8

ORLICZ SPACES AND ORLICZ-SOBOLEV SPACES

Introduction

8.1 In this final chapter we present results on generalizations of Lebesgue and Sobolev spaces in which the role usually played by the convex function t^p is assumed by a more general convex function $A(t)$. The spaces $L_A(\Omega)$, called *Orlicz spaces*, are studied in depth in the monograph by Krasnosel'skii and Rutickii [KR] and also in the doctoral thesis by Luxemburg [Lu], to either of which the reader is referred for a more complete development of the material outlined below. The former also contains examples of applications of Orlicz spaces to certain problems in nonlinear analysis.

It is of some interest to note that a gap in the Sobolev imbedding theorem (Theorem 4.12) can be filled by an Orlicz space. Specifically, if $mp = n$ and $p > 1$, then for suitably regular Ω we have

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad p \le q < \infty, \qquad \text{but} \qquad W^{m,p}(\Omega) \not\to L^\infty(\Omega);$$

there is no *best*, (i.e., smallest) target L^p-space for the imbedding. In Theorem 8.27 below we will provide an optimal imbedding of $W^{m,p}(\Omega)$ into a certain Orlicz space. This result is due to Trudinger [Td], with precedents in [Ju] and [Pz]. There has been much further work, for instance [Ms] and [Ad1].

Following [KR], we use the class of "N-functions" as defining functions A for Orlicz spaces. This class is not as wide as the class of Young's functions used in

[Lu]; for instance, it excludes $L^1(\Omega)$ and $L^\infty(\Omega)$ from the class of Orlicz spaces. However, N-functions are simpler to deal with, and are adequate for our purposes. Only once, in Theorem 8.39 below, do we need to refer to a more general Young's function.

If the role played by $L^p(\Omega)$ in the definition of the Sobolev space $W^{m,p}(\Omega)$ is assigned instead to the Orlicz space $L_A(\Omega)$, the resulting space is denoted by $W^m L_A(\Omega)$ and is called an *Orlicz-Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces by Donaldson and Trudinger [DT]. We present some of these results in this chapter.

N-Functions

8.2 (Definition of an N-Function) Let a be a real-valued function defined on $[0, \infty)$ and having the following properties:

(a) $a(0) = 0$, $a(t) > 0$ if $t > 0$, $\lim_{t\to\infty} a(t) = \infty$;

(b) a is nondecreasing, that is, $s > t$ implies $a(s) \geq a(t)$;

(c) a is right continuous, that is, if $t \geq 0$, then $\lim_{s\to t+} a(s) = a(t)$.

Then the real-valued function A defined on $[0, \infty)$ by

$$A(t) = \int_0^t a(\tau)\, d\tau \tag{1}$$

is called an *N-function.*

It is not difficult to verify that any such N-function A has the following properties:

(i) A is continuous on $[0, \infty)$;

(ii) A is strictly increasing that is, $s > t \geq 0$ implies $A(s) > A(t)$;

(iii) A is convex, that is, if $s, t \geq 0$ and $o < \lambda < 1$, then

$$A\big(\lambda s + (1-\lambda)t\big) \leq \lambda A(s) + (1-\lambda)A(t);$$

(iv) $\lim_{t\to 0} A(t)/t = 0$, and $\lim_{t\to\infty} A(t)/t = \infty$;

(v) if $s > t > 0$, then $A(s)/s > A(t)/t$.

Properties (i), (iii), and (iv) could have been used to define an N-function since they imply the existence of a representation of A in the form (1) with a having the properties (a)–(c).

The following are examples of N-functions:

$$\begin{aligned}
&A(t) = t^p, && 1 < p < \infty,\\
&A(t) = e^t - t - 1, &&\\
&A(t) = e^{(t^p)} - 1, && 1 < p < \infty,\\
&A(t) = (1+t)\log(1+t) - t. &&
\end{aligned}$$

Evidently, $A(t)$ is represented by the area under the graph $\sigma = a(\tau)$ from $\tau = 0$ to $\tau = t$ as shown in Figure 9. Rectilinear segments in the graph of A correspond to intervals on which a is constant, and angular points on the graph of A correspond to discontinuities (i.e., vertical jumps) in the graph of a.

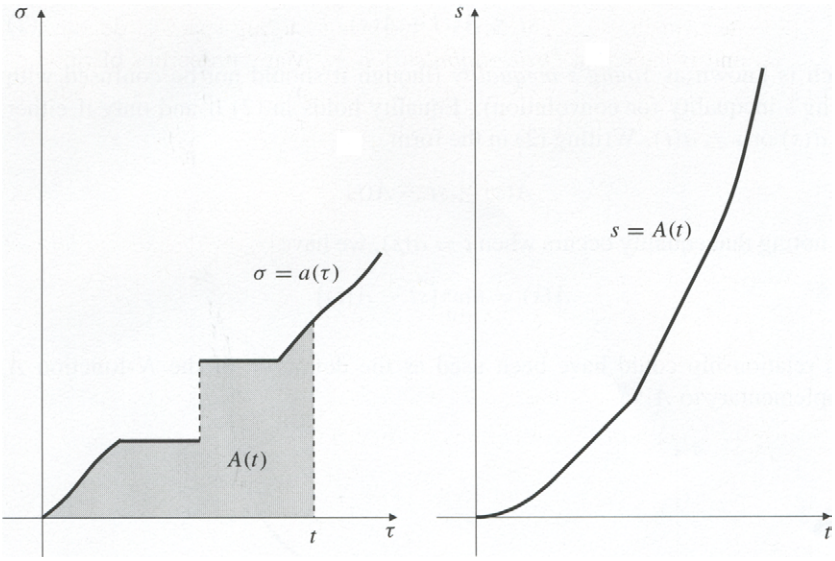

Fig. 9

8.3 (Complementary N-Functions) Given a function a satisfying conditions (a)–(c) of the previous Paragraph, we define

$$\tilde{a}(s) = \sup_{a(t)\leq s} t.$$

It is readily checked that the function $\tilde{a}$ so defined also satisfies (a)–(c) and that a can be recovered from $\tilde{a}$ via

$$a(t) = \sup_{\tilde{a}(s)\leq t} s.$$

If a is strictly increasing then $\tilde{a} = a^{-1}$. The N-functions A and $\tilde{A}$ given by

$$A(t) = \int_0^t a(\tau)\,d\tau, \qquad \tilde{A}(s) = \int_0^s \tilde{a}(\sigma)\,d\sigma$$

are said to be *complementary*; each is the *complement* of the other. Examples of such complementary pairs are:

$$A(t) = \frac{t^p}{p}, \qquad \tilde{A}(s) = \frac{s^{p'}}{p'}, \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and

$$A(t) = e^t - t - 1, \qquad \tilde{A}(s) = (1+s)\log(1+s) - s.$$

$\tilde{A}(s)$ is represented by the area to the left of the graph $\sigma = a(\tau)$ (or, more correctly, $\tau = \tilde{a}(\sigma)$) from $\sigma = 0$ to $\sigma = s$ as shown in Figure 10. Evidently, we have

$$st \le A(t) + \tilde{A}(s), \tag{2}$$

which is known as *Young's inequality* (though it should not be confused with Young's inequality for convolution). Equality holds in (2) if and only if either $t = \tilde{a}(s)$ or $s = a(t)$. Writing (2) in the form

$$\tilde{A}(s) \ge st - A(t)$$

and noting that equality occurs when $t = \tilde{a}(s)$, we have

$$\tilde{A}(s) = \max_{t \ge 0}\big(st - A(t)\big).$$

This relationship could have been used as the definition of the N-function $\tilde{A}$ complementary to A.

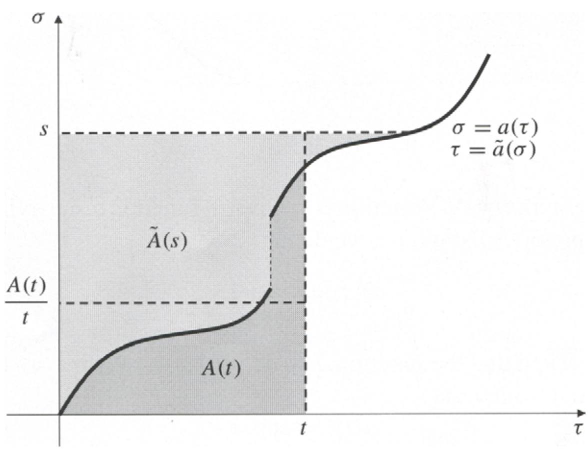

Fig. 10

Since A and $\tilde{A}$ are strictly increasing, they have inverses and (2) implies that for every $t \ge 0$

$$A^{-1}(t)\tilde{A}^{-1}(t) \le A\big(A^{-1}(t)\big) + \tilde{A}\big(\tilde{A}^{-1}(t)\big) = 2t.$$

On the other hand, $A(t) \le ta(t)$, so that, considering Figure 10 again, we have for every $t > 0$,

$$\tilde{A}\left(\frac{A(t)}{t}\right) < \frac{A(t)}{t}\, t = A(t). \tag{3}$$

Replacing $A(t)$ by t in inequality (3), we obtain

$$\tilde{A}\left(\frac{t}{A^{-1}(t)}\right) < t.$$

Therefore, for any $t > 0$,

$$t < A^{-1}(t)\tilde{A}^{-1}(t) \le 2t. \tag{4}$$

8.4 (Dominance and Equivalence of N-Functions) We shall require certain partial ordering relationships among N-functions. If A and B are two N-functions, we say that *B dominates A globally* if there exists a positive constant k such that

$$A(t) \le B(kt) \tag{5}$$

holds for all $t \ge 0$. Similarly, *B dominates A near infinity* if there exist positive constants t_0 and k such that (5) holds for all $t \ge t_0$. The two N-functions A and B are *equivalent globally* (or *near infinity*) if each dominates the other globally (or near infinity). Thus A and B are equivalent near infinity if there exist positive constants t_0, k_1, and k_2, such that if $t \ge t_0$, then $B(k_1 t) \le A(t) \le B(k_2 t)$. Such will certainly be the case if

$$0 < \lim_{t\to\infty} \frac{B(t)}{A(t)} < \infty.$$

If A and B have respective complementary N-functions $\tilde{A}$ and $\tilde{B}$, then B dominates A globally (or near infinity) if and only if $\tilde{A}$ dominates $\tilde{B}$ globally (or near infinity). Similarly, A and B are equivalent if and only if $\tilde{A}$ and $\tilde{B}$ are.

