Chapter 1

Sobolev inequalities in $\mathbb{R}^n$

1.1 Sobolev inequalities

1.1.1 Introduction

How can one control the size of a function in terms of the size of its gradient? The well-known Sobolev inequalities answer precisely this question in multidimensional Euclidean spaces. On the real line, the answer is given by a simple yet extremely useful calculus inequality. Namely, for any smooth function f with compact support on the line,

$$|f(t)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |f'(s)|ds. \tag{1.1.1}$$

The factor $1/2$ in this inequality comes from the fact that f vanishes at both $+\infty$ and $-\infty$. In this respect, note that if f is smooth but no other restriction is imposed the inequality above may fail.

It is natural to wonder if there is such an inequality for smooth compactly supported functions in higher-dimensional Euclidean spaces. More precisely, for each integer n, can one find $p, q > 0$ and $C > 0$ such that

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_q \leq C\|\nabla f\|_p? \tag{1.1.2}$$

Here and in the sequel $\mathcal{C}_0^\infty(\mathbb{R}^n)$ denotes the set of all smooth compactly supported functions in $\mathbb{R}^n$. For $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we set

$$\|f\|_q = \left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q}, \quad \|f\|_\infty = \sup_{\mathbb{R}^n}\{|f|\}$$

and

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx\right)^{1/p}$$

where $\nabla f = (\partial_1 f, \ldots, \partial_n f)$ is the gradient of f and $|\nabla f| = \sqrt{\sum_1^n |\partial_i f|^2}$ is the Euclidean length of the gradient. In $\mathbb{R}^n$, we denote by $\mu_n = \mu$ the

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Lebesgue measure and by μ_{n-1} the volume measure on smooth hypersurfaces of dimension $n-1$. When using coordinates $x = (x_1, \dots, x_n)$, we also write

$$d\mu(x) = dx = dx_1 \dots dx_n.$$

This question was first addressed in this form by Sobolev in [78] which appeared in Russian in 1938. Fixing a function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and replacing $x \mapsto f(x)$ by $x \mapsto f(tx)$, $t > 0$, in (1.1.2) yields

$$t^{-n/q}\|f\|_q \le C\, t^{1-n/p}\|\nabla f\|_p.$$

Letting t tend to zero and to infinity shows that (1.1.2) can only be satisfied if the exponents of t on both sides of the inequality above are the same. That is, (1.1.2) can only be satisfied if

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \text{ i.e., } q = \frac{np}{n-p}. \tag{1.1.3}$$

For instance, in $\mathbb{R}^2$, this says that one might possibly have

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^2), \quad \|f\|_\infty \le \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy. \tag{1.1.4}$$

The next example shows that this last inequality fails to be true.

EXAMPLE 1.1.1: Consider the function

$$f(x) = \begin{cases} \log|\log|x|| & \text{if } |x| \le 1/e \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\nabla f\|_2^2 = 2\pi \int_0^{1/e} \frac{dr}{r|\log r|^2} = 2\pi$ but f is not bounded. Of course, f is not smooth, but it can easily be approximated by smooth functions f_n such that $\|\nabla f_n\|_2 \to \|\nabla f\|_2$ and $f_n \to f$. This shows that that (1.1.4) cannot be true.

What is true is recorded in the following theorem.

Theorem 1.1.1 *Fix an integer $n \ge 2$ and a real p, $1 \le p < n$ and set $q = np/(n-p)$. Then there exists a constant $C = C(n,p)$ such that*

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_q \le C\|\nabla f\|_p. \tag{1.1.5}$$

This inequality is called the Sobolev inequality although the case $p = 1$ is not contained in [78]. Note that the case $p = n$ (i.e., $q = \infty$) is excluded in this result as should be the case according to the preceding example.

In the next few subsections we will give or outline several proofs of (1.1.5). As it turns out, when $p = 1$, (1.1.5) has a very simple proof based on

(1.1.1) and Hölder's inequality. This well-known proof (due independently to E. Gagliardo [28] and L. Nirenberg [68]) is presented in the next section. Moreover, as we shall see in 1.1.3 below, the case $p > 1$ follows from the case $p = 1$ by a simple trick.

We conclude this short introduction to Sobolev inequalities by recording a couple of useful remarks concerning the validity of (1.1.5). First, if (1.1.5) holds for all $f \in C_0^\infty(\mathbb{R}^n)$, it obviously also holds for a larger class of functions including for instance all C^1 functions with compact support or even Lipschitz functions vanishing at infinity. In fact, (1.1.5) holds for all functions vanishing at infinity whose gradient in the sense of distributions is in L^p. Second, (1.1.5) restricted to non-negative functions in $C_0^\infty(\mathbb{R}^n)$ suffices to prove (1.1.5) in its full generality. Indeed, (1.1.5) for such functions implies that it also holds true for non-negative Lipschitz functions with compact support and, if $f \in C_0^\infty(\mathbb{R}^n)$, $|f|$ is Lipschitz and satisfies $|\nabla|f|| \le |\nabla f|$ almost everywhere. It then follows that (1.1.5) holds for $f \in C_0^\infty(\mathbb{R}^n)$.

1.1.2 The proof due to Gagliardo and to Nirenberg

Recall that Hölder's inequality asserts that, for any positive measure μ,

$$\left| \int fg\,d\mu \right| \le \|f\|_p \|g\|_{p'}$$

for all $f \in L^p(\mu)$, $g \in L^{p'}(\mu)$, $1 \le p, p' \le \infty$ with $1 = 1/p + 1/p'$. By a simple induction we find that

$$\left| \int f_1 f_2 \dots f_k\,d\mu \right| \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k} \tag{1.1.6}$$

for all $f_i \in L^{p_i}$, $1 \le i \le k$, $1 \le p_i \le \infty$, $1/p_1 + 1/p_2 + \cdots + 1/p_k = 1$.

Now, fix $f \in C_0^\infty(\mathbb{R}^n)$. By (1.1.1), for any $x = (x_1, \dots, x_n)$ and any integer $1 \le i \le n$, we have

$$|f(x)| \le \frac{1}{2} \int_{-\infty}^{+\infty} |\partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)|\,dt$$

(with the obvious interpretation if $i = 1$ or n). Set

$$F_i(x) = \int_{-\infty}^{+\infty} |\partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)|\,dt$$

and

$$F_{i,m}(x) = \begin{cases} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\partial_i f(x)|\,dx_1 \dots dx_m & \text{if } i \le m \\ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_i(x)\,dx_1 \dots dx_m & \text{if } i > m. \end{cases}$$

Note that each F_i depends only on $n-1$ variables, i.e., all coordinates but the i^{th}. Similarly, $F_{i,m}$ depends on either $n - m$ or $n - m - 1$ variables

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depending on whether $i \leq m$ or $i > m$. In particular, for $m = n$, $F_{i,n}(x) = \int_{\mathbb{R}^n} |\partial_i f(y)| dy$ is a constant function. Now, we can estimate f by

$$|f| \leq (1/2)(F_1 \dots F_n)^{1/n}$$

so that

$$|f|^{n/(n-1)} \leq (1/2)^{n/(n-1)} (F_1 \dots F_n)^{1/(n-1)}.$$

Using (1.1.6) with $k = n-1$, $p_1 = p_2 = \cdots = p_k = n-1$ and induction on $m \leq n$, one easily proves that

$$\int \cdots \int |f(x)|^{n/(n-1)} dx_1 \dots dx_m \leq (1/2)^{n/(n-1)} \left(F_{1,m}(x) \dots F_{n,m}(x)\right)^{1/(n-1)}.$$

For $m = n$ this reads

$$\|f\|_{n/(n-1)} \leq (1/2) \left(\prod_1^n \|\partial_i f\|_1 \right)^{1/n}. \tag{1.1.7}$$

As $(\prod_1^n a_i)^{1/n} \leq \frac{1}{n} \sum_1^n a_i$ for any positive numbers a_i and integer n, we obtain

$$\|f\|_{n/(n-1)} \leq \frac{1}{2n} \sum_1^n \|\partial_i f(x)\|_1 dx \leq \frac{1}{2\sqrt{n}} \|\nabla f\|_1. \tag{1.1.8}$$

To see the last inequality, use $\sum_1^n |\partial_i f| \leq \sqrt{n} |\nabla f|$. This proves (1.1.5) for $p = 1$.

