to 0+} \int_{\mathbb{R}^n - B_\delta} D_i v(x-y) D^\gamma \xi_\alpha(y)\, dy.
\end{aligned}$$

We now integrate by parts in the last integral to free v and obtain $D^\beta\xi_\alpha$ under the integral. The integrated term is a surface integral over the spherical boundary Σ_δ of B_δ of the product of $v(x-\cdot)$ and a function homogeneous of degree $1-n$ near zero. This surface integral must therefore tend to $Kv(x)$ as $\delta \to 0+$, for some constant K. Noting that $D_i v(x-y) = -(\partial/\partial y_i)v(x-y)$, we now have

$$D^\beta(\xi_\alpha * v)(x) = \lim_{\delta\to 0+} \int_{\mathbb{R}^n} v(x-y) D^\beta\xi_\alpha(y)\, dy + Kv(x).$$

Now $D^\beta\xi_\alpha$ is homogeneous of degree $-n$ near the origin and so, by the last assertion of Theorem 5.27, $D^\beta\xi_\alpha$ satisfies all the conditions for the singular kernel

g of that theorem. Since $1 < p < \infty$, we have for any $v \in L^p(\Omega)$ (regarded as being identically zero outside Ω)

$$\left\| D^\beta \xi_\alpha * v \right\|_{0,p,\mathbb{R}^n} \le K_{\alpha,\beta} \left\| v \right\|_{0,p,\mathbb{R}^n} .$$

This completes the proof. ∎

As observed in the proof of the above theorem, the Calderón extension of a function $u \in W^{m,p}(\Omega)$ coincides with the zero extension $\tilde{u}$ of u if $\tilde{u}$ belongs to $W^{m,p}(\mathbb{R}^n)$. The following theorem (which could have been proved in Chapter 3) shows that in this case u must belong to $W_0^{m,p}(\Omega)$.

5.29 THEOREM (Characterization of $W_0^{m,p}(\Omega)$ by Exterior Extension) Let Ω have the segment property. Then a function u on Ω belongs to $W_0^{m,p}(\Omega)$ if and only if the zero extension $\tilde{u}$ of u belongs to $W^{m,p}(R^n)$.

Proof. Lemma 3.27 shows, with no hypotheses on Ω, that if $u \in W_0^{m,p}(\Omega)$, then $\tilde{u} \in W^{m,p}(R^n)$.

Conversely, suppose that Ω has the segment property and that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$. Proceed as in the proof of Theorem 3.22, first multiplying u by a suitable smooth cutoff function f_ϵ to approximate u in $W^{m,p}(\Omega)$ by a function in that space with a bounded support. Replace u by that approximation; then $\tilde{u}$ is replaced by $f_\epsilon \tilde{u}$, and so still belongs to $W^{m,p}(\mathbb{R}^n)$. Now split this u into finitely-many pieces u_j, where $0 \le i \le k$, with u_j supported in a set V_j and the union of the sets V_j covering the support of u. In the context of that theorem, u_0 already belongs to $W_0^{m,p}(\Omega)$.

For the other values of j, use a translate $u_{j,t}$ of $\widetilde{u_j}$ mapping x to $\widetilde{u_j}(x - ty)$ rather than to $\widetilde{u_j}(x + ty)$ as we did in the proof of Theorem 3.22. For small enough positive values of t, using $x - ty$ shifts the support of $\widetilde{u_j}$ strictly inside the domain Ω. Then $u_{j,t}$ belongs to $W^{m,p}(R^n)$ since $\widetilde{u_j}$ does. Since $u_{j,t}$ vanishes outside a compact subset of Ω, the restriction of $u_{j,t}$ to Ω belongs to $W_0^{m,p}(\Omega)$. As $t \to 0+$, these restrictions converge to u_j in $W^{m,p}(\Omega)$. Thus each piece u_j belongs to $W_0^{m,p}(\Omega)$, and so does u. ∎

5.30 There is a close connection between the existence of extension operators and imbeddings into spaces of Hölder continuous functions. For example, it is shown in [Ko] that the imbedding $W^{m,p}(\Omega) \to C^{0,1-(n/p)}(\overline{\Omega})$ implies the existence of a simple $(1,q)$-extension operator for Ω provided $q > p$.

A short survey of extension theorems for Sobolev spaces can be found in [Bu2].