8.5 If B dominates A near infinity and A and B are not equivalent near infinity, then we say that *A increases essentially more slowly than B* near infinity. This is the case if and only if for every positive constant k

$$\lim_{t\to\infty} \frac{A(kt)}{B(t)} = 0.$$

The reader may verify that this limit is equivalent to

$$\lim_{t\to\infty} \frac{B^{-1}(t)}{A^{-1}(t)} = 0.$$

Let $1 < p < \infty$ and let A_p denote the N-function

$$A_p(t) = \frac{t^p}{p}, \qquad 0 \le t < \infty.$$

If $1 < p < q < \infty$, then A_p increases essentially more slowly than A_q near infinity. However, A_q does not dominate A_p globally.

8.6 (The Δ_2 Condition) An N-function is said to satisfy a *global* Δ_2-*condition* if there exists a positive constant k such that for every $t \geq 0$,

$$A(2t) \leq kA(t). \tag{6}$$

This is the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t \geq 0$,

$$A(rt) \leq kA(t). \tag{7}$$

Similarly, A satisfies a Δ_2 *condition near infinity* if there exists $t_0 > 0$ such that (6) (or equivalently (7) with $r > 1$) holds for all $t \geq t_0$. Evidently, t_0 may be replaced with any smaller positive number t_1, for if $t_1 \leq t \leq t_0$, then

$$A(rt) \leq \frac{A(rt_0)}{A(t_1)} A(t).$$

If A satisfies a Δ_2-condition globally (or near infinity) and if B is equivalent to A globally (or near infinity), then B also satisfies such a Δ_2-condition. Clearly the N-function $A_p(t) = t^p/p$, $(1 < p < \infty)$, satisfies a global Δ_2-condition. It can be verified that A satisfies a Δ_2-condition globally (or near infinity) if and only if there exists a positive, finite constant c such that

$$\frac{1}{c}\, t\, a(t) \leq A(t) \leq t\, a(t)$$

holds for all $t \geq 0$ (or for all $t \geq t_0 > 0$), where A is given by (1).

Orlicz Spaces

8.7 (The Orlicz Class $K_A(\Omega)$) Let Ω be a domain in $\mathbb{R}^n$ and let A be an N-function. The *Orlicz class* $K_A(\Omega)$ is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions u defined on Ω that satisfy

$$\int_\Omega A(|u(x)|)\, dx < \infty.$$

Since A is convex $K_A(\Omega)$ is always a convex set of functions but it may not be a vector space; for instance, there may exist $u \in K_A(\Omega)$ and $\lambda > 0$ such that $\lambda u \notin K_A(\Omega)$.

We say that the pair (A, Ω) is Δ*-regular* if *either*

(a) A satisfies a global Δ_2-condition, *or*

(b) A satisfies a Δ_2-condition near infinity and Ω has finite volume.

8.8 LEMMA $K_A(\Omega)$ is a vector space (under pointwise addition and scalar multiplication) if and only if (A, Ω) is Δ-regular.

Proof. Since A is convex we have:

(i) $\lambda u \in K_A(\Omega)$ provided $u \in K_A(\Omega)$ and $|\lambda| \le 1$, and

(ii) if $u \in K_A(\Omega)$ implies that $\lambda u \in K_A(\Omega)$ for every complex λ, then $u, v \in K_A(\Omega)$ implies $u + v \in K_A(\Omega)$.

It follows that $K_A(\Omega)$ is a vector space if and only if $\lambda u \in K_A(\Omega)$ whenever $u \in K_A(\Omega)$ and $|\lambda| > 1$.

If A satisfies a global Δ_2-condition and $|\lambda| > 1$, then we have by (7) for $u \in K_A(\Omega)$

$$\int_\Omega A(|\lambda u(x)|)\, dx \le k(|\lambda|) \int_\Omega A(|u(x)|)\, dx < \infty.$$

Similarly, if A satisfies a Δ_2-condition near infinity and $\mathrm{vol}(\Omega) < \infty$, we have for $u \in K_A(\Omega)$, $|\lambda| > 1$, and some $t_0 > 0$,

$$\begin{aligned}\int_\Omega A(|\lambda u(x)|)\, dx &= \left(\int_{\{x\in\Omega:|u(x)|\ge t_0\}} + \int_{\{x\in\Omega:|u(x)|<t_0\}}\right) A(|\lambda u(x)|)\, dx \\ &\le k(|\lambda|) \int_\Omega A(|\lambda u(x)|)\, dx + A(|\lambda| t_0)\mathrm{vol}(\Omega) < \infty.\end{aligned}$$

In either case $K_A(\Omega)$ is seen to be a vector space.

Now suppose that (A, Ω) is not Δ-regular and, if $\mathrm{vol}(\Omega) < \infty$, that $t_0 > 0$ is given. There exists a sequence $\{t_j\}$ of positive numbers such that

(i) $A(2t_j) \ge 2^j A(t_j)$, and

(ii) $t_j \ge t_0 > 0$ if $\mathrm{vol}(\Omega) < \infty$.

Let $\{\Omega_j\}$ be a sequence of mutually disjoint, measurable subsets of Ω such that

$$\mathrm{vol}(\Omega)_j = \begin{cases} 1/[2^j A(t_j)] & \text{if } \mathrm{vol}(\Omega) = \infty \\ A(t_0)\mathrm{vol}(\Omega)/[2^j A(t_j)] & \text{if } \mathrm{vol}(\Omega) < \infty. \end{cases}$$

Let

$$u(x) = \begin{cases} t_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega - \left(\bigcup_{j=1}^\infty \Omega_j\right). \end{cases}$$

Then

$$\int_\Omega A(|u(x)|)\,dx = \sum_{j=1}^{\infty} A(t_j)\mathrm{vol}(\Omega)_j$$
$$= \begin{cases} 1 & \text{if } \mathrm{vol}(\Omega) = \infty \\ A(t_0)\mathrm{vol}(\Omega) & \text{if } \mathrm{vol}(\Omega) < \infty. \end{cases}$$

But

$$\int_\Omega A(|2u(x)|)\,dx \ge \sum_{j=1}^{\infty} 2^j A(t_j)\mathrm{vol}(\Omega)_j = \infty.$$

Thus $K_A(\Omega)$ is not a vector space. ∎

8.9 (The Orlicz Space $L_A(\Omega)$) The *Orlicz space* $L_A(\Omega)$ is the linear hull of the Orlicz class $K_A(\Omega)$, that is, the smallest vector space (under pointwise addition and scalar multiplication) that contains $K_A(\Omega)$. Evidently, $L_A(\Omega)$ contains all scalar multiples λu of elements $u \in K_A(\Omega)$. Thus $K_A(\Omega) \subset L_A(\Omega)$, these sets being equal if and only if (A, Ω) is Δ-regular.

The reader may verify that the functional

$$\|u\|_A = \|u\|_{A,\Omega} = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right)dx \le 1\right\}$$

is a norm on $L_A(\Omega)$. It is called the Luxemburg norm. The infimum is attained. In fact, if k decreases towards $\|u\|_A$ in the inequality

$$\int_\Omega A\left(\frac{|u(x)|}{k}\right)dx \le 1, \tag{8}$$

we obtain by monotone convergence

$$\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right)dx \le 1. \tag{9}$$

Equality may fail to hold in (9) but if equality holds in (8), then $k = \|u\|_A$.

8.10 THEOREM $L_A(\Omega)$ is a Banach space with respect to the Luxemburg norm.

The completeness proof is similar to that for the L^p spaces given in Theorem 2.16. The details are left to the reader. We remark that if $1 < p < \infty$ and $A_p(t) = t^p/p$, then

$$L^p(\Omega) = L_{A_p}(\Omega) = K_{A_p}(\Omega).$$

Moreover, $\|u\|_{A_p,\Omega} = p^{-1/p}\,\|u\|_{p,\Omega}$.

8.11 (A Generalized Hölder Inequality) If A and $\tilde{A}$ are complementary N-functions, a generalized version of Hölder's inequality

$$\left| \int_\Omega u(x)v(x)\,dx \right| \le 2\,\|u\|_{A,\Omega}\,\|v\|_{\tilde{A},\Omega} \tag{10}$$

can be obtained by applying Young's inequality (2) to $|u(x)|/\,\|u\|_A$ and $|v(x)|/\,\|v\|_{\tilde{A}}$ and integrating over Ω.

The following elementary imbedding theorem is an analog for Orlicz spaces of Theorem 2.14 for L^p spaces.

8.12 THEOREM (An Imbedding Theorem for Orlicz Spaces) The imbedding

$$L_B(\Omega) \to L_A(\Omega)$$

holds if and only if *either*

(a) B dominates A globally, *or*

(b) B dominates A near infinity and $\mathrm{vol}(\Omega) < \infty$.

Proof. If $A(t) \le B(kt)$ for all $t \ge 0$, and if $u \in L_B(\Omega)$, then

$$\int_\Omega A\left(\frac{|u(x)|}{k\,\|u\|_B}\right) dx \le \int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) dx \le 1.$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le k\,\|u\|_B$.

If $\mathrm{vol}(\Omega) < \infty$, let $t_1 = A^{-1}\big((2\mathrm{vol}(\Omega))^{-1}\big)$. If B dominates A near infinity, then there exists positive numbers t_0 and k such that $A(t) \le B(kt)$ for $t \ge t_0$. Evidently, for $t \ge t_1$ we have

$$A(t) \le \max\left\{1, \frac{A(t_0)}{B(kt_1)}\right\} B(kt) = k_1 B(kt).$$

If $u \in L_B(\Omega)$ is given, let $\Omega'(u) = \{x \in \Omega : |u(x)|/[2k_1 k\,\|u\|_B] < t_1\}$ and $\Omega''(u) = \Omega - \Omega'(u)$. Then

$$\begin{aligned}
\int_\Omega A\left(\frac{|u(x)|}{2k_1 k\,\|u\|_B}\right) dx &= \left(\int_{\Omega'(u)} + \int_{\Omega''(u)}\right) A\left(\frac{|u(x)|}{2k_1 k\,\|u\|_B}\right) dx \\
&\le \frac{1}{2\mathrm{vol}(\Omega)} \int_{\Omega'(u)} dx + k_1 \int_{\Omega''(u)} B\left(\frac{|u(x)|}{2k_1\,\|u\|_B}\right) dx \\
&\le \frac{1}{2} + \frac{1}{2}\int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) dx \le 1.
\end{aligned}$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le 2k_1 k\,\|u\|_B$.