1.1.3 $p = 1$ implies $p \geq 1$

Assume that (1.1.5) holds for $p = 1$, that is,

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_{n/(n-1)} \leq C \|\nabla f\|_1. \tag{1.1.9}$$

Fix $p > 1$. For any $\alpha > 1$ and $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, note that $|f|^\alpha$ is $\mathcal{C}^1$, has compact support, and satisfies

$$|\nabla |f|^\alpha| = \alpha |f|^{\alpha-1} |\nabla f|.$$

Since we can easily approximate $|f|^\alpha$ by a sequence (f_i) of smooth functions with compact support such that $\nabla f_i \to \nabla |f|^\alpha$, inequality (1.1.9) holds with f replaced by $|f|^\alpha$. This yields

$$\begin{aligned} \|f\|_{\alpha n/(n-1)}^\alpha &\leq C\alpha \int |f(x)|^{\alpha-1} |\nabla f(x)| dx \\ &\leq C\alpha \left(\int |f(x)|^{(\alpha-1)p'} dx \right)^{1/p'} \left(\int |\nabla f(x)|^p dx \right)^{1/p} \end{aligned}$$

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where $1/p + 1/p' = 1$. If we pick $\alpha = (n-1)p/(n-p)$, we find (rather miraculously) that $(\alpha - 1)q = n(p-1)p'/(n-p) = np/(n-p)$. Thus

$$\|f\|_{np/(n-p)}^{(n-1)p/(n-p)} \le C\,\frac{(n-1)p}{n-p}\|f\|_{np/(n-p)}^{n(p-1)/(n-p)}\|\nabla f\|_p.$$

Finally, $(n-1)p/(n-p) - n(p-1)/(n-p) = 1$, so that simplifying the last inequality yields

$$\|f\|_{np/(n-p)} \le C\,\frac{(n-1)p}{n-p}\|\nabla f\|_p.$$

Thus we have proved the following version of Theorem 1.1.1.

Theorem 1.1.2 *For any integer $n \ge 2$ and real p, $1 \le p < n$, set $q = np/(n-p)$. Then*

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \|f\|_{np/(n-p)} \le \frac{(n-1)p}{2(n-p)\sqrt{n}}\|\nabla f\|_p.$$

The Sobolev constant given by this theorem (i.e., the constant appearing in front of $\|\nabla f\|_p$) is not the best possible constant. This will be discussed in Section 1.3.1 below.

1.2 Riesz potentials

1.2.1 Another approach to Sobolev inequalities

Sobolev inequalities relate the size of ∇f to the size of f. In order to prove such inequalities, one may try to express f in terms of its gradient. We now derive such a representation formula. Using polar coordinates (r, θ), $r > 0$, $\theta \in \mathbb{S}^{n-1}$, in $\mathbb{R}^n$, write

$$f(x) = -\int_0^\infty \partial_r f(x + r\theta)\,dr$$

for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Integrating over the unit sphere $\mathbb{S}^{n-1}$ yields

$$\begin{aligned} f(x) &= -\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty \partial_r f(x + r\theta)\,dr\,d\theta \\ &= -\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{\partial_r f(x + r\theta)}{r^{n-1}} r^{n-1}\,dr\,d\theta. \end{aligned}$$

Here ω_{n-1} is the $(n-1)$-dimensional volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. That is, if Ω_n is the volume of the unit ball,

$$\omega_{n-1} = n\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$$

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where Γ is the gamma function ($\Gamma(n) = (n+1)!$ when n is an integer). Now, if $y = x + r\theta$, we have $r = |y - x|$ and

$$dy = r^{n-1} dr d\theta \quad \text{and} \quad \partial_r f(x + r\theta) = |y - x|^{-1} \sum_1^n (y_i - x_i) \partial_i f(y).$$

Hence

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle x - y, \nabla f(y) \rangle}{|y - x|^n} dy. \tag{1.2.1}$$

In particular

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|y - x|^{n-1}} dy. \tag{1.2.2}$$

In view of this formula, we are led to study the properties of the convolution operator associated with $x \mapsto |x|^{-n+1}$.

More generally, for $0 < \alpha < n$, consider the Riesz potential operator I_α defined on $C_0^\infty(\mathbb{R}^n)$ by

$$I_\alpha f(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|y - x|^{n-\alpha}} dy \tag{1.2.3}$$

where $c_\alpha = \pi^{n/2} 2^\alpha \Gamma(\alpha/2)/\Gamma((n - \alpha)/2)$. By Fourier transform arguments, one verifies that

$$I_\alpha f = \Delta^{-\alpha/2} f$$

where $\Delta = -\sum_1^n \partial_i^2 f$ is the Laplace operator. Here, $\Delta^{-\alpha/2}$ is defined using Fourier analysis. Namely, for all functions f in $C_0^\infty(\mathbb{R}^n)$,

$$\widehat{\Delta^{\beta/2} f} = (2\pi|x|)^\beta \hat{f}, \quad \hat{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i x.y} f(y) dy.$$

The identity $I_\alpha f = \Delta^{-\alpha/2} f$ amounts to the fact that the Fourier transform of $c_\alpha^{-1} |x|^{-n+\alpha}$ is precisely $(2\pi|\xi|)^{-\alpha}$ in the sense that

$$c_\alpha^{-1} \int_{\mathbb{R}^n} |x|^{-n+\alpha} f(x) dx = \int_{\mathbb{R}^n} (2\pi|\xi|)^{-\alpha} \hat{f}(\xi) d\xi$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, when $0 < \alpha < n$. The restriction $0 < \alpha < n$ corresponds to the requirement that both $|x|^{-n+\alpha}$ and $|x|^{-\alpha}$ must be locally integrable for the above identity to make sense. One can show that $I_\alpha I_\beta = I_{\alpha+\beta}$ for $\alpha, \beta > 0$, $\alpha + \beta < n$, and $\Delta I_\alpha f = I_\alpha \Delta f = I_{\alpha-2} f$ for $2 \leq \alpha < n$.

Theorem 1.2.1 *Fix $0 < \alpha < n$, $1 < p < n/\alpha$ and define q by $1/q = 1/p - \alpha/n$, i.e., $q = np/(n - \alpha p)$. Then there exists a constant $C = C(n, \alpha, p)$ such that*

$$\forall f \in C_0^\infty(\mathbb{R}^n), \quad \|I_\alpha f\|_q \leq C \|f\|_p.$$

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This theorem will be proved below in a more general form. As a corollary of Theorem 1.2.1 and (1.2.2), we obtain (1.1.5) for $1 < p < n$. Observe that the case $p = 1$ is excluded in Theorem 1.2.1. This has to be the case. Indeed, if we had $\|I_\alpha f\|_q \le C\|f\|_1$, we could let $f \in C_0^\infty(\mathbb{R}^n)$ tend to the Dirac mass. This would imply that the function $x \mapsto |x|^{-n+\alpha}$ is in L^q with $q = n/(n-\alpha)$. But this is clearly not the case.

Similarly, the case $p = n/\alpha$ must also be excluded. This follows from the case $p = 1$ by duality, or more directly by the following example.

EXAMPLE Consider

$$f(x) = \begin{cases} |x|^{-\alpha}(\log 1/|x|)^{-(\alpha/n)(1+\epsilon)} & \text{for } |x| \le 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in L^{n/\alpha}$ if $\epsilon > 0$, but

$$I_\alpha f(0) = \frac{1}{c_\alpha} \int_{|x|<1/2} |x|^{-n}(\log|x|)^{-(\alpha/n)(1+\epsilon)} = \infty$$

when $\epsilon > 0$ is chosen small enough so that $(\alpha/n)(1+\epsilon) < 1$ (recall that $\alpha/n < 1$).

1.2.2 Marcinkiewicz interpolation theorem

Consider a measure space (M, μ). Fix $1 \le p, q \le \infty$. A linear operator K defined on $L^1 \cap L^\infty$ is of weak type (p, q) if there exists a constant A such that

$$\forall t > 0,\ \ \forall f \in L^1 \cap L^\infty,\ \ \mu(\{x : |Kf(x)| > t\}) \le (A\|f\|_p/t)^q. \qquad (1.2.4)$$

If $q = \infty$, this must be understood as $\|Kf\|_\infty \le A\|f\|_p$.