An Approximation Theorem

5.31 (The Approximation Property) The following question is involved in the matter of interpolation of Sobolev spaces on order of smoothness that will play a central role in the development of Besov spaces and Sobolev spaces of fractional order in Chapter 7:

> If $0 < k < m$ does there exist a constant C such that for every $u \in W^{k,p}(\Omega)$ and every sufficiently small ϵ there exists $u_\epsilon \in W^{m,p}(\Omega)$ satisfying
>
> $$\|u - u_\epsilon\|_p \le C\epsilon^k \|u\|_{k,p}, \qquad \text{and} \qquad \|u_\epsilon\|_{m,p} \le C\epsilon^{k-m} \|u\|_{k,p}?$$

If the answer is "yes," we will say that the domain Ω has the *approximation property*. Combined with the interpolation Theorem 5.2, this property will show that $W^{k,p}(\Omega)$ is suitably intermediate between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ for purposes of interpolation. In Theorem 5.33 we prove that $\mathbb{R}^n$ itself has the approximation property. It will therefore follow that any domain Ω admitting a total extension operator will have the approximation property for any choice of k and m with $0 < k < m$. In particular, therefore, a domain satisfying the strong local Lipschitz condition has the approximation property.

There are domains with the approximation property that do not satisfy the strong local Lipschitz condition. The approximation property does not prevent a domain from lying on both sides of a boundary hypersurface. In [AF4] the authors obtain the property under the assumption that Ω satisfies the "smooth cone condition," which is essentially a cone condition with the added restriction that the cone must vary smoothly from point to point. Our proof of Theorem 5.33 is a simplified version of the proof in [AF4].

We begin by stating an elementary lemma.

5.32 LEMMA If $u \in L^p(\mathbb{R}^n)$ and $B_\epsilon(x)$ is the ball of radius ϵ about x, then

$$\int_{\mathbb{R}^n} \left(\int_{B_\epsilon(x)} |u(y)|\, dy \right)^p dx \le K_n^p \epsilon^{np} \|u\|_{p,\mathbb{R}^n}^p,$$

where K_n is the volume of the unit ball $B_1(0)$.

Proof. The proof is immediate using Hölder's inequality and a change or order of integration. ∎

5.33 THEOREM (An Approximation Theorem for $\mathbb{R}^n$) If $0 < k < m$, there exists a constant C such that for $u \in W^{k,p}(\mathbb{R}^n)$ and $0 < \epsilon \le 1$ there exists u_ϵ in $C^\infty(\mathbb{R}^n)$ such that the following seminorm inequalities hold:

$$\|u - u_\epsilon\|_p \le C\epsilon^k |u|_{k,p}, \qquad \text{and}$$

$$|u_\epsilon|_{j,p} \le \begin{cases} C\,\|u\|_{k,p} & \text{if } j \le k-1 \\ C\,\epsilon^{k-j}\,|u|_{k,p} & \text{if } j \ge k. \end{cases}$$

In particular, $\mathbb{R}^n$ has the approximation property.

Proof. It is sufficient to establish the inequalities for $u \in C_0^\infty(\mathbb{R}^n)$ which is dense in $W^{k,p}(\mathbb{R}^n)$. We apply Taylor's formula

$$f(1) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(t)\, dt$$

to the function $f(t) = u\big(tx + (1-t)y\big)$ to obtain

$$\begin{aligned} u(x) = &\sum_{|\alpha| \le k-1} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \\ &+ \sum_{|\alpha|=k} \frac{k}{\alpha!}(x-y)^\alpha \int_0^1 (1-t)^{k-1} D^\alpha u\big(tx + (1-t)y\big)\, dt. \end{aligned}$$

Now let $\phi \in C_0^\infty\big(B_1(0)\big)$ satisfy $0 \le \phi(x) \le K_0$ for all x and $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. We multiply the above Taylor formula by $\epsilon^{-n}\phi\big((x-y)/\epsilon\big)$ and integrate y over $\mathbb{R}^n$ to obtain $u(x) = u_\epsilon(x) + R(x)$ where

$$\begin{aligned} u_\epsilon(x) &= \epsilon^{-n} \sum_{|\alpha| \le k-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right)(x-y)^\alpha D^\alpha u(y)\, dy \\ R(x) &= \epsilon^{-n} \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right)(x-y)^\alpha\, dy \\ &\qquad \times \int_0^1 (1-t)^{k-1} D^\alpha u\big(tx + (1-t)y\big)\, dt. \end{aligned}$$

