Quaternions

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d, \quad a, b, c, d \in \mathbb{R}$$

Abstract Algebra

group

a set G with an operation $*$ which satisfies the axioms:

1. associative

$$(a * b) * c = a * (b * c)$$

2. element *zero*

$$\forall a \in G, \exists e, s.t. a * e = e * a$$

3. element *inverse*

$$\forall a, \exists a^{-1}, s.t. a * a^{-1} = a^{-1} * a = e$$

ring

a set A with operations called addition $+$ and multiplication $*$ which satisfy the following axioms:

1. A with addition alone is an abelian group
2. Multiplication is associative
3. Multiplication is distributive over addition

some terms

commutative ring multiplication is also commutative.

filed a commutative ring with unity in which every nonzero element is invertible.

unity neutral element for multiplication.

division ring non-commutative ring

Optimization

least square optimization

$$\min_{\mathbf{x} \in \chi} \sum_{i=1}^{N} r^2(\mathbf{y}_i, \mathbf{x})$$

where x is the variable to estimate, e.g. the pose in ICP problems,

χ is the domain of x, e.g. $SO(3)$ in ICP problems,

$\mathbf{y}_i$ is the i^{th} measurement,

the function $r(\mathbf{y}_i, \mathbf{x})$ is the residual for the i^{th} measurement.

Generally $(???)$ is difficult to solve globally, due to the nonlinearity of the residual function and non-convexity of χ. Further more, in the presence of outliers, $(???)$ gives false estimation. This call for a robust cost $\rho(\cdot)$

$$\min_{\mathbf{x}\in\chi}\sum_{i=1}^{N}\rho(r(\mathbf{y}_i,\mathbf{x}))$$

Graduated non-convexity (GNC)

Black-Rangarajan duality

Given a robust cost function $\rho(\cdot)$, define $\phi(z) := \rho(\sqrt{z})$. If $\phi(z)$ satisfies $\lim_{z\to 0}\phi'(z) = 1$, $\lim_{z\to\infty}\phi'(z) = 0$, and $\phi''(z) < 0$, then the robust estimation problem $(???)$ is equivalent to

$$\min_{\mathbf{x}\in\chi, w_i\in[0,1]}\sum_{i=1}^{N}[w_i r^2(\mathbf{y}_i,\mathbf{x}) + \Phi_\rho(w_i)]$$

where $w_i \in [0,1]$ are weight associated to measurement y_i, and the function $\Phi_\rho(w_i)$ defines a penalty on the weight w_i, which depends on the choice of robust cost function $\rho(\cdot)$.

GNC

GNC is a popular approach for the optimization of a generic non-convex cost function $\rho(\cdot)$.

The basic idea is to introduce a surrogate cost $\rho_\mu(\cdot)$, governed by a control parameter μ, which adjust the non-convexity.

Example 1 (Geman McClure and GNC)

GM as following, blue is original residual, red is residual reduced. Smaller c indicate larger penalty for outliers. Results in slower converge but more robust.

$$\rho_\mu(r) = \frac{\mu\bar{c}^2 r^2}{\mu\bar{c}^2 + r^2}$$

$$\lim_{\mu\to\infty}\rho_\mu = \rho$$

the surrogate function $\rho_\mu(r)$ satisfies (i) $\rho_\mu(r)$ becomes convex for large μ. (ii) $\mu = 1$ recovers the original form image

Example 2 (TLS)

TLS is defined as

$$\rho(r) = \begin{cases} r^2 & if\, r^2 \in [0, \bar{c}^2] \\ \bar{c}^2 & if\, r^2 \in [\bar{c}^2, +\infty) \end{cases}$$

GNC surrogate is1

$$\rho_\mu(r) = \begin{cases} r^2 & if\, r^2 \in [0, \frac{\mu}{\mu+1}\bar{c}^2] \\ 2\bar{c}\,|r|\sqrt{\mu(\mu+1)} - \mu(\bar{c}^2 + r^2) & if\, r^2 \in [\frac{\mu}{\mu+1}\bar{c}^2, \frac{\mu+1}{\mu}\bar{c}^2] \\ \bar{c}^2 & if\, r^2 \in [\bar{c}^2, +\infty) \end{cases}$$