Theorem 1.2.2 *Assume that K is of weak types (p_1, q_1) and (p_2, q_2) with $1 \le p_i, q_i \le \infty$, $p_1 < p_2$, $q_1 \ne q_2$. Then for each $0 < \theta < 1$ and $1/p = \theta/p_1 + (1-\theta)/p_2$, $1/q = \theta/q_1 + (1-\theta)/q_2$, there exists a constant $C = C_\theta$ such that*

$$\|Kf\|_q \le C\|f\|_p.$$

This is an interpolation result due to Marcinkiewicz. It says that boundedness of K at the end points $(1/p_1, 1/q_1)$, $(1/p_2, 1/q_2)$ implies boundedness along the segment joining these points in the $(1/p, 1/q)$ plane. See [79, 81] for a proof. Of course, one of the important aspects here is that weak boundedness at the end points is enough to prove strong boundedness in the interior.

We want to apply this result to the case where K is given by a kernel $K(x,y)$ of weak type r for some $1 < r \le \infty$, that is, which satisfies

$$\forall t > 0,\ \ \forall x, y \in M,\ \begin{cases} \mu(\{z : |K(x,z)| > t\}) \le (A/t)^r, \\ \mu(\{z : |K(z,y)| > t\}) \le (A/t)^r. \end{cases} \tag{1.2.5}$$

Again, if $r = \infty$, this must be understood as $\sup_{x,y} |K(x,y)| \le A < \infty$.

Theorem 1.2.3 *Assume that*

$$Kf(x) = \int_M K(x,y) f(y) d\mu(y)$$

where K is a kernel of weak type r for some fixed $1 < r \le \infty$. Then the operator K is of weak type (p,q) for all $1 \le p < \infty$ and $p < q < \infty$ such that $1 + 1/q = 1/p + 1/r$. Moreover, for each such p, q, there exists a constant $B = B(r,p)$ such that

$$\forall f \in L^p,\ \ \|Kf\|_q \le B\|f\|_p. \tag{1.2.6}$$

Without loss of generality we can assume that $K \ge 0$. For each $t > 0$, write $K = K_t + K^t$ where

$$K_t(x,y) = K(x,y)\mathbf{1}_{\{(u,v):K(u,v)\le t\}}(x,y).$$

Lemma 1.2.4 *Fix $p \ge 1$. There exists a constant B_1 such that, for all $t > 0$ and all $f \in L^p$,*

$$\|K^t f\|_p \le B_1 t^{-r+1} \|f\|_p.$$

Moreover, if $p/(p-1) < r$, there exists a constant B_2 such that, for all $t > 0$ and all $f \in L^p$,

$$\|K_t f\|_\infty \le B_2 t^{1-r(p-1)/p} \|f\|_p.$$

To prove the first inequality, observe that

$$\begin{aligned} \int_M |K^t(x,y)| d\mu(y) &= \int_0^\infty \mu(\{y : |K^t(x,y)| > s\}) ds \\ &\le\ t\mu(\{y : K(x,y) > t\}) + A \int_t^\infty s^{-r} ds \le B_1\, t^{1-r} \end{aligned} \tag{1.2.7}$$

because $r > 1$. Thus $\|K^t f\|_\infty \le B_1\, t^{1-r} \|f\|_\infty$ and, by duality, $\|K^t f\|_1 \le B_1\, t^{1-r} \|f\|_1$. The duality argument runs as follows. For $f \in L^1 \cap L^\infty$, we have

$$\|K^t f\|_1 = \sup_{\substack{g \in L^\infty \\ \|g\|_\infty \le 1}} \int (K^t f)\, g\, d\mu.$$

Moreover,

$$\int (K^t f)\, g\, d\mu = \int f\, (\widetilde{K}^t g)\, d\mu$$

where $\widetilde{K}(x,y) = K(y,x)$. From the x, y symmetry of our hypothesis it follows that (1.2.7) also holds for $\int_M |\widetilde{K}^t(x,y)| d\mu(y)$. Hence

$$\|\widetilde{K}^t g\|_\infty \le B_1 t^{1-r}$$

and

$$\begin{aligned} \int (K^t f)\, g\, d\mu &= \int f\, (\widetilde{K}^t g)\, d\mu \\ &\le \|\widetilde{K}^t g\|_\infty \|f\|_1 \\ &\le B_1 t^{1-r} \|g\|_\infty \|f\|_1. \end{aligned}$$

Thus

$$\|K^t f\|_1 = \sup_{\substack{g \in L^\infty \\ \|g\|_\infty \le 1}} \int (K^t f)\, g\, d\mu \le B_1 t^{1-r} \|f\|_1.$$

We still need to prove that

$$\|K^t f\|_p \le B_1 t^{1-r} \|f\|_p$$

for $1 < p < \infty$. To this end, use Jensen's inequality to obtain

$$|K^t f(x)|^p \le \left(\int_M K^t(x,y) d\mu(y) \right)^{p-1} \int_M K^t(x,y) |f(y)|^p d\mu(y).$$

Using the $L^1 \to L^1$ bound, we finally get $\|K^t f\|_p \le B_1\, t^{1-r} \|f\|_p$ as desired.

To prove the second inequality of Lemma 1.2.4, write $1/p + 1/p' = 1$ so that $p' = p/(p-1)$ and note that

$$\begin{aligned} \int_M |K_t(x,y)|^{p'} d\mu(y) &= p' \int_0^\infty s^{p'-1} \mu(\{z : |K_t(x,z)| > s\}) ds \\ &\le p'\, A \int_0^t s^{p'-1-r} ds = p'(p'-r)^{-1} A\, t^{p'-r} \end{aligned}$$

because $p' < r$. It follows that

$$|K_t f| \le B_2 t^{1-r/p'} \|f\|_p.$$

To prove the first assertion of Theorem 1.2.3, fix $t > 0$ to be chosen later. Then, for any $s > 0$ and $f \in L^1 \cap L^\infty$ with $\|f\|_p = 1$, write

$$\mu(\{z : |Kf(z)| \ge s\}) \le \mu(\{z : |K_t f(z)| \ge s/2\}) + \mu(\{z : |K^t f(z)| \ge s/2\}).$$

By Lemma 1.2.4,

$$\mu(\{z : |K^t f(z)| \geq s/2\}) \leq (2\|K^t f\|_p/s)^p \leq (2B_1 t^{1-r}/s)^p$$

and

$$|K_t f| \leq B_2 t^{1-r(p-1)/p}.$$

Pick t so that $B_2 t^{1-r(p-1)/p} = s/4$. Thus $t = (B_2\, s/4)^{p/(p+r-rp)}$. Then

$$\mu(\{z : |K^t f(z)| \geq s/2\}) = 0$$

and

$$\begin{aligned} \mu(\{z : |Kf(z)| \geq s\}) &\leq \mu(\{z : |K_t f(z)| \geq s/2\}) \leq (2B_1 t^{1-r}/s)^p \\ &\leq B_3\, s^{-p[1-(1-r)p/(p+r-rp)]} \\ &= B_3\, s^{-pr/(p+r-rp)} = B_3\, s^{-q} \end{aligned}$$

if $1/q = 1/p + 1/r - 1$, that is $q = pr/(p+r-rp)$. In words, the operator K is of weak type (p,q). This is true for all $1 < p < \infty$. The last assertion of Theorem 1.2.3 now follows from the Marcinkiewicz interpolation theorem, i.e., Theorem 1.2.2 with $1 < p_1 < p_2 < \infty$ arbitrary and $1/q_i = 1/p_i + 1/r - 1$. This ends the proof of Theorem 1.2.3. This is a typical use of the Marcinkiewicz interpolation theorem. We have turned a weak (p,q) boundedness result into a strong (p,q) boundedness result using the fact that the weak result holds for all p in a certain interval.

1.2.3 Proof of Sobolev Theorem 1.2.1

In order to prove Theorem 1.2.1, note that $K(x,y) = |x-y|^{-n+\alpha}$ is of weak type $n/(n-\alpha)$. Hence, by Theorem 1.2.3, I_α is of weak type $(1, n/(n-\alpha))$ and satisfies $\|I_\alpha f\|_q \leq C\|f\|_p$ for all $1 < p < \infty$ with $1/q = 1/p - \alpha/n$.