We can estimate $|u(x) - u_\epsilon(x)| = |R(x)|$ by reversing the order of the double integral, substituting $z = tx + (1-t)y$ (so that $z - x = (1-t)(y-x)$ and $dz = (1-t)^n\, dy$), and reversing the order of integration again:

$$\begin{aligned} |u(x) - u_\epsilon(x)| &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!} \epsilon^{-n} \int_0^1 (1-t)^{-1-n}\, dt \int_{B_{\epsilon(1-t)}(x)} |x-z|^k |D^\alpha u(z)|\, dz \\ &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!} \epsilon^{-n} \int_{B_\epsilon(x)} |x-z|^k |D^\alpha u(z)|\, dz \int_0^{1-|z-x|/\epsilon} (1-t)^{-n-1}\, dt \\ &< K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!} \epsilon^{-n} \int_{B_\epsilon(x)} |x-z|^k |D^\alpha u(z)|\, dz \\ &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!} \epsilon^{k-n} \int_{B_\epsilon(x)} |D^\alpha u(z)|\, dz. \end{aligned}$$

Estimating the L^p-norm of the last integral above by the previous lemma, we obtain

$$\|u(x) - u_\epsilon(x)\|_p \leq K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!} \epsilon^k \|D^\alpha u\|_p \leq C\epsilon^k |u|_{k,p}.$$

On the other hand, we have

$$u_\epsilon(x) = \epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-1}(u; x, y)\, dy,$$

where

$$P_j(u; x, y) = \sum_{i=0}^{j} T_i(u; x, y),$$

$$T_j(u; x, y) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha.$$

It is readily verified that

$$\frac{\partial}{\partial x_i} T_j(u; x, y) = \begin{cases} T_{j-1}(D_i u; x, y) & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial x_i} P_j(u; x, y) = \begin{cases} P_{j-1}(D_i u; x, y) & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial y_i} P_j(u; x, y) = T_j(D_i u; x, y) \qquad \text{for} \quad j \geq 0.$$

Since $\dfrac{\partial}{\partial x_i}\phi\left(\dfrac{x-y}{\epsilon}\right) = -\dfrac{\partial}{\partial y_i}\phi\left(\dfrac{x-y}{\epsilon}\right)$, integration by parts gives

$$D_i u_\epsilon(x) = \epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-2}(D_i u; x, y)\, dy + \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_{k-1}(D_i u; x, y)\, dy.$$

By induction, if $|\beta| = j \leq k$,

$$D^\beta u_\epsilon(x) = \epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-1-j}(D^\beta u; x, y)\, dy + j\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_{k-j}(D^\beta u; x, y)\, dy.$$

When $j = k$ the sums P_{k-1-j} are empty, leaving only the second line above, which becomes

$$k\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_0(D^\beta u; x, y)\, dy = k\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) D^\beta u(y)\, dy.$$

Write any multi-index γ with $|\gamma| > k$ in the form $\beta + \delta$ with $|\beta| = k$ to get that

$$D^{\gamma} u_{\epsilon}(x) = k\epsilon^{-n-|\delta|} \int_{\mathbb{R}^n} D^{\delta}\phi\left(\frac{x-y}{\epsilon}\right) D^{\beta}u(y)\, dy$$

in these cases. Apply the previous lemma to the various terms above to get that

$$|u_{\epsilon}|_{j,p} \le \begin{cases} C\,\|u\|_p & \text{if } j \le k-1 \\ C\epsilon^{k-j}\,|u|_{k,p} & \text{if } j \ge k. \end{cases}$$

In deriving this when $j < k$, expand the (nonempty) sums P_{k-1-j} to see that

$$|D^{\beta}u_{\epsilon}(x)| \le K_0\epsilon^{-n} \int_{B_{\epsilon}(x)} \left[\sum_{i=0}^{k-1-j} |T_i(D^{\beta}u; x, y)| + j|T_{k-j}(D^{\beta}u; x, y)|\right] dy.$$

This completes the proof. ∎

Boundary Traces

5.34 Of importance in the study of boundary value problems for differential operators defined on a domain Ω is the determination of spaces of functions defined on the boundary of Ω that contain the traces $u|_{\text{bdry}\,\Omega}$ of functions u in $W^{m,p}(\Omega)$. For example, if $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$, then clearly $u|_{\text{bdry}\,\Omega}$ belongs to $C(\text{bdry}\,\Omega)$. We outline below an L^q-imbedding result for such traces which can be obtained for domains with suitably smooth boundaries as a corollary of Theorem 4.12 via the use of an extension operator.