Implementation

variable update

For inner loop, first fix w_i and minimize (???) with respect to $\mathbf{x}$

$$\mathbf{x}^{(t)} = \arg\min_{\mathbf{x}\in\chi} \sum_{i=1}^{N} w_i^{(t-1)} r^2(\mathbf{y}_i, \mathbf{x})$$

which can be solved globally.

weight update

Then fix x and minimize (???) with respect to w_i.

$$\mathbf{w}^{(t)} = \arg\min_{\mathbf{x}\in\chi, w_i\in[0,1]} \sum_{i=1}^{N} [w_i r^2(\mathbf{y}_i, \mathbf{x}^{(t)}) + \Phi_\rho(w_i)]$$

we will now try to optimize $\Phi_\rho(w_i)$ in closed form.

μ update

GM-GNC

given

$$\Phi_{\rho_\mu}(w_i) = \mu\bar{c}^2(\sqrt{w_i} - 1)^2$$

then (???) can be solved in closed form

$$w_i^{(t)} = \left(\frac{\mu\bar{c}^2}{\hat{r}_i^2 + \mu\bar{c}^2}\right)^2$$

where $\hat{r} = r(\mathbf{y}_i, \mathbf{x}^{(t)})$

TLS-GNC

$$\Phi_{\rho_\mu}(w_i) = \frac{\mu(1 - w_i)}{\mu + w_i}\bar{c}^2$$

closed form as following:

$$w_i^{(t)} = \begin{cases} 0 & if\, \hat{r}^2 \in [\frac{\mu+1}{\mu}\bar{c}^2, +\infty] \\ \frac{\bar{c}}{\hat{r}_i}\sqrt{\mu(\mu+1)} - \mu & if\, r^2 \in [\frac{\mu}{\mu+1}\bar{c}^2, \frac{\mu+1}{\mu}\bar{c}^2] \\ 1 & if\, r^2 \in [0, \frac{\mu}{\mu+1}\bar{c}^2) \end{cases}$$

Residuals

notation

$$\mathbf{b}_i \in B$$
$$\mathbf{a}_i \in A$$

基本形式

$$\mathbf{T} = \arg\min_{\mathbf{T}\in SE(3)} \sum_{i=1}^{m} \|\mathbf{T} \otimes \mathbf{a}_i - \mathbf{b}_i\|_2^2$$

$$(\mathbf{R}, \mathbf{t}) = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathbb{R}^3} \sum_{i=1}^{m} \|\mathbf{R}\mathbf{a}_i + \mathbf{t} - \mathbf{b}_i\|_2^2$$

$$\frac{\partial E}{\partial \mathbf{t}} = \mathbf{0}$$

$$\mathbf{t} = \bar{\mathbf{b}} - \mathbf{R}\bar{\mathbf{a}}, \quad \bar{\mathbf{a}} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{a}_i, \bar{\mathbf{b}} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{b}_i$$

$$\mathbf{R} = \arg\min_{\mathbf{R}\in SO(3)} \sum_{i=1}^{m} \left\|\mathbf{R}\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i\right\|_2^2$$

$$\tilde{\mathbf{b}}_i = \mathbf{b}_i - \bar{\mathbf{b}}, \tilde{\mathbf{a}}_i = \mathbf{a}_i - \bar{\mathbf{a}}$$

$$\left\|\mathbf{R}\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i\right\|_2^2 = \tilde{\mathbf{a}}_i^T\tilde{\mathbf{a}}_i - 2\tilde{\mathbf{b}}_i^T\mathbf{R}\tilde{\mathbf{a}}_i + \tilde{\mathbf{b}}_i^T\tilde{\mathbf{b}}_i$$

$$\mathbf{S} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T, \quad \mathbf{R} = \mathbf{V}\mathbf{U}^T$$

$$\mathbf{S} = \mathbf{X}\mathbf{Y}^T, \quad \mathbf{X}_{i,:} = \tilde{\mathbf{a}}_i, \mathbf{Y}_{i,:} = \tilde{\mathbf{b}}_i$$

generalized distance function

$$r = d_B(\mathbf{T} * \mathbf{a}_i)$$

$d_B(\cdot)$ denote the minimum distance to B.