1.3 Best constants

1.3.1 The case $p = 1$: isoperimetry

Let $\mathbb{B}_n(r)$ and $\mathbb{S}^{n-1}(r)$ denote respectively the ball and the sphere of radius r centered at the origin in $\mathbb{R}^n$. Let $\Omega_n = \mu_n(\mathbb{B}^n(1))$ and $\omega_{n-1} = \mu_{n-1}(\mathbb{S}_{n-1}(1))$. The isoperimetric inequality in $\mathbb{R}^n$ asserts that among sets having a smooth boundary of given finite $(n-1)$-dimensional measure, the ball has the largest n-dimensional volume. Namely,

$$\mu_n(\Omega) \leq \mu_n(\mathbb{B}^n(r)) = \Omega_n\, r^n$$

where r is such that

$$\mu_{n-1}(\partial\Omega) = \mu_{n-1}(\mathbb{S}^{n-1}(r)) = \omega_{n-1} r^{n-1},$$

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that is

$$r = (\mu_{n-1}(\partial\Omega)/\omega_{n-1})^{1/(n-1)}.$$

Hence, for any compact set $\Omega \subset \mathbb{R}^n$ with smooth boundary,

$$[\mu_n(\Omega)]^{(n-1)/n} \le C_n \mu_{n-1}(\partial\Omega) \tag{1.3.1}$$

where

$$C_n = \frac{\Omega_n^{1-1/n}}{\omega_{n-1}} = \frac{[\Gamma((n-1)/2)]^{1/n}}{\sqrt{\pi}\, n}.$$

Indeed, recall that $n\Omega_n = \omega_{n-1}$ and $\Omega_n = \pi^{n/2}/\Gamma((n-1)/2)$. This inequality has been known to geometers for a very long time; in particular, it was known well before Sobolev's work in the 1930's.

Apparently, the discovery that (1.3.1) is equivalent to Sobolev inequality (1.1.5) for $p = 1$ with the same constant, that is,

$$\forall f \in C_0^\infty(\mathbb{R}^n), \quad \|f\|_{n/(n-1)} \le C_n \|\nabla f\|_1, \tag{1.3.2}$$

was only made much later. In fact, in [78], Sobolev only proved (1.1.5) for $p > 1$. The case $p = 1$ is attributed to Gagliardo and to Nirenberg who published the proof given in Section 1.1.2 in 1958 and 1959 respectively. The connection between (1.3.1) and (1.3.2) was made in 1960 by Maz'ja and by Federer and Fleming. See [60].

The fact that (1.3.1) follows from (1.3.2) is rather straightforward. One approximates the function $\mathbf{1}_\Omega$ by smooth functions f_n so that

$$\|f_n\|_{n/(n-1)} \to \mu_n(\Omega)^{(n-1)/n} \text{ and } \|\nabla f_n\|_1 \to \mu_{n-1}(\partial\Omega).$$

To prove the other direction one needs the following co-area formula. See, e.g., [60, 1.2.4] and the references therein.

Theorem 1.3.1 *For any $f, g \in C_0^\infty(\mathbb{R}^n)$,*

$$\int g|\nabla f| d\mu_n = \int_{-\infty}^{+\infty} \left(\int_{f(x)=t} g(x) d\mu_{n-1}(x) \right) dt.$$

Indeed, with this theorem at hand, for any smooth compactly supported $f \ge 0$ we have

$$\begin{aligned} \int |f(x)|^{n/(n-1)} dx &\le \int_0^\infty \mu_n(\{f > t\})^{(n-1)/n} dt \\ &\le C_n \int_0^\infty \mu_{n-1}(\{f = t\}) dt \\ &= C_n \int |\nabla f| d\mu_n = \|\nabla f\|_1. \end{aligned}$$

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To see the first inequality, write

$$f(x) = \int_0^\infty \mathbf{1}_{\{f(x)>t\}}(t)dt$$

and use the Minkowski inequality

$$\left\| \int f(\cdot, y)dy \right\|_q \leq \int \|f(\cdot, y)\|_q dy$$

with $q = n/(n-1) > 1$ to obtain

$$\begin{aligned} \left\| \int_0^\infty \mathbf{1}_{\{f(\cdot)>t\}}(t)dt \right\|_{n/(n-1)} &\leq \int_0^\infty \|\mathbf{1}_{\{f(\cdot)>t\}}\|_{n/(n-1)}dt \\ &= \int_0^\infty \mu_n(\{z : f(z) > t\})^{(n-1)/n}dt. \end{aligned}$$

This shows that the isoperimetric inequality (1.3.1) implies the Sobolev inequality (1.3.2) with the same constant C_n.

1.3.2 A complete proof with best constant for $p = 1$

According to M. Gromov [62], inequality (1.3.2) goes back to H. Brunn's inaugural dissertation in 1887. My understanding is that Brunn proved the celebrated Brunn-Minkowski inequality (for convex sets and without the case of equality due to Minkowski) from which the isoperimetric inequality easily follows. Whether or not Brunn dicussed the isoperimetric inequality is not clear to me. Of course, he did not discuss (1.3.2). As mention earlier, the observation that (1.3.2) is equivalent to (1.3.1) is usually attributed to Maz'ja and to Federer and Fleming.

Gromov gives the following beautiful proof of (1.3.2) which he attributes to H. Knothe [48]. As we shall see, this proof yields both a proof of the isoperimetric inequality (1.3.1) and a proof of the (equivalent) Sobolev inequality (1.3.2). Again, as far as I can tell, there is no discussion of (1.3.2) in the work of Knothe.

Let g be a non-negative, locally integrable function with compact support S. For $x \in S$, set

$$A_i(x) = \{z : z_j = x_j \text{ for } j < i \text{ and } z_i \leq x_i\}$$

and

$$B_i(x) = \{z : z_j = x_j \text{ for } j < i\}$$

with the convention that $B_1 = \mathbb{R}^n$. Consider the map $y_g : x \mapsto y$ defined by

$$y_i = \int_{A_i(x)} g(z)dz_i \dots dz_n \bigg/ \int_{B_i(x)} g(z)dz_i \dots dz_n.$$

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Since $g \geq 0$ and $A_i \subset B_i$, we have $0 \leq y_i \leq 1$ for all x. That is, y is a map from S to the cube $[0,1]^n$. Obviously, this map is triangular, that is, for each i, y_i is a function of $x_1, \ldots, x_i$ only. Clearly, for each i, the partial derivative $\partial y_i/\partial x_i$ is non-negative and equal to

$$\frac{\partial y_i}{\partial x_i}(x) = \begin{cases} \int_{B_{i+1}(x)} g(z)dz_{i+1}\ldots dz_n \Big/ \int_{B_i(x)} g(z)dz_i \ldots dz_n & \text{if } 1 \leq i < n \\ g(x) \Big/ \int_{B_n(x)} g(z)dz_n & \text{if } i = n \end{cases}$$

at each point x where g is continuous. Thus, at any such point, the Jacobian of $x \mapsto y$ is equal to

$$J(x) = \prod_1^n \frac{\partial y_i}{\partial x_i} = g(x) \Big/ \int g(z)dz.$$

First, apply this construction to the case where $g = \chi$ is the characteristic function of the unit ball and observe that $y_\chi : \mathbb{B}^n \to (0,1)^n$ is invertible. Let $z = y_\chi^{-1}$ be the inverse map, $z : (0,1)^n \to \mathbb{B}^n$. Clearly, this is a triangular map with Jacobian equal to $\Omega_n = \mu_n(\mathbb{B}^n)$ in $[0,1]^n$.

The fact that y_χ is invertible is one of the crucial points of this proof. In fact, y_χ is invertible as soon as χ is the characteristic function of a convex set. We urge readers to check this for themselves.

Now, fix $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$ and set $g = f^{n/(n-1)}$. We can assume that $\int g(x)dx = 1$. Construct the map y_g as above and consider the map

$$F = z \circ y_g : S = \operatorname{supp}(f) \to \mathbb{B}^n.$$

By construction, the map F has non-negative partial derivatives $\partial F_i/\partial x_i$ and its Jacobian satisfies

$$J_F(x) = \Omega_n\, g(x)$$

for all x in the interior of S. Thus, the divergence $\operatorname{div} F = \sum_1^n \partial F_i/\partial x_i$ satisfies

$$\frac{1}{n}\operatorname{div} F(x) \geq [J_F(x)]^{1/n} = [\Omega_n\, g(x)]^{1/n}. \tag{1.3.3}$$

Furthermore, by the divergence theorem, we have

$$\int f(x)\operatorname{div} F(x)dx = -\int \langle \nabla f(x), F(x)\rangle dx \leq \int |\nabla f(x)|dx \tag{1.3.4}$$

because $|F| \leq 1$. Hence

$$\begin{aligned} 1 = \int f(x)^{n/(n-1)}dx &= \int f(x) g(x)^{1/n} dx \\ &\leq \frac{1}{n\Omega_n^{1/n}} \int f(x)\operatorname{div} F(x)dx \\ &\leq \frac{1}{n\Omega_n^{1/n}} \int |\nabla f(x)|dx. \end{aligned}$$

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Removing the normalization $\int g(x)dx = 1$, we finally obtain

$$\|f\|_{n/(n-1)} \leq \frac{1}{n\Omega_n^{1/n}} \|\nabla f\|_1$$

which is exactly (1.3.2).