The more interesting problem of characterizing the image of $W^{m,p}(\Omega)$ under the mapping $u \to u|_{\text{bdry}\,\Omega}$ will be dealt with in Chapter 7. See, in particular, Theorem 7.39. The characterization is in terms of Besov spaces which are generalized Sobolev spaces of fractional order.

5.35 Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition of Paragraph 4.10. Thus there exists a locally finite open cover $\{U_j\}$ of bdry Ω, and corresponding m-smooth transformations Ψ_j mapping $B = \{y \in \mathbb{R}^n : |y| < 1\}$ onto U_j such that $U_j \cap \text{bdry}\,\Omega = \Psi_j(B_0)$, where $B_0 = \{y \in B : y_n = 0\}$. If f is a function having support in U_j, we may define the integral of f over bdry Ω via

$$\int_{\text{bdry}\,\Omega} f(x)\, d\sigma = \int_{U_j \cap \text{bdry}\,\Omega} f(x)\, d\sigma = \int_{B_0} f \circ \Psi_j(y', 0)\, J_j(y')\, dy',$$

where $d\sigma$ is the $(n-1)$-volume element on bdry Ω, $y' = (y_1, \dots, y_{n-1})$, and, if $x = \Psi_j(y)$, then

$$J_j(y') = \left[\sum_{k=1}^{n} \left(\frac{\partial(x_1, \dots, \hat{x}_k, \dots, x_n)}{\partial(y_1, \dots, y_{n-1})} \right)^2 \right]^{1/2} \Bigg|_{y_n=0} .$$

If f is an arbitrary function defined on $\mathbb{R}^n$, we may set

$$\int_{\text{bdry}\,\Omega} f(x)\, d\sigma = \sum_j \int_{\text{bdry}\,\Omega} f(x) v_j(x)\, d\sigma,$$

where $\{v_j\}$ is a partition of unity for bdry Ω subordinate to $\{U_j\}$.

5.36 THEOREM (A Boundary Trace Imbedding Theorem) Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition, and suppose there exists a simple (m, p)-extension operator E for Ω. Also suppose that $mp < n$ and $p \le q \le p^* = (n-1)p/(n-mp)$. Then

$$W^{m,p}(\Omega) \to L^q(\text{bdry}\,\Omega)\,. \tag{28}$$

If $mp = n$, then imbedding (28) holds for $p \le q < \infty$.

Proof. Imbedding (28) should be interpreted in the following sense. If $u \in W^{m,p}(\Omega)$, then Eu has a trace on bdry Ω in the sense described in Paragraph 4.2, and $\|Eu\|_{0,q,\text{bdry}\,\Omega} \le K\, \|u\|_{m,p,\Omega}$ with K independent of u. Note that since $C_0(R^n)$ is dense in $W^{m,p}(\Omega)$, $\|Eu\|_{0,q,\text{bdry}\,\Omega}$ is independent of the particular extension operator E used.

We prove the special case $mp < n$, $q = p^* = (n-1)p/(n-mp)$ of the theorem; the other cases are similar. We use the notations of the previous Paragraph.

There is a constant K_1 such that for every $u \in W^{m,p}(\Omega)$,

$$\|Eu\|_{m,p,\mathbb{R}^n} \le K_1\, \|u\|_{m,p,\Omega}\,.$$

By the uniform C^m-regularity condition (see Paragraph 4.10) there exists a constant K_2 such that for each j and every $y \in B$ we have $x = \Psi_j(y) \in U_j$,

$$|J_j(y')| \le K_2, \qquad \text{and} \qquad \left| \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} \right| \le K_2.$$

