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Supplementary Material for "Certifiably Globally Optimal Extrinsic Calibration from Per-Sensor Egomotion"

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Abstract

This document contains theorems and proofs that supplement the Robotics and Automation Letters paper titled "Certifiably Globally Optimal Extrinsic Calibration from Per-Sensor Egomotion" [2].

1 Introduction

We begin by repeating the main optimization problem presented in the main paper:

$$
\begin{aligned}
\underset{\mathbf{R},\mathbf{t}}{\text{minimize}} \quad & J_{\mathbf{R}} + J_{\mathbf{t}}, \\
\text{subject to} \quad & \mathbf{R} \in \mathrm{SO}(3),
\end{aligned}
\tag{1}
$$

where

$$
J_{\mathbf{R}} = \sum_{i=1}^{n} \kappa_i \| \mathbf{R}\mathbf{R}_{a_i} - \mathbf{R}_{b_i}\mathbf{R} \|_F^2
\tag{2}
$$

is the rotation cost and

$$
J_{\mathbf{t}} = \sum_{i=1}^{n} \tau_i \| \mathbf{R}\mathbf{t}_{a_i} + \mathbf{t} - \mathbf{R}_{b_i}\mathbf{t} - \mathbf{t}_{b_i} \|_2^2
\tag{3}
$$

is the translation cost. The theorem we seek to prove in this supplementary document is also repeated from [2] below.

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Theorem 1. *An instance of our extrinsic calibration from egomotion problem has a strictly convex cost only if the measurement data contains rotations of the sensor rig about at least two unique axes.*

To summarize, we will prove in Section 3 that a common observability criterion is a necessary condition for our cost function to be strictly convex. Then, we will use Theorem 2 to demonstrate that our optimization problem has a *zero-duality-gap region* [1] when it is strictly convex. This theoretical result supports the experiments in [2] that demonstrate a globally optimal primal solution can be extracted from the dual solution, even when significant measurement error is present.

2 Zero-Duality-Gap Theorem

In this section we include Theorem 8 from [1], which is used in Section 4 to demonstrate that our problem formulation (when strictly convex) has a tolerance of measurement error that allows the primal solution to be extracted from the dual solution. The statement of the theorem here is identical to the original in [1], with the exception of slight changes to notation.

Theorem 2. *Consider the problem*

$$\underset{\mathbf{y} \in Y}{\text{minimize}} \quad q_{\boldsymbol{\theta}}(\mathbf{y}), \tag{4}$$

where $Y := \{\mathbf{y} \in \mathbb{R}^n : f_1(\mathbf{y}) = \cdots = f_m(\mathbf{y}) = 0\}$, $q_{\boldsymbol{\theta}}, f_i$ *are quadratic, and the dependence on* $\boldsymbol{\theta}$ *is continuous. Let* $\bar{\boldsymbol{\theta}}$ *be such that* $q_{\boldsymbol{\theta}}$ *is strictly convex, and its minimizer* $\bar{\mathbf{y}}$ *satisfies* $\nabla q_{\bar{\boldsymbol{\theta}}}(\bar{\mathbf{y}}) = \mathbf{0}$, *or equivalently,* $\bar{\mathbf{y}}$ *is the unconstrained minimizer of* $q_{\bar{\boldsymbol{\theta}}}(\mathbf{y})$. *If* $ACQ_Y(\bar{\mathbf{y}})$ *holds, then there is zero-duality-gap whenever* $\boldsymbol{\theta}$ *is close enough to* $\bar{\boldsymbol{\theta}}$. *Moreover, the solution to the primal SDP relaxation of (4) recovers the minimizer of the original, unrelaxed problem.*

The interested reader can find the definition of the Abadie Constraint Qualification (ACQ) and a full proof of Theorem 2 in [1].

3 Proof of Theorem 1

Proof. In order to prove that the strict convexity of the objective function in (1) necessitates rotation about at least two axes, we will use the fact that the Hessian of a strictly convex function is positive definite. Since we are dealing with the non-homogenized or affine form of our cost function (as required by Theorem 8 in [1]), the Hessian has the symmetric block-matrix form

$$\mathbf{H} = \sum_{i=1}^{n} \begin{bmatrix} \mathbf{H}_{\mathbf{t}_i}^T \mathbf{H}_{\mathbf{t}_i} & \mathbf{H}_{\mathbf{t}_i}^T \mathbf{H}_{\mathbf{r}_i,\mathbf{t}_i} \\ \mathbf{H}_{\mathbf{r}_i,\mathbf{t}_i}^T \mathbf{H}_{\mathbf{t}_i} & \mathbf{H}_{\mathbf{r}_i}^T \mathbf{H}_{\mathbf{r}_i} + \mathbf{H}_{\mathbf{r}_i,\mathbf{t}_i}^T \mathbf{H}_{\mathbf{r}_i,\mathbf{t}_i} \end{bmatrix}, \tag{5}$$

where

$$\mathbf{H}_{\mathbf{t}_i} = \mathbf{I} - \mathbf{R}_{b_i}, \tag{6}$$

$$\mathbf{H}_{\mathbf{r}_i,\mathbf{t}_i} = \mathbf{t}_{a_i}^T \otimes \mathbf{I}, \tag{7}$$