generalized form

$$\begin{aligned} d_B(\mathbf{x}) &= \min_{\mathbf{b}_i\in B} \|\mathbf{x} - \mathbf{b}_i\|_{\mathbf{C}} \\ &= (\mathbf{x} - \mathbf{b}_i)^T * \mathbf{C} * (\mathbf{x} - \mathbf{b}_i) \end{aligned}$$

point to point

$$d_B(\mathbf{x}) = \min_{\mathbf{b}_i\in B} \|\mathbf{x} - \mathbf{b}_i\|_{\mathbf{I}}$$

point to line

$$d_B(\mathbf{x}) = \min_{\mathbf{b}_i\in B} \|\mathbf{x} - \mathbf{b}_i\|_{(\mathbf{I}-\mathbf{v}\mathbf{v}^T)}$$

where v is the unit direction vector for a line.

point to plane

$$d_B(\mathbf{x}) = \min_{\mathbf{b}_i \in B} \|\mathbf{x} - \mathbf{b}_i\|_{(\mathbf{n}\mathbf{n}^T)}$$

where n is the unit normal vector for a plane.

Gauss-Newton Least square

$$\mathbf{x} = \arg\min_{\mathbf{x}} \sum_i \mathbf{r}_i(\mathbf{x})^T \mathbf{\Omega}_i \mathbf{r}_i(\mathbf{x})$$

$$E(\mathbf{x}) = \sum_i \mathbf{r}_i(\mathbf{x})^T \mathbf{\Omega}_i \mathbf{r}_i(\mathbf{x})$$

$$E(\mathbf{x} + \Delta\mathbf{x}) = \sum_i \mathbf{r}_i(\mathbf{x} + \Delta\mathbf{x})^T \mathbf{\Omega}_i \mathbf{r}_i(\mathbf{x} + \Delta\mathbf{x})$$

$$\mathbf{r}_i(\mathbf{x}^* + \Delta\mathbf{x}) \simeq \underbrace{\mathbf{r}_i(\mathbf{x}^*)}_{\mathbf{r}_i} + \underbrace{\frac{\partial \mathbf{r}_i(\mathbf{x})}{\partial(\mathbf{x})}|_{\mathbf{x}=\mathbf{x}^*}}_{\mathbf{J}_i} \Delta\mathbf{x}$$

$$\mathbf{r}_i(\mathbf{x} + \Delta\mathbf{x})^T \mathbf{\Omega}_i \mathbf{r}_i(\mathbf{x} + \Delta\mathbf{x}) \simeq \Delta\mathbf{x}^T \underbrace{\mathbf{J}_i^T \mathbf{\Omega}_i \mathbf{J}_i}_{\mathbf{H}_i} \Delta\mathbf{x} + 2 \underbrace{\mathbf{J}_i^T \mathbf{\Omega}_i \mathbf{r}_i}_{\mathbf{b}_i^T} \Delta\mathbf{x} + \underbrace{\mathbf{r}_i^T \mathbf{\Omega}_i \mathbf{r}_i}_{c_i}$$

$$\mathbf{x} = \mathbf{x} + \Delta\mathbf{x}$$

$$E(\mathbf{x} + \Delta\mathbf{x}) \simeq \underbrace{\sum_i c_i}_{c} + 2 \underbrace{\sum_i \mathbf{b}_i^T}_{\mathbf{b}^T} \Delta\mathbf{x} + \Delta\mathbf{x}^T \underbrace{\sum_i \mathbf{H}_i}_{\mathbf{H}} \Delta\mathbf{x}$$

$$\Delta\mathbf{x} = \arg\min_{\Delta\mathbf{x}} E(\mathbf{x} + \Delta\mathbf{x})$$

$$\Delta\mathbf{x} \simeq \arg\min_{\Delta\mathbf{x}} \Delta\mathbf{x}^T \mathbf{H} \Delta\mathbf{x} + 2\mathbf{b}^T \Delta\mathbf{x} + c$$

Quadratic formulation

$$d_B^2(\mathbf{T} * \mathbf{a}_i) = (\mathbf{T} * \mathbf{a}_i - \mathbf{b}_i)^T * \mathbf{C} * (\mathbf{T} * \mathbf{a}_i - \mathbf{b}_i)$$