Noting that $0 \le v_j(x) \le 1$ on $\mathbb{R}^n$, and using imbedding (4) of Theorem 4.12 applied over B, we have, for $u \in W^{m,p}(\Omega)$,

$$\int_{\text{bdry}\,\Omega} |Eu(x)|^q\, d\sigma \le \sum_j \int_{U_j \cap \text{bdry}\,\Omega} |Eu(x)|^q\, d\sigma$$

$$
\begin{aligned}
&\leq K_2 \sum_j \left\| Eu \circ \Psi_j \right\|_{0,q,B_0}^q \\
&\leq K_3 \sum_j \left(\left\| Eu \circ \Psi_j \right\|_{m,p,B}^p \right)^{q/p} \\
&\leq K_4 \left(\sum_j \|Eu\|_{m,p,U_j}^p \right)^{q/p} \\
&\leq K_4 R \, \|Eu\|_{m,p,\mathbb{R}^n}^q \\
&\leq K_5 \, \|u\|_{m,p,\Omega}^q \, .
\end{aligned}
$$

The second last inequality above makes use of the finite intersection property possessed by the cover $\{U_j\}$. The constant K_4 is independent of j because $|D^\alpha \Psi_{j,i}(y)| \leq \text{const}$ for all i, j, where $\Psi_j = (\Psi_{j,1}, \ldots, \Psi_{j,n})$. This completes the proof. ∎

Finally, we show that functions in $W^{m,p}(\Omega)$ belong to $W_0^{m,p}(\Omega)$ if and only if they have suitably trivial boundary traces.

5.37 THEOREM (Trivial Traces) Under the same hypotheses as Theorem 5.36, a function u in $W^{m,p}(\Omega)$ belongs to $W_0^{m,p}(\Omega)$ if and only the boundary traces of its derivatives of order less than m all coincide with the 0-function.

Proof. Every function in $C_0^\infty(\Omega)$ has trivial boundary trace, and so do all derivatives of such functions. Since the trace mapping is a continuous linear operator from $W^{m,p}(\Omega)$ to $W^{m-1,p}(\text{bdry } \Omega)$, all functions in $W_0^{m,p}(\Omega)$ have trivial boundary traces, and so do their derivatives of order less than m.

To prove the converse, we suppose that $u \in W^{m,p}(\Omega)$ and that u and its derivatives or order less than m have trivial boundary traces. Localization and a suitable change of variables reduces matters to the case where Ω is the half-space $\{x \in R^n : x_n > 0\}$. We then show that the zero-extension $\tilde{u}$ must belong to $W^{m,p}(\mathbb{R}^n)$, forcing u to belong to $W_0^{m,p}(\Omega)$ by Theorem 5.29.

In fact, we claim that if $u \in W^{m,p}(\Omega)$ has trivial boundary traces for u and its derivatives of order less than m, then the distributional derivatives $D^\alpha \tilde{u}$ of order at most m coincide with the zero-extensions $\widetilde{D^\alpha u}$. To verify this, approximate the integrals

$$
\int_{\mathbb{R}^n} \tilde{u}(x) D^\alpha \phi(x)\, dx \qquad \text{and} \qquad (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widetilde{D^\alpha u}(x) \phi(x)\, dx \tag{29}
$$

by approximating u with functions v_j in $C^\infty(\overline{\Omega})$, without requiring that these approximations have trivial traces.

Let e_n be the unit vector $(0, \ldots, 0, 1)$. Since $v_j \in C^\infty(\overline{\Omega})$, integrating by parts with respect to the other variables and then with respect to x_n shows that the difference between the integrals

$$\int_{\mathbb{R}^n} \tilde{v}_j(x) D^\alpha \phi(x)\, dx \qquad \text{and} \qquad (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widetilde{D^\alpha v_j}(x) \phi(x)\, dx$$

is a finite alternating sum of integrals of the form

$$\int_{\mathbb{R}^{n-1}} D^{\alpha - k e_n} v_j(x_1, \dots, x_{n-1}, 0) D_n^{k-1} \phi(x_1, \dots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} \tag{30}$$

with $k > 0$. Choose the sequence $\{v_j\}$ to converge to u in $W^{m,p}(\Omega)$. For each multi-index β with $\beta < \alpha$, the trace of $D^\beta v_j$ will converge in $L^p(\mathbb{R}^{n-1})$ to the trace of $D^\beta u$, that is to 0 in that space. Since the restriction of $D_n^{k-1}\phi$ to R^{n-1} belongs to $L^{p'}(\mathbb{R}^{n-1})$, each of the integrals in (30) tends to 0 as $j \to \infty$.

It follows that the two integrals in (29) are equal, and that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$. This completes the proof. ∎