Conversely, suppose that neither of the hypotheses (a) and (b) holds. Then there exist numbers $t_j > 0$ such that

$$A(t_j) \geq B(jt_j), \qquad j = 1, 2, \ldots.$$

If $\operatorname{vol}(\Omega) < \infty$, we may assume, in addition, that

$$t_j \geq \frac{1}{j} B^{-1}\left(\frac{1}{\operatorname{vol}(\Omega)}\right).$$

Let Ω_j be a subdomain of Ω having volume $1/B(jt_j)$, and let

$$u_j(x) = \begin{cases} jt_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega - \Omega_j. \end{cases}$$

Then

$$\int_\Omega A\left(\frac{|u_j(x)|}{j}\right) dx \geq \int_\Omega B(|u_j(x)|)\, dx = 1$$

so that $\|u_j\|_B = 1$ but $\|u_j\|_A \geq j$. Thus $L_B(\Omega)$ is not imbedded in $L_A(\Omega)$. ∎

8.13 (Convergence in Mean) A sequence $\{u_j\}$ of functions in $L_A(\Omega)$ is said to *converge in mean* to $u \in L_A(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega A\big(|u_j(x) - u(x)|\big)\, dx = 0.$$

The convexity of A implies that for $0 < \epsilon \leq 1$ we have

$$\int_\Omega A\big(|u_j(x) - u(x)|\big)\, dx \leq \epsilon \int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx$$

from which it follows that norm convergence in $L_A(\Omega)$ implies mean convergence. The converse holds, that is, mean convergence implies norm convergence, if and only if (A, Ω) is Δ-regular. The proof is similar to that of Lemma 8.8 and is left to the reader.

8.14 (The Space $E_A(\Omega)$) Let $E_A(\Omega)$ denote the closure in $L_A(\Omega)$ of the space of functions u which are bounded on Ω and have bounded support in $\overline{\Omega}$. If $u \in K_A(\Omega)$, the sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq j \text{ and } |x| \leq j, \quad x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

converges a.e. on Ω to u. Since $A(|u(x) - u_j(x)|) \leq A(|u(x)|)$, we have by dominated convergence that u_j converges to u in mean in $L_A(\Omega)$. Therefore, if

(A, Ω) is Δ-regular, then $E_A(\Omega) = K_A(\Omega) = L_A(\Omega)$. If (A, Ω) is not Δ-regular, then we have

$$E_A(\Omega) \subset K_A(\Omega) \subsetneqq L_A(\Omega)$$

so that $E_A(\Omega)$ is a proper closed subspace of $L_A(\Omega)$ in this case. To verify the first inclusion above let $u \in E_A(\Omega)$ be given. Let v be a bounded function with bounded support such that $0 < \|u - v\|_A < 1/2$. Using the convexity of A and (9), we obtain

$$\frac{1}{\|2u - 2v\|_A} \int_\Omega A\big(|2u(x) - 2v(x)|\big)\,dx \le \int_\Omega A\left(\frac{|2u(x) - 2v(x)|}{\|2u - 2v\|_A}\right) dx \le 1,$$

whence $2u - 2v \in K_A(\Omega)$. Since $2v$ clearly belongs to $K_A(\Omega)$ and $K_A(\Omega)$ is convex, we have $u = (1/2)(2u - 2v) + (1/2)(2v) \in K_A(\Omega)$.

8.15 LEMMA $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$.

Proof. Let S be a linear subspace of $K_A(\Omega)$ and let $u \in S$. Then $\lambda u \in K_A(\Omega)$ for every scalar λ. If $\epsilon > 0$ and u_j is given by (11), then u_j/ϵ converges to u/ϵ in mean in $L_A(\Omega)$ as noted in Paragraph 8.14. Hence, for sufficiently large values of j we have

$$\int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx \le 1$$

and therefore u_j converges to u in norm in $L_A(\Omega)$. Thus $S \subset E_A(\Omega)$. ∎

8.16 THEOREM Let Ω have finite volume, and suppose that the N-function A increases essentially more slowly than the N-function B near infinity. Then

$$L_B(\Omega) \to E_A(\Omega).$$

Proof. Since $L_B(\Omega) \to L_A(\Omega)$ is already established we need only show that $L_B(\Omega) \subset E_A(\Omega)$. Since $L_B(\Omega)$ is the linear hull of $K_B(\Omega)$ and $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$, it is sufficient to show that $\lambda u \in K_A(\Omega)$ whenever $u \in K_B(\Omega)$ and λ is a scalar. But there exists a positive number t_0 such that $A(|\lambda|t) \le B(t)$ for all $t \ge t_0$. Thus

$$\begin{aligned}\int_\Omega A\big(|\lambda u(x)|\big)\,dx &= \left(\int_{\{x\in\Omega:|u(x)\le t_0\}} + \int_{\{x\in\Omega:|u(x)>t_0\}}\right) A\big(|\lambda u(x)|\big)\,dx \\ &\le A\big(|\lambda|t_0\big)\mathrm{vol}(\Omega) + \int_\Omega B\big(|u(x)|\big)\,dx < \infty\end{aligned}$$

whence the theorem follows. ∎

Duality in Orlicz Spaces

8.17 LEMMA Given $v \in L_{\tilde{A}}(\Omega)$, the linear functional F_v defined by

$$F_v(u) = \int_\Omega u(x)v(x)\,dx \tag{12}$$

belongs to the dual space $[L_A(\Omega)]'$ and its norm $\|F_v\|$ in that space satisfies

$$\|v\|_{\tilde{A}} \le \|F_v\| \le 2\,\|v\|_{\tilde{A}}\,. \tag{13}$$

Proof. It follows by Hölder's inequality (10) that

$$|F_v(u)| \le 2\,\|u\|_A\,\|v\|_{\tilde{A}}$$

holds for all $u \in L_A(\Omega)$, confirming the second inequality in (13).

To establish the other half of (13) we may assume that $v \neq 0$ in $L_{\tilde{A}}(\Omega)$ so that $\|F_v\| = K > 0$. Let

$$u(x) = \begin{cases} \tilde{A}\left(\dfrac{|v(x)|}{K}\right) \Big/ \dfrac{|v(x)|}{K} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0. \end{cases}$$

If $\|u\|_A > 1$, then for $0 < \epsilon \le \|u\|_A - 1$ we have

$$\frac{1}{\|u\|_A - \epsilon}\int_\Omega A\big(|u(x)|\big)\,dx \ge \int_\Omega A\left(\frac{|u(x)|}{\|u\|_A - \epsilon}\right)dx > 1.$$

Letting $\epsilon \to 0+$ we obtain, using (3),

$$\begin{aligned}\|u\|_A &\le \int_\Omega A\big(|u(x)|\big)\,dx = \int_\Omega A\left(\tilde{A}\left(\frac{|v(x)|}{K}\right)\Big/\frac{|v(x)|}{K}\right)dx \\ &< \int_\Omega \tilde{A}\left(\frac{|v(x)|}{K}\right)dx = \frac{1}{\|F_v\|}\int_\Omega u(x)v(x)\,dx \le \|u\|_A\,.\end{aligned}$$

This contradiction shows that $\|u\|_A \le 1$. Now

$$\|F_v\| = \sup_{\|u\|_A \le 1} |F_v(u)| \ge \|F_v\|\left|\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right)dx\right|$$

so that

$$\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right)dx \le 1. \tag{14}$$

Thus, $\|v\|_{\tilde{A}} \le \|F_v\|$. ∎

8.18 REMARK The above lemma also holds when F_v is restricted to act on $E_A(\Omega)$. To obtain the first inequality of (13) in this case take $\|F_u\|$ to be the norm of F_v in $[E_A(\Omega)]'$ and replace u in the above proof by $\chi_n u$ where χ_n is the characteristic function of $\Omega_n = \{x \in \Omega : |x| \le n \text{ and } |u(x)| \le n\}$. Evidently, $\chi_n u$ belongs to $E_A(\Omega)$, $\|\chi_n u\|_A \le 1$, and (14) becomes

$$\int_\Omega \chi_n(x)\tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right) dx \le 1.$$

Since $\chi_n(x)$ increases to unity a.e. on Ω as $n \to \infty$, we obtain (14) again, and $\|v\|_{\tilde{A}} \le \|F_v\|$ as before.

8.19 THEOREM (The Dual of $E_A(\Omega)$) The dual space of $E_A(\Omega)$ is isomorphic and homeomorphic to $L_{\tilde{A}}(\Omega)$.

Proof. We have already shown that any element $v \in L_{\tilde{A}}(\Omega)$ determines a bounded linear functional F_v via (12) on $L_A(\Omega)$ and also on $E_A(\Omega)$, and that in either case the norm of this functional differs from $\|v\|_{\tilde{A}}$ by at most a factor of 2. It remains to be shown that every bounded linear functional on $E_A(\Omega)$ is of the form F_v for some such v.

Let $F \in [E_A(\Omega)]'$ be given. We define a complex measure λ on the measurable subsets S of Ω having finite volume by setting

$$\lambda(S) = F(\chi_S),$$

χ_S being the characteristic function of S. Since

$$\int_\Omega A\left(|\chi_S(x)|A^{-1}\left[\frac{1}{\mathrm{vol}(S)}\right]\right) dx = \int_S \frac{1}{\mathrm{vol}(S)}\, dx = 1 \tag{15}$$

we have

$$|\lambda(S)| \le \|F\|\,\|\chi_S\|_A = \frac{\|F\|}{A^{-1}\big(1/\mathrm{vol}(S)\big)}.$$

Since the right side tends to zero with vol(S), the measure λ is absolutely continuous with respect to Lebesgue measure, and so by the Radon-Nikodym Theorem 1.52, λ can be expressed in the form

$$\lambda(S) = \int_S v(x)\, dx,$$

for some v that is integrable on Ω. Thus

$$F(u) = \int_\Omega u(x)v(x)\, dx$$

holds for measurable, simple functions u.

If $u \in E_A(\Omega)$, a sequence of measurable, simple functions u_j can be found that converges a.e. to u and satisfies $|u_j(x)| \le |u(x)|$ on Ω. Since $|u_j(x)v(x)|$ converges a.e. to $|u(x)v(x)|$, Fatou's Lemma 1.49 yields

$$\begin{aligned}\left|\int_\Omega u(x)v(x)\,dx\right| &\le \sup_j \int_\Omega |u_j(x)v(x)|\,dx = \sup_j \left|F\big(|u_j|\operatorname{sgn} v\big)\right| \\ &\le \|F\| \sup_j \left\|u_j\right\|_A \le \|F\|\,\|u\|_A .\end{aligned}$$

It follows that the linear functional

$$F_v(u) = \int_\Omega u(x)v(x)\,dx$$

is bounded on $E_A(\Omega)$ whence $v \in L_{\tilde{A}}(\Omega)$ by Remark 8.18. Since F_v and F assume the same values on the measurable, simple functions, a set that is dense in $E_A(\Omega)$ (see Theorem 8.21 below), they agree on $E_A(\Omega)$ and the theorem is proved. ∎

A simple application of the Hahn-Banach Theorem shows that if $E_A(\Omega)$ is a proper subspace of $L_A(\Omega)$ (that is, if (A, Ω) is *not* Δ-regular), then there exists a bounded linear functional F on $L_A(\Omega)$ that is not given by (12) for any $v \in L_{\tilde{A}}(\Omega)$. As an immediate consequence of this fact we have the following theorem.

