



General Sobolev Inequalities and Embeddings on Riemannian Manifolds

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Introduction



Our goal in this report is to introduce embeddings of various Sobolev spaces into others. The crucial analytic tools here will be certain so-called "Sobolev inequalities".

Though Riemannian manifolds are natural extensions of Euclidean space, several important questions still puzzle mathematicians today.

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Background Materials



Compact Manifolds without Boundary



Complete Manifolds without Boundary



Compact Manifolds with Boundary

Sobolev Spaces on \mathbb{R}^n ¹



General Sobolev Inequalities ($k p < n$)

Let U be a bounded open subset of \mathbb{R}^n , with a C^1 boundary. Assume $u \in W^{k,p}(U)$.

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We have in addition the estimate

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{k,p}(U)}$$

the constant C depending only on k, p, n and U .

¹Evans, Lawrence C. *Partial differential equations*. Interscience Publishers, 1964.

Sobolev Spaces on \mathbb{R}^n ²



General Sobolev Inequalities ($kp > n$)

(ii) If $k > \frac{n}{p}$, then $u \in C^{k - [\frac{n}{p}] - 1, \gamma}(\bar{U})$, where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have in addition the estimate

$$\|u\|_{C^{k - [\frac{n}{p}] - 1, \gamma}(\bar{U})} \leq C \|u\|_{W^{k, p}(U)}$$

the constant C depending only on k, p, n, γ and U .

²Evans, Lawrence C. *Partial differential equations*. Interscience Publishers, 1964.

Sobolev Spaces on \mathbb{R}^n ³



Rellich-Kondrachov Compactness Theorem (Compactness)

Assume U is a bounded open subset of \mathbb{R}^n , and ∂U is C^1 . Suppose $1 \leq p < n$. Then

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^*$, where $1 \leq p < n, p^* = \frac{pn}{n-p}$

Before we start on Riemannian manifolds, we may ask:

1. ∂U is C^1 is an important assumption, what about manifolds without boundary?
2. can we find better embedded spaces which have more compactness properties?
3. how to generalize the Sobolev inequalities to Riemannian manifolds?

³Evans, Lawrence C. *Partial differential equations*. Interscience Publishers, 1964.

Sobolev Spaces on \mathbb{R}^n



Let Ω be some open subset of \mathbb{R}^n , k an integer, $p \geq 1$ real, and $u : \Omega \rightarrow \mathbb{R}$ a smooth, real-valued function. We define then the Sobolev spaces

$$H_k^p(\Omega) = \text{the completion of } \{u \in C^\infty(\Omega), \|u\|_{k,p} < +\infty\} \text{ for } \|\cdot\|_{k,p}$$
$$W^{k,p}(\Omega) = \{u \in L^p(\Omega), \forall |\alpha| \leq k, D_\alpha u \text{ exists and belongs to } L^p(\Omega)\}$$

[Meyers-Senin] For any Ω , any k , and any $p \geq 1$, $H_k^p(\Omega) = W^{k,p}(\Omega)$. ⁴

⁴Adams, R. A. *Sobolev spaces*. Academic Press, San Diego, 1978.

Sobolev Spaces on (M, g) ⁵



Sobolev Spaces on (M, g)

Given (M, g) a smooth Riemannian manifold, k an integer, and $p \geq 1$ real, the Sobolev space $H_k^p(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect to $\|\cdot\|_{H_k^p}$, where

$$\mathcal{C}_k^p(M) = \left\{ u \in C^\infty(M), \forall j = 0, \dots, k, \int_M |\nabla^j u|^p dv(g) < +\infty \right\}$$

When M is compact, one clearly has that $\mathcal{C}_k^p(M) = C^\infty(M)$ for any k and any $p \geq 1$. For $u \in \mathcal{C}_k^p(M)$, set also

$$\|u\|_{H_k^p} = \sum_{j=0}^k \left(\int_M |\nabla^j u|^p dv(g) \right)^{1/p}$$

⁵Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Compact Manifolds without Boundary ⁶



General Sobolev Inequalities on Compact Manifolds without Boundary

Let (M, g) be a smooth, compact Riemannian n -manifold. Let $p \geq 1$ be real and $0 \leq m < k$ be two integers.