$$\mathbf{H}_{\mathbf{r}_i} = (\mathbf{R}_{a_i}^T \otimes \mathbf{I}) - (\mathbf{I} \otimes \mathbf{R}_{b_i}), \tag{8}$$

and we have assumed without loss of generality that $\kappa_i = 1, \tau_i = 1 \ \forall i$. Since $\mathbf{H} \succ \mathbf{0}$ it is nonsingular and therefore full rank, and the Guttman rank additivity formula [4] gives

$$\text{rank}(\mathbf{H}) = 12 = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{H}/\mathbf{A}), \tag{9}$$

where we have summarized the expressions forming $\mathbf{H}$ as

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix}. \tag{10}$$

for simplicity. We will proceed to demonstrate that if all $\mathbf{R}_{b_i}$ are rotations about the same axis $\hat{\mathbf{n}}$, then $\text{rank}(\mathbf{A}) = \text{rank}(\sum_{i=1}^n \mathbf{H}_{\mathbf{t}_i}^T \mathbf{H}_{\mathbf{t}_i}) < 3$ and $\mathbf{H}$ cannot in fact be full rank and the cost function is therefore not strictly convex. Without loss of generality, we will assume that $n = 2$ because the addition of positive semidefinite matrices $\mathbf{H}_{\mathbf{t}_i}^T \mathbf{H}_{\mathbf{t}_i}$ to $\hat{\mathbf{n}}$ cannot cause a decrease in rank.

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Suppose $\mathbf{R}_{b_1}$ and $\mathbf{R}_{b_2}$ are both rotation matrices that represent rotations about the same normalized non-zero axis $\hat{\mathbf{n}}$. This means that $\hat{\mathbf{n}}$ is the eigenvector corresponding to the eigenvalue 1 for both matrices [3], or equivalently $(\mathbf{I} - \mathbf{R}_{b_i})\hat{\mathbf{n}} = \mathbf{0}$. Therefore,

$$\mathbf{A}\hat{\mathbf{n}} = (\mathbf{H}_{\mathbf{t}_1}^T \mathbf{H}_{\mathbf{t}_1} + \mathbf{H}_{\mathbf{t}_2}^T \mathbf{H}_{\mathbf{t}_2})\hat{\mathbf{n}}, \tag{11}$$

$$= \mathbf{H}_{\mathbf{t}_1}^T(\mathbf{I} - \mathbf{R}_{b_1})\hat{\mathbf{n}} + \mathbf{H}_{\mathbf{t}_2}^T(\mathbf{I} - \mathbf{R}_{b_2})\hat{\mathbf{n}}, \tag{12}$$
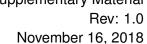
$$= \mathbf{0}. \tag{13}$$

Thus, $\mathbf{A}$ cannot be full rank and neither can $\mathbf{H}$, proving that it is necessary to have rotation about two distinct axes in order for the cost function to be strictly convex. $\qquad\square$

4   Proof of Zero-Duality-Gap Region

This section briefly demonstrates that (1) satisfies the requirements of Theorem 2 in Section 2 when the cost function is strictly convex, therefore proving that there is a zero-duality-gap whenever the measurements comprising the problem data are close to the ideal, noise-free case. The first requirement is that in the noise free case, the optimal solution is an unconstrained minimizer. Since the cost function (3) is a positively weighted sum of norms, it is clearly nonnegative. When there is no noise, the optimal solution is the true extrinsic calibration, meaning the value of the cost function is zero. Therefore, the true calibration is an unconstrained minimizer. The only other requirement is that ACQ holds for the optimal solution [1]. Like Example 7.3 in [1], we use Lemma 21 in [1] to prove that ACQ holds. Lemma 21 requires that the ideal of the polynomial system describing the constraints on the optimization variables (in our case $\mathbf{R}$ and $\mathbf{t}$) is radical. Since our only constraints are from $\mathbf{R} \in \mathrm{SO}(3)$, we conclude as in Example 7.3 that the ideal is radical and ACQ holds.

To summarize, this theorem guarantees the existence of some finite error tolerance ϵ between the noisy measurements and the true underlying values they are measuring such that if the realized error falls within this tolerance, then the duality-based optimization method presented in this paper is globally optimal.

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The experiments in Section V-A.1 of the main paper indicate that this bound is very large, and may be arbitrarily large when redundant constraints have been added.

References

[1] Diego Cifuentes, Sameer Agarwal, Pablo A Parrilo, and Rekha R Thomas. On the local stability of semidefinite relaxations. *arXiv preprint arXiv:1710.04287*, 2017.

[2] Matthew Giamou, Ziye Ma, Valentin Peretroukhin, and Jonathan Kelly. Certifiably globally optimal extrinsic calibration from per-sensor egomotion. *arXiv preprint arXiv:1809.03554*, 2018.

[3] KN Srinivasa Rao, Rao Srinivasa KN, and Srinivasa Rao Koneru. *The rotation and Lorentz groups and their representations for physicists*. John Wiley & Sons, 1988.

[4] Fuzhen Zhang. *The Schur complement and its applications*, volume 4. Springer Science & Business Media, 2006.