8.20 THEOREM (Reflexivity of Orlicz Spaces) $L_A(\Omega)$ is reflexive if and only if both (A, Ω) and $(\tilde{A}, \Omega)$ are Δ-regular.

We omit any discussion of uniform convexity of Orlicz spaces. This subject is treated in Luxemburg's thesis [Lu].

Separability and Compactness Theorems

We next generalize to Orlicz spaces the L^p approximation Theorems 2.19, 2.21, and 2.30.

8.21 THEOREM (Approximation of Functions in $E_A(\Omega)$)

(a) $C_0(\Omega)$ is dense in $E_A(\Omega)$.

(b) $E_A(\Omega)$ is separable.

(c) If J_ϵ is the mollifier of Paragraph 2.28, then for each $u \in E_A(\Omega)$ we have $\lim_{\epsilon\to 0+} J_\epsilon * u = u$ in norm in $E_A(\Omega)$.

(d) $C_0^\infty(\Omega)$ is dense in $E_A(\Omega)$.

Proof. Part (a) is proved by the same method used in Theorem 2.19. In approximating $u \in E_A(\Omega)$ first by simple functions we can assume that u is bounded on

Ω and has bounded support. Then a dominated convergence argument shows that the simple functions converge in norm to u in $E_A(\Omega)$. (The details are left to the reader.)

Part (b) follows from part (a) by the same proof given for Theorem 2.21.

Consider part (c). If $u \in E_A(\Omega)$, let u be extended to $\mathbb{R}^n$ so as to vanish identically outside Ω. Let $v \in L_{\tilde{A}}(\Omega)$. Then

$$\left|\int_\Omega \big(J_\epsilon * u(x) - u(x)\big)v(x)\,dx\right| \le \int_{\mathbb{R}^n} J(y)\,dy \int_\Omega |u(x-\epsilon y) - u(x)||v(x)|\,dx$$
$$\le 2\,\|v\|_{\tilde{A},\Omega} \int_{|y|\le 1} J(y)\,\left\|u_{\epsilon y} - u\right\|_{A,\Omega}\,dy$$

by Hölder's inequality (10), where $u_{\epsilon y}(x) = u(x-\epsilon y)$. Thus by (13) and Theorem 8.19,

$$\|J_\epsilon * u - u\|_{A,\Omega} = \sup_{\|v\|_{\tilde{A},\Omega}\le 1} \left|\int_\Omega \big(J_\epsilon * u(x) - u(x)\big)v(x)\,dx\right|$$
$$\le 2\int_{|y|\le 1} J(y)\,\left\|u_{\epsilon y} - u\right\|_{A,\Omega}\,dy.$$

Given $\delta > 0$ we can find $\tilde{u} \in C_0(\Omega)$ such that $\|u - \tilde{u}\|_{A,\Omega} < \delta/6$. Clearly, $\left\|u_{\epsilon y} - \tilde{u}_{\epsilon y}\right\|_{A,\Omega} < \delta/6$ and for sufficiently small ϵ, $\left\|\tilde{u}_{\epsilon y} - \tilde{u}\right\|_{A,\Omega} < \delta/6$ for every y with $|y| \le 1$. Thus $\|J_\epsilon * u - u\|_{A,\Omega} < \delta$ and (c) is established.

Part (d) is an immediate consequence of parts (a) and (c). ∎

8.22 REMARK $L_A(\Omega)$ is not separable unless $L_A(\Omega) = E_A(\Omega)$, that is, unless (A, Ω) is Δ-regular. A proof of this fact may be found in [KR] (Chapter II, Theorem 10.2).

8.23 (Convergence in Measure) A sequence $\{u_j\}$ of measurable functions is said to *converge in measure* on Ω to the function u provided that for each $\epsilon > 0$ and $\delta > 0$ there exists an integer M such that if $j > M$, then

$$\operatorname{vol}(\{x \in \Omega : |u_j(x) - u(x)| > \epsilon\}) \le \delta.$$

Clearly, in this case there also exists an integer N such that if $j, k \ge N$, then

$$\operatorname{vol}(\{x \in \Omega : |u_j(x) - u_k(x)| \ge \epsilon\}) \le \delta.$$

8.24 THEOREM Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. If the sequence $\{u_j\}$ is bounded in $L_A(\Omega)$ and convergent in measure on Ω, then it is convergent in norm in $L_B(\Omega)$.

Proof. Fix $\epsilon > 0$ and let $v_{j,k}(x) = \big(u_j(x) - u_k(x)\big)/\epsilon$. Clearly $\{v_{j,k}\}$ is bounded in $L_A(\Omega)$; say $\|v_{j,k}\|_{A,\Omega} \le K$. Now there exists a positive number t_0 such that if $t > t_0$, then

$$B(t) \le \frac{1}{4} A\left(\frac{t}{K}\right).$$

Let $\delta = 1/[4B(t_0)]$ and set

$$\Omega_{j,k} = \left\{x \in \Omega \,:\, |v_{j,k}(x)| \ge B^{-1}\left(\frac{1}{2\mathrm{vol}(\Omega)}\right)\right\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N such that if $j, k \ge N$, then $\mathrm{vol}(\Omega)_{j,k} \le \delta$. Set

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} \,:\, |v_{j,k}(x)| \ge t_0\}, \qquad \Omega''_{j,k} = \Omega_{j,k} - \Omega'_{j,k}.$$

For $j, k \ge N$ we have

$$\begin{aligned}\int_\Omega B(|v_{j,k}(x)|)\,dx &= \left(\int_{\Omega-\Omega_{j,k}} + \int_{\Omega'_{j,k}} + \int_{\Omega''_{j,k}}\right) B(|v_{j,k}(x)|)\,dx \\ &\le \frac{\mathrm{vol}(\Omega)}{2\mathrm{vol}(\Omega)} + \frac{1}{4}\int_{\Omega'_{j,k}} A\left(\frac{|v_{j,k}(x)|}{K}\right) dx + \delta B(t_0) \le 1.\end{aligned}$$

Hence $\|u_j - u_k\|_{B,\Omega} \le \epsilon$ and so $\{u_j\}$ converges in $L_B(\Omega)$. ∎

The following theorem will be useful when we wish to extend the Rellich-Kondrachov Theorem 6.3 to imbeddings of Orlicz-Sobolev spaces.

8.25 THEOREM (Precompact Sets in Orlicz Spaces) Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. Then any bounded subset S of $L_A(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_B(\Omega)$.

Proof. Evidently $L_A(\Omega) \to L^1(\Omega)$ since Ω has finite volume. If $\{u_j^*\}$ is a sequence in S, then it has a subsequence $\{u_j\}$ that converges in $L^1(\Omega)$; say $u_j \to u$ in $L^1(\Omega)$. Let $\epsilon, \delta > 0$. Then there exists an integer N such that if $j \ge N$, then $\|u_j - u\|_{1,\Omega} \le \epsilon\delta$. If follows that

$$\mathrm{vol}(\{x \in \Omega \,:\, |u_j(x) - u(x)| \ge \epsilon\}) \le \delta.$$

Thus $\{u_j\}$ converges to u in measure on Ω and hence also in $L_B(\Omega)$. ∎

A Limiting Case of the Sobolev Imbedding Theorem

8.26 If $mp = n$ and $p > 1$, the Sobolev Imbedding Theorem 4.12 provides no best (i.e., smallest) target space into which $W^{m,p}(\Omega)$ can be imbedded. In this case, for suitably regular Ω,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \le q < \infty,$$

but (see Example 4.43)

$$W^{m,p}(\Omega) \not\subset L^\infty(\Omega).$$

If the class of target spaces for the imbedding is enlarged to contain Orlicz spaces, then a best such target space can be found.

We first consider the case of bounded Ω and later extend our consideration to unbounded domains. The following theorem was established by Trudinger [Td]. For other proofs see [B+] and [Ta]; for refinements going beyond Orlicz spaces see [BW] and [MP].

8.27 THEOREM (Trudinger's Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let $mp = n$ and $p > 1$. Set

$$A(t) = \exp\big(t^{n/(n-m)}\big) - 1 = \exp\big(t^{p/(p-1)}\big) - 1. \tag{16}$$

Then there exists the imbedding

$$W^{m,p}(\Omega) \to L_A(\Omega).$$

Proof. If $m > 1$ and $mp = n$, then $W^{m,p}(\Omega) \to W^{1,n}(\Omega)$. Therefore it is sufficient to prove the theorem for $m = 1$, $p = n > 1$. Let $u \in C^1(\Omega) \cap W^{1,n}(\Omega)$ (a set that is dense in $W^{1,n}(\Omega)$) and let $x \in \Omega$. By the special case $m = 1$ of Lemma 4.15 we have, denoting by C a cone contained in Ω, having vertex at x, and congruent to the cone specifying the cone condition for Ω,

$$\begin{aligned}
|u(x)| &\le K_1 \left(\|u\|_{1,C} + \sum_{j=1}^{n} \int_C |D_j u(x)| |x-y|^{1-n}\, dy \right) \\
&\le K_1 \left(\|u\|_{1,\Omega} + \sum_{j=1}^{n} \int_\Omega |D_j u(y)| |x-y|^{1-n}\, dy \right).
\end{aligned}$$

We want to estimate the L^s-norm $\|u\|_s$ for arbitrary $s > 1$. If $v \in L^{s'}(\Omega)$ (where

$(1/s) + (1/s') = 1$), then

$$\begin{aligned}\int_\Omega |u(x)v(x)|\,dx &\le K_1 \left(\|u\|_1 \int_\Omega |v(x)|\,dx + \sum_{j=1}^n \int_\Omega \int_\Omega \frac{|D_j u(y)||v(x)|}{|x-y|^{n-1}}\,dy\,dx \right) \\ &\le K_1 \|u\|_1 \|v\|_{s'} \left(\mathrm{vol}(\Omega)\right)^{1/s} \\ &\quad + K_1 \sum_{j=1}^n \left(\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}\,dy\,dx \right)^{(n-1)/n} \\ &\quad \times \left(\int_\Omega \int_\Omega \frac{|D_j u(y)|^n |v(x)|}{|x-y|^{(n-1)/s}}\,dy\,dx \right)^{1/n}.\end{aligned}$$

By Lemma 4.64, if $0 \le \nu < n$,

$$\int_\Omega \frac{1}{|x-y|^\nu}\,dy \le \frac{K_2}{n-\nu} \left(\mathrm{vol}(\Omega)\right)^{1-(\nu/n)}.$$