The same argument gives (1.3.1) if we take f to be the characteristic function of a bounded domain Ω with smooth boundary and if we replace (1.3.4) by

$$\int_\Omega \operatorname{div} F d\mu_n = \int_{\partial\Omega} \langle F, \mathbf{n} \rangle d\mu_{n-1}$$

where $\mathbf{n}$ is the exterior normal along $\partial\Omega$.

1.3.3 The case $p > 1$

The following theorem gives the best constant in the Sobolev inequality for $1 \leq p < \infty$.

Theorem 1.3.2 *For $1 \leq p < n$, the Sobolev inequality*

$$\forall f \in C_0^\infty(\mathbb{R}^n), \quad \|f\|_q \leq C \|\nabla f\|_p$$

holds with C equal to

$$C(n,p) = \frac{p-1}{n-p} \left(\frac{n-p}{n(p-1)} \right)^{1/q} \left(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}} \right)^{1/n} \tag{1.3.5}$$

for $1 < p < n$ and

$$C(n,1) = \frac{1}{n} \left(\frac{n}{\omega_{n-1}} \right)^{1/n}. \tag{1.3.6}$$

These are the best possible constants and the functions

$$x \mapsto \left(a + b|x-y|^{p/(p-1)} \right)^{1-n/p},$$

$a, b > 0$, $y \in \mathbb{R}^n$, are the extremal functions when $1 < p < n$.

We only briefly sketch the proof. See [5, 85] for details.

The proof is in two steps. First, one shows that it suffices to treat the case where f is a non-negative radial decreasing function. This follows from the following classic rearrangement inequality. For any function $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, let f^* be the radial decreasing function such that

$$\mu_n(\{z : f^*(z) > t\}) = \mu_n(\{z : f(z) > t\}).$$

That is, $f^*(x) = f_*(|x|)$ where

$$f_*(t) = \sup\{s : \mu_n(\{z : f(z) > s\}) > \Omega_n t^n\},$$

$\Omega_n = \omega_{n-1}/n$ being the volume of the unit ball.

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Theorem 1.3.3 *For all $f \in C_0^\infty(\mathbb{R}^n)$ and all $1 \le p < \infty$, we have*

$$\|\nabla f^*\|_p \le \|\nabla f\|_p.$$

One proof of this theorem uses the isoperimetric inequality (1.3.1) and the co-area formula of Theorem 1.3.1. Talenti [85] gives a nice account. See also [5, Proposition 2.17].

Theorem 1.3.3 reduces the proof of Theorem 1.3.2 to the following 1-dimensional statement.

Lemma 1.3.4 *Fix $1 \le p < n$ and set $q = pn/(n-p)$. Let h be a decreasing function which is absolutely continuous on $[0, \infty)$ and equal to zero at infinity. Then*

$$\left(\int_0^\infty |h(t)|^q t^{n-1} dt\right)^{1/q} \le C'(n,p) \left(\int_0^\infty |h'(t)|^p t^{n-1} dt\right)^{1/p}$$

where

$$\begin{aligned} C'(n,p) &= \frac{p-1}{n-p}\left(\frac{n-p}{n(p-1)}\right)^{1/q}\left(\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)}\right)^{1/n} \\ &= C(n,p)\omega_{n-1}^{1/n}. \end{aligned}$$

Moreover, for $1 < p < n$, equality is attained for the functions $t \mapsto (a + bt^{p/(p-1)})^{1-n/p}$.

See [5, 85] for a proof and earlier references.

1.4 Some other Sobolev inequalities

1.4.1 The case $p > n$

What can be said about the size of smooth functions with compact support in terms of $\|\nabla f\|_p$ when $p > n$? The following theorem gives a partial answer.

Theorem 1.4.1 *For $p > n$ there exists a constant $C = C(n,p)$ such that for any set Ω of finite volume we have*

$$\forall f \in C_0^\infty(\Omega), \quad \|f\|_\infty \le C\mu_n(\Omega)^{1/n-1/p}\|\nabla f\|_p. \tag{1.4.1}$$

Start with (1.2.2), that is,

$$|f(x)| \le \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy.$$

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Define p' by $1/p+1/p'=1$ and note that $(n-1)(p'-1)=(n-1)/(p-1)<1$. Let also R be such that $\mu_n(\Omega)=\mu_n(\mathbb{B}(R))$, that is $R=(\mu_n(\Omega)/\Omega_n)^{1/n}$.

Then use the following computation. Write

$$\begin{aligned}\int_\Omega \frac{1}{|x-y|^{p'(n-1)}}dy &\le \int_{\mathbb{B}(R)} \frac{1}{|x-y|^{p'(n-1)}}dy\\ &\le \omega_{n-1}\int_0^R r^{(1-n)p'+n-1}dr\\ &= \omega_{n-1}(1-(n-1)(p'-1))^{-1}R^{1-(n-1)(p'-1)}\\ &= \omega_{n-1}(1-(n-1)(p'-1))^{-1}R^{(p-n)/(p-1)}\\ &= \frac{\omega_{n-1}\mu_n(\Omega)^{(p-n)/n(p-1)}}{\Omega_n^{(p-n)/n(p-1)}(1-(n-1)(p'-1))} \qquad (1.4.2)\\ &= B\,\mu_n(\Omega)^{(p-n)/n(p-1)}.\end{aligned}$$

Now, by (1.2.2),

$$\begin{aligned}\|f\|_\infty &\le \left(\frac{1}{\omega_{n-1}}\int_\Omega \frac{1}{|x-y|^{p'(n-1)}}dy\right)^{1/p'}\|\nabla f\|_p\\ &\le C\mu_n(\Omega)^{1/n-1/p}\|\nabla f\|_p.\end{aligned}$$

This proves Theorem 1.4.1.

One crucial difference between Theorem 1.4.1 and Sobolev inequality (1.1.5) for $1\le p<n$ is that the right-hand side of (1.4.1) depends on the set Ω on which the function f is supported. As the volume of Ω tends to infinity, the term $\mu_n(\Omega)^{1/n-1/p}$ also tends to infinity since $n<p$. In fact, when $n\le p$, there is no way to control the size of f purely in terms $\|\nabla f\|_p$. To see this, consider the function $f_r : x\mapsto (r-|x|)_+$. This function is supported in $\mathbb{B}(r)$ and

$$\|\nabla f_r\|_p=\mu_n(\mathbb{B}(r))^{1/p}=\Omega_n^{1/p}r^{n/p}.$$

Also,

$$\|f_r\|_q\ge \mu_n(\mathbb{B}(r/2))(r/2)=\Omega_n^{1/q}(r/2)^{n/q+1}.$$

For any fixed q, the ratio $\|f_r\|_q/\|\nabla f\|_p$ tends to infinity as r tends to infinity if $n\le p$.

Theorem 1.4.1 can be complemented as follows.

Theorem 1.4.2 *For $p>n$, there exists a constant $C=C(n,p)$ such that any function $f\in C^\infty(\mathbb{R}^n)$ with $\|\nabla f\|_p<\infty$ satisfies*

$$\sup_{\substack{x,y\in\mathbb{R}^n\\ x\ne y}}\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\}\le C\|\nabla f\|_p$$

with $\alpha=1-n/p$.

Note that this result does not require that f vanishes at infinity. For the proof we need a localized version of the representation formula (1.2.2).