$\mathbf{T} * \mathbf{a}_i$ is in fact linear in the elements of $\mathbf{T}$

$$\mathbf{T} * \mathbf{a}_i = \mathbf{R}\mathbf{a}_i + \mathbf{t} = \underbrace{(\tilde{\mathbf{a}}_i \otimes \mathbf{I}_3)}_{\mathbf{A}_i} vec(\mathbf{T})$$

where $\tilde{\mathbf{a}}_i = [\mathbf{a}_i^T, 1]^T$, $vec(T) = \begin{bmatrix} vec(\mathbf{R}) \\ \mathbf{t} \end{bmatrix}$.

we name $\tau = vec(\mathbf{T})$, the generalized distance is a quadratic function of τ

$$d_B^2(\mathbf{T} * \mathbf{a}_i) = \tilde{\tau}^T \underbrace{\mathbf{N}_i^T \mathbf{C}_i \mathbf{N}_i}_{\tilde{\mathbf{M}}_i} \tilde{\tau}$$

with $\mathbf{N}_i = [\tilde{\mathbf{a}}_i \otimes \mathbf{I}_3 | -\mathbf{b}_i]$ and $\tilde{\tau} = \begin{bmatrix} vec(\mathbf{T} \\ 1 \end{bmatrix}^T$.

compression for the whole point cloud

$$f(\mathbf{T}) = \sum_{i=1}^{m} d_{B_i}^2(\mathbf{T} \otimes \mathbf{a}_i) = \tilde{\tau}^T \underbrace{\left(\sum_{i=1}^{m} \tilde{\mathbf{M}}_i \right)}_{\tilde{\mathbf{M}}} \tilde{\tau}$$

$\mathbf{t}$ can be derived in terms of $\mathbf{R}$

$$\mathbf{t}(\mathbf{R}) = -\tilde{\mathbf{M}}_{\mathbf{t},\mathbf{t}}^{-1} \tilde{\mathbf{M}}_{\mathbf{t},!t} \tilde{\mathbf{r}}, \quad \tilde{\mathbf{r}} = \begin{bmatrix} vec(\mathbf{R}) \\ 1 \end{bmatrix}$$

the marginalized optimization problem is then

$$f = \min_{\mathbf{R} \in SO(3)} \underbrace{\tilde{\mathbf{r}}^T \tilde{\mathbf{Q}} \tilde{\mathbf{r}}}_{q(\tilde{\mathbf{r}})}, \quad \tilde{\mathbf{r}} = \begin{bmatrix} vec(\mathbf{R}) \\ 1 \end{bmatrix}$$

where $\tilde{\mathbf{Q}} = \tilde{\mathbf{M}} / \tilde{\mathbf{M}}_{\mathbf{t},\mathbf{t}}$.

SO(3) constraints

in ???, the $SO(3)$ constraints are as follows

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} : \mathbf{R}^T\mathbf{R} = \mathbf{I}_3, det(\mathbf{R}) = +1\}$$

the orthonormality is quadratic, but the determinant constraint is cubic.

TIMs

translation invariant measurement where $\bar{\mathbf{b}}_{ij} = \mathbf{b}_i - \mathbf{b}_j$, $\bar{\mathbf{a}}_{ij} = \mathbf{a}_i - \mathbf{a}_j$. (Construct with complete graph, can be simplified with max clique)

$$\bar{\mathbf{b}}_{ij} = R\bar{\mathbf{a}}_{ij} + \mathbf{o}_{ij} + \epsilon_{ij}$$

in (???), set $\mathbf{x}$ as $\mathbf{R}$, $\mathbf{y}_i$ as $\bar{\mathbf{b}}_{ij}, \bar{\mathbf{a}}_{ij}$.

$$r(\bar{\mathbf{b}}_{ij}, \bar{\mathbf{a}}_{ij}, \mathbf{R}) = \bar{\mathbf{b}}_{ij} - \mathbf{R} * \bar{\mathbf{a}}_{ij}$$

Robustness Test

$$d(R, R_0) = |arccos((tr(R^T R_0) - 1)/2)|$$