Hence

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}\,dy\,dx &\le K_2 s \left(\mathrm{vol}(\Omega)\right)^{1/(sn)} \int_\Omega |v(x)|\,dx \\ &\le K_3 s \left(\mathrm{vol}(\Omega)\right)^{1/(sn)+1/s} \|v\|_{s'}.\end{aligned}$$

Also,

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|D_j u(y)|^n |v(x)|}{|x-y|^{(n-1)/s}}\,dy\,dx &\le \int_\Omega |D_j u(y)|^n\,dy \;\|v\|_{s'} \left(\int_\Omega \frac{1}{|x-y|^{n-1}}\,dx \right)^{1/s} \\ &\le \left\| D_j u \right\|_p^p \|v\|_{s'} \left(K_2 \left(\mathrm{vol}(\Omega)\right)^{1/n} \right)^{1/s} \\ &= K_4 \left\| D_j u \right\|_n^n \|v\|_{s'} \left(\mathrm{vol}(\Omega)\right)^{1/(ns)}.\end{aligned}$$

It follows from these estimates that

$$\begin{aligned}\int_\Omega |u(x)v(x)|\,dx &\le K_1 \|u\|_1 \|v\|_{s'} \left(\mathrm{vol}(\Omega)\right)^{1/s} \\ &\quad + K_4 \sum_{j=1}^n s^{(n-1)/n} \left\| D_j u \right\|_n \|v\|_{s'} \left(\mathrm{vol}(\Omega)\right)^{1/s}.\end{aligned}$$

Since $s^{(n-1)/n} > 1$ and since $W^{1,n}(\Omega) \to L^1(\Omega)$, we now have

$$\|u\|_s = \sup_{v \in L^{s'}(\Omega)} \frac{1}{\|v\|_{s'}} \int_\Omega |u(x)v(x)|\,dx \le K_5 s^{(n-1)/n} \left(\mathrm{vol}(\Omega)\right)^{1/s} \|u\|_{1,n}.$$

The constant K_5 depends only on n and the cone determining the cone condition for Ω. Setting $s = nk/(n-1)$, we obtain

$$\begin{aligned}\int_\Omega |u(x)|^{nk/(n-1)}\,dx &\le \operatorname{vol}(\Omega)\left(\frac{nk}{n-1}\right)^k \left(K_5 \|u\|_{1,n}\right)^{nk/(n-1)}\\ &= \operatorname{vol}(\Omega)\left(\frac{k}{e^{n/(n-1)}}\right)^k \left(eK_5\left[\frac{n}{n-1}\right]^{(n-1)/n} \|u\|_{1,n}\right)^{nk/(n-1)}\end{aligned}$$

Since $e^{n/(n-1)} > e$, the series $\sum_{k=1}^\infty (1/k!)\left(k/e^{n/(n-1)}\right)^k$ converges to a finite sum $K6$. Let $K_7 = \max\{1, K_6\operatorname{vol}(\Omega)\}$ and put

$$K_8 = eK_7K_5\left(\frac{n}{n-1}\right)^{(n-1)/n} \|u\|_{1,n} = K_9 \|u\|_{1,n}\,.$$

Then

$$\int_\Omega \left(\frac{|u(x)|}{K_8}\right)^{nk/(n-1)} dx \le \frac{\operatorname{vol}(\Omega)}{K_7^{nk/(n-1)}}\left(\frac{k}{e^{n/(n-1)}}\right)^k < \frac{\operatorname{vol}(\Omega)}{K_7}\left(\frac{k}{e^{n/(n-1)}}\right)^k$$

since $K_7 \ge 1$ and $nk/(n-1) > 1$. Expanding $A(t)$ in a power series, we now obtain

$$\begin{aligned}\int_\Omega A\left(\frac{|u(x)|}{K_8}\right) dx &= \sum_{k=1}^\infty \frac{1}{k!}\int_\Omega \left(\frac{|u(x)|}{K_8}\right)^{nk/(n-1)} dx\\ &< \frac{\operatorname{vol}(\Omega)}{K_7}\sum_{k=1}^\infty \frac{1}{k!}\left(\frac{k}{e^{n/(n-1)}}\right)^k \le 1.\end{aligned}$$

Hence $u \in L_A(\Omega)$ and

$$\|u\|_A \le K_8 = K_9 \|u\|_{m,p}\,,$$

where K_9 depends on n, $\operatorname{vol}(\Omega)$, and the cone C determining the cone condition for Ω. ∎

8.28 REMARK The imbedding established in Theorem 8.27 is "best possible" in the sense that if there exist an imbedding of the form

$$W_0^{m,p}(\Omega) \to L_B(\Omega),$$

then A dominates B near infinity. A proof of this fact for the case $m = 1$, $p = n > 1$ can be found in [HMT]. The general case is left to the reader as an exercise.

Trudinger's theorem can be generalized to fractional-order spaces. For results in this direction the reader is referred to [Gr] and [P].

Recent efforts have identified non-Orlicz function spaces that are smaller than Trudinger's space into which $W^{m,p}(\Omega)$ can be imbedded in the limiting case $mp = n$. See [MP] in this regard.

8.29 (Extension to Unbounded Domains) If Ω is unbounded and so (satisfying the cone condition) has infinite volume, then the N-function A given by (16) may not decrease rapidly enough at zero to to allow membership in $L_A(\Omega)$ of every $u \in W^{m,p}(\Omega)$ (where $mp = n$). Let k_0 be the smallest integer such that $k_0 \geq p - 1$ and define a modified N-function A_0 by

$$A_0(t) = \exp\bigl(t^{p/(p-1)}\bigr) - \sum_{j=0}^{k_0-1} \frac{1}{j!} t^{jp/(p-1)}.$$

Evidently A_0 is equivalent to A near infinity so for any domain Ω having finite volume, $L_A(\Omega)$ and $L_{A_0}(\Omega)$ coincide and have equivalent norms. However, A_0 enjoys the further property that for $0 < r \leq 1$,

$$A_0(rt) \leq r^{k_0 p/(p-1)} A_0(t) \leq r^p A_0(t). \tag{17}$$

We show that if $mp = n$, $p > 1$, and Ω satisfies the cone condition (but may be unbounded), then

$$W^{m,p}(\Omega) \to L_{A_0}(\Omega).$$

Lemma 4.22 implies that even an unbounded domain Ω satisfying the cone condition can be written as a union of countably many subdomains Ω_j each satisfying the cone condition specified by a cone independent of j, each having volume satisfying

$$0 < K_1 \leq \operatorname{vol}(\Omega_j) \leq K_2$$

with K_1 and K_2 independent of j, and such that any $M + 1$ of the subdomains have empty intersection. It follows from Trudinger's theorem that

$$\|u\|_{A_0,\Omega_j} \leq K_3 \|u\|_{m,p,\Omega_j}$$

with K_3 independent of j. Using (17) with $r = M^{1/p} \|u\|_{m,p,\Omega_j}^{-1} \|u\|_{m,p,\Omega}$ and the finite intersection property of the domains Ω_j, we have

$$\begin{aligned} \int_\Omega A_0\left(\frac{|u(x)|}{M^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx &\leq \sum_{j=1}^{\infty} \int_{\Omega_j} A_0\left(\frac{|u(x)|}{M^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx \\ &\leq \sum_{j=1}^{\infty} \frac{\|u\|_{m,p,\Omega_j}^p}{M \|u\|_{m,p,\Omega}^p} \leq 1. \end{aligned}$$

Hence $\|u\|_{A_0,\Omega} \le M^{1/p} K_3 \|u\|_{m,p,\Omega}$ as required.

We remark that if $k_0 > p - 1$, the above result can be improved slightly by using in place of A_0 the N-function $\max\{t^p, A_0(t)\}$.

Orlicz-Sobolev Spaces

8.30 (Definitions) For a given domain Ω in $\mathbb{R}^n$ and a given N-function A the *Orlicz-Sobolev space* $W^m L_A(\Omega)$ consists of those (equivalence classes of) functions u in $L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ also belong to $L_A(\Omega)$ for all α with $|\alpha| \le m$. The space $W^m E_A(\Omega)$ is defined in an analogous fashion. It may be checked by the same method used for ordinary Sobolev spaces in Chapter 3 that $W^m L_A(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,A} = \|u\|_{m,A,\Omega} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_{A,\Omega},$$

and that $W^m E_A(\Omega)$ is a closed subspace of $W^m L_A(\Omega)$ and hence also a Banach space with the same norm. It should be kept in mind that $W^m E_A(\Omega)$ coincides with $W^m L_A(\Omega)$ if and only if (A, Ω) is Δ-regular. If $1 < p < \infty$ and $A_p(t) = t^p$, then $W^m L_{A_p}(\Omega) = W^m E_{A_p}(\Omega) = W^{m,p}(\Omega)$, the latter space having norm equivalent to those of the former two spaces.

As in the case of ordinary Sobolev spaces, $W_0^m L_A(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^m L_A(\Omega)$. (An analogous definition for $W_0^m E_A(\Omega)$ clearly leads to the same spaces in all cases.)

Many properties of Orlicz-Sobolev spaces are obtained by very straightforward generalization of the proofs of the same properties for ordinary Sobolev spaces. We summarize these in the following theorem and refer the reader to the corresponding results in Chapter 3 for the method of proof. The details can also be found in the article by Donaldson and Trudinger [DT].

8.31 THEOREM (Basic Properties of Orlicz-Sobolev Spaces)

(a) $W^m E_A(\Omega)$ is separable (Theorem 3.6).

(b) If (A, Ω) and $(\tilde{A}, \Omega)$ are Δ-regular, then $W^m E_A(\Omega) = W^m L_A(\Omega)$ is reflexive (Theorem 3.6).

(c) Each element F of the dual space $[W^m E_A(\Omega)]'$ is given by

$$F(u) = \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha u(x)\, v_\alpha(x)\, ds$$

for some functions $v_\alpha \in L_{\tilde{A}}(\Omega)$, $0 \le |\alpha| \le m$ (Theorem 3.9).

(d) $C^\infty(\Omega) \cap W^m E_A(\Omega)$ is dense in $W^m E_A(\Omega)$ (Theorem 3.17).

(e) If Ω satisfies the segment condition, then $C^\infty(\overline{\Omega})$ is dense in $W^m E_A(\Omega)$ (Theorem 3.22).

(f) $C_0^\infty(\mathbb{R}^n)$ is dense in $W^m E_A(\mathbb{R}^n)$. Thus $W_0^m L_A(\mathbb{R}^n) = W^m E_A(\mathbb{R}^n)$ (Theorem 3.22).