Lemma 1.4.3 *Let B be a ball of radius $r > 0$. Then,*

$$\forall f \in C^\infty(B), \quad \forall x \in B, \ |f(x) - f_B| \leq \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|x-y|^{n-1}} dy$$

where

$$f_B = \frac{1}{\mu_n(B)} \int_B f(z)dz.$$

For $x, y \in B$ write

$$f(x) - f(y) = -\int_0^{|x-y|} \partial_\rho f\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho.$$

It follows that

$$|f(x) - f(y)| \leq \int_0^\infty F\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho$$

where

$$F(z) = \begin{cases} |\nabla f(z)| & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Integrating with respect to $y \in B$ yields

$$\begin{aligned}
|f(x) - f_B| &= \left| f(x) - \frac{1}{\mu_n(B)} \int_B f(y)dy \right| \\
&\leq \frac{1}{\mu_n(B)} \int_B |f(x) - f(y)|dy \\
&\leq \frac{1}{\Omega_n r^n} \int_B dy \left\{ \int_0^\infty F\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho \right\} \\
&\leq \frac{1}{\Omega_n r^n} \int_{\{y:|x-y|\leq 2r\}} dy \left\{ \int_0^\infty F\left(x + \rho\frac{y-x}{|y-x|}\right) d\rho \right\} \\
&= \frac{1}{\Omega_n r^n} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^{2r} F\left(x + \rho\theta\right) s^{n-1} ds d\theta d\rho \\
&= \frac{2^n}{n\Omega_n} \int_0^\infty \int_{\mathbb{S}^{n-1}} F\left(x + r\theta\right) d\theta dr \\
&= \frac{2^n}{\omega_{n-1}} \int_B \frac{|\nabla f(y)|}{|y-x|^{n-1}} dy.
\end{aligned}$$

This proves Lemma 1.4.3.

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To obtain Theorem 1.4.2, it now suffices to apply Lemma 1.4.3 and the argument of the proof of Theorem 1.4.1 to obtain

$$|f(x) - f_B| \le C\mu_n(B)^{1/n-1/p}\left(\int_B |\nabla f|^p d\mu_n\right)^{1/p} \le C\mu_n(B)^{1/n-1/p}\|\nabla f\|_p$$

for all $x \in B$ and all balls $B \subset \mathbb{R}^n$. Thus, for all x, y such that $|x - y| \le r$, we get

$$|f(x) - f(y)| \le 2C\Omega_n r^{1-n/p}\|\nabla f\|_p \le 2C\Omega_n|x-y|^{1-n/p}\|\nabla f\|_p.$$

This proves Theorem 1.4.2.

1.4.2 The case $p = n$

To treat the case $p = n$, we first compute

$$\int_\Omega \frac{1}{|x-y|^{r(n-1)}}dy$$

when $r < n/(n-1)$. Actually, this has already been done in (1.4.2) where we have shown that

$$\int_\Omega \frac{1}{|x-y|^{r(n-1)}}dy \le \frac{\omega_{n-1}}{1-(r-1)(n-1)}[\mu_n(\Omega)/\Omega_n]^{-(n+r-nr)/n}. \tag{1.4.3}$$

For any $n < q < \infty$, set $1/n - 1/q = \delta$ and $1/r = 1 + 1/q - 1/n = 1 - \delta$. Now, write

$$\begin{aligned} |f(x)| &\le \frac{1}{\omega_{n-1}}\int \frac{|\nabla f(y)|}{|x-y|^{n-1}}dy \\ &\le \frac{1}{\omega_{n-1}}\int \frac{|\nabla f(y)|^{n/q}}{|x-y|^{r(n-1)/q}} \times |\nabla f(y)|^{n\delta} \times \frac{1}{|x-y|^{r(n-1)(1-1/n)}}dy. \end{aligned}$$

Notice that $1/q + \delta + (1 - 1/n) = 1$, and use the Hölder inequality (1.1.6) with $p_1 = q$, $p_2 = 1/\delta$, $p_3 = n/(n-1)$ to get

$$\begin{aligned} |f(x)| \le \frac{1}{\omega_{n-1}}&\left(\int \frac{|\nabla f(y)|^n}{|x-y|^{r(n-1)}}dy\right)^{1/q} \times \\ &\left(\int |\nabla f(y)|^n dy\right)^\delta \left(\int_{\mathrm{supp}(f)} \frac{1}{|x-y|^{r(n-1)}}dy\right)^{1-1/n}. \end{aligned}$$

It follows that, if f is supported in Ω,

$$\begin{aligned} \|f\|_q &\le \frac{1}{\omega_{n-1}}\|\nabla f\|_n^{n/q+n\delta}\left(\int_\Omega \frac{1}{|x-y|^{r(n-1)}}dy\right)^{1/q+1-1/n} \\ &\le \frac{1}{\omega_{n-1}}\|\nabla f\|_n\left(\int_\Omega \frac{1}{|x-y|^{r(n-1)}}dy\right)^{1/r}. \end{aligned}$$

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Thanks to (1.4.3), this yields

$$\|f\|_q \leq \frac{\omega_{n-1}^{-1+1/r}}{[1-(r-1)(n-1)]^{1/r}\Omega_n^{(n+r-nr)/nr}}\mu_n(\Omega)^{(n+r-nr)/nr}\|\nabla f\|_n.$$

As $1/r = 1 + 1/q - 1/n$, we get

$$\begin{aligned} \|f\|_q &\leq \frac{\omega_{n-1}^{1/q-1/n}}{[1-(r-1)(n-1)]^{1/r}\Omega_n^{1/q}}\mu_n(\Omega)^{1/q}\|\nabla f\|_n \\ &= \frac{n^{1/q}}{[1-(r-1)(n-1)]^{1/r}\omega_{n-1}^{1/n}}\mu_n(\Omega)^{1/q}\|\nabla f\|_n. \end{aligned}$$

Note that $1-(r-1)(n-1) = n(n+1)/(nq+n-q) \geq (n+1)/q$ because $q > n$. Hence, for $q > n$, we get

$$\|f\|_q^q \leq q^{1+q(n-1)/n}\omega_{n-1}^{-q/n}\mu_n(\Omega)\|\nabla f\|_n^q. \tag{1.4.4}$$

It follows that for any integer $k = n, n+1, \ldots,$

$$\int_\Omega \left(\frac{|f(x)|}{\|\nabla f\|_n}\right)^{kn/(n-1)} dx \leq [kn/(n-1)]^{1+k}\omega_{n-1}^{-k/(n-1)}\mu_n(\Omega).$$

For $k = 0, 1, \ldots, n-1$, the left-hand side is easily bounded by $C\mu_n(\Omega)$ by Jensen's inequality.

Clearly, the series

$$\sum_0^\infty \frac{\alpha^k k^k}{(k-1)!}\left(\frac{n}{(n-1)\omega_{n-1}^{1/(n-1)}}\right)^k$$

converges if $\alpha > 0$ is small enough, e.g., $\alpha < (n-1)\omega_{n-1}^{1/(n-1)}/en$, and for such α we have

$$\begin{aligned} \int_\Omega \exp\left(\alpha\left(\frac{|f(x)|}{\|\nabla f\|_n}\right)^{n/(n-1)}\right)dx &\leq \sum_0^\infty \frac{\alpha^k}{k!}\int_\Omega \left(\frac{|f(x)|}{\|\nabla f\|_n}\right)^{kn/(n-1)} dx \\ &\leq C\mu_n(\Omega). \end{aligned}$$

This inequality is often attributed to N. Trudinger [86] but it appears in an earlier paper of V. Yudovich [90]. See [61]. We record it in the following theorem.

Theorem 1.4.4 *There exist two constants $C_n, c_n > 0$ such that, for any $0 < c \leq c_n$, for any bounded set $\Omega \subset \mathbb{R}^n$ and any function $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega \exp\left[c\left(\frac{|f(x)|}{\|\nabla f\|_n}\right)^{n/(n-1)}\right]dx \leq C_n\mu_n(\Omega).$$

A more precise result is known [66]. Namely, let $\alpha_n = n\omega_{n-1}^{1/(n-1)}$. Then, for each bounded domain $\Omega \subset \mathbb{R}^n$,

$$\forall\, f \in C_0^\infty(\Omega), \quad \int_\Omega \exp\left(\alpha\left(\frac{|f(x)|}{\|\nabla f\|_n}\right)^{n/(n-1)}\right) dx \le C\mu_n(\Omega)$$

for all $0 < \alpha \le \alpha_n$ whereas, if $\alpha > \alpha_n$,

$$\sup\left\{\int_\Omega \exp\left(\alpha|f(x)|^{n/(n-1)}\right) dx : f \in C_0^\infty(\Omega),\ \ \|\nabla f\|_n = 1\right\} = \infty.$$

This is proved by reducing the problem to a 1-dimensional inequality (thanks to Theorem 1.3.3) and then studying this 1-dimensional inequality. See, e.g., [5].