(A1) If $1/q = 1/p - (k - m)/n > 0$, then $H_k^p(M) \subset H_m^q(M)$.

(A2) If $1/q = 1/p - (k - m)/n < 0$, then $H_k^p(M) \subset C^m(M)$.

where

$$\|u\|_{H_k^p} = \sum_{j=0}^k \left(\int_M |\nabla^j u|^p dv(g) \right)^{1/p} \quad \text{and} \quad \|u\|_{C^m} = \sum_{j=0}^m \max_{x \in M} |(\nabla^j u)(x)|$$

⁶Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Compact Manifolds without Boundary ⁷



General Sobolev Embeddings on Compact Manifolds without Boundary

Let (M, g) be a smooth, compact Riemannian n -manifold. Let $p \geq 1$ be real and $0 \leq m < k$ be two integers.

(A1) For any real q such that $1 \leq q < \frac{np}{(n-(k-m)p)}$, $H_k^p(M) \subset\subset H_m^q(M)$.

Take $k = 1, m = 0$, for any $p < n$ real and any $1 \leq q < \frac{np}{n-p}$, $H_1^p(M) \subset\subset L^q(M)$.

(A2) Take $k = 1, m = 0$, for $p > n$ and for any $\lambda \in (0, 1)$, such that $(1 - \lambda)p > n$, $H_1^p(M) \subset\subset C^\lambda(M)$. Particularly, $H_1^p(M) \subset\subset C^0(M)$.

$$\|u\|_{C^\lambda} = \max_{x \in M} |u(x)| + \max_{x \neq y \in M} \frac{|u(y) - u(x)|}{d_g(x, y)^\lambda}$$

⁷Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Complete Manifolds without Boundary ⁸



General Sobolev Inequalities on Complete Manifolds without Boundary

Let (M, g) be a smooth, complete Riemannian n -manifold with Ricci curvature bounded from below. Assume that

$$\inf_{x \in M} \text{Vol}_g(B_x(1)) > 0$$

where $\text{Vol}_g(B_x(1))$ stands for the volume of $B_x(1)$ with respect to g . Then the Sobolev embeddings in their first part (A1) are valid for (M, g) . i.e.

(B1) If $1/q = 1/p - (k - m)/n > 0$, then $H_k^p(M) \subset H_m^q(M)$.

The assumption of Ricci curvature is satisfactory but not necessary. e.g. $H_1^p \subset L^q$

⁸Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Complete Manifolds without Boundary ⁹



General Sobolev Inequalities on Complete Manifolds without Boundary

Let (M, g) be a smooth, complete Riemannian n -manifold with Ricci curvature bounded from below and positive injectivity radius.

For $p \geq 1$ real and $0 \leq m < k$ two integers,

(B2) If $1/q = 1/p - (k - m)/n < 0$, then $H_k^p(M) \subset C_B^m(M)$.

where $C_B^m(\Omega)$ consists of the functions $u \in C^m(\Omega)$ and $\nabla^j u$ is bounded on M for $0 \leq |j| \leq m$. Particularly, for $p \geq 1$ real and $\lambda \in (0, 1)$ real, if $1/p \leq (1 - \lambda)/n$, then $H_1^p(M) \subset C_B^\lambda(M)$.

⁹Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Compact Manifold with Boundary¹⁰



Sobolev Inequalities on Compact Manifolds with Boundary

Let (M, g) be a smooth, compact, n -dimensional Riemannian manifold with boundary. For $p < n$ real, set $p^* = np/(n - p)$. Then for any $q \in [1, p^*]$, $H_1^p(M) \subset L^q(M)$

Sobolev Embeddings on Compact Manifolds with Boundary

If $q \in [1, p^*)$, the embedding above is compact, i.e. $H_1^p(M) \subset\subset L^q(M)$.

¹⁰Emmanuel Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. 1999.

Thank You



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