Imbedding Theorems for Orlicz-Sobolev Spaces

8.32 Imbedding results analogous to those obtained for the spaces $W^{m,p}(\Omega)$ in Chapters 4 and 6 can be formulated for the Orlicz-Sobolev spaces $W^m L_A(\Omega)$ and $W^m E_A(\Omega)$. The first results in this direction were obtained by Dankert [Da]. A fairly general imbedding theorem along the lines of Theorems 4.12 and 6.3 was presented by Donaldson and Trudinger [DT] and we develop it below.

As was the case with ordinary Sobolev spaces, most of these imbedding results are obtained for domains satisfying the cone condition. Exceptions are those yielding (generalized) Hölder continuity estimates; these require the strong local Lipschitz condition. Some results below are proved only for bounded domains. The method used in extending the analogous results for ordinary Sobolev spaces to unbounded domains does not seem to extend in a straightforward manner when general Orlicz spaces are involved. In this sense the imbedding picture we present here is incomplete. Best possible Orlicz-Sobolev imbeddings, involving a careful study of rearrangements, have been found recently by Cianchi [Ci]. We settle here for results that follow by methods we used earlier for imbeddings of $W^{m,p}(\Omega)$ and for weighted spaces; that is also how we proved Trudinger's theorem.

8.33 (A Sobolev Conjugate) We concern ourselves for the time being with imbeddings of $W^1 L_A(\Omega)$; the imbeddings of $W^m L_A(\Omega)$ are summarized in Theorem 8.43. As usual, Ω is assumed to be a domain in $\mathbb{R}^n$.

Let A be an N-function. We shall always suppose that

$$\int_0^1 \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty, \tag{18}$$

replacing, if necessary, A by another N-function equivalent to A near infinity. (If Ω has finite volume, (18) places no restrictions on A from the point of view of imbedding theory since N-functions equivalent near infinity determine identical Orlicz spaces in that case.)

Suppose also that

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt = \infty. \tag{19}$$

For instance, if $A(t) = A_p(t) = t^p$, $p > 1$, then (19) holds precisely when $p \le n$. With (19) satisfied, we define the *Sobolev conjugate N-function* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau, \qquad t \ge 0. \tag{20}$$

It may readily be checked that A_* is an N-function. If $1 < p < n$, we have, setting $q = np/(n-p)$ (the normal Sobolev conjugate exponent for p),

$$A_{p*}(t) = q^{1-q} p^{-q/p} A_q(t).$$

It is also readily seen for the case $p = n$ that $A_{n*}(t)$ is equivalent near infinity to the N-function $e^t - t - 1$. In [Ci] a different Sobolev conjugate is used; it is equivalent when $p = n$ to the N-function in Trudinger's theorem.

8.34 LEMMA Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and let f satisfy a Lipschitz condition on $\mathbb{R}$. If $g(x) = f(|u(x)|)$, then $g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and

$$D_j g(x) = f'(|u(x)|)\operatorname{sgn} u(x) \cdot D_j u(x).$$

Proof. Since $|u| \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $D_j|u(x)| = \operatorname{sgn} u(x) \cdot D_j u(x)$ it is sufficient to establish the lemma for positive, real-valued functions u so that $g(x) = f\big(u(x)\big)$. Let $\phi \in \mathscr{D}(\Omega)$ and let $\{e_j\}_{j=1}^n$ be the standard basis in $\mathbb{R}^n$. Then

$$\begin{aligned} -\int_\Omega f\big(u(x)\big) D_j\phi(x)\,dx &= -\lim_{h\to 0}\int_\Omega f\big(u(x)\big)\,\frac{\phi(x) - \phi(x - he_j)}{h}\,dx \\ &= \lim_{h\to 0}\int_\Omega \frac{f\big(u(x+he_j)\big) - f\big(u(x)\big)}{h}\,\phi(x)\,dx \\ &= \lim_{h\to 0}\int_\Omega Q(x,h)\,\frac{u(x+he_j) - u(x)}{h}\,\phi(x)\,dx, \end{aligned}$$

where, since f satisfies a Lipschitz condition, for each h the function $Q(\cdot,h)$ is defined a.e. on Ω by

$$Q(x,h) = \begin{cases} \dfrac{f\big(u(x+he_j)\big) - f\big(u(x)\big)}{u(x+he_j) - u(x)} & \text{if } u(x+he_j) \neq u(x) \\ f'\big(u(x)\big) & \text{otherwise.} \end{cases}$$

Moreover, $\|Q(\cdot,h)\|_{\infty,\Omega} \le K$ for some constant K independent of h. A well-known theorem in functional analysis tells us that for some sequence of values of h tending to zero, $Q(\cdot,h)$ converges to $f'\big(u(\cdot)\big)$ in the weak-star topology of $L^\infty(\Omega)$. On the other hand, since $u \in W^{1,1}(\operatorname{supp}(\phi))$ we have

$$\lim_{h\to 0}\frac{u(x+he_j) - u(x)}{h}\,\phi(x) = D_j u(x)\cdot\phi(x)$$

in $L^1(\operatorname{supp}(\phi))$. It follows that

$$-\int_\Omega f\big(u(x)\big) D_j\phi(x)\,dx = \int_\Omega f'\big(u(x)\big)\,D_j u(x)\,\phi(x)\,dx,$$

which implies the lemma. ∎

8.35 THEOREM (Imbedding Into an Orlicz Space) Let Ω be bounded and satisfying the cone condition in $\mathbb{R}^n$. If (18) and (19) hold, then

$$W^1 L_A(\Omega) \to L_{A_*}(\Omega),$$

where A_* is given by (20). Moreover, if B is any N-function increasing essentially more slowly than A_* near infinity, then the imbedding

$$W^1 L_A(\Omega) \to L_B(\Omega)$$

(exists and) is compact.

Proof. The function $s = A_*(t)$ satisfies the differential equation

$$A^{-1}(s)\,\frac{ds}{dt} = s^{(n+1)/n}, \tag{21}$$

and hence, since $s < A^{-1}(s)\tilde{A}^{-1}(s)$ (see (4)),

$$\frac{ds}{dt} \le s^{1/n}\tilde{A}^{-1}(s).$$

Therefore $\sigma(t) = \big(A_*(t)\big)^{(n-1)/n}$ satisfies the differential inequality

$$\frac{d\sigma}{dt} \le \frac{n-1}{n}\tilde{A}^{-1}\Big(\big(\sigma(t)\big)^{n/(n-1)}\Big). \tag{22}$$

Let $u \in W^1 L_A(\Omega)$ and suppose, for the moment, that u is bounded on Ω and is not zero in $L_A(\Omega)$. Then $\int_\Omega A_*(|u(x)|/\lambda)\,dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, and, accordingly, assumes the value unity for some positive value K of λ. Thus

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx = 1, \qquad K = \|u\|_{A_*}. \tag{23}$$

Let $f(x) = \sigma\big(|u(x)|/K\big)$. Evidently, $u \in W^{1,1}(\Omega)$ and σ is Lipschitz on the range of $|u|/K$ so that, by the previous lemma, f belongs to $W^{1,1}(\Omega)$. By Theorem 4.12 we have $W^{1,1}(\Omega) \to L^{n/(n-1)}(\Omega)$ and so

$$\begin{aligned}\|f\|_{n/(n-1)} &\le K_1\left(\sum_{j=1}^n \big\|D_j U\big\|_1 + \|f\|_1\right)\\ &= K_1\left[\sum_{j=1}^n \frac{1}{K}\int_\Omega \sigma'\left(\frac{|u(x)|}{K}\right)|D_j u(x)|\,dx + \int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx\right].\end{aligned} \tag{24}$$

By (23) and Hölder's inequality (10), we obtain

$$1 = \left(\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx\right)^{(n-1)/n} = \|f\|_{n/(n-1)}$$
$$\leq \frac{2K_1}{K}\sum_{j=1}^{n}\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}}\left\|D_j u\right\|_A + K_1\int_\Omega \sigma'\left(\frac{|u(x)|}{K}\right) dx. \tag{25}$$

Making use of (22), we have

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \leq \frac{n-1}{n}\left\|\tilde{A}^{-1}\left(\left(\sigma\left(\frac{|u|}{K}\right)\right)^{n/(n-1)}\right)\right\|_{\tilde{A}}$$
$$= \frac{n-1}{n}\inf\left\{\lambda > 0 : \int_\Omega \tilde{A}\left(\frac{\tilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right) dx \leq 1\right\}.$$

Suppose $\lambda > 1$. Then

$$\int_\Omega \tilde{A}\left(\frac{\tilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right) dx \leq \frac{1}{\lambda}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx = \frac{1}{\lambda} < 1.$$

Thus

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \leq \frac{n-1}{n}. \tag{26}$$

Let $g(t) = A_*(t)/t$ and $h(t) = \sigma(t)/t$. It is readily checked that h is bounded on finite intervals and $\lim_{t\to\infty} g(t)/h(t) = \infty$. Thus there exists a constant t_0 such that $h(t) \leq g(t)/(2K)$ if $t \geq t_0$. Putting $K_2 = K_2 \sup_{0\leq t\leq t_0} h(t)$, we have, for all $t \geq 0$,

$$\sigma(t) \leq \frac{1}{2K_1} A_*(t) + \frac{K_2}{K_1} t.$$

Hence

$$K_1\int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx \leq \frac{1}{2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_2}{K_1}\int_\Omega |u(x)|\, dx$$
$$\leq \frac{1}{2} + \frac{K_3}{K}\|u\|_A, \tag{27}$$

where $K_3 = 2K_2\|1\|_{\tilde{A}} < \infty$ since Ω has finite volume.

Combining (25)–(27), we obtain

$$1 \leq \frac{2K_1}{K}(n-1)\|u\|_{1,A} + \frac{1}{2} + \frac{K_3}{K}\|u\|_A,$$

so that

$$\|u\|_{A_*} = K \le K_4 \|u\|_{1,A}, \tag{28}$$

where K_4 depends only on n, A, $\mathrm{vol}(\Omega)$, and the cone determining the cone condition for Ω.

To extend (28) to arbitrary $u \in W^1 L_A(\Omega)$ let

$$u_k(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \le k \\ k \operatorname{sgn} u(x) & \text{if } |u(x)| > k. \end{cases} \tag{29}$$

Clearly u_k is bounded and it belongs to $W^1 L_A(\Omega)$ by the previous lemma. Moreover, $\|u_k\|_{A_*}$ increases with k but is bounded by $K_4 \|u\|_A$. Therefore, $\lim_{k\to\infty} \|u_k\|_{A_*} = K$ exists and $K \le K_4 \|u\|_{1,A}$. By Fatou's lemma 1.49

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \le \lim_{k\to\infty} \int_\Omega A_*\left(\frac{|u_k(x)|}{K}\right) dx \le 1$$

whence $u \in L_{A_*}(\Omega)$ and (28) holds.