1.4.3 Higher derivatives

This short section describes Sobolev inequalities involving higher order derivatives. Most of the results easily follow from the first order case, but some additional arguments are needed to obtain optimal statements.

For a function $f \in C_0^\infty(\mathbb{R}^n)$, let

$$\nabla_k f = (\partial_{i_1} \ldots \partial_{i_k} f)_{(i_1,\ldots,i_k)},$$

$$|\nabla_k f| = \sqrt{\sum_{(i_1,\ldots,i_k)} |\partial_{i_1} \ldots \partial_{i_k} f|^2},$$

and

$$\|\nabla_k f\|_p = \left(\int_{\mathbb{R}^n} |\nabla_k f(x)|^p dx\right)^{1/p}$$

with the obvious interpretation if $p = \infty$. By induction based on the case $k = 1$ (which is treated in the previous sections), one easily obtains the following statement.

Theorem 1.4.5 *Fix two integers n, k, and $1 \le p < \infty$.*

- *If $1 \le kp \le n$ and $q = np/(n - kp)$, there exists a constant $C = C(n, k, p)$ such that*

$$\forall\, f \in C_0^\infty(\mathbb{R}^n), \quad \|f\|_q \le C\|\nabla_k f\|_p.$$

- *If $kp = n$, there exist $c = c(n, k)$ and $C' = C(n, k)$ such that for all bounded subsets $\Omega \subset \mathbb{R}^n$,*

$$\forall\, f \in C_0^\infty(\Omega), \quad \int_\Omega \exp\left[c\left(\frac{|f(x)|}{\|\nabla_k f\|_p}\right)^{n/(n-1)}\right] dx \le C'\mu_n(\Omega).$$

- *If $kp > n$, let m be the integer such that $m \leq k - n/p < m + 1$ and set $\alpha = k - n/p - m$. If $\alpha > 0$, there exists $B = B(n, k, p)$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$ and all m-tuples $(i_1, \ldots, i_m)$,*

$$\sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \left\{ \frac{|\partial_{i_1} \ldots \partial_{i_m} f(x) - \partial_{i_1} \ldots \partial_{i_m} f(y)|}{|x - y|^\alpha} \right\} \leq B \|\nabla_k f\|_p.$$

The result given above for $kp = n$, $k \geq 2$ is not optimal. The optimal result is as follows.

Theorem 1.4.6 *If $kp = n$, there exist $c = c(n, k)$ and $C' = C(n, k)$ such that for all bounded subsets $\Omega \subset \mathbb{R}^n$,*

$$\forall\, f \in C_0^\infty(\Omega), \quad \int_\Omega \exp\left[c \left(\frac{|f(x)|}{\|\nabla_k f\|_p} \right)^{n/(n-k)} \right] dx \leq C' \mu_n(\Omega).$$

For the proof, recall the representation formula

$$f(x) = \int \langle P_{n,k}(x - y), \nabla_k f(y) \rangle d(y)$$

where $P_{n,k}$ is homogeneous of degree $-n + k$, $0 < k < n$. More precisely, using multi-indices notation, we have

$$f(x) = \frac{(-1)^k k}{\omega_{n-1}} \sum_{|\alpha| = k} \int_{\mathbb{R}^n} \frac{\theta^\alpha \partial_\alpha f(y)}{\alpha! |x - y|^{n-k}} dy$$

where $\theta = (\theta_i)_1^n = (x - y)/|x - y|$. Starting from this higher order representation formula and proceeding as for Theorem 1.4.4, one proves Theorem 1.4.6.

The case $p = 1$, $k = n$ is a very special case. Indeed, we obviously have

$$|f(x)| \leq \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\partial_1 \ldots \partial_n f(y)| dy.$$

Hence

$$\|f\|_\infty \leq \|\nabla_n f\|_1.$$

This is (at last) the higher-dimensional version of inequality (1.1.1).

The statement given above in the case $kp > n$ is also not satisfactory because the case where n/p is an integer (i.e., $\alpha = 0$) is excluded. For instance the case $p = 2, k = 2, n = 2$ is not treated. When $n/p = \ell$ is an integer, the optimal result is as follows.

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Theorem 1.4.7 *Fix $n < p < \infty$ with $n/p = \ell$ an integer. There exists a constant $C = C(n,k,p)$ such that, for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and any $(\ell-1)$-tuple $(i_1,\dots,i_{\ell-1})$, the function $g = \partial_{i_1}\dots\partial_{i_{\ell-1}}f$ satisfies*

$$\sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \left\{ \frac{|g(x+y)+g(x-y)-2g(x)|}{|y|} \right\} \le C\|\nabla_k f\|_p. \tag{1.4.5}$$

This is proved by the technique discussed above with the help of the following inequality.

Lemma 1.4.8 *Fix $n > 2$. For any smooth function in a ball B, there exists a linear function P_f such that, for all $x \in B$,*

$$|f(x) - P_f(x)| \le C(n) \int_B \frac{|\nabla_2 f(y)|}{|x-y|^{n-2}} dy. \tag{1.4.6}$$

To prove this, for any $x, y \in B$, write $y - x = \rho\theta$ and

$$\begin{aligned} f(y) - f(x) &= \int_0^\rho \partial_s f(x+s\theta)ds \\ &= -\int_0^\rho \partial_s^2 f(x+s\theta)sds + \rho\partial_s f(x+s\theta)\big|_{s=\rho}. \end{aligned}$$

As $\partial_s f(x+s\theta) = \langle\theta, \nabla f(x+s\theta)\rangle$, this yields

$$|f(y) - f(x) - \langle y-x, \nabla f(y)\rangle| \le \int_0^\rho |\nabla_2 f(x+s\theta)|\, s\, ds.$$

Setting

$$F(z) = \begin{cases} |\nabla_2 f(z)| & \text{if } z \in B \\ 0 & \text{otherwise} \end{cases}$$

and integrating in polar coordinates around x in the ball B gives

$$\begin{aligned} |f(x) - P_f(x)| &\le \frac{1}{\Omega_n r^n} \int_0^{2r} \int_{\mathbb{S}^n} \int_0^\infty F(x+r\theta)sds\rho^{n-1}d\rho \\ &\le \frac{2^n}{\Omega_n} \int_B \frac{|\nabla_2 f(y)|}{|x-y|^{n-2}} dy \end{aligned}$$

where

$$P_f(x) = \frac{1}{\Omega_n r^n} \int_B [f(y) - \langle y-x, \nabla f(y)\rangle] dy.$$

With (1.4.6) at hand, the argument used for Theorem 1.4.2 yields

$$\forall x \in B, \quad \forall f \in \mathcal{C}^\infty(B), \quad |f(x) - P_f(x)| \le C(n,p) r^{2-n/p} \|\nabla_2 f\|_p$$

where B is a ball of radius r and $2p > n$. Note that

$$P_f(x+y) + P_f(x-y) - 2P_f(x) = 0$$

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because P_f is linear. Hence, for any $x, y \in B$,

$$|f(x+y) + f(x-y) - 2f(x)| \leq 4C(n,p)r^{2-n/p}\|\nabla_2 f\|_p.$$

For any $f \in C^\infty(\mathbb{R}^n)$ and any x, y such that $|x - y| < r$, we can use this estimate in the ball of center x and radius r to obtain

$$|f(x+y) + f(x-y) - 2f(x)| \leq 4C(n,p)|x-y|^{2-n/p}\|\nabla_2 f\|_p.$$

This obviously gives (1.4.5) in the case where $k = 2$ and n/p is an integer with $0 < n/p < 2$, i.e., $n/p = 1$. The case where $k \geq 3$ follows from this by induction, using the first statement in Theorem 1.4.5.

1.5 Sobolev–Poincaré inequalities on balls

1.5.1 The Neumann and Dirichlet eigenvalues

Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Classically, one considers the following two eigenvalue problems:

(1) The Neumann eigenvalue problem

$$\begin{cases} \Delta u = \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{n}$ is the exterior normal along $\partial\Omega$.