Since Ω has finite volume we have

$$W^1 L_A(\Omega) \to W^{1,1}(\Omega) \to L^1(\Omega),$$

the latter imbedding being compact by Theorem 6.3. A bounded subset of $W^1 L_A(\Omega)$ is bounded in $L_{A_*}(\Omega)$ and precompact in $L^1(\Omega)$, and hence precompact in $L_B(\Omega)$ by Theorem 8.25 whenever B increases essentially more slowly than A_* near infinity. ∎

Theorem 8.35 extends to arbitrary (even unbounded) domains Ω provided W is replaced by W_0.

8.36 THEOREM Let Ω be an arbitrary domain in $\mathbb{R}^n$. If the N function A satisfies (18) and (19), then

$$W_0^m L_A(\Omega) \to L_{A_*}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbedding

$$W_0^m L_A(\Omega) \to L_B(\Omega_0)$$

exists and is compact for any N-function B increasing essentially more slowly that A_* near infinity.

Proof. If $u \in W_0^m L_A(\Omega)$, then the function f in the proof of Theorem 8.35 can be approximated in $W^{1,1}(\Omega)$ by elements of $C_0^\infty(\Omega)$. By Sobolev's inequality

(Theorem 4.31), (24) holds with the term $\|f\|_1$ absent from the right side. Therefore (27) is not needed and the proof does not require that Ω have finite volume. The cone condition is not required either, since Sobolev's inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$. The compactness arguments are similar to those above. ∎

8.37 REMARK Theorem 8.35 is not optimal in the sense that for some A, L_{A_*} is not necessarily the smallest Orlicz space in which $W^1L_A(\Omega)$ can be imbedded. For instance, if $A(t) = A_n(t) = t^n/n$, then, as noted earlier, $A_*(t)$ is equivalent near infinity to $e^t - t - 1$, an N-function that increases essentially more slowly near infinity than does $\exp\big(t^{n/(n-1)}\big) - 1$. Thus Theorem 8.27 gives a sharper result than Theorem 8.35 in this case. In [DT] Donaldson and Trudinger state that Theorem 8.35 can be improved by the methods of Theorem 8.27 provided A dominates near infinity every t^p with $p < n$, but that Theorem 8.35 gives optimal results if for some $p < n$, t^p dominates A near infinity. The former cases are those where [Ci] improves on Theorem 8.35.

There are also some unbounded domains [Ch] for which some Orlicz-Sobolev imbeddings are compact.

The following theorem generalizes (the case $m = 1$ of) the part of Theorem 4.12 dealing with traces on lower dimensional hyperplanes.

8.38 THEOREM (Traces on Planes) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition, and let Ω_k denote the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Let A be an N-function for which (18) and (19) hold, and let A_* be given by (20). Let $1 \le p < n$ where p is such that the function B defined by $B(t) = A(t^{1/p})$ is an N-function. If either $n - p < k \le n$ or $p = 1$ and $n - 1 \le k \le n$, then

$$W^1L_A(\Omega) \to L_{A_*^{k/n}}(\Omega_k),$$

where $A_*^{k/n}(t) = [A_*(t)]^{k/n}$.

Moreover, if $p > 1$ and C is an N-function increasing essentially more slowly than $A_*^{k/n}$ near infinity, then the imbedding

$$W^1L_A(\Omega) \to L_C(\Omega_k) \tag{30}$$

is compact.

Proof. The problem of verifying that $A_*^{k/n}$ is an N-function is left to the reader. Let $u \in W^1L_A(\Omega)$ be a bounded function. Then

$$\int_{\Omega_k} A_*^{k/n}\left(\frac{|u(y)|}{K}\right) dy = 1, \qquad K = \|u\|_{A_*^{k/n},\Omega_k}. \tag{31}$$

We wish to show that

$$K \leq K_1 \|u\|_{1,A,\Omega} \tag{32}$$

with K_1 independent of u. Since this inequality is known to hold for the special case $k = n$ (Theorem 8.35) we may assume without loss of generality that

$$K \geq \|u\|_{A_*,\Omega} = \|u\|_{A_*^{n/n},\Omega_n}. \tag{33}$$

Let $\omega(t) = [A_*(t)]^{1/q}$ where $q = np/(n-p)$. By (case $m = 1$ of) Theorem 4.12 we have

$$\begin{aligned}
\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega_k} &\leq K_2\left[\sum_{j=1}^n \left\|D_j\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega} + \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}\right] \\
&= K_2\left[\frac{1}{K^p}\sum_{j=1}^n \int_\Omega \left|\omega'\left(\frac{|u(x)|}{K}\right)\right|^p |D_j u(x)|^p\,dx \right.\\
&\qquad \left. + \int_\Omega \left|\omega\left(\frac{|u(x)|}{K}\right)\right|^p dx\right].
\end{aligned}$$

Using (31) and noting that $\||v|^p\|_{B,\Omega} \leq \|v\|^p_{A,\Omega}$, we obtain

$$\begin{aligned}
1 &= \left[\int_{\Omega_k}\left(A_*\left(\frac{|u(y)|}{K}\right)\right)^{k/n} dy\right]^{(n-p)/k} = \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega_k} \\
&\leq \frac{2K_2}{K^p}\sum_{j=1}^n \left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\tilde{B},\Omega} \||D_j u|^p\|_{B,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega} \\
&\leq \frac{2nK_2}{K^p}\left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\tilde{B},\Omega} \|u\|^p_{1,A,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}.
\end{aligned} \tag{34}$$

Now $B^{-1}(t) = \left(A^{-1}(t)\right)^p$ and so, using (21) and (4), we have

$$\begin{aligned}
\left(\omega'(t)\right)^p &= \frac{1}{q^p}(A_*(t))^{p(1-q)/q}\left(A_*'(t)\right)^p \\
&= \frac{1}{q^p}A_*(t)\frac{1}{B^{-1}\left(A_*(t)\right)} \leq \frac{1}{q^p}\tilde{B}^{-1}\left(A_*(t)\right).
\end{aligned}$$

It follows by (33) that

$$\int_\Omega \tilde{B}\left(\left(\frac{\omega'(|u(x)|/K)}{1/q}\right)^p\right) dx \leq \int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \leq 1.$$

So

$$\left\| \left(\omega' \left(\frac{|u|}{K} \right) \right)^p \right\|_{\tilde{B},\Omega} \le \frac{1}{q^p}. \tag{35}$$

Now set $g(t) = A_*(t)/t^p$ and $h(t) = \big(\omega(t)/t\big)^p$. It is readily checked that $\lim_{t\to\infty} g(t)/h(t) = \infty$. In order to see that $h(t)$ is bounded near zero let $s = A_*(t)$ and consider

$$\big(h(t)\big)^{1/p} = \frac{\big(A_*(t)\big)^{1/q}}{t} = \frac{s^{(1/p)-(1/n)}}{\displaystyle\int_0^s \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau} \le \frac{s^{1/p}}{\displaystyle\int_0^s \frac{\big(B^{-1}(\tau)\big)^{1/p}}{\tau}}\,d\tau.$$

Since B is an N-function $\lim_{\tau\to\infty} B^{-1}(\tau)/\tau = \infty$. Hence, for sufficiently small values of t we have

$$\big(h(t)\big)^{1/p} \le \frac{s^{1/p}}{\displaystyle\int_0^s \tau^{-1+(1/p)}\,d\tau} = \frac{1}{p}.$$

Therefore, there exists a constant K_3 such that for $t \ge 0$

$$\big(\omega(t)\big)^p \le \frac{1}{2K_2} A_*(t) + K_3 t^p.$$

Using (33) we now obtain

$$\begin{aligned}
\left\| \omega\left(\frac{|u|}{K}\right) \right\|_{p,\Omega}^p &\le \frac{1}{2K_2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_3}{K^p}\int_\Omega |u(x)|^p\,dx \\
&\le \frac{1}{2K_2} + \frac{2K_3}{K^p}\left\| |u|^p \right\|_{B,\Omega} \|1\|_{\tilde{B},\Omega} \\
&\le \frac{1}{2K_2} + \frac{K_4}{K^p}\|u\|_{A,\Omega}^p .
\end{aligned} \tag{36}$$

From (34)–(36) there follows the inequality

$$1 \le \frac{2nK_2}{K^p}\cdot\frac{1}{q^p}\|u\|_{1,A,\Omega}^p + \frac{1}{2} + \frac{K_4 K_2}{K^p}\|u\|_{A,\Omega}^p$$

and hence (32). The extension of (32) to arbitrary $u \in W^1L_A(\Omega)$ now follows as in the proof of Theorem 8.35.

Since $B(t) = A(t^{1/p})$ is an N-function and Ω is bounded, we have

$$W^1L_A(\Omega) \to W^{1,p}(\Omega) \to L^1(\Omega_k),$$

the latter imbedding being compact by Theorem 6.3 provided $p > 1$. The compactness of (30) now follows from Theorem 8.25. ∎

8.39 THEOREM (Imbedding Into a Space of Continuous Functions) Let Ω satisfy the cone condition in $\mathbb{R}^n$. Let A be an N-function for which

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty. \tag{37}$$

Then

$$W^1L_A(\Omega) \to C_B^0(\Omega) = C(\Omega) \cap L^\infty(\Omega).$$

Proof. Let C be a finite cone contained in Ω. We shall show that there exists a constant K_1 depending on n, A, and the dimensions of C such that

$$\|u\|_{\infty,C} \le K_1 \|u\|_{1,A,C}\,. \tag{38}$$

In doing so, we may assume without loss of generality that A satisfies (18), for if not, and if B is an N-function satisfying (18) and equivalent to A near infinity, then $W^1L_A(C) \to W^1L_B(C)$ with imbedding constant depending on A, B, and $\operatorname{vol}(C)$ by Theorem 8.12. Since B satisfies (37) we would have

$$\|u\|_{\infty,C} \le K_2 \|u\|_{1,B,C} \le K_3 \|u\|_{1,A,C}\,.$$

Now Ω is a union of congruent copies of some such finite cone C so that (38) clearly implies

$$\|u\|_{\infty,\Omega} \le K_1 \|u\|_{1,A,\Omega}\,. \tag{39}$$

Since A is assumed to satisfy (18) and (37) we have

$$\int_0^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = K_4 < \infty.$$

Let

$$\Lambda^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau.$$

The Λ^{-1} maps $[0,\infty)$ in a one-to-one manner onto $[0, K_4)$ and has a convex inverse Λ. We extend the domain of definition of Λ to $[0,\infty)$ by defining $\Lambda(t) = \infty$ for $t \ge K_4$. The function Λ is a *Young's function*. (See Luxemburg [Lu] or O'Neill [O].) Although it is not an N-function in the sense defined early in this chapter, nevertheless the Luxemburg norm

$$\|u\|_{\Lambda,C} = \inf\left\{k > 0 : \int_C \Lambda\left(\frac{|u(x)|}{k}\right) dx \le 1\right\}$$

is easily seen to be a norm on $L^\infty(C)$ equivalent to the usual norm; in fact,

$$\frac{1}{K_4}\|u\|_{\infty,C} \le \|u\|_{\Lambda,C} \le \frac{1}{\Lambda^{-1}(1/\mathrm{vol}(C))}\|u\|_{\infty,C}\,.$$

Moreover, $s = \Lambda(t)$ satisfies the differential equation (21), so that the proof of Theorem 8.35 can be carried over in this case to yield, for $u \in W^1L_A(C)$,

$$\|u\|_{\Lambda,C} \le K_5\|u\|_{1,A,C}$$

and inequality (38) follows.