(2) The Dirichlet eigenvalue problem

$$\begin{cases} \Delta u = \lambda u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The boundary condition in (1) (resp. (2)) is known as the Neumann (resp. Dirichlet) boundary condition. Solutions of these problems are pairs (u, λ) with u a smooth function and λ a real. In both cases, integrating $\Delta u = \lambda u$ against u on Ω with the normalization $\int_\Omega u^2 d\mu = 1$ and integrating by parts, we obtain

$$\begin{aligned} \lambda &= \lambda \int_\Omega u^2 d\mu = \int_\Omega u \Delta u d\mu \\ &= \int_\Omega |\nabla u|^2 d\mu + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} u d\mu_{n-1} \\ &= \int_\Omega |\nabla u|^2 d\mu \geq 0. \end{aligned}$$

For the Neumann problem, $u \equiv 1$, $\lambda = 0$, is an obvious solution. In view of this, it is natural to set

$$\lambda^N(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu} : u \neq 0,\ \int_\Omega u d\mu = 0,\ u \in \mathcal{C}^\infty(\Omega)\right\}$$

and

$$\lambda^D(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu} : u \neq 0,\ u \in \mathcal{C}_0^\infty(\Omega)\right\}.$$

Indeed, one can show that $\lambda^N(\Omega)$ (resp. λ^D) is the smallest real λ such that (1) (resp. (2)) admits a non-constant solution. Observe that, by definition, the inequality $\lambda^N \geq c$ (resp. $\lambda^D \geq c$) is equivalent to saying that, for all $u \in \mathcal{C}^\infty(\Omega)$ (resp. $u \in \mathcal{C}_0^\infty(\Omega)$),

$$\int_\Omega u^2 d\mu \leq \frac{1}{c}\int_\Omega |\nabla u|^2 d\mu.$$

This type of inequality is known as a (L^2) Poincaré inequality.

1.5.2 Poincaré inequalities on Euclidean balls

There are two sets of Poincaré inequalities on Euclidean balls, corresponding when $p = 2$ to the Dirichlet and Neumann eigenvalue problems.

Theorem 1.5.1 *Let $B = B(z, r)$ be a Euclidean ball of radius r and center z in $\mathbb{R}^n$. For any $1 \leq p < \infty$, we have*

$$\forall f \in \mathcal{C}_0^\infty(B),\ \left(\int_B |f|^p d\mu\right)^{1/p} \leq r\left(\int_B |\nabla f|^p d\mu\right)^{1/p} \tag{1.5.1}$$

and also

$$\forall f \in \mathcal{C}^\infty(B),\ \left(\int_B |f - f_B|^p d\mu\right)^{1/p} \leq 2^n r\left(\int_B |\nabla f|^p d\mu\right)^{1/p} \tag{1.5.2}$$

where $f_B = \mu(B)^{-1}\int_B f d\mu$ is the mean of f over B.

Clearly, we can assume that $B = \mathbb{B}$ is the unit ball. For the proof of (1.5.1), we use (1.2.2), that is

$$|f(x)| \leq \frac{1}{\omega_{n-1}}\int \frac{|\nabla f(y)|}{|y - x|^{n-1}} dy.$$

This yields

$$\int_{\mathbb{B}} |f| d\mu \leq \frac{1}{\omega_{n-1}}\int_{\mathbb{B}} |\nabla f(y)|\left(\int_{\mathbb{B}} \frac{dx}{|x - y|^{n-1}}\right) dy.$$

As

$$\int_{\mathbb{B}} \frac{dx}{|x-y|^{n-1}} \leq \int_{\mathbb{B}} \frac{dx}{|x|^{n-1}} = \omega_{n-1},$$

we get

$$\int_{\mathbb{B}} |f| d\mu \leq \int_{\mathbb{B}} |\nabla f| d\mu,$$

which is the case $p = 1$ of (1.5.1). The case $p > 1$ can be obtained in a number of ways. We will use Jensen's inequality for the measure $c(x)^{-1}|x - y|^{-n+1}\mathbf{1}_{\mathbb{B}}(y)dy$ where $x \in \mathbb{B}$ is fixed and $c(x) = \int_{\mathbb{B}} |x-y|^{1-n}dy$. Observe that $c(x) \leq \omega_{n-1}$. By Jensen's inequality,

$$|f(x)|^p \leq \frac{c(x)^{p-1}}{\omega_{n-1}^p} \int_{\mathbb{B}} \frac{|\nabla f(y)|^p}{|y-x|^{n-1}} dy \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{B}} \frac{|\nabla f(y)|^p}{|y-x|^{n-1}} dy.$$

Integrating over $x \in \mathbb{B}$ as in the case $p = 1$ gives the desired result. The proof of (1.5.2) is similar but uses Lemma 1.4.3 instead of (1.2.2). Let us note that the constants in (1.5.1), (1.5.2) are not optimal.

1.5.3 Sobolev–Poincaré inequalities

For any open set Ω and $1 \leq p \leq \infty$, we set

$$\|f\|_{p,\Omega} = \left(\int_{\Omega} |f|^p d\mu \right)^{1/p}.$$

With this notation the inequalities of Theorem 1.5.1 read

$$\forall\, f \in \mathcal{C}_0^\infty(\mathbb{B}), \quad \|f\|_{p,B} \leq r\|\nabla f\|_{p,B},$$

$$\forall\, f \in \mathcal{C}^\infty(B), \quad \|f - f_B\|_{p,B} \leq 2^n r\|\nabla f\|_{p,B}.$$

Sobolev inequalities localized in a given ball can be obtained by using the representation formula (1.2.1) and Lemma 1.4.3. Indeed, the kernel

$$K(x,y) = \mathbf{1}_B(x)\mathbf{1}_B(y)\frac{1}{|x-y|^{n-1}}$$

is a kernel of weak type $(n-1)$ as defined in Section 1.2.2. Thus, the proof of Theorem 1.2.1 given in Section 1.2.3 applies here and yields the following inequalities. Note that the case $s = p$ in Theorem 1.5.2 below reduces to Theorem 1.5.1 and only the case $s = q$ needs to be proved. Note also that it suffices to treat the case where B is the unit ball.

Theorem 1.5.2 *Fix $1 \leq p < n$ and set $q = np/(n-p)$. There exists a constant $C = C(n,p)$ such that for any smooth function with compact support in a ball $B \subset \mathbb{R}^n$ of radius $r > 0$, i.e., $f \in \mathcal{C}_0^\infty(B)$, we have*

$$\|f\|_{s,B} \leq C\, r^{1+n(1/s-1/p)}\, \|\nabla f\|_{p,B} \tag{1.5.3}$$

for all $1 \le s \le q$.

When f *is a smooth function on the ball* B *which does not necessarily vanish on the boundary, i.e.,* $f \in \mathcal{C}^\infty(B)$, *we have instead*

$$\|f - f_B\|_{s,B} \le C\, r^{1+n(1/s-1/p)}\, \|\nabla f\|_{p,B} \tag{1.5.4}$$

for all $1 \le s \le q$. *Here,* f_B *is the mean of* f *over the ball* B.

It is natural to wonder whether the ball B can be replaced by some more general bounded domain. Let Ω be a bounded domain in $\mathbb{R}^n$. On the one hand, there is no difficulty with the case of functions with compact support in Ω because any $f \in \mathcal{C}_0^\infty(\Omega)$ can be extended to a function in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ by setting $f = 0$ outside Ω. Thus, Jensen's inequality and the usual Sobolev inequality in $\mathbb{R}^n$, i.e., $\forall\, f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\|f\|_q \le C\|\nabla f\|_p$ with $q = np/(n-p)$, yield

$$\forall\, f \in \mathcal{C}_0^\infty(\Omega),\quad \|f\|_{s,\Omega} \le C\, \mu(\Omega)^{1/s-1/q}\, \|\nabla f\|_{p,\Omega} \tag{1.5.5}$$

for all $1 \le p < n$, $q = np/(n-p)$ and $1 \le s \le q$. If Ω has diameter d, we can bound $\mu(\Omega)$ in this inequality by $\Omega_n d^n$.

On the other hand, consider the problem of whether or not the inequality

$$\forall\, f \in \mathcal{C}^\infty(\Omega),\quad \|f - f_\Omega\|_{p,\Omega} \le C(p,\Omega)\|\nabla f\|_{p,\Omega} \tag{1.5.6}$$

holds true for some finite constant $C(p,\Omega)$. It turns out that the answer depends in a subtle way on the regularity of the boundary of Ω. Inequality (1.5.6) does hold on domains with smooth (or even Lipschitz) boundary but there are domains on which it does not hold. The same is true for Sobolev–Poincaré inequalities of type (1.5.4) with $s > p$. The book [61] gives an account of what is known and references to the literature.

Finally, note that it is easy to treat the case of bounded convex domains by adapting the argument given above in the case of Euclidean balls. All the results described above hold for bounded convex domains with the radius r of B replaced by the diameter d of the domain.