By Theorem 8.31(d) an element $u \in W^m E_A(\Omega)$ can be approximated in norm by functions continuous on Ω. It follows from (39) that u must coincide a.e. on Ω with a continuous function. (See Paragraph 4.16.)

Suppose that an N-function B can be constructed such that the following conditions are satisfied:

(a) $B(t) = A(t)$ near zero.

(b) B increases essentially more slowly than A near infinity.

(c) B satisfies

$$\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt \le 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty.$$

Then, by Theorem 8.16, $u \in W^1L_A(C)$ implies $u \in W^1E_B(C)$ so that we have $W^1L_A(\Omega) \subset C(\Omega)$ as required.

It remains, therefore, to construct an N-function B having the properties (a)–(c). Let $1 < t_1 < t_2 < \cdots$ be such that

$$\int_{t_k}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = \frac{1}{2^{2k}}\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt.$$

We define a sequence $\{s_k\}$ with $s_k \ge t_k$, and the function $B^{-1}(t)$, inductively as follows.

Let $s_1 = t_1$ and $B^{-1}(t) = A^{-1}(t)$ for $0 \le t \le s_1$. Having chosen $s_1, s_2, \dots, s_k$ and defined $B^{-1}(t)$ for $0 \le t \le s_{k-1}$, we continue $B^{-1}(t)$ to the right of s_{k-1} along a straight line with slope $(A^{-1})'(s_{k-1}-)$ (which always exists since A^{-1} is concave) until a point t_k' is reached where $B^{-1}(t_k') = 2^{k-1}A^{-1}(t_k')$. Such t_k' exists because $\lim_{t\to\infty} A^{-1}(t)/t = 0$. If $t_k' \ge t_k$, let $s_k = t_k'$. Otherwise let $s_k = t_k$ and extend B^{-1} from t_k' to s_k by setting $B^{-1}(t) = 2^{k-1}A^{-1}(t)$. Evidently B^{-1} is concave and B is an N-function. Moreover, $B(t) = A(t)$ near zero and since

$$\lim_{t\to\infty} \frac{B^{-1}(t)}{A^{-1}(t)} = \infty,$$

B increases essentially more slowly than A near infinity. Finally,

$$\begin{aligned}\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt &\le \int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty} \int_{s_{k-1}}^{s_k} \frac{2^{k-1}A^{-1}(t)}{t^{(n+1)/n}}\,dt \\ &\le \int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty} 2^{k-1} \int_{t_{k-1}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt \\ &= 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt,\end{aligned}$$

as required. ∎

8.40 THEOREM (Uniform Continuity) Let Ω be a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition. If the N-function A satisfies

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty, \tag{40}$$

then there exists a constant K such that for any $u \in W^1L_A(\Omega)$ (which may be assumed continuous by the previous theorem) and all $x, y \in \Omega$ we have

$$|u(x) - u(y)| \le K\,\|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt. \tag{41}$$

Proof. We establish (41) for the case where Ω is a cube of unit edge; the extension to more general strongly Lipschitz domains can then be carried out just as in the proof of Lemma 4.28. As in that lemma we let Ω_σ denote a parallel subcube of Ω having edge σ and obtain for $x \in \overline{\Omega_\sigma}$

$$\left| u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\,dz \right| \le \frac{\sqrt{n}}{\sigma^{n-1}} \int_0^1 t^{-n}\,dt \int_{\Omega_{t\sigma}} |\operatorname{grad} u|\,dz.$$

By (15), $\|1\|_{\tilde{A},\Omega_{t\sigma}} = 1/\tilde{A}^{-1}(t^{-n}\sigma^{-n})$. It follows by Hölder's inequality and (4) that

$$\begin{aligned}\int_{\Omega_{t\sigma}} |\operatorname{grad} u|\,dz &\le 2\,\|\operatorname{grad} u\|_{A,\Omega_{t\sigma}}\,\|1\|_{\tilde{A},\Omega_{t\sigma}} \\ &\le \frac{2}{\tilde{A}^{-1}(t^{-n}\sigma^{-n})}\,\|u\|_{1,A,\Omega} \\ &\le 2\sigma^n t^n A^{-1}(t^{-n}\sigma^{-n})\,\|u\|_{1,A,\Omega}\,.\end{aligned}$$

Hence

$$\begin{aligned}\left| u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\,dz \right| &\le 2\sqrt{n}\sigma\,\|u\|_{1,A,\Omega} \int_0^1 A^{-1}\left(\frac{1}{t^n\sigma^n}\right)dt \\ &= \frac{2}{\sqrt{n}}\,\|u\|_{1,A,\Omega} \int_{\sigma^{-n}}^{\infty} \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau.\end{aligned}$$

If $x, y \in \Omega$ and $\sigma = |x-y| < 1$, then there exists a subcube Ω_σ with $x, y \in \overline{\Omega_\sigma}$, and it follows from the above inequality applied to both x and y that

$$|u(x) - u(y)| \le \frac{4}{\sqrt{n}} \|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt.$$

For $|x-y| \ge 1$, (41) follows directly from (39) and (40). ∎

8.41 (Generalization of Hölder Continuity) Let M denote the class of positive, continuous, increasing functions of $t > 0$. If $\mu \in M$, the space $C_\mu(\overline{\Omega})$, consisting of those functions $u \in C(\overline{\Omega})$ for which the norm

$$\|u\,;C_\mu(\overline{\Omega})\| = \|u\,;C(\overline{\Omega})\| + \sup_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|u(x)-u(y)|}{\mu(|x-y|)}$$

is finite, is a Banach space under that norm. The theorem above asserts that if (40) holds, then

$$W^1L_A(\Omega) \to C_\mu(\overline{\Omega}), \qquad \text{where} \quad \mu(t) = \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt. \tag{42}$$

If $\mu, \nu \in M$ are such that $\mu/\nu \in M$, then for bounded Ω we have, as in Theorem 1.34, that the imbedding

$$C_\mu(\overline{\Omega}) \to C_\nu(\overline{\Omega})$$

exists and is compact. Hence the imbedding

$$W^1L_A(\Omega) \to C_\nu(\overline{\Omega})$$

is also compact if μ is given as in (42).

8.42 (Generalization to Higher Orders of Smoothness) We now prepare to state the general Orlicz-Sobolev imbedding theorem of Donaldson and Trudinger [DT] by generalizing the framework used for imbeddings of $W^1L_A(\Omega)$ considered above so that we can formulate imbeddings of $W^mL_A(\Omega)$.

For a given N-function A we define a sequence of N-functions $B_0, B_1, B_2, \ldots$ as follows:

$$B_0(t) = A(t)$$

$$(B_k)^{-1}(t) = \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau, \qquad k = 1, 2, \ldots.$$

(Observe that $B_1 = A_*$.) At each stage we assume that

$$\int_0^1 \frac{(B_k)^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty, \tag{43}$$

replacing B_k, if necessary, with another N-function equivalent to it near infinity and satisfying (43).

Let $J = J(A)$ be the smallest nonnegative integer such that

$$\int_1^\infty \frac{(B_J)^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty.$$

Evidently, $J(A) \le n$. If μ belongs to the class M defined in the previous Paragraph, we define the space $C_\mu^m(\overline{\Omega})$ to consist of those functions $u \in C(\overline{\Omega})$ for which $D^\alpha u \in C_\mu(\overline{\Omega})$ whenever $|\alpha| \le m$. The space $C_\mu^m(\overline{\Omega})$ is a Banach space with respect to the norm

$$\left\| u\,;C_\mu^m(\overline{\Omega})\right\| = \max_{|\alpha|\le m} \left\| D^\alpha u\,;C_\mu(\overline{\Omega})\right\|.$$

8.43 THEOREM (A General Orlicz-Sobolev Imbedding Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let A be an N-function.

(a) If $m \le J(A)$, then $W^m L_A(\Omega) \to L_{B_m}(\Omega)$. Moreover, if B is an N-function increasing essentially more slowly than B_m near infinity, then the imbedding $W^m L_A(\Omega) \to L_B(\Omega)$ exists and is compact.

(b) If $m > J(A)$, then $W^m L_A(\Omega) \to C_B^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$.

(c) If $m > J(A)$ and Ω satisfies the strong local Lipschitz condition, then $W^m L_A(\Omega) \to C_\mu^{m-J-1}(\overline{\Omega})$ where

$$\mu(t) = \int_{t^{-n}}^\infty \frac{(B_J)^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau.$$

Moreover, the imbedding $W^m L_A(\Omega) \to C^{m-J-1}(\overline{\Omega})$ is compact and so is $W^m L_A(\Omega) \to C_\nu^{m-J-1}(\overline{\Omega})$ provided $\nu \in M$ and $\mu/\nu \in M$.

8.44 REMARK Theorem 8.43 follows in a straightforward way from the special cases with $m = 1$ provided earlier. Also, if we replace L_A by E_A in part (a) we get $W^m E_A(\Omega) \to E_{B_m}(\Omega)$ since the sequence $\{u_k\}$ defined by (29) converges to u if $u \in W^1 E_A(\Omega)$. Theorem 8.43 holds without any restrictions on Ω if $W^m L_A(\Omega)$ is replaced with $W_0^m L_A(\Omega)$.

8.45 REMARK Since Theorem 8.43 implies that $W^m L_A(\Omega) \to W^1 L_{B_{m-1}}(\Omega)$, we will also have $W^m L_A(\Omega) \to L_{[(B_m)^{k/n}]}(\Omega_k)$, where Ω_k is the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$, provided that (using Theorem 8.38) there exists p satisfying $1 \le p < n$ for which $n - p < k \le n$ (or $n - 1 \le k \le n$ if $p = 1$) and $B(t) = B_m(t^{1/p})$ is an N-function.